SEMICOSIMPLICIAL LIE ALGEBRAS

DEFORMATION THEORY 2011-12; M. M.

1. SEMICOSIMPLICIAL TOT

Let $\mathbf{\Delta}_{\text{mon}}$ be the category whose objects are the finite ordinal sets $[n] = \{0, 1, \dots, n\}$, $n = 0, 1, \dots$, and whose morphisms are order-preserving injective maps among them. Every morphism in $\mathbf{\Delta}_{\text{mon}}$, different from the identity, is a finite composition of *coface* morphisms:

$$\partial_k \colon [i-1] \to [i], \qquad \partial_k(p) = \begin{cases} p & \text{if } p < k \\ p+1 & \text{if } k \le p \end{cases}, \qquad k = 0, \dots, i.$$

The relations about compositions of them are generated by

$$\partial_l \partial_k = \partial_{k+1} \partial_l \,, \qquad \text{for every } l \le k.$$

A *semicosimplicial* object in a category $\mathbf{C}$ is a covariant functor $A^\Delta \colon \mathbf{\Delta}_{\text{mon}} \to \mathbf{C}$. Equivalently, a semicosimplicial object A^Δ is a diagram in $\mathbf{C}$:

$$A_0 \rightrightarrows A_1 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; A_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots ,$$

where each A_i is in $\mathbf{C}$, and, for each $i > 0$, there are $i+1$ morphisms

$$\partial_k \colon A_{i-1} \to A_i, \qquad k = 0, \dots, i,$$

such that $\partial_l \partial_k = \partial_{k+1} \partial_l$, for any $l \le k$.

Given a semicosimplicial abelian group

$$V^\Delta : \qquad V_0 \rightrightarrows V_1 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; V_2 \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots ,$$

the map

$$\delta \colon V_i \to V_{i+1}, \qquad \delta = \sum_i (-1)^i \partial_i$$

satisfies $\delta^2 = 0$ and then give a complex of vector spaces denoted

$$\operatorname{Tot}(V^\Delta) : \qquad V_0 \xrightarrow{\delta} V_1 \xrightarrow{\delta} V_2 \to \cdots$$

Example 1.1. Let $\mathcal{L}$ be a sheaf of abelian groups on a topological space X, and $\mathcal{U} = \{U_i\}$ an open covering of X; it is naturally defined the semicosimplicial abelian group of Čech alternating cochains:

$$\mathcal{L}(\mathcal{U}) : \qquad \prod_i \mathcal{L}(U_i) \rightrightarrows \prod_{i<j} \mathcal{L}(U_{ij}) \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; \prod_{i<j<k} \mathcal{L}(U_{ijk}) \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots ,$$

where the coface maps $\partial_h \colon \displaystyle\prod_{i_0 < \cdots < i_{k-1}} \mathcal{L}(U_{i_0 \cdots i_{k-1}}) \to \prod_{i_0 < \cdots < i_k} \mathcal{L}(U_{i_0 \cdots i_k})$ are given by

$$\partial_h(x)_{i_0 \dots i_k} = x_{i_0 \dots \widehat{i_h} \dots i_k \,|U_{i_0 \cdots i_k}}, \qquad \text{for } h = 0, \dots, k.$$

Thus $\operatorname{Tot}(\mathcal{L}(\mathcal{U}))$ is the Čech complex of $\mathcal{L}$ in the covering $\mathcal{U}$.

2. Maurer-Cartan and deformation functor

Consider a semicosimplicial Lie algebra $\mathfrak{g}^\Delta$ is a diagram

$$\mathfrak{g}^\Delta: \qquad \mathfrak{g}_0 \rightrightarrows \mathfrak{g}_1 \overrightarrow{\rightrightarrows} \mathfrak{g}_2 \overrightarrow{\overrightarrow{\rightrightarrows}} \cdots,$$

Denoting by $\alpha_i = \partial_i \colon \mathfrak{g}_0 \to \mathfrak{g}_1$, $\beta_j = \partial_j \colon \mathfrak{g}_1 \to \mathfrak{g}_2$ and $\gamma_k = \partial_k \colon \mathfrak{g}_2 \to \mathfrak{g}_3$ the coface maps we have:

$$\beta_1\alpha_0 = \beta_0\alpha_0, \qquad \beta_2\alpha_0 = \beta_0\alpha_1, \qquad \beta_2\alpha_1 = \beta_1\alpha_1, \tag{2.1}$$

$$\gamma_0\beta_0 = \gamma_1\beta_0, \quad \gamma_0\beta_1 = \gamma_2\beta_0, \quad \gamma_0\beta_2 = \gamma_3\beta_0, \quad \gamma_1\beta_1 = \gamma_2\beta_1, \quad \gamma_1\beta_2 = \gamma_3\beta_1, \quad \gamma_2\beta_2 = \gamma_3\beta_2.$$

Define the Maurer-Cartan functor

$$\mathrm{MC}_{\mathfrak{g}^\Delta} \colon \mathbf{Art} \to \mathbf{Set},$$

$$\mathrm{MC}_{\mathfrak{g}^\Delta}(A) = \{e^x \in \exp(\mathfrak{g}_1 \otimes \mathfrak{m}_A) \mid e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = 1\},$$

or equivalently, using the BCH product $\bullet$,

$$\mathrm{MC}_{\mathfrak{g}^\Delta}(A) = \{x \in \mathfrak{g}_1 \otimes \mathfrak{m}_A \mid (\beta_0(x)) \bullet (-\beta_1(x)) \bullet (\beta_2(x)) = 0\}.$$

Lemma 2.1. *The action*

$$\exp(\mathfrak{g}_0 \otimes \mathfrak{m}_A) \times \exp(\mathfrak{g}_1 \otimes \mathfrak{m}_A) \to \exp(\mathfrak{g}_1 \otimes \mathfrak{m}_A), \qquad (e^a, e^x) \mapsto e^{\alpha_1(a)}e^x e^{-\alpha_0(a)}$$

preserves Maurer-Cartan elements.

Proof. Let $e^x \in \mathrm{MC}_{\mathfrak{g}^\Delta}(A)$, $e^a \in \exp(\mathfrak{g}_0 \otimes \mathfrak{m}_A)$ and denote $e^y = e^{\alpha_1(a)}e^x e^{-\alpha_0(a)}$. Then

$$\begin{aligned} e^{\beta_0(y)} &= e^{\beta_0\alpha_1(a)}e^{\beta_0(x)}e^{-\beta_0\alpha_0(a)}, \\ e^{-\beta_1(y)} &= e^{\beta_1\alpha_0(a)}e^{-\beta_0(x)}e^{-\beta_1\alpha_1(a)}, \\ e^{\beta_2(y)} &= e^{\beta_2\alpha_1(a)}e^{\beta_2(x)}e^{-\beta_2\alpha_0(a)}, \end{aligned}$$

and the proof follows from equations (2.1). □

Moreover, we can define the quotient functor

$$\mathrm{Def}_{\mathfrak{g}^\Delta} \colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{Def}_{\mathfrak{g}^\Delta}(A) = \frac{\mathrm{MC}_{\mathfrak{g}^\Delta}(A)}{\exp(\mathfrak{g}_0 \otimes \mathfrak{m}_A)}$$

Proposition 2.2. *The projectioon $\mathrm{MC}_{\mathfrak{g}^\Delta} \to \mathrm{Def}_{\mathfrak{g}^\Delta}$ is a smooth morphism of deformation functors.*

Proof. Immediate consequence of **??** (altra dispensa). □

Notice that if $\mathfrak{g}_2$ is abelian, then

$$e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = e^{\beta_0(x)-\beta_1(x)+\beta_2(x)}$$

and therefore $\mathrm{MC}_{\mathfrak{g}^\Delta}$ is a smooth functor, since

$$\mathrm{MC}_{\mathfrak{g}^\Delta}(A) = Z^1(\mathrm{Tot}(\mathfrak{g}^\Delta)) \otimes \mathfrak{m}_A.$$

Finally **every morphism of semicosimplicial Lie algebras induce a natural transformation of associated MC and Def functors.**

Let's now compute tangent and obstruction space of $\mathrm{MC}_{\mathfrak{g}^\Delta}$ and $\mathrm{Def}_{\mathfrak{g}^\Delta}$.

$$T^1\,\mathrm{MC}_{\mathfrak{g}^\Delta} = \mathrm{MC}_{\mathfrak{g}^\Delta}(\mathbb{K}[\varepsilon]) = \{x \in \mathfrak{g}_1 \otimes \mathbb{K}\varepsilon \mid e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = 1\} =$$
$$\{x \in \mathfrak{g}_1 \otimes \varepsilon \mid \beta_0(x) - \beta_1(x) + \beta_2(x) = 0\} = \ker(\delta) = Z^1(\mathrm{Tot}(\mathfrak{g}^\Delta)).$$

Next

$$\mathrm{Def}_{\mathfrak{g}^\Delta}(\mathbb{K}[\varepsilon]) = \frac{\mathrm{MC}_{\mathfrak{g}^\Delta}(\mathbb{K}[\varepsilon])}{\exp(\mathfrak{g}_0 \otimes \mathbb{K}\varepsilon)} =$$

$$= \frac{Z^1(\mathrm{Tot}(\mathfrak{g}^\Delta))}{\{-\alpha_1(a)+\alpha_0(a) \mid \; a\in\mathfrak{g}_0\}} = H^1(\mathrm{Tot}(\mathfrak{g}^\Delta)).$$

Next goal is to determine a complete obstruction theory for $\mathrm{MC}_{\mathfrak{g}^\Delta}$; by Lemma **??** (altra dispensa), this will be also an obstruction theory for $\mathrm{Def}_{\mathfrak{g}^\Delta}$.

Let $0\to J\to A\to B\to 0$ be a small extension and let $x\in\mathfrak{g}_1\otimes\mathfrak{m}_A$ be any lifting of an element $y\in\mathrm{MC}_{\mathfrak{g}^\Delta}(B)$, then

$$e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = e^r, \qquad \text{where} \quad r\in\mathfrak{g}_2\otimes J.$$

Since J is annihilated by maximal ideals, the element e^r belongs to the center of the group $\exp(\mathfrak{g}_2\otimes\mathfrak{m}_A)$ and then we have

$$e^r = e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)} = e^{\beta_2(x)}e^{\beta_0(x)}e^{-\beta_1(x)} = e^{-\beta_1(x)}e^{\beta_2(x)}e^{\beta_0(x)}.$$

Lemma 2.3. *In the notation above r is a cocycle in $\mathrm{Tot}(\mathfrak{g}^\Delta)$, i.e. $\sum_k(-1)^k\gamma_k(r)=0$.*

Proof. Notice that

$$\gamma_i(e^r)\gamma_j(e^r) = e^{\gamma_i(r)+\gamma_j(r)} = \gamma_j(e^r)\gamma_i(e^r)$$

for every i,j. It is therefore sufficient to prove that

$$\gamma_0(e^r)\gamma_2(e^r) = \gamma_1(e^r)\gamma_3(e^r).$$

We have

$$\gamma_k(e^r) = e^{\gamma_k\beta_0(x)}e^{-\gamma_k\beta_1(x)}e^{\gamma_k\beta_2(x)}, \qquad k=0,1,2,3,$$

and then

$$(\gamma_1(e^r))^{-1}\gamma_0(e^r) = e^{-\gamma_1\beta_2(x)}e^{\gamma_1\beta_1(x)}e^{-\gamma_0\beta_1(x)}e^{\gamma_0\beta_2(x)},$$

$$\gamma_2(e^r)(\gamma_3(e^r))^{-1} = e^{\gamma_2\beta_0(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)}e^{-\gamma_3\beta_0(x)} = e^{-\gamma_3\beta_0(x)}e^{\gamma_2\beta_0(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)},$$

$$\begin{aligned}
(\gamma_1(e^r))^{-1}\gamma_0(e^r)\gamma_2(e^r)(\gamma_3(e^r))^{-1} &= \\
&= e^{-\gamma_1\beta_2(x)}e^{\gamma_1\beta_1(x)}e^{-\gamma_0\beta_1(x)}e^{\gamma_0\beta_2(x)}e^{-\gamma_3\beta_0(x)}e^{\gamma_2\beta_0(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)} \\
&= e^{-\gamma_1\beta_2(x)}e^{\gamma_1\beta_1(x)}e^{-\gamma_0\beta_1(x)}e^{\gamma_2\beta_0(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)} \\
&= e^{-\gamma_1\beta_2(x)}e^{\gamma_1\beta_1(x)}e^{-\gamma_2\beta_1(x)}e^{\gamma_3\beta_1(x)} \\
&= e^{-\gamma_1\beta_2(x)}e^{\gamma_3\beta_1(x)} = 1.
\end{aligned}$$

□

Therefore, the element r defines a cohomology class $[r]\in H^2(\mathrm{Tot}(\mathfrak{g}^\Delta))\otimes J$ depending only by the class of y in $\mathfrak{g}_1\otimes\mathfrak{m}_B$. In fact, any other lifting is equal to $x+u$, with $u\in\mathfrak{g}_1\otimes J$. Every $e^{\beta_j(u)}$ belongs to the center of $\exp(\mathfrak{g}_2\otimes\mathfrak{m}_A)$ and so

$$e^{\beta_0(x+u)}e^{-\beta_1(x+u)}e^{\beta_2(x+u)} = e^{\beta_0(x)}e^{-\beta_1(x)}e^{\beta_2(x)}e^{\beta_0(u)-\beta_1(u)+\beta_2(u)} = e^{r+\delta(u)}.$$

The same argument proves that $[r]\in H^2(\mathrm{Tot}(\mathfrak{g}^\Delta))\otimes J$ is a complete obstruction.

Corollary 2.4. *Notation as above, if $H^2(\mathrm{Tot}(\mathfrak{g}^\Delta))=0$ then $\mathrm{Def}_{\mathfrak{g}^\Delta}$ is smooth.*

Corollary 2.5. *Let $f\colon\mathfrak{g}^\Delta\to\mathfrak{h}^\Delta$ be a morphism of semicosimplicial Lie algebras. If:*

1. *$f\colon H^1(\mathrm{Tot}(\mathfrak{g}^\Delta))\to H^1(\mathrm{Tot}(\mathfrak{h}^\Delta))$ is surjective,*
2. *$f\colon H^2(\mathrm{Tot}(\mathfrak{g}^\Delta))\to H^2(\mathrm{Tot}(\mathfrak{h}^\Delta))$ is injective,*

then the morphism $f\colon \mathrm{Def}_{\mathfrak{g}^\Delta}\to\mathrm{Def}_{\mathfrak{h}^\Delta}$ is smooth.

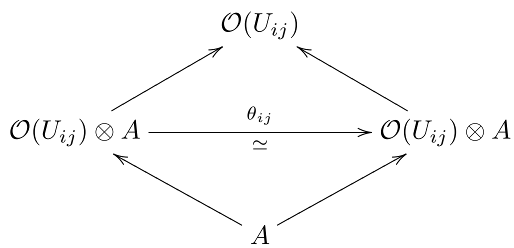
Proof. Apply standard smoothness criterion. □

3. Deformations of manifolds

Let $\mathcal{U} = \{U_i\}$ be an affine open cover of a smooth variety X, defined over an algebraically closed field of characteristic 0; denote by Θ_X the tangent sheaf of X. Then, we can define the Čech semicosimplicial Lie algebra $\Theta_X(\mathcal{U})$ as the semicosimplicial Lie algebra

$$\Theta_X(\mathcal{U}): \quad \prod_i \Theta_X(U_i) \rightrightarrows \prod_{i<j} \Theta_X(U_{ij}) \rightrightarrows \prod_{i<j<k} \Theta_X(U_{ijk}) \rightrightarrows \cdots,$$

Since every infinitesimal deformation of a smooth affine scheme is trivial [?, Lemma II.1.3], every infinitesimal deformation X_A of X over $\operatorname{Spec}(A)$ is obtained by gluing the trivial deformations $U_i \times \operatorname{Spec}(A)$ along the double intersections U_{ij}, and therefore it is determined by the sequence $\{\theta_{ij}\}_{i<j}$ of automorphisms of sheaves of A-algebras



satisfying the cocycle condition

$$\theta_{jk}\theta_{ik}^{-1}\theta_{ij} = Id_{\mathcal{O}(U_{ijk})\otimes A}, \qquad \forall\ i<j<k \in I. \tag{3.1}$$

Since we are in characteristic zero, we can take the logarithms and write $\theta_{ij} = e^{d_{ij}}$, where $d_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$. Therefore, the Equation (3.1) is equivalent to

$$e^{d_{jk}}e^{-d_{ik}}e^{d_{ij}} = 1 \in \exp(\Theta_X(U_{ijk}) \otimes \mathfrak{m}_A), \qquad \forall\ i<j<k \in I.$$

Next, let X'_A be another deformation of X over $\operatorname{Spec}(A)$, defined by the cocycle θ'_{ij}. To give an isomorphism of deformations $X'_A \simeq X_A$ is the same to give, for every i, an automorphism α_i of $\mathcal{O}(U_i) \otimes A$ such that $\theta_{ij} = {\alpha_i}^{-1}{\theta'_{ij}}^{-1}\alpha_j$, for every $i<j$. Taking again logarithms, we can write $\alpha_i = e^{a_i}$, with $a_i \in \Theta_X(U_i) \otimes \mathfrak{m}_A$, and so $e^{-a_i}e^{d'_{ij}}e^{a_j} = e^{d_{ij}}$.

Theorem 3.1. *Let $\mathcal{U}$ be an affine open cover of a smooth algebraic variety X defined over an algebraically closed field of characteristic 0. Denoting by Def_X the functor of infinitesimal deformations of X, there exist isomorphisms of functors*

$$\operatorname{Def}_X \cong \operatorname{Def}_{\Theta_X(\mathcal{U})}$$

where $\Theta_X(\mathcal{U})$ is the semicosimplicial Lie algebra defined above.

Proof. By definition,

$$\operatorname{MC}_{\Theta_X(\mathcal{U})}(A) = \{\{x_{ij}\} \in \prod_{i<j} \Theta_X(U_{ij}) \otimes \mathfrak{m}_A \mid e^{x_{jk}}e^{-x_{ik}}e^{x_{ij}} = 1\ \forall\ i<j<k\},$$

for each $A \in \mathbf{Art}$. Moreover, given $x = \{x_{ij}\}$ and $y = \{y_{ij}\} \in \prod_{i<j} \Theta_X(U_{ij}) \otimes \mathfrak{m}_A$, we have $x \sim y$ if and only if there exists $a = \{a_i\} \in \prod_i \Theta_X(U_i) \otimes \mathfrak{m}_A$ such that $e^{-a_j}e^{x_{ij}}e^{a_i} = e^{y_{ij}}$ for all $i<j$. □

In particular this proves that if $H^2(\Theta_X) = 0$ then X has unobstructed deformations.