Deformations of singularities via differential graded Lie algebras

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1 Introduction

Let $\mathbb{K}$ be a fixed algebraically closed field of characteristic 0, $X \subset \mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ a closed subscheme. Denote by **Art** the category of local artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$.

Definition 1.1. *An* infinitesimal deformation *of X over $A \in$ **Art** is a commutative diagram of schemes*

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X_A \\ \downarrow & & \downarrow{\scriptstyle f_A} \\ \operatorname{Spec}(\mathbb{K}) & \longrightarrow & \operatorname{Spec}(A) \end{array}$$

such that f_A is flat and the induced morphism $X \to X_A \times_{\operatorname{Spec}(A)} \operatorname{Spec}(\mathbb{K})$ is an isomorphism.

It is not difficult to see (cf. [1]) that X_A is affine and more precisely it is isomorphic to a closed subscheme of $\mathbb{A}^n \times \operatorname{Spec}(A)$. Two deformations $X \xrightarrow{i} X_A \xrightarrow{f_A} \operatorname{Spec}(A)$, $X \xrightarrow{j} \tilde{X}_A \xrightarrow{g_A} \operatorname{Spec}(A)$ are isomorphic if there exists a commutative diagram of schemes

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & X_A \\ {\scriptstyle j}\downarrow & \swarrow{\scriptstyle \theta} & \downarrow{\scriptstyle f_A} \\ \tilde{X}_A & \xrightarrow[g_A]{} & \operatorname{Spec}(A) \end{array}$$

It is easy to prove that necessarily θ is an isomorphism (cf. [5]). Since flatness commutes with base change, for every deformations $X \xrightarrow{i} X_A \xrightarrow{f_A} \operatorname{Spec}(A)$ and every morphism $A \to B$ in the category **Art**, the diagram

$$\begin{array}{ccc} X & \longrightarrow & X_A \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathbb{K}) & \longrightarrow & \operatorname{Spec}(B) \end{array}$$

is a deformation of X over $\operatorname{Spec}(B)$; it is then defined a covariant functor $\operatorname{Def}_X \colon$ **Art** $\to$ **Set**,

$$\operatorname{Def}_X(A) = \{\text{ isomorphism classes of deformations of } X \text{ over } A\,\}.$$

The set $\operatorname{Def}_X(\mathbb{K})$ contains only one point.

In a similar way we can define the functor $\operatorname{Hilb}_X \colon$ **Art** $\to$ **Set** of embedded deformations of X into $\mathbb{A}^n$: $\operatorname{Hilb}_X(A)$ is the set of closed subschemes $X_A \subset \mathbb{A}^n \times \operatorname{Spec}(A)$ such that the restriction to X_A of the projection on the second factor is a flat map $X_A \to \operatorname{Spec}(A)$ and $X_A \cap (\mathbb{A}^n \times \operatorname{Spec}(\mathbb{K})) = X \times \operatorname{Spec}(\mathbb{K})$.

In these notes we give a recipe for the construction of two differential graded Lie algebras $\mathcal{H}$, $\mathcal{L}$ together two isomorphism of functors

$$\operatorname{Def}_{\mathcal{L}} = \frac{MC_{\mathcal{L}}}{\text{gauge}} \to \operatorname{Def}_X, \qquad \operatorname{Def}_{\mathcal{H}} = \frac{MC_{\mathcal{H}}}{\text{gauge}} \to \operatorname{Hilb}_X.$$

The DGLAs $\mathcal{L}$, $\mathcal{H}$ are unique up to quasiisomorphism and their cohomology can be computed in terms of the cotangent complex of X. For the notion of differential graded Lie algebra, Maurer-Cartan functors and gauge equivalence we refer to [3], [5], [2].

Moreover we can choose $\mathcal{H}$ as a differential graded Lie subalgebra of $\mathcal{L}$ such that $\mathcal{H}^i = \mathcal{L}^i$ for every $i > 0$.

2 Flatness and relations

In this section $A \in \mathbf{Art}$ is a fixed local artinian $\mathbb{K}$-algebra with residue field $\mathbb{K}$.

Lemma 2.1. *Let M be an A-module, if $M \otimes_A \mathbb{K} = 0$ then $M = 0$.*

Proof. If M is finitely generated this is Nakayama's lemma. In the general case consider a filtration of ideals $0 = I_0 \subset I_1 \subset \ldots \subset I_n = A$ such that $I_{i+1}/I_i = \mathbb{K}$ for every i. Applying the right exact functor $\otimes_A M$ to the exact sequences of A-modules

$$0 \longrightarrow \mathbb{K} = \frac{I_{i+1}}{I_i} \longrightarrow \frac{A}{I_i} \longrightarrow \frac{A}{I_{i+1}} \longrightarrow 0$$

we get by induction that $M \otimes_A (A/I_i) = 0$ for every i. □

The following is a special case of the *local flatness criterion* [6, Thm. 22.3]

Theorem 2.2. *For an A-module M the following conditions are equivalent:*

1. *M is free.*
2. *M is flat.*
3. $\operatorname{Tor}_1^A(M, \mathbb{K}) = 0$.

Proof. The only nontrivial assertion is 3) ⇒ 1). Assume $\operatorname{Tor}_1^A(M, \mathbb{K}) = 0$ and let F be a free module such that $F \otimes_A \mathbb{K} = M \otimes_A \mathbb{K}$. Since $M \to M \otimes_A \mathbb{K}$ is surjective there exists a morphism $\alpha\colon F \to M$ such that its reduction $\overline{\alpha}\colon F \otimes_A \mathbb{K} \to M \otimes_A \mathbb{K}$ is an isomorphism. Denoting by K the kernel of α and by C its cokernel we have $C \otimes_A \mathbb{K} = 0$ and then $C = 0$; $K \otimes_A \mathbb{K} = \operatorname{Tor}_1^A(M, \mathbb{K}) = 0$ and then $K = 0$. □

Corollary 2.3. *Let $h\colon P \to L$ be a morphism of flat A-modules, $A \in \mathbf{Art}$. If $\overline{h}\colon P \otimes_A \mathbb{K} \to L \otimes_A \mathbb{K}$ is injective (resp.: surjective) then also h is injective (resp.: surjective).*

Proof. Same proof of Theorem 2.2. □

Corollary 2.4. *Let $0 \to M \to N \to P \to 0$ be an exact sequence of A-modules with N flat. Then:*

1. *$M \otimes_A \mathbb{K} \to N \otimes_A \mathbb{K}$ injective ⇒ P flat.*
2. *P flat ⇒ M flat and $M \otimes_A \mathbb{K} \to N \otimes_A \mathbb{K}$ injective.*

Proof. Take the associated long $\mathrm{Tor}^A_*(-, \mathbb{K})$ exact sequence and apply 2.2 and 2.3. ☐

Corollary 2.5. *Let*

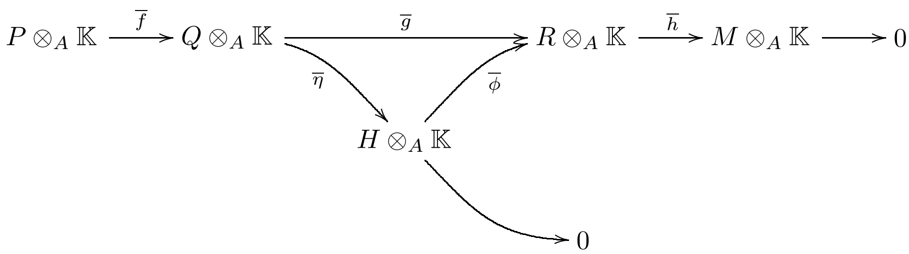
$$P \xrightarrow{f} Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0 \qquad (1)$$

be a complex of A-modules such that:

1. *P, Q, R are flat.*
2. *$Q \xrightarrow{g} R \xrightarrow{h} M \longrightarrow 0$ is exact.*
3. *$P \otimes_A \mathbb{K} \xrightarrow{\overline{f}} Q \otimes_A \mathbb{K} \xrightarrow{\overline{g}} R \otimes_A \mathbb{K} \xrightarrow{\overline{h}} M \otimes_A \mathbb{K} \longrightarrow 0$ is exact.*

Then M is flat and the sequence (1) is exact.

Proof. Denote by $H = \ker h = \operatorname{Im} g$ and $g = \phi\eta$, where $\phi\colon H \to R$ is the inclusion and $\eta\colon Q \to H$; by assumption we have an exact diagram



which allows to prove, after an easy diagram chase, that $\overline{\phi}$ is injective. According to Corollary 2.4 H and M are flat modules. Denoting $L = \ker g$ we have, since H is flat, that also L is flat and $L \otimes_A K \to Q \otimes_A \mathbb{K}$ injective. This implies that $P \otimes_A \mathbb{K} \to L \otimes_A \mathbb{K}$ is surjective. By Corollary 2.3 $P \to L$ is surjective. ☐

Corollary 2.6. *Let $n > 0$ and*

$$0 \longrightarrow I \longrightarrow P_0 \xrightarrow{d_1} P_1 \longrightarrow \dots \xrightarrow{d_n} P_n,$$

a complex of A-modules with $P_0, \dots, P_n$ flat. Assume that

$$0 \longrightarrow I \otimes_A \mathbb{K} \longrightarrow P_0 \otimes_A \mathbb{K} \xrightarrow{\overline{d_1}} P_1 \otimes_A \mathbb{K} \longrightarrow \dots \xrightarrow{\overline{d_n}} P_n \otimes_A \mathbb{K}$$

is exact; then $I, \operatorname{coker}(d_n)$ are flat modules and the natural morphism $I \to \ker(P_0 \otimes_A \mathbb{K} \to P_1 \otimes_A \mathbb{K})$ is surjective.

Proof. Induction on n and Corollary 2.5. ☐

3 Differential graded algebras, I

Unless otherwise specified by the symbol $\otimes$ we mean the tensor product $\otimes_{\mathbb{K}}$ over the field $\mathbb{K}$. We denote by:

- **G** the category of $\mathbb{Z}$-graded $\mathbb{K}$-vector space; given an object $V = \oplus V_i$, $i \in \mathbb{Z}$, of **G** and a homogeneous element $v \in V_i$ we denote by $\overline{v} = i$ its degree.
- **DG** the category of $\mathbb{Z}$-graded differential $\mathbb{K}$-vector space (also called complexes of vector spaces).

Given (V, d) in $\mathbf{DG}$ we denote as usual by $Z(V) = \ker d$, $B(V) = d(V)$, $H(V) = Z(V)/B(V)$.

Given an integer n, the shift functor $[n] \colon \mathbf{DG} \to \mathbf{DG}$ is defined by setting $V[n] = \mathbb{K}\,[n] \otimes V$, $V \in \mathbf{DG}$, $f[n] = Id_{\mathbb{K}\,[n]} \otimes f$, $f \in \mathrm{Mor}_{\mathbf{DG}}$, where

$$\mathbb{K}\,[n]_i = \begin{cases} \mathbb{K} & \text{if } i+n=0 \\ 0 & \text{otherwise} \end{cases}$$

More informally, the complex $V[n]$ is the complex V with degrees shifted by n, i.e. $V[n]_i = V_{i+n}$, and differential multiplied by $(-1)^n$.

Given two graded vector spaces V, W, the "graded Hom" is the graded vector space

$$\mathrm{Hom}^*_{\mathbb{K}}(V, W) = \oplus_n \mathrm{Hom}^n_{\mathbb{K}}(V, W) \in \mathbf{G},$$

where by definition $\mathrm{Hom}^n_{\mathbb{K}}(V, W)$ is the set of $\mathbb{K}$-linear map $f \colon V \to W$ such that $f(V_i) \subset W_{i+n}$ fore every $i \in \mathbb{Z}$. Note that $\mathrm{Hom}^0_{\mathbb{K}}(V, W) = \mathrm{Hom}_{\mathbf{G}}(V, W)$ is the space of morphisms in the category $\mathbf{G}$ and there exist natural isomorphisms

$$\mathrm{Hom}^n_{\mathbb{K}}(V, W) = \mathrm{Hom}_{\mathbf{G}}(V[-n], W) = \mathrm{Hom}_{\mathbf{G}}(V, W[n]).$$

A morphism in $\mathbf{DG}$ is called a quasiisomorphism if it induces an isomorphism in homology. A commutative diagram in $\mathbf{DG}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow g & & \big\downarrow f \\ C & \longrightarrow & D \end{array}$$

is called cartesian if the morphism $A \to C \times_D B$ is an isomorphism; it is an easy exercise in homological algebra to prove that if f is a surjective (resp.: injective) quasiisomorphism, then g is a surjective (resp.: injective) quasiisomorphism.

Definition 3.1. *A graded (associative, $\mathbb{Z}$-commutative) algebra is a graded vector space $A = \oplus A_i \in \mathbf{G}$ endowed with a product $A_i \times A_j \to A_{i+j}$ satisfying the properties:*

1. *$a(bc) = (ab)c$.*
2. *$a(b+c) = ab + ac$, $(a+b)c = ac + bc$.*
3. *(Koszul sign convention) $ab = (-1)^{\overline{a}\,\overline{b}} ba$ for a, b homogeneous.*

The algebra A is unitary if there exists $1 \in A_0$ such that $1a = a1 = a$ for every $a \in A$.

Let A be a graded algebra, then A_0 is a commutative $\mathbb{K}$-algebra in the usual sense; conversely every commutative $\mathbb{K}$-algebra can be considered as a graded algebra concentrated in degree 0. If $I \subset A$ is a homogeneous left (resp.: right) ideal then I is also a right (resp.: left) ideal and the quotient A/I has a natural structure of graded algebra.

Example 3.2. *Polynomial algebras.* Given a set $\{x_i\}$, $i \in I$, of homogeneous indeterminates of integral degree $\overline{x_i} \in \mathbb{Z}$ we can consider the graded algebra $\mathbb{K}\,[\{x_i\}]$. As a $\mathbb{K}$-vector space $\mathbb{K}\,[\{x_i\}]$ is generated by monomials in the indeterminates x_i. Equivalently $\mathbb{K}\,[\{x_i\}]$ can be defined as the symmetric algebra $\bigoplus_{n \geq 0} \bigodot^n V$, where $V = \oplus_{i \in I} \mathbb{K}\, x_i \in \mathbf{G}$. In some cases, in order to avoid confusion about terminology, for a monomial $x_{i_1}^{\alpha_1} \ldots x_{i_n}^{\alpha_n}$ it is defined:

- The *internal degree* $\sum_h \overline{x_{i_h}} \alpha_h$.
- The *external degree* $\sum_h \alpha_h$.

In a similar way it is defined $A[\{x_i\}]$ for every graded algebra A.

Definition 3.3. *A* dg-algebra *(differential graded algebra) is the data of a graded algebra A and a $\mathbb{K}$-linear map $s\colon A \to A$, called* differential, *with the properties:*

1. *$s(A_n) \subset A_{n+1}$, $s^2 = 0$.*
2. *(graded Leibnitz rule) $s(ab) = s(a)b + (-1)^{\overline{a}} a s(b)$.*

A morphism of dg-algebras is a morphism of graded algebras commuting with differentials; the category of dg-algebras is denoted by **DGA**.

In the sequel, for every dg-algebra A we denote by $A_\sharp$ the underlying graded algebra.

Exercise 3.4. Let (A, s) be a unitary dg-algebra; prove:

1. $1 \in Z(A)$.
2. $1 \in B(A)$ if and only if $H(A) = 0$.
3. $Z(A)$ is a graded subalgebra of A and $B(A)$ is a homogeneous ideal of $Z(A)$.
4. If A is local with maximal ideal M then $s(M) \subset M$ if and only if $H(A) \neq 0$.

△

A differential ideal of a dg-algebra (A, s) is a homogeneous ideal I of A such that $s(I) \subset I$; there exists an obvious bijection between differential ideals and kernels of morphisms of dg-algebras.

On a polynomial algebra $\mathbb{K}[\{x_i\}]$ a differential s is uniquely determined by the values $s(x_i)$.

Example 3.5. Let t, dt be inderminates of degrees $\overline{t} = 0$, $\overline{dt} = 1$; on the polynomial algebra $\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$ there exists an obvious differential d such that $d(t) = dt$, $d(dt) = 0$. Since $\mathbb{K}$ has characteristic 0, we have $H(\mathbb{K}[t, dt]) = \mathbb{K}$. More generally if (A, s) is a dg-algebra then $A[t, dt]$ is a dg-algebra with differential $s(a \otimes p(t)) = s(a) \otimes p(t) + (-1)^{\overline{a}} a \otimes p'(t)dt$, $s(a \otimes q(t)dt) = s(a) \otimes q(t)dt$.

Definition 3.6. *A morphism of dg-algebras $B \to A$ is a* quasiisomorphism *if the induced morphism $H(B) \to H(A)$ is an isomorphism.*

Given a morphism of dg-algebras $B \to A$ the space $\mathrm{Der}^n_B(A, A)$ of B-derivations of degree n is by definition

$$\mathrm{Der}^n_B(A, A) = \{\phi \in \mathrm{Hom}^n_{\mathbb{K}}(A, A) \mid \phi(ab) = \phi(a)b + (-1)^{n\overline{a}} a\phi(b),\ \phi(B) = 0\}.$$

We also consider the graded vector space

$$\mathrm{Der}^*_B(A, A) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Der}^n_B(A, A) \in \mathbf{G}.$$

There exists a structure of differential graded Lie algebra on $\mathrm{Der}^*_B(A, A)$ with differential

$$d\colon \mathrm{Der}^n_B(A, A) \to \mathrm{Der}^{n+1}_B(A, A), \qquad d\phi = d_A\phi - (-1)^n \phi d_A$$

and bracket

$$[f, g] = fg - (-1)^{\overline{f}\,\overline{g}} gf.$$

Exercise 3.7. Verify that $d[f,g] = [df,g] + (-1)^{\overline{f}}[f,dg]$. △

Exercise 3.8. Let A be graded algebra: if every $a \neq 0$ is invertible then $A = A_0$ is a field. △

Exercise 3.9. Let A be a graded algebra and let $I \subset A$ be a left ideal. Then the following conditions are equivalent:

1. I is the unique left maximal ideal.
2. A_0 is a local ring with maximal ideal M and $I = M \oplus_{i \neq 0} A_i$.

△

4 The DG-resolvent

Let $X \subset \mathbb{A}^n$ be a closed subscheme, $R_0 = \mathbb{K}[x_1, \ldots, x_n]$ the ring of regular functions on $\mathbb{A}^n$, $I_0 \subset R_0$ the ideal of X and $\mathcal{O}_X = R_0/I$ the function ring of X.

Our aim is to construct a dg-algebra (R,d) and a quasiisomorphism $R \to \mathcal{O}_X$ such that $R = R_0[y_1, y_2, \ldots]$ is a countably generated graded polynomial R_0-algebra, every indeterminate y_i has negative degree and, if $R = \oplus_{i \leq 0} R_i$, then R_i is a finitely generated free R_0 module.

Choosing a set of generators $f_1, \ldots, f_{s_1}$ of the ideal I_0 we first consider the graded-commutative polynomial dg-algebra

$$R(1) = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_{s_1}] = R_0[y_1, \ldots, y_{s_1}], \qquad \overline{x_i} = 0, \quad \overline{y_i} = -1$$

with differential d defined by $dx_i = 0$, $dy_j = f_j$. Note that $(R(1), d)$, considered as a complex of R_0 modules, is the Koszul complex of the sequence $f_1, \ldots, f_{s_1}$. By construction the complex of R_0-modules

$$\ldots \longrightarrow R(1)_{-2} \xrightarrow{d} R(1)_{-1} \xrightarrow{d} R_0 \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0$$

is exact in R_0 and $\mathcal{O}_X$. If $(R(-1), d) \to \mathcal{O}_X$ is a quasiisomorphism of dg-algebras (e.g. if X is a complete intersection) the construction is done. Otherwise let $f_{s_1+1}, \ldots, f_{s_2} \in \ker d \cap R(1)_{-1}$ be a set of generators of the R_0 module $(\ker d \cap R(1)_{-1})/dR(1)_{-2}$ and define

$$R(2) = R(1)[y_{s_1+1}, \ldots, y_{s_2}], \qquad \overline{y_j} = -2, \quad dy_j = f_j, \quad j = s_1 + 1, \ldots, s_2.$$

Repeating in a recursive way the above argument (step by step killing cycles) we get a chain of polynomial dg-algebras

$$R_0 = R(0) \subset R(1) \subset \ldots \subset R(i) \subset \ldots$$

such that $(R(i), d) \to \mathcal{O}_X$ is a quasiisomorphism in degree $> -i$. Setting

$$R = \cup R(i) = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m, \ldots] = \bigoplus_{i \leq 0} R_i,$$

the projection $\pi \colon R \to \mathcal{O}_X$ is a quasiisomorphism of dg-algebras; in particular

$$\ldots \xrightarrow{d} R_{-i} \xrightarrow{d} \ldots \xrightarrow{d} R_{-2} \xrightarrow{d} R_{-1} \xrightarrow{d} R_0 \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0$$

is a free resolution of the R_0 module $\mathcal{O}_X$.

We denote by:

1. $Z_i = \ker d \cap R_i$.

2. $\mathcal{L} = \mathrm{Der}^*_{\mathbb{K}}(R, R)$.

3. $\mathcal{H} = \mathrm{Der}^*_{R_0}(R, R) = \{g \in \mathcal{L} \,|\, g(R_0) = 0\}$.

It is clear that, since $gR_i \subset R_{i+j}$ for every $g \in \mathcal{L}^j$, $\mathcal{L}^i = \mathcal{H}^i$ for every $i > 0$ and then the DGLAs $\mathcal{L}$, $\mathcal{H}$ have the same Maurer-Cartan functor $MC_{\mathcal{H}} = MC_{\mathcal{L}}$. Moreover R is a free graded algebra and then $\mathcal{L}^j$ is in bijection with the maps of "degree j" $\{x_i, y_h\} \to R$.

Consider a fixed $\eta \in MC_{\mathcal{H}}(A)$. Recalling the definition of $MC_{\mathcal{H}}$ we have that $\eta = \sum \eta_i \otimes a_i \in \mathrm{Der}^1_{R_0}(R, R) \otimes m_A$ and the A-derivation

$$d + \eta \colon R \otimes A \to R \otimes A, \qquad (d + \eta)(x \otimes a) = dx \otimes a + \sum \eta_i(x) \otimes a_i a$$

is a differential. Denoting by $\mathcal{O}_A$ the cokernel of $d + \eta \colon R_{-1} \otimes A \to R_0 \otimes A$ we have by Corollary 2.5 that $(R \otimes A, d + \eta) \to \mathcal{O}_A$ is a quasiisomorphism, $\mathcal{O}_A$ is flat and $\mathcal{O}_A \otimes \mathbb{K} = \mathcal{O}_X$. Therefore we have natural transformations of functors

$$MC_{\mathcal{H}} = MC_{\mathcal{L}} \to \mathrm{Hilb}_X \to \mathrm{Def}_X\,.$$

Lemma 4.1. *The above morphisms of functors are surjective.*

Proof. Let $\mathcal{O}_A$ be a flat A-algebra such that $\mathcal{O}_A \otimes_A \mathbb{K} = \mathcal{O}_X$; since R_0 is a free $\mathbb{K}$-algebra, the projection $R_0 \xrightarrow{\pi} \mathcal{O}_X$ can be extended to a morphism of flat A-algebras $R_0 \otimes A \xrightarrow{\pi_A} \mathcal{O}_A$. According to Corollary 2.3 π_A is surjective; this proves that $\mathrm{Hilb}_X(A) \to \mathrm{Def}_X(A)$ is surjective (in effect it is possible to prove directly that $\mathrm{Hilb}_X \to \mathrm{Def}_X$ is smooth, cf. [1]). An element of $\mathrm{Hilb}_X(A)$ gives an exact sequence of flat A-modules

$$R_0 \otimes A \xrightarrow{\pi_A} \mathcal{O}_A \longrightarrow 0.$$

Denoting by $I_{0,A} \subset R_0 \otimes A$ the kernel of π_A we have that $I_{0,A}$ is A-flat and the projection $I_{0,A} \to I_0$ is surjective. We can therefore extend the restriction to $R(1)$ of the differential d to a differential d_A on $R(1) \otimes A$ by setting $d_A(y_j) \in I_{0,A}$ a lifting of $d(y_j)$, $j = 1, \ldots, s_1$. Again by local flatness criterion the kernel $Z_{-1,A}$ of $R_{-1} \otimes A = R(1)_{-1} \otimes A \xrightarrow{d_A} R_0 \otimes A$ is flat and surjects onto Z_{-1}. The same argument as above, with $I_{0,A}$ replaced by $Z_{-1,A}$ shows that d can be extended to a differential d_A on $R(2)$ and then by induction to a differential d_A on $R \otimes A$ such that $(R \otimes A, d_A) \to \mathcal{O}_A$ is a quasiisomorphism. If $a_1, \ldots, a_r$ is a $\mathbb{K}$-basis of the maximal ideal of A we can write $d_A(x \otimes 1) = dx \otimes 1 + \sum \eta_i(x) \otimes a_i$ and then $\eta = \sum \eta_i \otimes a_i \in MC_{\mathcal{H}}(A)$. $\square$

If $\xi \in \mathrm{Der}^0_{R_0}(R, R) \otimes m_A$, $A \in$ **Art**, then $e^{\xi} \colon R \otimes A \to R \otimes A$ is an automorphism inducing the identity on R and $R_0 \otimes A$. Therefore the morphism $MC_{\mathcal{H}}(A) \to \mathrm{Hilb}_X(A)$ factors through $\mathrm{Def}_{\mathcal{H}}(A) \to \mathrm{Hilb}_X(A)$. Similarly the morphism $MC_{\mathcal{L}}(A) \to \mathrm{Def}_X(A)$ factors through $\mathrm{Def}_{\mathcal{L}}(A) \to \mathrm{Def}_X(A)$.

Theorem 4.2. *The natural transformations*

$$\mathrm{Def}_{\mathcal{H}} \to \mathrm{Hilb}_X, \qquad \mathrm{Def}_{\mathcal{L}} \to \mathrm{Def}_X$$

are isomorphisms of functors.

Proof. We have already proved the surjectivity. The injectivity follows from the following lifting argument. Given $d_A, d'_A \colon R \otimes A \to R \otimes A$ two liftings of the differential d and $f_0 \colon R_0 \otimes A \to R_0 \otimes A$ a lifting of the identity on R_0 such that $f_0 d_A(R_{-1} \otimes A) \subset d'_A(R_{-1} \otimes A)$ there exists an isomorphism $f \colon (R \otimes A, d_A) \to (R \otimes A, d'_A)$ extending f_0 and the identity on R. This is essentially trivial because $R \otimes A$ is a free $R_0 \otimes A$ graded algebra and $(R \otimes A, d'_A)$ is exact in degree < 0. Thinking f as an automorphism of the graded algebra $R \otimes A$ we have, since $\mathbb{K}$ has characteristic 0, that $f = e^{\xi}$ for some $\xi \in \mathcal{L}^0$ and $\xi \in \mathcal{H}^0$ if and only if $f_0 = Id$. By the definition of gauge action $d'_A - d = exp(\xi)(d_A - d)$; the injectivity follows. $\square$

Proposition 4.3. *If $I \subset R_0$ is the ideal of $X \subset \mathbb{A}^n$ then:*

1. $H^i(\mathcal{H}) = H^i(\mathcal{L}) = 0$ *for every* $i < 0$.
2. $H^0(\mathcal{H}) = 0$, $H^0(\mathcal{L}) = \operatorname{Der}_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X)$.
3. $H^1(\mathcal{H}) = \operatorname{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X)$ *and* $H^1(\mathcal{L})$ *is the cokernel of the natural morphism*

$$\operatorname{Der}_{\mathbb{K}}(R_0, \mathcal{O}_X) \xrightarrow{\alpha} \operatorname{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X).$$

Proof. There exists a short exact sequence of complexes

$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{L} \longrightarrow \operatorname{Der}^*_{\mathbb{K}}(R_0, R) \longrightarrow 0.$$

Since R_0 is free and R is exact in degree < 0 we have:

$$H^i(\operatorname{Der}^*_{\mathbb{K}}(R_0, R)) = \begin{cases} 0 & i \neq 0, \\ \operatorname{Der}_{\mathbb{K}}(R_0, \mathcal{O}_X) & i = 0. \end{cases}$$

Moreover $\operatorname{Der}_{\mathbb{K}}(\mathcal{O}_X, \mathcal{O}_X)$ is the kernel of α and then it is sufficient to compute $H^i(\mathcal{H})$ for $i \leq 1$.

Every $g \in Z^i(\mathcal{H})$, $i \leq 0$, is a R_0-derivation $g\colon R \to R$ such that $g(R) \subset \oplus_{i<0} R_i$ and $gd = \pm dg$. As above R is free and exact in degree < 0, a standard argument shows that g is a coboundary. If $g \in Z^1(\mathcal{H})$ then $g(R_{-1}) \subset R_0$ and, since $gd + dg = 0$, g induces a morphism

$$\overline{g}\colon \frac{R_{-1}}{dR_{-2}} = I \longrightarrow \frac{R_0}{dR_{-1}} = \mathcal{O}_X.$$

The easy verification that $Z^1(\mathcal{H}) \to \operatorname{Hom}_{R_0}(I, \mathcal{O}_X)$ induces an isomorphism $H^1(\mathcal{H}) \to \operatorname{Hom}_{R_0}(I, \mathcal{O}_X)$ is left to the reader. ☐

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