

Deformation theory via differential graded Lie algebras

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This paper concerns the basic philosophy that, over a field of characteristic 0, every deformation problem is governed by a differential graded Lie algebra (DGLA) via solutions of Maurer-Cartan equation modulo gauge action.

The classical approach (Grothendieck-Mumford-Schlessinger) to infinitesimal deformation theory is described by the procedure (see e.g [2])

$$\text{Deformation problem} \rightsquigarrow \text{Deformation functor}$$

The above picture is rather easy and suffices for many applications; it is however clear that in this way we forget information which can be useful. The other classical approach, which consider categories fibred in groupoids instead of deformation functors, is not much better. A possible and useful way to preserve information is to consider a factorization

$$\text{Deformation problem} \rightsquigarrow \text{DGLA} \rightsquigarrow \text{Deformation functor}$$

where by *DGLA* we mean a differential graded Lie Algebra depending from the data of the deformation problem and the arrow $\text{DGLA} \rightsquigarrow \text{Deformation functor}$ is a well defined, functorial procedure explained in Section 3 of this notes. Moreover we prove that every quasiisomorphism of differential graded Lie algebras induces an isomorphism of deformation functors.

Given a deformation problem, in general it is not an easy task to find a factorization as above; some general technics of this “art” (cf. [12, p. 5]), will be discussed elsewhere. Here we only point out that in general the correct DGLA is only defined up to quasiisomorphism and then this note represents the necessary background for the whole theory.

Although the interpretation of deformation problems in terms of solutions of Maurer-Cartan equation is very useful on its own, in many situation it is unavoidable to recognize that the category of DGLA is too rigid for a “good” theory. The appropriate way of extending this category will be the introduction of homotopy Lie algebras and L_∞ -algebras; these new objects will be described in next lectures.

The results of this paper are, more or less, known to experts; if some originality is present in this notes then it is only contained in the proofs.

1 Differential graded Lie algebras and the Maurer-Cartan equation

Unless otherwise specified we shall follow the notation of [11].

\mathbb{K} is a field of characteristic 0.

$\text{Graded}^{\mathbb{K}}$ is the category of \mathbb{Z} -graded vector space over \mathbb{K} .

$[n]: \text{Graded}^{\mathbb{K}} \rightarrow \text{Graded}^{\mathbb{K}}$ is the shift operator, $V[n]^i = V^{i+n}$; for example $\mathbb{K}[1]^{-1} = \mathbb{K}$, $\mathbb{K}[1]^i = 0$ for $i \neq -1$.

If V is a \mathbb{Z} -graded vector space and $v \in V$ is a homogeneous element we write $\bar{v} \in \mathbb{Z}/2\mathbb{Z}$ for the class modulo 2 of the degree of v .

Definition 1.1. A *Differential graded Lie algebra* (DGLA for short) $(L, [,], d)$ is the data of a \mathbb{Z} -graded vector space $L = \bigoplus_{i \in \mathbb{Z}} L^i$ together a bilinear bracket $[,]: L \times L \rightarrow L$ and a linear map $d: L \rightarrow L$ satisfying the following condition:

1. $[,]$ is homogeneous skewsymmetric; this means $[L^i, L^j] \subset L^{i+j}$ and $[a, b] + (-1)^{\bar{a}\bar{b}}[b, a] = 0$ for every a, b homogeneous.
2. Every a, b, c homogeneous satisfy the Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}}[b, [a, c]].$$

3. $d(L^i) \subset L^{i+1}$, $d \circ d = 0$ and $d[a, b] = [da, b] + (-1)^{\bar{a}}[a, db]$. The map d is called the differential of L .

Note that L^0 and $L^{even} = \bigoplus L^{2n}$ are Lie algebras in the usual sense.

Definition 1.2. A linear map $f: L \rightarrow L$ is called a *derivation of degree n* if $f(L^i) \subset L^{i+n}$ and satisfies the graded Leibniz rule $f([a, b]) = [f(a), b] + (-1)^{n\bar{a}}[a, f(b)]$.

We note that if $a \in L^i$ then $ad(a): L \rightarrow L$, $ad(a)(b) = [a, b]$, is a derivation of degree i and d is a derivation of degree 1.

By following the standard notation we denote by $Z^i(L) = \ker(d: L^i \rightarrow L^{i+1})$, $B^i(L) = \text{image}(d: L^{i-1} \rightarrow L^i)$, $H^i(L) = Z^i(L)/B^i(L)$.

Definition 1.3. The Maurer-Cartan equation (also called the deformation equation) of a DGLA L is

$$da + \frac{1}{2}[a, a] = 0, \quad a \in L^1.$$

There is also an obvious notion of morphisms of DGLA's; every morphism of DGLA induces a morphism between cohomology groups. It is also evident that morphisms of DGLA preserves solutions of the Maurer-Cartan equation.

A *quasiisomorphism* is a morphism inducing isomorphisms in cohomology. Two DGLA's are *quasiisomorphic* if they are equivalent under the equivalence relation generated by quasiisomorphisms.

The cohomology of a DGLA is itself a differential graded Lie algebra with the induced bracket and zero differential:

Definition 1.4. A DGLA L is called *Formal* if it is quasiisomorphic to its cohomology DGLA $H^*(L)$.

EXERCISE: Let $D: L \rightarrow L$ be a derivation, then the kernel of D is a graded Lie subalgebra.

EXERCISE: Let $L = \bigoplus L^i$ be a DGLA and $a \in L^i$. Prove that:

1. If i is even then $[a, a] = 0$.
2. If i is odd then $[a, [a, b]] = \frac{1}{2}[[a, a], b]$ for every $b \in L$ and $[[a, a], a] = 0$.

In nature there exists several examples of DGLA, most of which govern deformation problems via Maurer-Cartan equation; here we present some of the most interesting ones.

Example 1.5. Every Lie algebra is a DGLA concentrated in degree 0.

Example 1.6. Let $A = \oplus A^i$ be an associative graded-commutative \mathbb{K} -algebra (this means that $ab = (-1)^{\overline{a}\overline{b}}ba$ for a, b homogeneous) and $L = \oplus L^i$ a DGLA. Then $L \otimes_{\mathbb{K}} A$ has a natural structure of DGLA by setting:

$$(L \otimes_{\mathbb{K}} A)^n = \oplus_i (L^i \otimes_{\mathbb{K}} A^{n-i}), \quad d(x \otimes a) = dx \otimes a, \quad [x \otimes a, y \otimes b] = (-1)^{\overline{a}\overline{y}}[x, y] \otimes ab.$$

In the next examples, when M is a complex variety and E a holomorphic vector bundle on M , we denote by $\mathcal{A}^{p,q}(E)$ the sheaf of C^∞ (p, q) -differential forms with values in E .

Example 1.7. Let E be a holomorphic vector bundle on a complex variety M . We have a DGLA $L = \oplus \Gamma(M, \mathcal{A}^{0,p}(End(E)))[-p]$ with the Dolbeault differential and the natural bracket. More precisely if e, g are local holomorphic sections of $End(E)$ and ϕ, ψ differential forms we define $d(\phi e) = (\overline{\partial}\phi)e$, $[\phi e, \psi g] = \phi \wedge \psi[e, g]$.

Example 1.9. Let T_M be the holomorphic tangent bundle of a complex variety M . The Kodaira-Spencer DGLA is defined as $KS(M) = \oplus \Gamma(M, \mathcal{A}^{0,p}(T_M))[-p]$ with the Dolbeault differential; if z_1, \dots, z_n are local holomorphic coordinates we have $[\phi d\overline{z}_I, \psi d\overline{z}_J] = [\phi, \psi] d\overline{z}_I \wedge d\overline{z}_J$ for $\phi, \psi \in \mathcal{A}^{0,0}(T_M)$, $I, J \in \wedge^* \{1, \dots, n\}$.

Example 1.10. Let $\mathbb{K} = \mathbb{R}$ and X be a smooth differentiable variety. The algebra of polyvector fields is given by

$$T_{Poly}(X) = \oplus \Gamma(X, \bigwedge^{n+1} T_X)[-n], \quad n \geq -1$$

with zero differential and the *Schouten-Nijenhuis* bracket defined in the following way (cf. 1.12):

For every open subset $U \subset X$, every function $h \in T_{Poly}^{-1}(U) = C^\infty(U)$ and every vector fields $\xi_0, \dots, \xi_n, \zeta_0, \dots, \zeta_m \in \Gamma(U, T_U)$ we set

$$[\xi_0 \wedge \dots \wedge \xi_n, h] = \sum_{i=0}^n (-1)^{n-i} \xi_i(h) \xi_0 \wedge \dots \wedge \hat{\xi}_i \wedge \dots \wedge \xi_n$$

$$[\xi_0 \wedge \dots \wedge \xi_n, \zeta_0 \wedge \dots \wedge \zeta_m] = \sum_{i=0}^n \sum_{j=0}^m (-1)^{i+j} [\xi_i, \zeta_j] \wedge \xi_0 \wedge \dots \wedge \hat{\xi}_i \wedge \dots \wedge \xi_n \wedge \zeta_0 \wedge \dots \wedge \hat{\zeta}_j \wedge \dots \wedge \zeta_m.$$

Example 1.11. Let A be an associative \mathbb{K} -algebra, the DGLA of Hochschild cochains is defined by

$$G = \oplus \text{Hom}_{\mathbb{K}}(A^{\otimes(n+1)}, A)[-n]$$

where by $\text{Hom}_{\mathbb{K}}$ we mean homomorphisms of \mathbb{K} -vector spaces. The differential is the usual differential of Hochschild cohomology: $\phi \in G^n$

$$(d\phi)(a_0 \otimes \dots \otimes a_{n+1}) = a_0 \phi(a_0 \otimes \dots \otimes a_{n+1}) + (-1)^n \phi(a_0 \otimes \dots \otimes a_n) a_{n+1} + \sum_{i=0}^n (-1)^i \phi(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_{n+1})$$

The bracket is the *Gerstenhaber* one:

$$[\phi, \psi] = \phi \circ \psi - (-1)^{\overline{\phi}\overline{\psi}} \psi \circ \phi$$

where the (non-associative) product \circ is defined, for $\phi \in G^n$ and $\psi \in G^m$, by the formula

$$\phi \circ \psi(a_0 \otimes \cdots \otimes a_{n+m}) = \sum_{i=0}^n (-1)^{im} \phi(a_0 \otimes \cdots \otimes a_{i-1} \otimes \psi(a_i \otimes \cdots \otimes a_{i+m}) \otimes a_{i+m+1} \otimes \cdots \otimes a_{n+m}).$$

Example 1.12. Let A be a commutative \mathbb{K} -algebra and $V \subset \text{Der}_{\mathbb{K}}(A, A)$ an A -submodule such that $[V, V] \subset V$.

We can define a DGLA $L = \bigoplus_{n \geq -1} L^n$, where

$$L^{-1} = A, \quad L^0 = V, \dots, L^n = \bigwedge_A^{n+1} V, \dots$$

with zero differential and the bracket uniquely characterized by the properties:

1. $[\cdot, \cdot]: L^0 \times L^0 \rightarrow L^0$ is the usual bracket on V .
2. If $a \in L^0$, $f \in L^{-1} = A$ then $[a, f] = a(f) \in A$.
3. For every $a \in L^n$, $b \in L^m$, $c \in L^h$

$$[a, b \wedge c] = [a, b] \wedge c + (-1)^{n(m+1)} b \wedge [a, c], \quad [a \wedge b, c] = a \wedge [b, c] + (-1)^{h(m+1)} [a, c] \wedge b.$$

EXERCISE: Prove that the bracket in 1.12 is well defined. (the unicity is obvious, the existence is easy when V is free; in the general case it is convenient to think L^n as the quotient of the \mathbb{K} -vector space generated by $\xi_0 \wedge \cdots \wedge \xi_n$, with $\xi_i \in V$, by the subspace generated by the skewsymmetric A -multilinear relation).

Prove moreover that for every $h \in L^{-1}$, $\xi_0, \dots, \xi_n, \zeta_0, \dots, \zeta_m \in L^0$ we have (cf. 1.10)

$$[\xi_0 \wedge \cdots \wedge \xi_n, h] = \sum_{i=0}^n (-1)^{n-i} \xi_i(h) \xi_0 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_n$$

$$[\xi_0 \wedge \cdots \wedge \xi_n, \zeta_0 \wedge \cdots \wedge \zeta_m] = \sum_{i=0}^n \sum_{j=0}^m (-1)^{i+j} [\xi_i, \zeta_j] \wedge \xi_0 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \xi_n \wedge \zeta_0 \wedge \cdots \wedge \hat{\zeta}_j \wedge \cdots \wedge \zeta_m.$$

For a better understanding of some of next topics it is useful to consider the following functorial construction. Given a DGLA $(L, [\cdot, \cdot], d)$ we can construct a new DGLA $(L_d, [\cdot, \cdot]_d, d_d)$ by setting $L_d^i = L^i$ for every $i \neq 1$, $L_d^1 = L^1 \oplus \mathbb{K}d$ with the bracket defined by extending by bilinearity the relation $[a + vd, b + wd]_d = [a, b] + vd(b) + (-1)^{\bar{a}} wd(a)$ and differential $d_d(a + vd) = [d, a + vd]_d = d(a)$.

The natural inclusion $L \subset L_d$ is a morphism of DGLA; in the manage of Maurer-Cartan equation it is convenient to consider the affine embedding $\phi: L \rightarrow L_d$, $\phi(a) = a + d$.

For $a \in L^1$ we have

$$d(a) + \frac{1}{2}[a, a] = 0 \quad \iff \quad [\phi(a), \phi(a)]_d = 0$$

Example 1.13. (Small variations of almost complex structures).

Let M be a compact complex variety, $\mathcal{A}^{p,q}$ the vector bundle of differential forms of type (p, q) , $\mathcal{A}^{p,q}$ the fine sheaf of its smooth sections and $KS(M) = \bigoplus \Gamma(M, \mathcal{A}^{0,p}(T_M))[-p]$ the Kodaira-Spencer algebra of M . Note that $\Gamma(M, \mathcal{A}^{0,p}(T_M))$ is exactly the set of morphisms of vector bundles $A^{1,0} \rightarrow A^{0,p}$. The choice of a hermitian metric on M induce a structure of pre-Hilbert space to $\Gamma(M, \mathcal{A}^{0,p}(T_M))$. Given a sufficiently small section $\alpha \in \Gamma(M, \mathcal{A}^{0,1}(T_M))$ we can consider the perturbed differential $\bar{\partial}_\alpha = \bar{\partial} + \alpha\partial: \mathcal{A}^{0,p} \rightarrow \mathcal{A}^{0,p+1}$; more precisely for $U \subset M$ open subset with holomorphic coordinates z_1, \dots, z_n and $f \in C^\infty(U)$ we have

$$\bar{\partial}_\alpha(f d\bar{z}_I) = (\bar{\partial}f + \alpha(\partial f)) \wedge d\bar{z}_I.$$

To α we can associate also a direct sum decomposition

$$A_\alpha^{1,0} \oplus A_\alpha^{0,1} = A^{1,0} \oplus A^{0,1}$$

where $A_\alpha^{1,0}$ is the graph of $-\alpha$ and $A_\alpha^{0,1} = \overline{A_\alpha^{0,1}}$. Therefore we can consider α as a small variation of the almost complex structure, as easy computation (exercise) show that:

1. The sheaf of almost holomorphic functions of the almost complex structure $A_\alpha^{i,j}$ is exactly the kernel of $\bar{\partial}_\alpha: \mathcal{A}^{0,0} \rightarrow \mathcal{A}^{0,1}$.
2. α satisfies the Maurer-Cartan equation in the Kodaira-Spencer algebra if and only if $\bar{\partial}_\alpha^2 = 0$.
3. By Newlander-Nirenberg theorem the almost complex structure $A_\alpha^{i,j}$ is integrable if and only if α satisfies the Maurer-Cartan equation in $KS(M)$.

Let's now introduce the notion of gauge action on a DGLA:

There exists a functor exp from the category of nilpotent Lie algebras (i.e. with descending central series definitively $=0$) to the category of groups. For every nilpotent Lie algebra N there exists a natural bijection $e: N \rightarrow exp(N)$ satisfying the following properties:

1. $e^a e^b = e^{(a+b)}$ if $[a, b] = 0$. More generally $e^a e^b = e^{(a*b)}$ where $a * b \in N$ is given by the Campbell-Baker-Hausdorff formula (cf [1], [9], [10]).
2. For every vector space V and every homomorphism of Lie algebras $\rho: N \rightarrow End(V)$ such that $\rho(N)$ is a nilpotent subalgebra, the morphism

$$exp(\rho): exp(N) \rightarrow Aut(V), \quad exp(\rho)(e^a) = e^{\rho(a)},$$

is a homomorphism of groups (here $e^{\rho(a)}$ denotes the usual exponential of endomorphisms).

3. If $\rho: N \rightarrow End(V) = P$ is a representation of N as above and $ad_\rho: N \rightarrow End(P)$ is the adjoint representation, then for every $a \in N$, $f \in End(V)$

$$exp(ad_\rho)(e^a)f = e^{\rho(a)} \circ f \circ e^{-\rho(a)}.$$

EXERCISE: Prove the above items.

Lemma 1.14. *Let V, W be vector spaces and $\rho: N \rightarrow End(V)$, $\eta: N \rightarrow End(W)$ representation of a nilpotent Lie algebra N .*

Let $b: V \times V \rightarrow W$ be a bilinear symmetric form such that $2b(v, \rho(n)v) = \eta(n)b(v, v)$ for every $n \in N$, $v \in V$ and $q(v) = b(v, v)$ the associated quadratic form.

Then the cone $Z = q^{-1}(0)$ is invariant under the exponential action $exp(\rho)$.

Proof. Let $n \in N$ be a fixed element, for every $v \in V$ define the polynomial function $F_v: \mathbb{K} \rightarrow W$ by

$$F_v(t) = exp(\eta(-tn))q(exp(\rho(tn))v)$$

For every $s, t \in \mathbb{K}$, if $u = exp(\rho(sn))v$ then

$$F_v(t+s) = e^{\eta(-sn)}F_w(t), \quad \frac{\partial F_v}{\partial t}(0) = -\eta(n)q(v) + 2b(v, \rho(n)v) = 0$$

$$\frac{\partial F_v}{\partial t}(s) = e^{\eta(-sn)}\frac{\partial F_w}{\partial t}(0) = 0$$

As the field \mathbb{K} has characteristic 0 every function F_v is constant, in particular for every $exp(\eta)$ invariant subspace $T \subset W$ the space $q^{-1}(T)$ is $exp(\rho)$ invariant. \square

Corollary 1.15. *Let L be a nilpotent DGLA, then the quadratic cone*

$$Z = \{a \in L^1 \mid [v, v] = 0\}$$

is stable under the exponential of the adjoint action of L^0 .

Proof. By Jacobi identity $[v, [a, v]] = -[v, [v, a]] = \frac{1}{2}[a, [v, v]] = 0$ and we can apply 1.14. \square

EXERCISE: In the notation of 1.14, if \mathbb{K} is algebraically closed and $q^{-1}(0)$ is not a double plane then 1.14 holds under the weaker assumption $b(v, \rho(n)v) = 0$ for every $n \in N, v \in Z$.

Again if L is a DGLA with L^0 nilpotent, the adjoint action of L^0 over L_d^1 preserves the cone $[v, v]_d = 0$ and the affine hyperplane $\{v + d \mid v \in L^1\}$; this gives an action of $\exp(L^0)$ over L^1 preserving the solution of the Maurer-Cartan equation. The infinitesimal generator of this action is given by $L^0 \ni a \mapsto [a, v + d]_d = [a, v] - d(a) \in T_v L^1$.

REMARK. It is often convenient to think the elements of L_d as operators on a \mathbb{Z} -graded vector space $V = \oplus V^i$, this means that the map $\rho: L_d \rightarrow \text{End}(V) = P$ is a morphism of DGLA. For example the elements of the extended Kodaira-Spencer algebra

$$KS(M)_\partial = \oplus \Gamma(M, \mathcal{A}^{0,p}(T_M))[-p] \oplus \mathbb{C}\bar{\partial}$$

act in a natural way on the graded vector space $V = \oplus \Gamma(M, \mathcal{A}^{0,p})[-q]$.

If we call $ad_\rho: L_d \rightarrow \text{End}(P)$ and $Ad_\rho: GL(L_d) \rightarrow GL(P)$ the the adjoint actions

$$ad_\rho(a)B = [\rho(a), B] = \rho(a)B - B\rho(a), \quad Ad_\rho(e^a)B = e^{\rho(a)}B e^{-\rho(a)}$$

by the properties of \exp we have $\exp(ad_\rho(a)) = Ad_\rho(\exp(a))$.

It is natural to consider two operator $A, B \in \text{End}(V)$ gauge equivalent if there exists $a \in L^0$ such that $A = e^{\rho(a)}B e^{-\rho(a)}$.

Therefore if $x, y \in L_d^1$ are gauge equivalent then $\rho(x)$ is gauge equivalent to $\rho(y)$, independently from the particular representation.

Example 1.16. A pair (L, d) is a *Polarized graded algebra* (PGA) if $L = \oplus L^i, i \in \mathbb{Z}$, is a graded associative \mathbb{K} -algebra, $d \in L^1$ and $d^2 = 0$.

The typical example is, for a fixed differential graded vector space $(V, d), d: V \rightarrow V[1]$, the algebra $\text{End}(V) = \oplus L^i$, where L^i is the space of morphisms $V \rightarrow V[i]$ in the category $\text{Graded}^{\mathbb{K}}$.

Every PGA admits natural structures of DGA (differential graded algebra) with differential $\delta(a) = da - (-1)^{\bar{a}}ad$ and DGLA with bracket $[a, b] = ab - (-1)^{\bar{a}\bar{b}}ba$.

If L^0 is a nilpotent \mathbb{K} -algebra then we can define the gauge action over L^1 in a characteristic free way. First we define a group structure on L^0 by setting $g * h = g + h + gh$ and the gauge action

$$g * v = v + \sum_{i=0}^{\infty} (-1)^i [g, d + v] g^i = v + \sum_{i=0}^{\infty} (-1)^i ([g, v] - \delta(g)) g^i.$$

EXERCISE: Prove that in characteristic 0 the action of 1.16 is equivalent to the usual gauge action.

EXERCISE: Let L be a DGLA with differential d , then the universal enveloping algebra of L_d is a polarized graded algebra.

In some cases (as in [11, p. 9]) it is useful to describe a DGLA as a suitable subset of a concrete PGA.

2 Quickstart guide to functors of Artin rings

In this section \mathbb{K} is a fixed field of arbitrary characteristic. We denote by:

\hat{Art} the category of local complete Noetherian rings with residue field \mathbb{K} .

$Art \subset \hat{Art}$ the full subcategory of Artinian rings.

For a given $S \in \hat{Art}$, Art_S is the category of Artinian S -algebras with residue field \mathbb{K} .

\hat{Art}_S the category of local complete Noetherian S -algebras with residue field \mathbb{K} .

Set the category of sets (in a fixed universe). $*$ is a fixed 1-point set.

Grp the category of groups. (\emptyset is not a group)

Definition 2.1. A *functor of Artin rings* is a covariant functor $F: Art_S \rightarrow Set$, $S \in \hat{Art}$ such that $F(\mathbb{K}) = *$.

The main interest to functors of Artin rings comes from deformation theory and moduli problems; from this point of view the notion of prorepresentability is one of the most important.

Given $R \in \hat{Art}_S$ we define a functor $h_R: Art_S \rightarrow Set$ by setting $h_R(A)$ as the set of S -algebra homomorphisms $R \rightarrow A$.

Definition 2.2. A functor $F: Art_S \rightarrow Set$ is *prorepresentable* if it is isomorphic to h_R for some $R \in \hat{Art}_S$.

The functors of Artin rings $F: Art_S \rightarrow Set$, with their natural transformation form a category denoted by Fun_S .

We left as an exercise to prove that the Yoneda functor $\hat{Art}_S^0 \rightarrow Fun_S$, $R \rightarrow h_R$, is fully faithful.

A necessary condition for a functor F to be prorepresentable is homogeneity. We first note that on Art_S there exist fibred products

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array} \quad (1)$$

Applying a functor $F \in Fun_S$ to the cartesian diagram (1) we get a map

$$\eta: F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C)$$

Definition 2.3. The functor F is *homogeneous* if η is an isomorphism whenever $B \rightarrow A$ is surjective.

Since the diagram (1) is cartesian, every prorepresentable functor is homogeneous.

Example 2.4. Other examples of homogeneous functors are:

2.4.0) The trivial functor $F(A) = *$ For every $A \in Art_S$.

2.4.1) Let M be a flat S -module, define $\hat{M}: Art_S \rightarrow Set$ as $\hat{M}(A) = M \otimes_S m_A$, where $m_A \subset A$ is the maximal ideal.

2.4.2) Assume $char \mathbb{K} = 0$ and let L^0 be a Lie Algebra over \mathbb{K} . We can define a group functor $exp(L^0): Art_{\mathbb{K}} \rightarrow Grp$ by setting $exp(L^0)(A) = exp(L^0 \otimes m_A)$ the exponential of the nilpotent Lie algebra $L^0 \otimes m_A$.

2.4.3) Let $X \rightarrow Spec(S)$ be a flat scheme over S , we can define $Aut(X/S): Art_S \rightarrow Grp$ by setting $Aut(X/S)(A)$ as the group of automorphisms of $X_A = X \times_{Spec(S)} Spec(A)$, commuting with the projection $X_A \rightarrow Spec(A)$ which are the identity on $X_{\mathbb{K}}$.

EXERCISE: Prove that the above functors \hat{M} , $\exp(L^0)$ and $\text{Aut}(X/S)$ are homogeneous. Show also by an example that if M is not S -flat then the functor \hat{M} is not homogeneous in general.

Definition 2.5. A functor F is called a *deformation functor* if:

1. η is surjective whenever $B \rightarrow A$ is surjective.
2. η is an isomorphism whenever $A = \mathbb{K}$.

The name comes from the fact that most functors arising in deformation theory are deformation functors.

EXERCISE: Let X be a scheme over \mathbb{K} , the deformation functor of X is defined as $\text{Def}_X: \text{Art}_{\mathbb{K}} \rightarrow \text{Set}$, where $\text{Def}_X(A)$ is the set of isomorphism classes of commutative cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{i} & X_A \\ \downarrow p & & \downarrow p_A \\ \text{Spec}(\mathbb{K}) & \longrightarrow & \text{Spec}(A). \end{array}$$

with i closed embedding and p_A flat morphism. Prove that Def_X is a deformation functor.

We denote $\mathbb{K}[\epsilon] = \mathbb{K} \oplus \mathbb{K}\epsilon = \mathbb{K}[t]/(t^2)$ with the trivial structure of S -algebra given by $S \rightarrow \mathbb{K} \rightarrow \mathbb{K}[\epsilon]$. More generally by ϵ and ϵ_i we will always mean indeterminates annihilated by the maximal ideal, and in particular of square zero (e.g., the algebra $\mathbb{K}[\epsilon]$ has dimension 2 and $\mathbb{K}[\epsilon_1, \epsilon_2]$ has dimension 3 as a \mathbb{K} -vector space).

Proposition 2.6 (Schlessinger, [13]). *Let F be a deformation functor, the set $t_F = F(\mathbb{K}[\epsilon])$ has a natural structure of \mathbb{K} -vector space. If $\phi: F \rightarrow G$ is a morphism of deformation functors the map $\phi: t_F \rightarrow t_G$ is linear.*

Proof. Let $\alpha \in \mathbb{K}$, the scalar multiplication by α is induced by the morphism in Art_S , $\hat{\alpha}: \mathbb{K}[\epsilon] \rightarrow \mathbb{K}[\epsilon]$ given by $\hat{\alpha}(a + b\epsilon) = a + \alpha b\epsilon$. Since $t_F \times t_F = F(\mathbb{K}[\epsilon_1] \times_{\mathbb{K}} \mathbb{K}[\epsilon_2])$ the sum is induced by the map $\mathbb{K}[\epsilon_1] \times_{\mathbb{K}} \mathbb{K}[\epsilon_2] = \mathbb{K}[\epsilon_1, \epsilon_2] \rightarrow \mathbb{K}[\epsilon]$ defined by $a + b\epsilon_1 + c\epsilon_2 \rightarrow a + (b+c)\epsilon$. \square

EXERCISE: Prove that if $R \in \hat{\text{Art}}_{\mathbb{K}}$, $F = h_R$ then t_F is isomorphic to the Zariski tangent space of $\text{Spec}(R)$; more generally if $R \in \hat{\text{Art}}_S$ then the tangent space of $h_R \in \text{Fun}_S$ is equal to the \mathbb{K} -dual of $m_R/(m_R^2 + m_S R)$.

EXERCISE: Let $A \in \text{Art}_S$, $F \in \text{Fun}_S$. Prove that there exists a natural bijection between $F(A)$ and morphisms $\text{Mor}(h_A, F)$.

Definition 2.7. A morphism $\phi: F \rightarrow G$ in the category Fun_S is called:

1. *unramified* if $\phi: t_F \rightarrow t_G$ is injective.
2. *smooth* if for every surjection $B \rightarrow A$ in Art_S the map $F(B) \rightarrow G(B) \times_{G(A)} F(A)$ is also surjective.
3. *étale* if it is smooth and unramified.

EXERCISE: If $\phi: F \rightarrow G$ is smooth then $F(A) \rightarrow G(A)$ is surjective for every A .

EXERCISE: Let $\phi: F \rightarrow G$, $\psi: H \rightarrow G$ be morphisms of deformation functors. If ϕ is smooth and H is prorepresentable then there exists a morphism $\tau: H \rightarrow F$ such that $\psi = \phi\tau$.

Definition 2.8. A functor F is *smooth* if the morphism $F \rightarrow *$ is smooth, i.e. if $F(A) \rightarrow F(B)$ is surjective for every surjective morphism of S -algebras $A \rightarrow B$.

Lemma 2.9. A prorepresentable functor h_R , $R \in \hat{Art}_S$, is smooth if and only if $R = S[[x_1, \dots, x_n]]$.

Proof. Exercise, cf. [13]. □

EXERCISE: Let X be a scheme over \mathbb{K} . Prove that if for every $A \in Art_{\mathbb{K}}$ and every deformation $X_A \rightarrow Spec(A)$ the functor $Aut(X/A)$ is smooth then Def_X is homogeneous.

Lemma 2.10. Let $\phi: F \rightarrow G$ be an unramified morphism of deformation functors. If G is homogeneous then $\phi: F(A) \rightarrow G(A)$ is injective for every A .

Proof. We prove by induction on the length of A that $F(A) \rightarrow G(A)$ is injective. Let $\epsilon \in A$ be an element such that $\epsilon m_A = 0$ and $B = A/(\epsilon)$; by induction $F(B) \rightarrow G(B)$ is injective. We note that

$$A \times_{\mathbb{K}} \mathbb{K}[\epsilon] \rightarrow A \times_B A, \quad (a, \bar{a} + \alpha\epsilon) \mapsto (a, a + \alpha\epsilon)$$

is an isomorphism of S -algebras, in particular for every deformation (resp. homogeneous) functor F there exists a natural surjective (resp. bijective) map

$$v: F(A) \times_{t_F} \rightarrow F(A) \times_{F(B)} F(A)$$

such that $v(F(A) \times \{0\}) = \Delta$ is the diagonal. Let $\xi, \eta \in F(A)$ such that $\phi(\xi) = \phi(\eta)$; since ϕ is injective over $F(B)$ the pair (ξ, η) belongs to the fibred product $F(A) \times_{F(B)} F(A)$ and there exists $h \in t_F$ such that $v(\xi, h) = (\xi, \eta)$. Therefore $v(\phi(\xi), \phi(h)) = (\phi(\xi), \phi(\eta)) \in \Delta$ and since G is homogeneous $\phi(h) = 0$. By the definition of unramified morphism $\phi: t_F \rightarrow t_G$ is injective and then $h = 0$, $\xi = \eta$. □

Corollary 2.11. Let $\phi: F \rightarrow G$ be an étale morphism of deformation functors. If G is homogeneous then ϕ is an isomorphism.

Proof. Evident. □

Let \sim the equivalence relation on the category of deformation functors generated by the étale morphisms. It is an easy consequence (left as exercise) of the above results that $h_R \sim h_T$ if and only if R, T are isomorphic S -algebras.

By a small extension e in Art_S we mean an exact sequence

$$e: \quad 0 \longrightarrow M \longrightarrow B \xrightarrow{\phi} A \longrightarrow 0$$

where ϕ is a morphism in Art_S and M is an ideal of B annihilated by the maximal ideal m_B . In particular M is a finite dimensional vector space over $B/m_B = \mathbb{K}$.

EXERCISE: Let $0 \longrightarrow M \longrightarrow B \xrightarrow{\phi} A \longrightarrow 0$ be a small extension and F a deformation functor. Then there exists a natural transitive action of $F(\mathbb{K} \oplus M) = t_F \otimes M$ on every nonempty fibre of $F(A) \xrightarrow{\phi} F(B)$. (Hint: look at the proof of 2.10).

Definition 2.12. Let F be a functor of Artin rings; an *obstruction theory* (V, v_e) for F is the data of a \mathbb{K} -vector space V , called *obstruction space*, and for every small extension in Art_S

$$e: \quad 0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

of an *obstruction map* $v_e: F(A) \rightarrow V \otimes_{\mathbb{K}} M$ satisfying the following properties:

1. If $\xi \in F(A)$ can be lifted to $F(B)$ then $v_e(a) = 0$.
2. (base change) For every morphism $\alpha: e_1 \rightarrow e_2$ of small extension, i.e. for every commutative diagram

$$\begin{array}{ccccccccc}
e_1: & 0 & \longrightarrow & M_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\
& & & \downarrow \alpha_M & & \downarrow \alpha_B & & \downarrow \alpha_A & & \\
e_2: & 0 & \longrightarrow & M_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0.
\end{array}$$

we have $v_{e_2}(\alpha_A(a)) = (Id_V \otimes \alpha_M)(v_{e_1}(a))$ for every $a \in F(A_1)$.

EXERCISE: If F is smooth then all the obstruction maps are trivial.

Definition 2.13. An obstruction theory (V, v_e) for F is called *complete* if the converse of item i) in 2.12 holds; i.e. the lifting exists if and only if the obstruction vanish.

Clearly if F admits a complete obstruction theory then it admits infinitely ones; it is in fact sufficient to embed V in a bigger vector space. One of the main interest is to look for the “smallest” complete obstruction theory.

Definition 2.14. A morphism of obstruction theories $(V, v_e) \rightarrow (W, w_e)$ is a linear map $\theta: V \rightarrow W$ such that $w_e = \theta v_e$ for every small extension e . An obstruction theory (O_F, ob_e) for F is called *universal* if for every obstruction theory (V, v_e) there exist an unique morphism $(O_F, ob_e) \rightarrow (V, v_e)$.

If a universal obstruction theory exists then it is unique up to isomorphisms. An important result is

Theorem 2.15. *Let F be a deformation functor, then there exists an universal obstruction theory (O_F, ob_e) for F . Moreover the universal obstruction theory is complete and every element of the vector space O_F is of the form $ob_e(\xi)$ for some principal extension*

$$e: 0 \longrightarrow \mathbb{K} \longrightarrow B \longrightarrow A \longrightarrow 0$$

and some $\xi \in F(A)$.

Proof. This is quite long and not easy. The interested reader can found a proof in [5]. \square

We note that if F is not a deformation functor then in general F doesn't have any complete obstruction theory even if F satisfies Schlessinger's conditions H1, H2, H3 of [13].

Example 2.16. (The primary obstruction map, $char \mathbb{K} \neq 2$).

Let (V, v_e) be a complete obstruction theory for a deformation functor F and let $[\cdot, \cdot]: t_F \times t_F \rightarrow V$ be the obstruction map associated to the small extension

$$0 \longrightarrow \mathbb{K} \xrightarrow{xy} \mathbb{K}[x, y]/(x^2, y^2) \longrightarrow \mathbb{K}[x, y]/(x^2, xy, y^2) \longrightarrow 0$$

(Note that $\mathbb{K}[x, y]/(x^2, xy, y^2) = \mathbb{K}[x]/(x^2) \times_{\mathbb{K}} \mathbb{K}[y]/(y^2)$). Then $[\cdot, \cdot]$ is a symmetric bilinear map (Exercise).

The substitution $\alpha(t) = x + y$ gives a morphism of small extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{K} & \xrightarrow{t^2} & \mathbb{K}[t]/(t^3) & \longrightarrow & \mathbb{K}[t]/(t^2) & \longrightarrow & 0 \\
& & \downarrow 2 & & \downarrow \alpha & & \downarrow \alpha & & \\
0 & \longrightarrow & \mathbb{K} & \xrightarrow{xy} & \mathbb{K}[x, y]/(x^2, y^2) & \longrightarrow & \mathbb{K}[x, y]/(x^2, xy, y^2) & \longrightarrow & 0.
\end{array}$$

From this and base change axiom it follows that the obstruction of lifting $\xi \in t_F$ to $F(\mathbb{K}[t]/(t^3))$ is equal to $\frac{1}{2}[\xi, \xi]$ “the quadratic part of Maurer-Cartan equation ”!!

Let $\phi: F \rightarrow G$ be a morphism of deformation functors and $(V, v_e), (W, w_e)$ obstruction theories for F and G respectively; a linear map $\phi': V \rightarrow W$ is *compatible* with ϕ if $w_e \phi = \phi' v_e$ for every small extension e .

Proposition 2.17 (Standard smoothness criterion). *Let $\nu: F \rightarrow G$ be a morphism of deformation functors and $(V, v_e) \xrightarrow{\nu'} (W, w_e)$ a compatible morphism between obstruction theories. If (V, v_e) is complete, $V \rightarrow W$ injective and $t_F \rightarrow t_G$ surjective then ϕ is smooth.*

Proof. Let $e: 0 \rightarrow \mathbb{K} \rightarrow B \rightarrow A \rightarrow 0$ be a small extension and let $(a, b') \in F(A) \times_{G(A)} G(B)$; let $a' \in G(A)$ be the common image of a and b' . Then $w_e(a') = 0$, as a' lifts to $G(B)$, hence $v_e(a) = 0$ by injectivity of ν' . Therefore a lifts to some $b \in F(B)$. In general $b'' = \nu(b)$ is not equal to b' . However, $(b'', b') \in G(B) \times_{G(A)} G(B)$ and therefore b'' differs from b' by the action of an element $v \in t_G$ (v need not be unique). As $t_F \rightarrow t_G$ is surjective, v lifts to a $w \in t_F$; acting with w on b produces a lifting of a which maps to b' , as required. \square

The composition $(V, v_e \phi)_{o(\phi)}$ is an obstruction theory for F and therefore there exists a unique compatible map $O_F \xrightarrow{o(\phi)} V$. In particular this happens for the universal obstruction of G . Therefore every morphism of deformation functors $F \rightarrow G$ induces linear morphisms both in tangent $t_F \rightarrow t_G$ and obstruction spaces $O_F \rightarrow O_G$. As an immediate consequence of 2.17 we have

Proposition 2.18. *A morphism of deformation functors $\nu: F \rightarrow G$ is smooth if and only if $t_F \rightarrow t_G$ is surjective and $o(\nu): O_F \rightarrow O_G$ is injective. In particular F is smooth if and only if $O_F = 0$.*

Proof. One implication is contained in 2.17. On the other side, if the morphism is smooth then $t_F \rightarrow t_G$ is surjective; since every $x \in O_F$ is the obstruction to lifting some element and (O_G, ob_e) is complete we have $o(\nu)(x) = 0$ if and only if $x = 0$. \square

EXERCISE: Let $F \xrightarrow{\phi} G \xrightarrow{\psi} H$ be morphisms of deformation functors. Prove that:

1. If ϕ, ψ are smooth then the composition $\psi \phi$ is smooth.
2. If $\psi \phi, \phi$ are smooth then ψ is smooth.
3. If $\psi \phi$ is smooth and $t_F \rightarrow t_G$ is surjective then ϕ is smooth.

EXERCISE: If $\phi: F \rightarrow G$ is smooth then $o(\phi): O_F \rightarrow O_G$ is an isomorphism.

In most concrete cases it is very difficult to calculate the universal obstruction space, while it is easy to describe complete obstruction theories and compatible morphism between obstruction spaces.

Consider now the following situation. $F: Art_S \rightarrow Set$ a deformation functor, $G: Art_S \rightarrow Grp$ a group functor of Artin rings which is a smooth deformation functor. (A theorem in [5] asserts that smoothness is automatic if S is a field of characteristic 0).

We assume that G acts on F ; this means that for every $A \in Art_S$ there exists an action $G(A) \times F(A) \xrightarrow{*} F(A)$, all these actions must be compatible with morphisms in Art_S .

In particular there exists an action $t_G \times t_F \xrightarrow{*} t_F$, denote by $\nu: t_G \rightarrow t_F$ the map $h \mapsto h * 0$.

Lemma 2.19. *The map ν is linear and t_G acts on t_F by translations, $h * v = \nu(h) + v$.*

Proof. Since the vector space structure on t_F and t_G is defined functorially by using morphisms in Art_S , it is easy to see that for every $a, b \in t_F, g, h \in t_G, t \in \mathbb{K}$ we have $(g + h) * (a + b) = (g * a) + (h * b), t(g * a) = (tg) * (ta)$. Setting $a = b = 0$ we get the linearity of ν and setting $a = 0, h = 0$ we have $g * b = (g * 0) + (0 * b) = \nu(g) + b$. \square

Lemma 2.20. *In the notation above the quotient functor $D = F/G$ is a deformation functor, $t_D = \text{Coker } \nu$, the projection $F \rightarrow D$ is smooth and for every obstruction theory (V, v_e) of F the group functor G acts trivially on the obstruction maps v_e . In particular the natural map $O_F \rightarrow O_D$ is an isomorphism.*

Proof. We left as exercise the (very easy) proof that D is a deformation functor and that $F \rightarrow D$ is a smooth morphism. The statement about obstruction follow easily from 2.15 and 2.18. Since this result will be fundamental in Section 3, in order to make this lecture selfcontained we give here an alternative proof which do not use the existence of O_F, O_D .

Let $a \in F(A)$, $g \in G(A)$ and $e : 0 \rightarrow \mathbb{K} \xrightarrow{j} B \rightarrow A \rightarrow 0$ be a small extension. We need to prove that $v_e(a) = v_e(g * a)$.

Let $\pi_i : A \times_{\mathbb{K}} A \rightarrow A$, $i = 1, 2$ be the projections, $\delta : A \rightarrow A \times_{\mathbb{K}} A$ be the diagonal and

$$\nabla e : 0 \rightarrow \mathbb{K} \xrightarrow{(j,0)} C \rightarrow A \times_{\mathbb{K}} A \rightarrow 0$$

where C is the quotient of $B \times_{\mathbb{K}} B$ by the ideal generated by (j, j) .

Let $c = \delta(a) \in F(A \times_{\mathbb{K}} A)$ and let $\tau \in G(A \times_{\mathbb{K}} A)$ such that $\pi_1(\tau) = 1$, $\pi_2(\tau) = g$, as δ lifts to a morphism $A \rightarrow C$ and G is smooth we have that c and τ lift to $F(C)$ and $G(C)$ respectively and therefore also $\tau * c$ lifts to $F(C)$. Since $\pi_1(\tau * c) = \pi_1(\tau) * \pi_1(c) = a$, $\pi_2(\tau * c) = \pi_2(\tau) * \pi_2(c) = g * a$.

On the other hand it is an easy consequence of the base change axiom that $0 = v_{\nabla e}(\tau * c) = v_e(\pi_1(\tau * c)) - v_e(\pi_2(\tau * c)) = v_e(a) - v_e(g * a)$. \square

Let now $R \in \text{Art}_S$ e $a, b \in F(R)$, we define a functor $\text{Iso}(a, b) : \text{Art}_R \rightarrow \text{Set}$ by setting

$$\text{Iso}(a, b)(R \xrightarrow{f} A) = \{g \in G(A) \mid g * f(a) = f(b)\}.$$

Proposition 2.21. *If F is homogeneous then $\text{Iso}(a, b)$ is a deformation functor with tangent space $\ker \nu$ and complete obstruction space $\text{Coker } \nu = t_D$.*

Proof. For simplicity of notation let's denote $H = \text{Iso}(a, b)$. Assume it is given a commutative diagram in $\text{Art}_{\mathbb{K}}$

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & A \\ \downarrow \beta & & \downarrow \delta \\ B & \xrightarrow{\gamma} & C. \end{array}$$

with γ surjective and let $(g_1, g_2) \in H(B) \times_{H(C)} H(A)$; since G is a deformation functor there exists $g \in G(B \times_C A)$ which lifts (g_1, g_2) . Let a', b' be the images of a, b on $F(B \times_C A)$, the elements $g * a'$ and b' have the same images in $F(A), F(B)$, since F is homogeneous it follows that $g \in H(B \times_C A)$; this proves that H is a deformation functor.

It is a trivial consequence of Lemma 2.19 that $t_H = \ker \nu$.

Let consider a small extension in Art_S

$$0 \rightarrow \mathbb{K} \xrightarrow{\epsilon} A \rightarrow B \rightarrow 0$$

and $g \in H(B)$, we want to determine the obstruction to lifting g to $H(A)$, by assumption G is smooth and then there exists $g' \in G(A)$ lifting g . Let $a', b' \in F(A)$ be the images of a, b , then

$$(g' * a', b') \in F(A) \times_{F(B)} F(A) = F(A \times_B A) = F(A) \times_{t_F}$$

It is easy to see that, changing g' inside the liftings of g , the projection over t_F change by an element of the image of ν , therefore the natural projection of the pair $(g' * a', b')$ into $\text{Coker } \nu$ gives a complete obstruction (some details are left to the reader). \square

The above construction of the obstruction theory for $Iso(a, b)$ is natural, in particular if G' is a smooth group functors acting on a homogeneous functors F' and $G \rightarrow G'$, $\phi: F \rightarrow F'$ are compatible morphisms then the induced morphism $\text{Coker } \nu \rightarrow \text{Coker } \nu'$ is a morphism of obstruction spaces compatible with the morphisms $Iso(a, b) \rightarrow Iso(\phi(a), \phi(b))$. We have therefore the following corollary.

Corollary 2.22. *In the above situation if $\ker \nu \rightarrow \ker \nu'$ is surjective and $\text{Coker } \nu \rightarrow \text{Coker } \nu'$ is injective then $F/G \rightarrow F'/G'$ is injective.*

Proof. (cf. [13]) We need to prove that, given $a, b \in F(A)$, if $Iso(\phi(a), \phi(b))(A) \neq \emptyset$ then $Iso(a, b)(A) \neq \emptyset$, by standard criterion of smoothness the morphism $Iso(a, b) \rightarrow Iso(\phi(a), \phi(b))$ is smooth, in particular $Iso(a, b)(A) \rightarrow Iso(\phi(a), \phi(b))(A)$ is surjective. \square

REMARK. The hypothesis F homogeneous in Proposition 2.21 can be weakened by taking F deformation functor and assuming also the existence of a complete relative obstruction theory for the morphism $G \rightarrow F$, $g \mapsto g * F(\mathbb{K})$, see [5]. This last condition seems usually verified in concrete cases, we only know counterexamples in positive characteristic.

EXERCISE: In the same notation of Corollary 2.22, if in addition G' is homogeneous and $\ker \nu \rightarrow \ker \nu'$ is bijective then $G \times F \rightarrow G' \times F'$ is a fully faithful morphism of groupoids.

3 Deformation functors associated to a DGLA

Let $L = \oplus L^i$ be a DGLA over a field \mathbb{K} of characteristic 0, we can define the following three functors:

1. The Gauge functor. $G_L: Art_{\mathbb{K}} \rightarrow Group$, defined by $G_L(A) = exp(L^0 \otimes m_A)$. It is immediate to see that G_L is smooth and homogeneous.
2. The Maurer-Cartan functor. $MC_L: Art_{\mathbb{K}} \rightarrow Set$ defined by

$$MC_L(A) = \{x \in L^1 \otimes m_A \mid dx + \frac{1}{2}[x, x] = 0\}.$$

MC_L is a homogeneous functor. If L is abelian then MC_L is smooth.

3. Since $L \otimes m_A$ is a DGLA with $(L \otimes m_A)^0$ nilpotent, by the result of Section 1 there exists an action of the group functor G_L over MC_L , we call $Def_L = MC_L/G_L$ the corresponding quotient. Def_L is called the **deformation functor** associated to the DGLA L .

In general Def_L is not homogeneous but it is only a deformation functor.

It is also evident that every morphism $\alpha: L \rightarrow N$ of DGLA induces morphisms of functors $G_L \rightarrow G_N$, $MC_L \rightarrow MC_N$. These morphisms are compatible with the gauge action and therefore induce a morphism between the deformation functors $Def_L \rightarrow Def_M$.

We are now ready to compute tangent and complete obstructions for the above functors.

- a) G_L is smooth, its tangent space is $G_L(\mathbb{K}[\epsilon]) = L^0 \otimes \mathbb{K}\epsilon$, where $\epsilon^2 = 0$.
- b) The tangent space of MC_L is

$$t_{MC_L} = \{x \in L^1 \otimes \mathbb{K}\epsilon \mid dx + \frac{1}{2}[x, x] = 0\} = Z^1(L) \otimes \mathbb{K}\epsilon$$

If $b \in L^0 \otimes \mathbb{K}\epsilon$ and $x \in Z^1 \otimes \mathbb{K}\epsilon$ we have $\exp(b)(x+d) = \exp(ad(b))(x+d) = x+d+[b, x+d]_d = x+d+db$ and therefore the action $t_{G_L} \times t_{MC_L} \rightarrow t_{MC_L}$ it is given by $(b, x) \mapsto x+db$.

Let's consider a small extension in $Art_{\mathbb{K}}$

$$e: \quad 0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

and let $x \in MC_L(B) = \{x \in L^1 \otimes m_B \mid dx + \frac{1}{2}[x, x] = 0\}$; we define an obstruction $v_e(x) \in H^2(L \otimes J) = H^2(L) \otimes J$ in the following way:

First take a lifting $\tilde{x} \in L^1 \otimes m_A$ of x and consider $h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] \in L^2 \otimes J$; we have

$$dh = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h, \tilde{x}] - \frac{1}{2}[[\tilde{x}, \tilde{x}], \tilde{x}]$$

Since $[L^2 \otimes J, L^1 \otimes m_A] = 0$ we have $[h, \tilde{x}] = 0$, by Jacobi identity $[[\tilde{x}, \tilde{x}], \tilde{x}] = 0$ and then $dh = 0$.

We define $v_e(x)$ as the class of h in $H^2(L \otimes J) = H^2(L) \otimes J$; the first thing to prove is that $v_e(x)$ is independent from the choice of the lifting \tilde{x} ; every other lifting is of the form $y = \tilde{x} + z$, $z \in L^1 \otimes J$ and then

$$d\tilde{y} + \frac{1}{2}[y, y] = h + dz.$$

It is evident from the above computation that:

1. (H^2, v_e) is a complete obstruction theory for the functor MC_L .
2. If $\phi: L \rightarrow M$ is a morphism of DGLA then the linear map $H^2(\phi): H^2(L) \rightarrow H^2(M)$ is a morphism of obstruction spaces compatible with the morphism $\phi: MC_L \rightarrow MC_M$.

Let's compute the primary obstruction map $v_2: Z^1(L) = t_{MC_L} \rightarrow H^2(L)$ associated to the small extension

$$0 \longrightarrow \mathbb{K} \xrightarrow{t^2} \mathbb{K}[t]/(t^3) \longrightarrow \mathbb{K}[t]/(t^2) \longrightarrow 0$$

Let $x \in MC_L(\mathbb{K}[t]/(t^2)) = Z^1(L) \otimes \mathbb{K}t$ and let \tilde{x} be a lifting of x to $L^1 \otimes (t)/(t^3)$, we can choose $\tilde{x} \in Z^1(L) \otimes (t)/(t^3)$ and therefore the primary obstruction $v_2(x)$ is the class of $h = d\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}] = \frac{1}{2}[x, x]$ into $H^2(L) \otimes \mathbb{K}t^2$.

EXERCISE: Assume either $H^2(L) = 0$ or $[L^1, L^1] \subset B^2$, then MC_L is smooth. If MC_L is smooth then $[Z^1, Z^1] \subset B^2$.

EXERCISE: If $[Z^1, Z^1] = 0$ then MC_L is smooth.

We shall prove later that if L is formal then MC_L is smooth if and only if $[Z^1, Z^1] \subset B^2$.

c) The tangent space of Def_L is simply the quotient of $Z^1(L)$ under the translation action of L^0 defined by the map $d: L^0 \rightarrow Z^1(L)$ and then $t_{Def_L} = H^1(L)$.

The projection $MC_L \rightarrow Def_L$ is smooth and then induces an isomorphism between universal obstruction theories; by 2.20 we can define naturally a complete obstruction theory $(H^2(L), o_e)$ by setting $o_e(x) = v_e(x')$ for every small extension e as above, $x \in Def_L(B)$ and $x' \in MC_L(B)$ a lifting of x .

In particular the primary obstruction map $o_2: H^1(L) \rightarrow H^2(L)$ is equal to $o_2(x) = \frac{1}{2}[x, x]$.

Theorem 3.1. *Let $\phi: L \rightarrow M$ be a morphism of DGLA and denote by $H^i(\phi): H^i(L) \rightarrow H^i(N)$ the induced maps in cohomology.*

1. *If $H^1(\phi)$ is bijective and $H^2(\phi)$ injective then the morphism $Def_L \rightarrow Def_N$ is étale.*
2. *If, in addition to 1), the map $H^0(L) \rightarrow H^0(N)$ is surjective then $Def_L \rightarrow Def_N$ is an isomorphism.*

Proof. In case 1) the morphism $Def_L \rightarrow Def_M$ is bijective on tangent spaces and injective on obstruction spaces, by the standard smoothness criterion it is étale.

In case 2), since étale morphisms are surjective, it is sufficient to prove that, for every $S \in Art_{\mathbb{K}}$ the map $Def_L(S) \rightarrow Def_N(S)$ is injective.

Let $a, b \in MC_L(S)$, as in Section 2 we define the functor

$$Iso(a, b): Art_S \rightarrow Set, \quad Iso(a, b)(S \xrightarrow{\eta} A) = \{g \in exp(L \otimes m_A) \mid g * \eta(a) = \eta(b)\}$$

Since MC_L is homogeneous, $Iso(a, b)$ is a deformation functor with tangent space $Z^0(L)$ and complete obstruction space $H^1(L)$.

Let $K_a: Art_S \rightarrow Grp$ the group functor defined by

$$K_a(S \xrightarrow{\eta} A) = exp(\{[\eta(a), b] + db \mid b \in L^{-1} \otimes m_A\}) \subset exp(L^0 \otimes m_A)$$

The above definition makes sense since it is easy to see that $\{[\eta(a), b] + db \mid b \in L^{-1} \otimes m_A\}$ is a Lie subalgebra of $L^0 \otimes m_A$. Since $L^{-1} \otimes m_A \rightarrow L^{-1} \otimes m_B$ is surjective for every surjective $A \rightarrow B$, the functor K_a is smooth.

Moreover K_a is a subfunctor of $Iso(a, a)$ and therefore acts by right multiplication on $Iso(a, b)$. Again by the result of Section 2 the quotient functor $Iso(a, b)/K_a$ is a deformation functor with tangent space $H^0(L)$ and complete obstruction space $H^1(L)$.

Now the proof follows as in 2.22; the morphism $Iso(a, b)/K_a \xrightarrow{\phi} Iso(\phi(a), \phi(b))/K_{\phi(a)}$ is smooth and in particular surjective. \square

Corollary 3.2. *Let $L \rightarrow N$ be a quasiisomorphism of DGLA. Then the induced morphism $Def_L \rightarrow Def_N$ is an isomorphism.*

Proof. Evident. \square

EXERCISE: Let L be a formal DGLA, then Def_L is smooth if and only if the quadratic map $[\cdot, \cdot]: H^1 \times H^1 \rightarrow H^2$ is zero.

Example 3.3. Let $N^1 \oplus B^1 = L^1$ be a vector space decomposition and consider the DGLA $N = \oplus N^i$ where

$$\begin{cases} N^i = 0 & \text{for } i \leq 0 \\ N^1 = N^1 \\ N^i = L^i & \text{for } i \geq 2 \end{cases}$$

with bracket and differential induced from L . Then the natural inclusion $N \rightarrow L$ gives isomorphisms $H^i(N) \rightarrow H^i(L)$ for every $i \geq 1$. In particular the morphism $Def_N \rightarrow Def_L$ is étale; note that $Def_N = MC_N$ is homogeneous.

EXERCISE: Let $z \in MC_L(S)$, $S \in Art_{\mathbb{K}}$ and consider the functor $P_z: Art_S \rightarrow Set$ given by

$$P_z(A) = \{x \in L^1 \otimes m_A \mid dx + [z, x] = 0\}.$$

Compute tangent and obstruction spaces of P_z .

REMARK. The abstract T^1 -lifting theorem (cf. [6]) implies that if the functor P_z is smooth (informally this means that the linear operator $d + ad(z)$ has constant rank) for every $n > 0$ and every $z \in MC_L(\mathbb{K}[t]/(t^n))$ then MC_L is also smooth.

Corollary 3.4. *Let L be a DGLA; if $H^0(L) = 0$ then Def_L is homogeneous.*

Proof. It is sufficient to apply 3.1 at the Example 3.3. □

Example 3.5. (The simplest example of deformation problem governed by a DGLA)
Let V be a \mathbb{K} -vector space and $J: V \rightarrow V$ linear such that $J^2 = -I$ (if $\mathbb{K} = \mathbb{R}$ then J is a complex structure on V). Let $M = \text{Hom}(V, V)$ and $d, \delta: M \rightarrow M$ defined by

$$d(A) = JA + AJ, \quad \delta(A) = JA - AJ, \quad A \in M$$

Since $\text{char}\mathbb{K} \neq 2$ it is easy to prove that $\ker d = \text{Image}(\delta)$, $\text{Image}(d) = \ker \delta$. Consider the DGLA $L = \bigoplus_{n \geq 1} M[-n]$ with differentials

$$0 \longrightarrow M[-1] \xrightarrow{d} M[-2] \xrightarrow{\delta} M[-3] \xrightarrow{d} M[-4] \xrightarrow{\delta} \dots$$

and the bracket defined by

$$[A, B] = AB - (-1)^{\overline{AB}} BA$$

Clearly $H^i(L) = 0$ for every $i \neq 1$ and then the associated functor $Def_L = MC_L$ is smooth homogeneous with tangent space $\ker d$.

The elements of $MC_L(A)$ are in natural bijection with the deformation of the solution of $J^2 = -I$ over the Artinian ring A ; in fact given $H \in M \otimes m_A$ we have $H \in MC_L(A)$ if and only if $(J + H)^2 = -I$. Since the gauge group is trivial, this example makes sense over arbitrary fields of $\text{char}\mathbb{K} \neq 2$.

4 The Kuranishi map and the Kuranishi functor

By a well known theorem of Schlessinger [13], if $F: \text{Art}_{\mathbb{K}} \rightarrow \text{Set}$ is a deformation functor with finite dimensional tangent vector space then there exists a prorepresentable functor $h_R \rightarrow F$ and an étale morphism $h_R \rightarrow F$.

In particular Schlessinger theorem applies to the functor Def_L for every DGLA L such that $H^1(L)$ is finite dimensional. In this section we give a explicit construction of a prorepresentable functor h_R and of an étale map $h_R \rightarrow Def_L$ by introducing Kuranishi maps.

Let $L = \bigoplus L^i$ be a DGLA, for every $i \in \mathbb{Z}$ choose direct sum decomposition

$$Z^i = B^i \oplus \mathcal{H}^i, \quad L^i = Z^i \oplus C^i$$

Let $\delta: L^{i+1} \rightarrow L^i$ be the linear map composition of

1. The projection $L^{i+1} \rightarrow B^{i+1}$ of kernel $C^{i+1} \oplus \mathcal{H}^{i+1}$.
2. $d^{-1}: B^{i+1} \rightarrow C^i$, the inverse of d .
3. The inclusion $C^i \rightarrow L^i$.

We note that $x \in \mathcal{H}^i$ if and only if $dx = \delta x = 0$ and $d\delta + \delta d = Id - H$, where $H: L^1 \rightarrow \mathcal{H}^1$ is the projection of kernel $B^1 \oplus C^1$.

Using the same notation of 2.4.1, for a vector space V we call $\hat{V}: \text{Art}_{\mathbb{K}} \rightarrow \text{Set}$ the homogeneous functor $\hat{V}(A) = V \otimes m_A$.

Definition 4.1. The Kuranishi map $F: \hat{L}^1 \rightarrow \hat{L}^1$ is the morphism of functors given by

$$x \in L^1 \otimes m_A, \quad F(x) = x + \frac{1}{2}\delta[x, x] \in L^1 \otimes m_A.$$

Lemma 4.2. *The Kuranishi map F is an isomorphism of functors.*

Proof. F is clearly a morphism of functors, since \hat{L}^1 is homogeneous, by 2.11 it is sufficient to prove that F is étale.

By construction F is the identity on the tangent space of \hat{L}^1 . Since \hat{L}^1 is smooth, the trivial obstruction theory $(0, v_e)$ is complete; by standard smoothness criterion 2.17 F is étale. \square

Definition 4.3. The Kuranishi functor $Kur: Art_{\mathbb{K}} \rightarrow Set$ is defined by

$$Kur(A) = \{x \in \mathcal{H}^1 \otimes m_A \mid H([F^{-1}(x), F^{-1}(x)]) = 0\}.$$

In other words Kur is the kernel the morphism of homogeneous functors $\hat{q}: \hat{\mathcal{H}}^1 \rightarrow \hat{\mathcal{H}}^2$ induced by the map of formal pointed schemes $q: \mathcal{H}^1 \rightarrow \mathcal{H}^2$ which is composition of: the inclusion $\mathcal{H}^1 \rightarrow L^1$, the inverse of F , the bracket $[\cdot, \cdot]: L^1 \rightarrow L^2$ and the projection $L^2 \rightarrow \mathcal{H}^2$.

Lemma 4.4. *Kur is homogeneous; if moreover the dimension of $H^1(L)$ is finite then $Kur = h_R$ is prorepresented by a \mathbb{K} -algebra $R \in \hat{Art}_{\mathbb{K}}$ such that $Spec(R) = (q^{-1}(0), 0)$ as formal scheme.*

Proof. Exercise (easy). \square

Lemma 4.5. *Let $x \in \hat{L}^1(A)$, $y \in \hat{L}^1(A)$, $c \in \mathbb{K}$ such that $y = c\delta[y, x]$, then $y = 0$.*

Proof. Exercise (easy). \square

Clearly MC_L and Kur can be considered as subfunctor of \hat{L}^1 .

Proposition 4.6. *The isomorphism $F: \hat{L}^1 \rightarrow \hat{L}^1$ induces an isomorphism*

$$F: MC_L \cap (C^1 \hat{\oplus} \mathcal{H}^1) \rightarrow Kur.$$

Proof. If $x \in MC_L(A) \cap (C^1 \hat{\oplus} \mathcal{H}^1) \otimes m_A$, then $dx + \frac{1}{2}[x, x] = 0$, $\delta x = 0$ and $H([x, x]) = H(-2dx) = 0$. Moreover we have:

$$[x, x] = [x, x] - H([x, x]) = d\delta[x, x] + \delta d[x, x] = d\delta[x, x]$$

$$\delta F(x) = \delta x + \frac{1}{2}\delta^2[x, x] = \delta x$$

$$dF(x) = dx + \frac{1}{2}d\delta[x, x] = dx + \frac{1}{2}[x, x] = 0$$

This prove that $F(x) \in Kur(A)$.

Conversely if $x \in L^1 \otimes m_A$ and $F(x) \in Kur(A)$ then $\delta x = \delta F(x) = 0$, $H([x, x]) = 0$.

$$dx + \frac{1}{2}[x, x] = dx + \frac{1}{2}(d\delta + \delta d)[x, x] = dF(x) + \frac{1}{2}\delta d[x, x] = \frac{1}{2}\delta d[x, x].$$

Therefore it is sufficient to prove that $\delta d[x, x] = 0$;

$$\delta d[x, x] = 2\delta[dx, x] = -\delta[d\delta[x, x], x] = -\delta[[x, x] - H([x, x]) - \delta d[x, x], x]$$

Since $H([x, x]) = 0$ and by Jacoby $[[x, x], x] = 0$ we have

$$\delta d[x, x] = \delta[\delta d[x, x], x]$$

and the proof follows from Lemma 4.5. \square

Proposition 4.6 says that, under the Kuranishi map F , the Kuranishi functor Kur is isomorphic to MC_N , where $N = \oplus N^i$ is the DGLA defined by

$$\begin{cases} N^i = 0 & \text{for } i \leq 0 \\ N^1 = C^1 \oplus \mathcal{H}^1 \\ N^i = L^i & \text{for } i \geq 2 \end{cases}$$

By Example 3.3 the morphism $MC_N \rightarrow Def_L$ is étale. In conclusion what we have proved is the following

Theorem 4.7. *For every DGLA L the morphism $Kur \xrightarrow{F^{-1}} MC_L \rightarrow Def_L$ is étale.*

EXERCISE: In the notation of 3.3, if $H^1(L)$ is finite dimensional then the isomorphism class of MC_N is independent from the choice of the complement N^1 . (This is always true but a proof of this fact without assumptions on $H^1(L)$ requires a nontrivial application of the factorization theorem [5, 6.2]).

5 Homotopy equivalence versus gauge equivalence

We introduce here a new equivalence relation in the set of solutions of the Maurer-Cartan equation in a differential graded Lie algebra. Let $(L, [,], \delta)$, $L = \oplus L^i$, be a fixed DGLA and let $\mathbb{K}[t, dt] = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$ be the differential graded algebra with t of degree 0, dt of degree 1, endowed with the expected differential $d(p(t) + q(t)dt) = \frac{dp(t)}{dt}dt$.

We denote by $\Omega = L \otimes_{\mathbb{K}} \mathbb{K}[t, dt] = \oplus \Omega^i$; it is a DGLA with $\Omega^i = L^i[t] \oplus L^{i-1}[t]dt$, differential $\delta_{\Omega}(a(t) + b(t)dt) = \delta a(t) + (-1)^{\overline{a(t)}} \frac{\partial a(t)}{\partial t} dt + \delta b(t)dt$ and bracket

$$[a(t) + b(t)dt, p(t) + q(t)dt] = [a(t), b(t)] + [a(t), q(t)]dt + (-1)^{\overline{p(t)}} [b(t), q(t)]dt.$$

For every $s \in \mathbb{K}$ there exists an evaluation morphism of DGLA $v_s: \Omega \rightarrow L$, $v_s(t) = s$, $v_s(dt) = 0$. In particular we have morphisms of functors of Artin rings $v_s: MC_{\Omega} \rightarrow MC_L$ which are left inverses of the natural inclusion $MC_L \subset MC_{\Omega}$.

Lemma 5.1. *The morphism $v_s: MC_{\Omega} \rightarrow MC_L$ is smooth for every $s \in \mathbb{K}$.*

Proof. We first give a more explicit description of the solutions of the Maurer-Cartan equation in Ω . Take $\omega \in \Omega^1$, $\omega = a(t) + b(t)dt$, we have

$$\delta_{\Omega}\omega + \frac{1}{2}[\omega, \omega] = \delta a(t) + \frac{1}{2}[a(t), a(t)] + \left(-\frac{\partial a(t)}{\partial t} + \delta b(t) + [a(t), b(t)] \right) dt$$

and then $\omega \in MC_{\Omega}$ if and only if

1. $a(t) \in MC_L$ for every $t \in \mathbb{K}$, and
2. $\frac{\partial a(t)}{\partial t} = \delta b(t) + [a(t), b(t)]$.

Let $0 \rightarrow J \rightarrow A \xrightarrow{\pi} B \rightarrow 0$ be a small extension of Artin rings, $\omega_B = a_B(t) + b_B(t)dt \in MC_{\Omega}(B)$ and $a_A(s) \in MC_L$ such that $\pi(a_A(s)) = a_B(s)$.

Let $\tilde{a}_A(t)$, $b_A(t)$ be liftings of $a_B(t)$, $b_B(t)$ in $L[t] \otimes m_A$ such that $\tilde{a}_A(s) = a_A(s)$; since $Jm_A = 0$ the bracket $[\tilde{a}_A(t), b_A(t)]$ is independent from the choice of the liftings.

Setting

$$\gamma(t) = \delta b_A(t) + [\tilde{a}_A(t), b_A(t)] - \frac{\partial \tilde{a}_A(t)}{\partial t} \in L^1[t] \otimes J$$

and

$$a_A(t) = \tilde{a}_A(t) + \int_s^t \gamma(\tau) d\tau$$

it is evident that $\omega_A = a_A(t) + b_A(t)dt$ satisfies the differential equation *ii*) and $v_s(\omega_A) = a_A(s)$. Moreover $\delta a_A(t) + \frac{1}{2}[a_A(t), a_A(t)] \in L^2[t] \otimes J$ and then

$$\begin{aligned} \frac{d}{dt}(\delta a_A(t) + \frac{1}{2}[a_A(t), a_A(t)]) &= \delta \frac{da_A(t)}{dt} + \left[\frac{da_A(t)}{dt}, a_A(t) \right] = \\ &= \delta[a_A(t), b_A(t)] + [\delta b_A(t) + [a_A(t), b_A(t)], a_A(t)] = \\ &= \left[-\frac{1}{2}[a_A(t), a_A(t)], b_A(t) \right] + [[a_A(t), b_A(t)], a_A(t)] = 0 \end{aligned}$$

By assumption $a_A(s) \in MC_L$ and then $a_A(t) \in MC_L$ for every $t \in \mathbb{K}$. \square

Definition 5.2. We shall say that $x \in MC_L(A)$ is homotopic to $y \in MC_L(A)$ if there exists $\omega \in MC_\Omega$ such that $x = v_0(\omega)$, $y = v_1(\omega)$; it is easy to see that the relation “ x is homotopic to y ” is reflexive and symmetric (cf. exercise below). The equivalence relation generated is called *homotopy equivalence*.

EXERCISE: There exists a natural action of the group of affine isomorphisms of \mathbb{K} on MC_Ω .

Let's momentarily denote by $F_L: Art_{\mathbb{K}} \rightarrow Set$ the quotient functor of MC_L by the homotopy equivalence (at the end of the trip F_L will be equal to Def_L).

Proposition 5.3. F_L is a deformation functor and the natural projection $MC_L \rightarrow F_L$ is smooth.

Proof. Easy exercise (Hint: use that v_0, v_1 are smooth left inverses of the natural inclusion $MC_L \subset MC_\Omega$.) \square

Lemma 5.4. The tangent space of F_L is the quotient of t_{MC_L} by the image of the linear map $v_1 - v_0: t_{MC_\Omega} \rightarrow t_{MC_L}$.

Proof. It is evident that if $x \in t_{MC_L}$ is homotopy equivalent to y then $x - y$ belongs to the image of $v_1 - v_0$. Conversely let $x \in t_{MC_L}$, $\omega \in t_{MC_\Omega}$, $h = v_1(\omega) - v_0(\omega)$ and $y = x + y$. We can write $x = v_0(x - v_0(\omega) + \omega)$, $y = v_1(x - v_0(\omega) + \omega)$ and therefore x is homotopy equivalent to y . \square

We note that $\omega = a(t) + b(t)dt \in t_{MC_\Omega}$ if and only if $\delta a(t) = 0$ and $\frac{da(t)}{dt} = \delta b(t)$; in particular

$$v_1(\omega) - v_0(\omega) = a(1) - a(0) = \int_0^1 \delta b(t) dt \in B^1(L)$$

On the other hand for every $b \in L^0$, $\delta(b)t + bdt \in t_{MC_\Omega}$ and therefore the image of $v_1 - v_0$ is exactly $B^1(L)$.

Theorem 5.5. $x \in MC_L(A)$ is homotopic to $y \in MC_L(A)$ if and only if x is gauge equivalent to y .

Before proving Theorem 5.5 we need some preliminary results about the exponential map: let N be a nilpotent Lie algebra and let $a \rightarrow e^a$ be its exponential map $N \rightarrow \exp(N)$. Given a polynomial $p(t) \in N[t]$ we have

$$e^{p(t+h)} e^{-p(t)} = e^{h(p'(t) + \gamma_p(t)) + h^2 \eta_p(t,h)}$$

where $p'(t) = \frac{\partial p(t)}{\partial t}$ and γ_p, η_p are polynomials determined exactly by Taylor expansion $p(t+h) = p(t) + hp'(t) + h^2 \dots$ and Campbell-Baker-Hausdorff formula. For later use we point out that $\gamma_p(t)$ is a linear combination with rational coefficients of terms of the following types:

$$ad(p(t))^{\alpha_1} ad(p'(t))^{\beta_1} \dots ad(p(t))^{\alpha_n} ad(p'(t))^{\beta_n} Q$$

with $Q = p(t), p'(t)$ and $\sum \alpha_i + \sum \beta_i > 0$.

Lemma 5.6. *In the notation above, for every $b(t) \in N[t]$ there exists an unique polynomial $p(t) \in N[t]$ such that $p(0) = 0$ and $p'(t) + \gamma_p(t) = b(t)$.*

Proof. Let $N = N^{(1)} \supset N^{(2)} \supset \dots \supset N^{(m)} = 0$ be the descending central series, $N^{(i+1)} = [N, N^{(i)}]$, and denote $N_i = N/N^{(i+1)}$. By construction $N_{i+1} \rightarrow N_i$ is a central extension for every $i \geq 0$.

The lemma is trivially true over N_0 ; assume that $p_i(t) \in N_i(t)$ is the unique solution of the differential equation $p_i'(t) + \gamma_i(t) = b(t)$ ($\gamma_i = \gamma_{p_i}$) on the vector space $N_i[t]$.

Let $\tilde{p}_{i+1}(t) \in N_{i+1}$ be a lifting of $p_i(t)$ such that $\tilde{p}_{i+1}(0) = 0$, as the kernel K_{i+1} of the projection $N_{i+1} \rightarrow N_i$ is contained in the centre the polynomial $\gamma_{i+1} = \gamma_{\tilde{p}_{i+1}}$ does not depend from the choice of the lifting.

We introduce the polynomials

$$\chi(t) = b(t) - \tilde{p}_{i+1}'(t) - \gamma_{i+1}(t) \in K_{i+1}[t]$$

$$p_{i+1}(t) = \tilde{p}_{i+1}(t) + \int_0^t \chi(\tau) d\tau \in N_{i+1}[t]$$

It is clear that $p_{i+1}(t)$ is the unique solution of the differential equation. □

If $N \times V \xrightarrow{[\cdot]} V$ is a representation of N into a vector space V , for every $v \in V$ we have

$$\frac{d}{dt}(e^{p(t)}v) = [p'(t) + \gamma_p(t), e^{p(t)}v]$$

As an immediate consequence of 5.6 we have

Proposition 5.7. *Let N be a nilpotent Lie algebra, for every $b(t) \in N[t]$ there exists unique $p(t) \in N[t]$ such that $p(0) = 0$ and for every representation $N \times V \xrightarrow{[\cdot]} V$, for every $v \in V$ holds*

$$\frac{d}{dt}(e^{p(t)}v) + [b(t), e^{p(t)}v] = 0.$$

Proof. Evident. □

Proof of 5.5. The gauge equivalence and the notion of homotopy extends naturally to the space $V = (L^1 \otimes m_A) \oplus \mathbb{K}\delta$:

1. i) $x, y \in V$ are gauge equivalent if and only if there exists $p \in L^0 \otimes m_A$ such that $y = e^p x$.
2. ii) $x \in V$ is homotopic to $y \in V$ if and only if there exists $a(t) \in V[t], b(t) \in L^0 \otimes m_A[t]$ such that $a(0) = x, a(1) = y$ and $\frac{da(t)}{dt} + [b(t), a(t)] = 0$.

Now 5.5 follows from 5.7. □

REMARK. The notion of homotopy equivalence extends naturally to L_∞ -algebras see [4], while the gauge action requires the structure of DGLA; this motivates this section and the redundancy of proofs given here.

References

- [1] E. Arbarello: *Introduction to Kontsevich's Result on Deformation-Quantization of Poisson Structures*. In: *Seminari di Geometria Algebrica 1998-1999* Scuola Normale Superiore (1999).
- [2] M. Artin: *Deformations of singularities*. Tata institute of fundamental research Bombay (1976).
- [3] S. Barannikov, M. Kontsevich: *Frobenius manifolds and formality of Lie algebras of polyvector fields*. alg-geom/9710032.
- [4] A. Canonaco: *L_∞ -algebras and quasi-isomorphisms*. In: *Seminari di Geometria Algebrica 1998-1999* Scuola Normale Superiore (1999).
- [5] B.Fantechi, M. Manetti: *Obstruction calculus for functors of Artin rings, I* Journal of Algebra **202** (1998) 541-576.
- [6] B.Fantechi, M. Manetti: *On the T^1 -lifting theorem*. J. Alg. Geom. (1999) 31-39.
- [7] W.M. Goldman, J.J. Millson: *The deformation theory of representations of fundamental groups of compact kähler manifolds*. Publ. Math. I.H.E.S. **67** (1988) 43-96.
- [8] W.M. Goldman, J.J. Millson: *The homotopy invariance of the Kuranishi space*. Ill. J. Math. **34** (1990) 337-367.
- [9] G. Hochschild: *The structure of Lie groups*. Holden-Day San Francisco (1965).
- [10] N. Jacobson: *Lie algebras*. Wiley & Sons (1962).
- [11] M. Kontsevich: *Deformation quantization of poisson manifolds, I*. q-alg/9709040.
- [12] M. Kontsevich: *Topics in algebra deformation theory*. Unpublished notes based on a course given at the University of Berkeley (1994).
- [13] M. Schlessinger: *Functors of Artin rings*. Trans. Amer. Math. Soc. **130** (1968) 208-222.