Lectures on algebraic geometry

Aldo Andreotti (1978)

Latex, footnotes, appendices and references by Marco Manetti and Sara Pirozzi (2011)

Premessa

Questo lavoro fa parte di un progetto di riscrittura e parziale revisione di alcune note, scritte a mano da Aldo Andreotti ed utilizzate per un corso di Geometria Algebrica che tenuto per vari anni in diverse università: dall'anno accademico 1973/1974 in Oregon per tre anni, nell'anno accademico 1977/1978 a Strasburgo e nell'anno accademico 1978/1979 alla Scuola Normale Superiore di Pisa.

Aldo Andreotti (Firenze, 15.3.1924 - Pisa, 21.2.1980) è stato un importante matematico italiano. Fu assistente di Francesco Severi dal quale ereditò gran parte dei metodi della scuola di geometria algebrica italiana, metodi che negli anni 50 risultavano troppo intuitivi e troppo poco rigorosi per riscuotere gli stessi successi del passato e per poter resistere all'esigenza di una rifondazione su basi analitiche e algebriche più precise.

Queste circostanze portarono Andreotti ad allontanarsi dallo studio della geometria algebrica e a dedicarsi alla geometria e all'analisi complessa. All'inizio degli anni 70 Andreotti si riavvicinò alla geometria algebrica, ed è proprio in quegli anni che scrive queste note, con un approccio basato su avanzati strumenti sia analitici che algebrici. Questo riavvicinamento va inteso come un invito a riprendere lo studio della geometria algebrica in Italia che per troppo tempo era stato abbandonato.

È accertato che Andreotti ritenesse queste note un lavoro in continua evoluzione e per questo non ne esiste una versione univoca e definitiva. Oltre alle note scritte a mano, conservate presso la biblioteca della Scuola Normale Superiore di Pisa, e prese come sorgente per questa dispensa, dovrebbero esistere da qualche parte anche delle copie dattiloscritte in francese del corso tenuto a Strasburgo.

Remark 0.1. Il progetto di cui fa parte questa dispensa è ancora incompleto: in particolare sono trattati solamente i primi 9 capitoli delle note originali.

Le modifiche apportate rispetto all'originale sono di carattere puramente cosmetico. Abbiamo integrato le note con dimostrazioni mancanti e chiarimenti riguardanti i punti piú oscuri a mezzo delle note a pie' di pagina, delle appendici e di opportuni riferimenti bibliografici (totalmente assenti nell'originale).

2

2. Riemann domains of a holomorphic function. a) Let X be a connected complex manifold.

For every open set $U \subset X$ we consider the space

$$\mathcal{O}(U) = \{ f : U \to \mathbb{C} \mid f \text{ holomorphic in } U \}$$

We have thus a law which associate to every open set $U \subset X$ a vector space (and ring) $\mathcal{O}(U)$ over $\mathbb{C}$:

$$(1) \qquad U \rightsquigarrow \mathcal{O}(U)$$

If $V \subset U$ are open we have a natural restriction map

$$(2) \qquad r^U_V : \mathcal{O}(U) \longrightarrow \mathcal{O}(V)$$

such that if

$W \subset V \subset U$ are open sets in X

the diagram

$$\begin{array}{ccc} \mathcal{O}(U) & \xrightarrow{r^U_V} & \mathcal{O}(V) \\ & {\scriptstyle r^U_W}\searrow \quad \swarrow{\scriptstyle r^V_W} & \\ & \mathcal{O}(W) & \end{array}$$

commutes: $r^U_W = r^V_W \circ r^U_V$

Moreover if Ω is any open set in X and $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of Ω, the following sequence is exact

$$(3) \qquad 0 \longrightarrow \mathcal{O}(\Omega) \xrightarrow{\alpha} \prod \mathcal{O}(U_i) \xrightarrow{\beta} \prod \mathcal{O}(U_i \cap U_j)$$

where:

for $s \in \mathcal{O}(\Omega)$ $\qquad \alpha(s) = \{ r^{\Omega}_{U_i} s \}$

for $\{s_i\} \in \prod \mathcal{O}(U_i)$ $\qquad \beta(\{s_i\}) = \{ r^{U_j}_{U_i \cap U_j} s_j - r^{U_i}_{U_i \cap U_j} s_i \}$.

Una pagina delle note di Aldo Andreotti.

Contents

CHAPTER 1

Riemann's domains

1. The value of heuristic arguments

Heuristic arguments have played a great role in the discovery of scientific truth. They are usually banned from the presentation of a subject for reason of conciseness. Still sometimes part of the guiding pieces of discovery are lost with them. We will strive to present them by side of the rigorous proof when the case present itself. Here is an historical example: in Lucretius "De rerum naturae" the following experiment is described.

(1) Take a box full of sand; plan a piece of iron on the top of the sand; shake the box. The piece of iron descends trough the sand to the bottom of the box.
(2) Take the same box; plan a piece of wood on the bottom of it: full the box of sand; shake the box. The piece of wood raises to the top of the sand in the box.
(3) Replacing sand with water the same phenomena take place.

Conclusion: liquid are made of microscopic particles in continuous motion.

The rigorous conclusion takes from Lucretius to Boltzmann. Algebraic geometry had in some aspect the same sort of slow development.

2. Riemann domains of a holomorphic function

a Let X be a connected complex manifold. For every open set $U \subset X$ we consider the space

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ holomorphic in } U\}.$$

We have thus a law which associate to every open set $U \subset X$ a vector space (and ring) $\mathcal{O}(U)$ over $\mathbb{C}$:

$$U \to \mathcal{O}(U). \tag{1.1}$$

If $V \subset U$ are open sets we have a natural restriction map

$$\tau_V^U : \mathcal{O}(U) \to \mathcal{O}(V) \tag{1.2}$$

such that if $W \subset V \subset U$ are open sets in X the diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[rr, "\tau_V^U"] \arrow[dr, "\tau_W^U"'] & & \mathcal{O}(V) \arrow[dl, "\tau_W^V"] \\ & \mathcal{O}(W) & \end{tikzcd}$$

commutes:

$$\tau_W^U = \tau_W^V \circ \tau_V^U.$$

Moreover if Ω is any open set in X and $\mathcal{U} = \{U_i\}_{i\in I}$ an open covering of Ω, the following sequence is exact

$$\mathcal{O}(\Omega) \xrightarrow{\alpha} \prod_i \mathcal{O}(U_i) \xrightarrow{\beta} \prod_{i,j} \mathcal{O}(U_i \cap U_j) \tag{1.3}$$

where for $s \in \mathcal{O}(\Omega)$ and for $\{s_i\} \in \prod_i \mathcal{O}(U_i)$:

$$\alpha(s) = \{\tau_{U_i}^{\Omega} s\}$$

$$\beta(\{s_i\}) = \{\tau_{U_i\cap U_j}^{U_j} s_j - \tau_{U_i\cap U_j}^{U_i} s_i\}.$$

Note: this mean two facts

(1) **unicity**: if a holomorphic function $s \in \mathcal{O}(\Omega)$ has zero restriction to the open sets U_i of $\mathcal{U}$ then $s = 0$;
(2) **gluing**: if a collection of holomorphic functions $s_i \in \mathcal{O}(U_i)$ are given with the property that if $U_i \cap U_j \neq \emptyset$

$$\tau^{U_i}_{U_i \cap U_j} s_i = \tau^{U_j}_{U_i \cap U_j} s_j$$

then the functions $\{s_i\}$ can be glued together into a holomorphic function $s \in \mathcal{O}(\Omega)$ with:

$$\tau^{\Omega}_{U_i} s = s_i \qquad \forall\, i \in I$$

Remark 1.1. In general an assignment to every open set U of a topological space X of an abelian group $\mathcal{S}(U)$ (or vector space, or ring)

$$U \rightsquigarrow \mathcal{S}(U)$$

verifying the condition mentioned above is called a **sheaf of abelian groups** (of vector spaces, of rings) on X .

b) Given $x \in X$ we can consider the vector space and ring

$$\mathcal{O}_x = \varinjlim_{x \in U} \mathcal{O}(U).$$

$\mathcal{O}_x$ is the ring of germs of holomorphic functions at x i.e. the equivalence class of functions, defined in a neighborhood of x, with respect to the equivalence relation

$$f \in \mathcal{O}(U) \qquad x \in U, \qquad g \in \mathcal{O}(V) \qquad x \in V$$

$$f \sim g$$

if and only if for some neighborhood $W \subset U \cap V$ we have

$$f_{|W} = g_{|W}.$$

Let

$$\mathcal{O} = \bigcup_{x \in X} \mathcal{O}_x$$

we have a natural map of the set $\mathcal{O}$ on X

$$\pi \colon \mathcal{O} \to X$$

which associate to every $f_x \in \mathcal{O}_x$ its base point $x = \pi(f_x)$.

c) Topology of $\mathcal{O}$ Given U open in X and $f \in \mathcal{O}(U)$ consider the subset of $\mathcal{O}$

$$A(f, U) = \{f_x\}_{x \in U} \qquad \text{where } f_x = \tau^U_x f$$

$\tau^U_x \colon \mathcal{O}(U) \to \mathcal{O}_x$ being the natural limit map. We take the sets $A(f, U)$ as a basis for open sets in $\mathcal{O}$ i.e. a subset $B \subset \mathcal{O}$ is open if and only if $B = \bigcup_{\alpha \in J} A(f_\alpha, V_\alpha)$ for some collection J of f_α and V_α. One verifies that in this way the projection map $\pi \colon \mathcal{O} \to X$ is a local homeomorphism i.e. given $\alpha \in \mathcal{O}$ there exist open neighborhoods $A(\alpha)$ of α in $\mathcal{O}$ and $B(\pi(\alpha))$ of $\pi(\alpha)$ in X such that

$$\pi(A(\alpha)) = B(\pi(\alpha)).$$

and the induced map

$$\pi_{|_{A(\alpha)}} \colon A(\alpha) \to B(\pi(\alpha))$$

is a homeomorphism.

Remark 1.2. In general given a sheaf $\{\mathcal{S}(U), \tau^U_V\}$ on a topological space X with the same construction we can build a topological space $\mathcal{S}$ with a map

$$\omega \colon \mathcal{S} \to X$$

surjective and a local homeomorphism (stack).

REMARK 1.3. Conversely given a triple $(\mathcal{S},\omega,X)$ where $\mathcal{S}$ and X are topological spaces, ω is surjective and a local homeomorphism, setting

$$\Gamma(U,\mathcal{S}) = \{s \text{ continuous}, \ \pi \circ s = id_U\}$$

we obtain a sheaf $U \to \Gamma(U,\mathcal{S})$ of set. It will be a sheaf of abelian groups, vector spaces etc. if additional structure is grow on the stalks $\mathcal{S}_x = \pi^{-1}(x)$ with obvious continuity conditions with respect to the base point x.

REMARK 1.4. Given a sheaf of abelian group $\{\mathcal{S}(U), \tau_V^U\}$ on X; if we construct the stack $(\mathcal{S},\omega,X)$ associated to it and the sheaf $U \to \Gamma(U,\mathcal{S})$ we obtain back the original sheaf, this by virtue of the exactness of the sequence analogous to (1.3).

d) The following theorem is particular to the sheaf $\mathcal{O}$.

THEOREM 1.5. *Let $\pi\colon \mathcal{O} \to X$ be the stalk associated to the sheaf of germs of holomorphic function on the complex manifold X. The topology of $\mathcal{O}$ is a Hausdorff topology.*

PROOF. Let $\alpha, \beta \in \mathcal{O}$ with $\alpha \neq \beta$. If $\pi(\alpha) \neq \pi(\beta)$ we can find holomorphic functions $f \in \mathcal{O}(U(\pi(\alpha)))$ $g \in \mathcal{O}(U(\pi(\beta)))$ defined in some open neighborhoods $U(\pi(\alpha))$, $U(\pi(\beta))$ of $\pi(\alpha)$ and $\pi(\beta)$ respectively such that

$$\alpha = f_{\pi(\alpha)} \qquad \beta = g_{\pi(\beta)}.$$

We can suppose $U(\pi(\alpha)) \cap U(\pi(\beta)) = \emptyset$. Then $A(U(\pi(\alpha)), f)$ and $A(U(\pi(\beta)), g)$ are open neighborhoods of α and β respectively with empty intersection. If $\pi(\alpha) = \pi(\beta) = x_0$, as before we can choose f, g and $U(\pi(\alpha)) = U(\pi(\beta)) \subset W$ open and connected. We claim that

$$A(W,f) \cap A(W,g) = \emptyset.$$

Otherwise for same $y \in W$ $f_y = g_y$. Since the set

$$B = \{y \in W | f_y = g_y\}$$

is open and closed and not empty $B = W$ and $\alpha = \beta$, which is impossible. □

e) Let X be a complex and connected manifold. Let $x_0 \in X$ and $\alpha = f_{x_0} \in \mathcal{O}_{x_0}$ a germ of a holomorphic function at x_0. Then α is a point of the stack $\mathcal{O}$ and we can consider Σ_α the connected component of α in $\mathcal{O}$. We have the following properties:

(1) $\omega = \pi_{|\Sigma_\alpha}$ is a local homeomorphism on X therefore Σ_α inherits a natural structure of a connected complex manifold in which ω becomes a local isomorphism of complex structures;
(2) for every point $\beta = g_x \in \Sigma_\alpha$ we consider the function

$$F\colon \Sigma_\alpha \to \mathbb{C}$$

defined by $F(\beta) = g_x(x)$. Then F is a holomorphic function defined on Σ_α and has the property $F(\alpha) = \alpha$ (via ω). We call F the analytic function defined by the germ α;
(3) any other germ of F defines the same manifold Σ and the same analytic function F on it.

DEFINITION 1.6. We call (Σ_F, ω, X) the Riemann domain of the analytic function F.

HISTORICAL NOTE 1.7. If would be more appropriate to call $(\Sigma_\alpha, \omega, X)$ the Weierstrass domain of the analytic function F. However the usage calls these spreaded manifolds over X Riemann's domain.

EXAMPLE 1.8. Let $X = \mathbb{C}$, $z_0 \neq 0$ and set $f = \sqrt{z} = e^{\frac{1}{2}\log z}$. Over z_0 f has two germs

$$f_{z_0} = e^{\frac{1}{2}\log z_0} e^{\frac{1}{2}\log(1+\frac{z-z_0}{z_0})} = e^{\frac{1}{2}\log z_0} \sum_n \binom{\frac{1}{2}}{n} (\frac{z-z_0}{z_0})^n$$

which depends on the choice of the two square root of z_0. Over every point $z \neq 0$ f has two germs and these are determined if we go around the origin once. Then Σ_F is in this case a 2-sheeted[1] covering of $\mathbb{C}^*$. Every two sheeted covering of $\mathbb{C}^*$ is isomorphic to $\mathbb{C}^*$ and a holomorphic coordinate ω on $\Sigma_F \cong \mathbb{C}^*$ can be chosen so that ω is given by

$$z = \omega^2.$$

EXAMPLE 1.9. Let $X = \mathbb{C}$, $z_0 \neq 0$, set f $\alpha \in \mathbb{R}$

$$f = z^\alpha = e^{\alpha \log z} = z_0^\alpha \big(1 + \frac{z - z_0}{z_0}\big)^\alpha = z_0^\alpha \sum_m \binom{\alpha}{m} \big(\frac{z - z_0}{z_0}\big)^m.$$

Choose a determination of $\log z_0$ and set $f_0 = e^{\alpha \log z_0} \sum_m \binom{\alpha}{m} \big(\frac{z-z_0}{z_0}\big)^m$. The various branches of f over z_0 are thus given by

$$f_k = e^{\pi i k \alpha} f_0 \qquad k \in \mathbb{Z}.$$

If $\alpha \in \mathbb{Q}$ $\alpha = \frac{b}{q}$ $\quad M.C.D.(p, q) = 1$ then f presents over z_0 q branches that are circularly permuted if we go around the origin. Now Σ_F is a q-sheeted covering of $\mathbb{C}^*$ and the coordinate ω on $\Sigma_F \cong \mathbb{C}^*$ can be chosen so that ω is given by

$$z = \omega^q$$

If $\alpha \notin \mathbb{Q}$ then the branches f_k are all distinct and permuted when we go around the origin. In this case $\Sigma_F \cong \mathbb{C}$ and a coordinate ω on Σ_F can be chosen so that the map ω is given by

$$z = e^\omega.$$

3. The theorem of Poincaré-Volterra

a) Given a complex manifold X, connected, we will call an **abstract Riemann domain** Y over X the set of the following data

(1) a complex manifold Y;
(2) a holomorphic map $\omega\colon Y \to X$ which is a local isomorphism.

THEOREM 1.10 (Poincaré-Volterra). *If X has a countable topology then every Riemann domain Y over X has also a countable topology.*

REMARK 1.11. A topological space X has a countable topology if there exist a countable collection $\{A_\alpha\}_{\alpha \in \mathbb{N}}$ of open subset of X such that any open subset of X can be written as union of the sets A_α.

COROLLARY 1.12. *Let $\Sigma_F \to X$ be the Riemann domain of the analytic function F. Over any point $x_0 \in X$, F has at most countably many germs.*

b) The above mentioned theorem is a consequence of the following more general.

THEOREM 1.13. *Let X be a differentiable and connected manifold with countable topology. Let Σ be a connected differentiable manifold. If we can find a differentiable map $\omega\colon \Sigma \to X$ which is a local diffeomorphism, then Σ has a countable topology.*

PROOF. As $\omega(\Sigma)$ is open in X we may as well assume that $X = \omega(\Sigma)$. We divide the proof in two steps.
Step 1. We first give the proof of the theorem in the particular case that X is an open connected subset of $\mathbb{R}^n$. Let I be the set of rational point in X and let $A = \omega^{-1}(I)$. For every $\alpha \in \Sigma$ we define the number $\epsilon(\alpha) = \sup\{r > 0|$ the ball with center $\omega(\alpha)$ and radius r is contained in X and can be isomorphically lifted to a ball in Σ centered at $\alpha\}$. Clearly $\epsilon(\alpha) > 0$ and is a continuos functions of $\alpha \in \Sigma$. Let for all $\alpha \in \Sigma$ denote by $B(\alpha, \rho)$ the open ball with center α and radius ρ, $0 < \rho < \epsilon(\alpha)$. The set

$$\mathcal{B} = \bigcup_{\alpha \in A} \bigcup_{\substack{0<\rho<\epsilon(\alpha) \\ \rho \in \mathbb{Q}}} B(\alpha, \rho)$$

[1] A covering $\pi\colon X \to Y$ is called an n-sheeted covering if each $\pi^{-1}(x)$ has cardinality n.

is a union of open balls. We claim that $\mathcal{B}$ is a basis for open sets in Σ and that $\mathcal{B}$ is countable. The first statement is obvious as any point $\beta \in \Sigma$ is contained in some open ball with center $\alpha \in A$ and rational radius. We have to show that $\mathcal{B}$ is countable. For this it is enough to show that A is countable. Let $\alpha_0 \in A$ be fixed. As Σ is connected we can find a continuos arc

$$\gamma\colon [0,1] \to \Sigma$$

with $\gamma(0) = \alpha$ and $\gamma(1) = \alpha_0$. It is not restrictive assume that

(1) $\omega(\gamma)$ is a broken line joining $\omega(\alpha)$ to $\omega(\alpha_0)$;
(2) the edges of the broken line are in I.

Now given an arc

$$\sigma\colon [0,1] \to X$$

with $\sigma(0) = \omega(\alpha)$ and $\sigma(1) = \omega(\alpha_0)$, there exists at most a unique[2] lifting $\gamma\colon [0,1] \to \Sigma$ with $\gamma(1) = \alpha_0$. Therefore the set A is in one to one correspondence with a subset of the set Λ of all broken lines with edges at rational point and ending at $\omega(\alpha_0)$. As any one of these broken lines is determined by the sequence of its edges points we have

$$\Lambda \cong \bigcup_{m=1}^{\infty} (\mathbb{Q}^n)^m$$

this shows that Λ is countable and thus $A \subset \Lambda$ is also countable.

Step 2. We now drop the assumption that X is an open subset of $\mathbb{R}^n$. We can, as X has a countable topology, find a proper embedding[3]

$$j\colon X \to \mathbb{R}^N$$

for some sufficiently large N. We can extend this embedding to a diffeomorphism

$$X \times D^{N-n} \xrightarrow{\lambda} \Omega \subset \mathbb{R}^N, \qquad D^{N-n} = \{t \in \mathbb{R}^{N-n} \mid \|t\| < 1\},$$

onto a connected open tubular neighborhood Ω of $j(X)$, $n = \dim_{\mathbb{R}} X$. The natural map

$$\omega \times id\colon \Sigma \times D^{N-n} \to X \times D^{N-n}$$

is a local diffeomorphism and $\Sigma \times D^{N-n}$ is connected. Then $\mu\colon \Sigma \times D^{N-n} \to \Omega$, $\mu = \lambda \circ (\omega \times id)$, is a local diffeomorphism. By the previous argument $\Sigma \times D^{N-n}$ has a countable topology. Therefore $\Sigma \cong \Sigma \times \{0\} \subset \Sigma \times D^{N-n}$ has a countable topology[4]. □

Problem 1.14. *In which class of topological spaces can an analog of Theorem 1.13 be formulated and proved? In particular is the theorem true for topological manifolds?*

4. Riemann's existence problem

a) Let X be a complex connected manifold with countable topology and let

$$\Sigma \xrightarrow{\lambda} X$$

be an abstract Riemann domain over X. Does there exist an analytic function F over X such that the Riemann domain of F

$$\Sigma_F \xrightarrow{\pi} X$$

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[2] Since Σ is Hausdorff and ω is a local homeomorphism, the diagonal is open and closed in the fiber product $\Sigma \times_X \Sigma$.

[3] By Whitney embedding theorem, see e.g. [**9**, Thm.2.14].

[4] This proof requires that the normal bundle of $j(X)$ in $\mathbb{R}^N$ is trivial. Without using this assumption we may argue as follows. According to [**9**, Thm. 5.1] there exists a vector bundle $E \to X$ whose total space E is diffeomorphic to an open neighborhood of $j(X)$ in $\mathbb{R}^N$. Then the natural map $\omega^* E \to E$ is a local diffeomorphism and then by step 1 the manifold $\omega^* E$ has a countable topology. Now Σ has a countable topology since it is isomorphic to the zero section of $\omega^* E$.

is isomorphic to the given one? This mean that we have a holomorphic isomorphism $\lambda\colon \Sigma \to \Sigma_F$ such that the diagram

$$\begin{tikzcd} \Sigma \arrow[rr, "\lambda"] \arrow[dr] & & \Sigma_F \arrow[dl] \\ & X & \end{tikzcd}$$

commutes.

b) Case Σ compact Then if λ exists Σ_F is also compact. Therefore $F\colon \Sigma_F \to \mathbb{C}$ is constant then $X = \omega(\Sigma)$, as we may assume, is compact. Therefore $\Sigma \cong X$ is necessary and also sufficient condition for the solvability of Riemann's problem.

c) Case Σ non compact We shall prove the following

PROPOSITION 1.15. *Let Σ be no compact. Assume that the following condition is satisfied:*

P: *for any divergent sequence $\{x_\nu\}$ in Σ (divergent means without accumulation points in Σ) and for any sequence $\{c_\nu\} \subset \mathbb{C}$ we can find a holomorphic function*

$$f\colon \Sigma \to \mathbb{C}$$

with

$$f(x_\nu) = c_\nu.$$

Then $\Sigma \xrightarrow{\omega} X$ is the Riemann's domain of an analytic function over X.

PROOF. Choose a complete Riemannian metric ds^2 on X and lift this metric to Σ. Let $\{K_i\}_{i\in\mathbb{N}\setminus\{0\}}$ be a sequence of compact sets in Σ such that

$$K_i \subset \dot{K}_{i+1} \quad \forall i, \qquad \cup_i K_i = \Sigma.$$

Set

$$\Delta_1 = K_2 \setminus \dot{K}_1$$
$$\Delta_2 = K_3 \setminus \dot{K}_2$$
$$\dots$$

These are compact set. Choose an ε-net[5] $x_1^{(1)}, \dots, x_{x_1}^{(1)}$ in Δ_1, choose an $\frac{\epsilon}{2}$-net $x_1^{(2)}, \dots, x_{x_2}^{(2)}$ in Δ_2, choose an $\frac{\epsilon}{2^2}$-net $x_1^{(3)}, \dots, x_{x_3}^{(3)}$ in Δ_3, ...

Order these points successively in a sequence $\{x_\mu\}$. This sequence is divergent in Σ as any compact set contains finite many of these points. Moreover any $y \in \Sigma \setminus K_j$ has a distance from $\{x_\mu\}$ which is less than or equal to $\frac{\epsilon}{2^{j-1}}$. Let $a \in \omega(\Sigma)$ and let $\omega^{-1}(a) = \{b_\nu\}_{\nu=1,2,\dots}$. This last is at most a countable set by the theorem of Poincaré-Volterra. By the assumption we can find a holomorphic function $f\colon \Sigma \to \mathbb{C}$ such that:

$$\lim |f(x_\mu)| = \infty \qquad f(b_i) \neq f(b_j) \quad \text{ if } i \neq j.$$

This because $\{x_\mu\} \cup \{b_\nu\}$ is a divergent sequence in Σ. Let f_a one of the germs of f at one of the points b's (transplanted on X) and let $\Sigma_f \xrightarrow{\pi} X$ the Riemann's domain of the analytic function f. We have a natural holomorphic map (compatible with ω and π)

$$\Sigma \xrightarrow{\lambda} \Sigma_f$$

which associate to every point $\alpha \in \Sigma$ the germ f_α of f at α. We show that λ is surjective. If $\lambda(\Sigma) \neq \Sigma_f$ there exists a point $x_0 \in \partial\lambda(\Sigma)$ in Σ_f. There exists a ball $B(x_0, \delta)$ of a certain radius $\delta > 0$, in Σ_f, which is mapped isomorphically into a ball of X of the same radius. There exists $y \in \lambda(\Sigma)$ with $dist(y, x_0) < \frac{\delta}{3}$. There exists $x_{\mu_0} \in \{x_\mu\}$ with $dist(y, \lambda(x_{\mu_0})) < \frac{\delta}{3}$. Hence $\lambda(x_{\mu_0}) \in B(x_0, \delta)$ and therefore $x_0 \in \overline{\{\lambda(x_\mu)\}}$ as δ can be taken arbitrarily small. But then f cannot be holomorphic at x_0 and this is a contradiction as $x_0 \in \Sigma_f$. We now show that Σ is a covering space of Σ_f, for this it is enough to prove that, given a path

$$\gamma\colon [0,1] \to \Sigma_f$$

[5] An ε-net is a finite or infinite set of points in a metric space such that each point of the space is within distance ε of same point in the set.

and a point $\alpha \in \lambda^{-1}(\gamma(0))$, there exists a path

$$\sigma\colon [0,1] \to \Sigma$$

with $\sigma(0) = \alpha$ and $\gamma = \lambda \circ \sigma$[6]. This is proved by the same argument used before.

Therefore Σ is the manifold obtained by dividing the universal covering space $\widetilde{\Sigma}_f$ of Σ_f by the action of a subgroup $G \subset \pi_1(\Sigma_f)$. Hence above any point $\beta \in \Sigma_f$

$$card(\lambda^{-1}(\beta)) = card(\frac{\pi_1(\Sigma_f)}{G}).$$

By construction

$$\lambda\colon \omega^{-1}(a) \to \pi^{-1}(a)$$

is one to one. Therefore λ is one to one, i.e. λ is an isomorphism of Σ onto Σ_f compatible with ω and π. $\square$

d) Additional remarks One can prove that if Σ is a connected non compact complex manifold of $\dim_{\mathbb{C}} \Sigma = 1$ then property **P** is satisfied. Therefore in complex dimension 1 the Riemann problem is solvable.

If Σ is a complex connected non compact manifold of $\dim_{\mathbb{C}} \Sigma \geq 2$ then the condition **P** may not be satisfied.

Definition 1.16. A complex manifold X with countable topology satisfying the following conditions:

(1) for every divergent sequence $\{x_\mu\} \subset X$ we can find a holomorphic function $f \in \Gamma(X, \mathcal{O})$ such that
$$\sup |f(x_\mu)| = \infty;$$
(2) the ring $\Gamma(X, \mathcal{O})$ separates points (i.e. for $x \neq y$ in X there exists $f \in \Gamma(X, \mathcal{O})$ with $f(x) \neq f(y)$)

is called a **Stein manifold**.

One can prove that if X is a Stein manifold then property **P** is satisfied. Let X be a Stein connected manifold and let $\Sigma_F \xrightarrow{\omega} X$ be the Riemann domain of an analytic function on X. Then by a theorem of Oka and Docquier-Grauert one can show that Σ_F is a Stein manifold. Oka theorem proves this fact when X is a domain of holomorphy in the space $\mathbb{C}^n$. By a theorem of Docquier-Grauert the general case can be reduced to this one. One uses the following facts:

(1) X can be imbedded by a proper biholomorphic map into a numerical space $\mathbb{C}^N$;
(2) $X \subset \mathbb{C}^N$ has a fundamental system of particular neighborhoods U (with holomorphic retraction $\lambda_U\colon U \to X$) which are Stein open set. Then by an argument as in the proof of Poincaré-Volterra theorem we deduced that Σ_F is a closed submanifold of a Stein manifold $\Sigma_{*\lambda F}$ and therefore it is Stein.

These facts being established then in complex dimension ≥ 2 the Riemann existence problem can be so formulated.

Let X be a Stein manifold and let Σ be a connected Stein manifold. Let $\pi\colon \Sigma \to X$ be a holomorphic map which is a local biholomorphism. Then the abstract Riemann domain (Σ, π, X) is isomorphic to the Riemann domain of an analytic function on X.

The case of a Riemann domain X over a complex manifold X which is not a Stein manifold is still open to investigation. One should thus formulate the following problem.

Problem 1.17. *Let X be a connected complex manifold of $\dim_{\mathbb{C}} X \geq 2$. Let $\Sigma \xrightarrow{\pi} X$ be a Riemann abstract domain over X. To find the necessary and sufficient condition for Σ to be isomorphic to the Riemann domain of an analytic function over X.*

[6] See Section 2 of Chapter 2

CHAPTER 2

Algebraic functions

1. The graph of an analytic function

a) Let X be a connected complex manifold and let $f_{x_0} \in \mathcal{O}_{x_0}$ be the germ of a holomorphic function. Let[1]

$$\Sigma_F \xrightarrow{\omega} X$$

be the Riemann domain defined by f_{x_0}. We have shown that there exists a holomorphic function $F\colon \Sigma_F \to \mathbb{C}$ such that

$$F_{f_{x_0}} = \omega^* f_{x_0}$$

i.e. F extends the germ f_{x_0} to the whole of Σ_F. We therefore have a holomorphic map

$$\omega \times F\colon \Sigma_F \to X \times \mathbb{C}.$$

The set

$$(\omega \times F)(\Sigma_F) \subset X \times \mathbb{C}$$

is called the **graph of the analytic function** F.

Example 2.1. Let $X = \mathbb{C}^*$ and $z_0 \in \mathbb{C}^*$. Take $\alpha \in \mathbb{R}$, $\alpha \notin \mathbb{Q}$, and consider the germ of the analytic function

$$f_{z_0} = z^\alpha(z-1) = z_0^\alpha(z-1)\sum_{m=0}^{\infty}\binom{\alpha}{m}\left(\frac{z-z_0}{z_0}\right)^m \qquad z \in \mathbb{C}$$

Then the Riemann domain Σ_F defined by a f_{z_0} is an infinitely sheeted covering space of $\mathbb{C}^*$. We have[2] $\Sigma_F \cong \mathbb{C}$, and indeed the Riemann domain Σ_F defined by f_{z_0} is isomorphic to the Riemann domain Σ_G defined by $g_{z_0} = z^\alpha = z_0^\alpha \sum_{m=0}^{\infty}\binom{\alpha}{m}\left(\frac{z-z_0}{z_0}\right)^m$.

The graph of Σ_F in $\mathbb{C}^* \times \mathbb{C}$ admits infinitely many branches through the point $(0,1)$ thus the graph of Σ_F is not an analytic set (i.e. we cannot find a holomorphic function g in a neighborhood U of $(1,0) \in \mathbb{C}^* \times \mathbb{C}$ such that $(\omega \times F)(\Sigma_F) \cap U = \{w \in U \mid g(w) = 0\}$).

<u>Moral</u> The graph of an analytic function has the great merit to exist. However it is not in general an analytic subset of the product $\omega(\Sigma) \times \mathbb{C}$.

Remark 2.2. An analytic subset A of a complex space Z is a set having the following property: for all $a \in A$ there exist an open neighborhood $U(a)$ of a and finite many holomorphic functions $f_\alpha \in \Gamma(U(a), \mathcal{O})$, $1 \le \alpha \le k$, such that

$$A \cap U(a) = \{w \in U(a) \mid f_\alpha(w) = 0 \; 1 \le \alpha \le k\}.$$

Remark 2.3. Different analytic functions may have isomorphic Riemann domain.

Remark 2.4. Consider in $\Sigma_F \times \Sigma_F$ the set

$$A = \{(x,y) \in \Sigma_F \times \Sigma_F \mid F(x) = F(y)\}$$

The set A is analytic and contains the diagonal Δ_{Σ_F}. Let $B = \overline{A \setminus \Delta_{\Sigma_F}}$. This is also an analytic set (the union of the irreducible components of A not contained in Δ_{Σ_F}). If p_1, p_2 are the two projection of the product $\Sigma_F \times \Sigma_F$ on each one of its factors, then $p_1(B) = p_2(B) \subset \Sigma_F$. Then $\omega \times F$ is one to one on $\Sigma_F \setminus p_1(B)$.

[1] F is the analytic function defined by f_{x_0}, see Definition 1.6 at page 3.

[2] More precisely there exists an isomorphism $\Sigma_F \cong \mathbb{C}$ such that $\omega(u) = e^u$ and $F(u) = e^{\alpha u}(e^u - 1)$; see Example 1.9 at page 4

2. Analysis of singularities of analytic functions of one variable

a) Unrestricted covering manifolds Let X, Σ be connected topological manifolds and let $\omega\colon \Sigma \to X$ be a continuous map which is a local homeomorphism. Set $\Omega = \omega(\Sigma)$. This is an open set.

DEFINITION 2.5. We say that $\Sigma \xrightarrow{\omega} X$ is an **unrestricted covering** of $\Omega = \omega(\Sigma)$ if the following condition is satisfied:

$$\forall\, x_0 \in \Omega, \qquad \forall\, \alpha \in \Sigma \qquad \text{with } \omega(\alpha) = x_0, \qquad \forall \text{ path } \gamma\colon [0,1] \to \Omega \qquad \text{with } \gamma(0) = x_0$$

there exist a lifting

$$\sigma\colon [0,1] \to \Sigma \qquad \sigma(0) = \alpha$$

i.e. such that

$$\begin{tikzcd} & \Sigma \arrow[d, "\omega"] \\ {[0,1]} \arrow[ur, "\sigma"] \arrow[r, "\gamma"] & \Omega \end{tikzcd}$$

commutes.[3]

REMARK 2.6. (1) If a lifting σ of γ exists with $\sigma(0) = \alpha$ this lifting is unique.
(2) If the above lifting property is satisfied at a point $x_0 \in \Omega$ then it is satisfied at any other point of Ω.

[3]Every surjective unrestricted covering is a covering space in the usual sense [**18**]. The proof of this fact is divided in several steps: here $\Sigma \xrightarrow{\omega} \Omega$ is a fixed unrestricted covering of Ω.

a) According to Footnote 2 of Page 5 the theorem of unicity of lifting holds with the same statement of usual covering.

b) Let $U \subset \Sigma$ be a open connected subset such that $\omega\colon U \to \Omega$ is injective. Then U is a connected component of $\omega^{-1}(\omega(U))$. It is sufficient to prove that $\omega^{-1}(\omega(U)) - U$ is open: let $x \notin U$ such that $\omega(x) \in \omega(U)$ and let $y \in U$ such that $\omega(x) = \omega(y)$. Since Σ is Hausdorff there exists two disjoint open subsets $A, B \subset \omega^{-1}(\omega(U))$ such that $x \in A$, $y \in B \subset U$ and $\omega(A) \subset \omega(B)$. Then $\omega(A \cap U) = \omega(A \cap (U - B)) \subset \omega(A) \cap \omega(U - B) = \omega(A) \cap (\omega(U) - \omega(B)) = \emptyset$ and therefore $A \cap U = \emptyset$.

c) Let $f\colon X \to \Sigma$ be a map such that the composition ωf is continuous. Then the set of points $x \in X$ where f is continuous is open. In fact if f is continuous in x there exists two open subsets $x \in V \subset X$, $f(x) \in U \subset \Sigma$ such that $f(V) \subset U$ and $\omega\colon U \to \omega(U)$ is a homeomorphism. Then the restriction of f to V must be equal to the composition $V \xrightarrow{\omega f} \omega(U) \xrightarrow{\omega^{-1}} U$ and therefore f is continuous on V.

d) Let $f\colon [0,1]^2 \to \Sigma$ be a map with the following properties:

(1) The composition $\omega f\colon [0,1]^2 \to \Omega$ is continuous.
(2) The map $s \mapsto f(s,0)$ is continuous.
(3) The map $t \mapsto f(s,t)$ is continuous for every $s \in [0,1]$.

Then f is continuous. Consider first the particular case in which there exists an open connected subset $U \subset \Sigma$ such that $\omega\colon U \to \omega(U)$ is a homeomorphism, $\omega f([0,1]^2) \subset \omega(U)$ and $f(s_0,t_0) \in U$ for some $t_0, s_0 \in [0,1]$. We have seen that U is a connected component of $\omega^{-1}(\omega(U))$ and then $f(s_0,t) \in U$ for every t. In particular $f(s_0,0) \in U$, therefore $f(s,0) \in U$ for every s and then $f(s,t) \in U$ for every s,t. This implies that f is the composition of ωf and $\omega^{-1}\colon \omega(U) \to U$.

In the general case the set of points where f is not continuous is compact and, if it is non empty there exists a point (s_0,t_0) where f is not continuous and f is continuous on the open subset $[0,1] \times [0,t_0)$. Assume for simplicity of exposition that $s_0, t_0 \in (0,1)$; then we can choose a connected open subset $U \subset \Sigma$ such that $f(s_0,t_0) \in U$ and $\omega\colon U \to \omega(U)$ is a homeomorphism. Let $\varepsilon > 0$ such that $\omega f([s_0-\varepsilon, s_0+\varepsilon] \times [t_0-\varepsilon, t_0+\varepsilon]) \subset \omega(U)$. The above particular case gives that f is continuous of the rectangle $[s_0-\varepsilon, s_0+\varepsilon] \times [t_0-\varepsilon, t_0+\varepsilon]$ and this gives a contradiction.

e) For proving that ω is a covering it is sufficient to prove that if $V \subset \Omega$ is a simply connected open subset and $U \subset \omega^{-1}(V)$ is a connected component, then $\omega\colon U \to V$ is a homeomorphism. Let $x_0 \in U$ and $q \in V$ and $\alpha\colon [0,1] \to V$ be a path such that $\alpha(0) = \omega(x_0)$, $\alpha(1) = q$. Then the lifting of α with base point x_0 is contained in the connected component U and then $\omega\colon U \to V$ is surjective. Assume now that $y = \omega(x_0) = \omega(x_1)$ for some $x_0, x_1 \in U$ and choose a path $\alpha\colon [0,1] \to U$ such that $\alpha(0) = x_0$, $\alpha(1) = x_1$. Since V is simply connected there exists a homotopy of paths $F\colon [0,1]^2 \to V$ such that $F(0,t) = \alpha(t)$, $F(1,t) = y$. Define $f\colon [0,1]^2 \to U$ by setting, for every s, $t \mapsto f(s,t)$ as the lifting of $t \mapsto F(s,t)$ with base point x_1. By the previous step F is continuous and then is a homotopy between the path α and the constant path. Therefore $x_1 = x_2$.

(3) If the above property of lifting is satisfied for a point $x_0 \in \Omega$ and on $\alpha \in \omega^{-1}(x_0)$ then it is satisfied at all $\alpha \in \omega^{-1}(x_0)$[4].

Let $\Sigma \xrightarrow{\omega} \Omega$ be an unrestricted covering of Ω. There exists a universal unrestricted covering manifold

$$\widetilde{\Omega} \xrightarrow{\pi} \Omega$$

on which the group $\pi_1(\Omega)$ operate as a group of automorphism, having the following property:

(1) (universal property of $\widetilde{\Omega} \xrightarrow{\pi} \Omega$) for all unrestricted covering $\Sigma \xrightarrow{\omega} \Omega$ we have a commutative diagram of unrestricted covering maps

$$\begin{tikzcd} \widetilde{\Omega} \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma \arrow[dl, "\omega"] \\ \Omega & \end{tikzcd}$$

(2) if $G = \{\gamma \in \pi_1(\Omega) = Aut(\widetilde{\Omega} \xrightarrow{\pi} \Omega) \mid \sigma \circ \gamma = \sigma\}$ (i.e. G is the subgroup of $Aut(\widetilde{\Omega} \xrightarrow{\pi} \Omega)$ which preserves the fibers of G) then $\Sigma \cong \widetilde{\Omega}/G$ and σ reduced to the natural projection $\widetilde{\Omega} \to \widetilde{\Omega}/G$. We have $\pi_1(\widetilde{\Omega}) = id$, $\pi_1(\Sigma) = G$.

(3) $\widetilde{\Omega} \xrightarrow{\pi} \Omega$ unrestricted and $\pi_1(\widetilde{\Omega}) = id$ characterize the universal covering manifold.

b) Pure ramification points of an analytic function of one variable

Let X be a complex connected manifold on which we make the drastic assumption

$$\dim_{\mathbb{C}} X = 1.$$

EXAMPLE 2.7. $X = \mathbb{C}$ or $X = \mathbb{P}_1(\mathbb{C})$ the Riemann sphere, this last is obtain from two copies of $\mathbb{C}$, where z and w are respectively the holomorphic coordinates, by gluing $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$ on the first copy to $\mathbb{C} \setminus \{0\} = \mathbb{C}^*$ on the second copy by the map

$$z = \frac{1}{w}.$$

This manifold $\mathbb{P}_1(\mathbb{C})$ is homeomorphic to a 2-sphere.

Let $\Sigma_F \xrightarrow{\omega} X$ be the Riemann domain of an analytic function F over X. Let $\Omega = \omega(\Sigma_F)$ and $C = X \setminus \Omega$. Let z_0 be a point of Ω or an isolated point of C. Then we can choose a local coordinate z centered at z_0 (i.e. such $z(z_0) = 0$) in X and $\varepsilon > 0$ such that

$$D_{z_0}(\varepsilon) = \{z \mid 0 < |z| < \varepsilon\} \subset \Omega \cap (\text{ chart of } z).$$

Consider the set

$$\omega^{-1}(D_{z_0}(\varepsilon)) \xrightarrow{\omega} D_{z_0}(\varepsilon).$$

Each connected component Λ of $\omega^{-1}(D_{z_0})$ is thus a Riemann domain over $D_{z_0}(\varepsilon)$.

DEFINITION 2.8. We say that z_0 is a **pure ramification point of the analytic function** F if for some $\varepsilon > 0$ each connected component Λ of $\omega^{-1}(D_{z_0}(\varepsilon))$ is an unrestricted covering of the punctured disc $D_{z_0}(\varepsilon)$.[5]

Replacing z by $\zeta = \frac{z}{\varepsilon}$ we may assume that $\varepsilon = 1$. We will be mainly interested in analytic function F defined over a compact manifold X connected (as $\mathbb{P}_1(\mathbb{C})$) and presenting everywhere on X only pure ramification points.

In this case $C = X \setminus \omega(\Sigma_F)$ will be at most a finite set. This because X is compact and if z_0 is a pure ramification point, z_0 by assumption is an isolated point of C.

c) Let $\Sigma_F \xrightarrow{\omega} X$ be the Riemann domain of an analytic function F over X; one says that given a point $a \in X$, F has a singularity over a if and only if we can find $x_0 \in \Omega$ $\alpha_0 \in \omega^{-1}(x_0)$ and a path

$$\gamma\colon [0,1] \to X \qquad \gamma(0) = x_0 \quad \gamma(1) = a$$

such that

[4]This means that $\Sigma \xrightarrow{\omega} X$ is an unrestricted covering if there exists $x_0 \in X$ and $\alpha \in \omega^{-1}(x_0)$ for what the above lifting property is satisfied.

[5]Example: $0 \in \mathbb{C}$ is not a pure ramification point for the analytic function $(\sin(\log z))^{-1}$.

(1) for all α, $0 < \alpha < 1$, the path $\gamma_\alpha \colon [0, \alpha] \to X$ admits a lifting $\sigma_\alpha \colon [0, \alpha] \to \Sigma_F$, $\sigma(0) = \alpha_0$;
(2) there is not a lifting $\sigma_1 \colon [0, 1] \to \Sigma$, $\sigma(0) = \alpha_0$, of γ.

EXAMPLE 2.9. Let $X = \mathbb{C}$ and let F be the analytic function defined by one of the germ $y(x)$ defined by the equation
$$y^2(y - 1) = x.$$
Then $x = 0$ is a singular point of F, it is a pure ramification point however and is contained in $\omega(\Sigma_F)$.

3. Power series expansions of pure ramification points

a) Let $D^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ be the standard punctured disc. Then

(1) $\pi_1(D^*) = \mathbb{Z}$ and is generated by the closed path
$$\gamma \colon [0, 1] \to D^* \qquad \gamma(t) = e^{2\pi i t}.$$
(2) Set $\widetilde{D}^* = \{w \in \mathbb{C} \mid \Re(w) < 0\}$ and consider the map $\pi \colon \widetilde{D}^* \to D^*$ given by
$$z = e^w.$$
The map π is surjective and $\widetilde{D}^* \xrightarrow{\pi} D^*$ is an unrestricted covering of D^*. It is the universal covering and the group of automorphism is the cyclic group generated by the translation
$$\pi \colon w \to w + 2\pi i.$$
$\pi_1(D^*) \cong \{\tau^m\}_{m \in \mathbb{Z}}$ as automorphism group of the universal covering of D^*. Any subgroup $G \subset \mathbb{Z}$ is of the form
$$G = m\mathbb{Z} \qquad \text{for some } m \in \mathbb{Z}.$$
We have therefore two type of possible unrestricted covering of D^*
$$\Lambda \to D^*.$$
<u>Case $m \neq 0$</u>. Then $\Lambda = \Lambda_m = \widetilde{D}^*/m\mathbb{Z}$ obtained by dividing $\widetilde{D}^*$ by the subgroup $\{\tau^{mk}\}_{k \in \mathbb{Z}} = G$. The function
$$\zeta = e^{\frac{w}{m}}$$
is G-invariant on $\widetilde{D}^*$ and therefore it gives a holomorphic coordinate covering Λ_m. The natural map $\Lambda_m \to D^*$ reduces then to
$$z = \zeta^m \tag{2.1}$$
and
$$\Lambda_m \cong D^* = \{\zeta \in \mathbb{C} \mid 0 < |\zeta| < 1\}.$$
<u>Case $m = 0$</u>. Then $\Lambda = \widetilde{D}^*$ is the universal covering of D^* with the natural map
$$z = e^w. \tag{2.2}$$

b) Let X be of $\dim_{\mathbb{C}} X = 1$ connected and let
$$\Sigma_F \xrightarrow{\omega} X$$
be the Riemann domain of an analytic function over X. Let $a \in X$ be a pure ramification point, so that a chart (z) at a can be chosen so that
$$D^* = D_a^* = \{z \mid 0 < |z| < 1\} \subset \Omega = \omega(\Sigma_F)$$
and each connected component Λ of $\omega^{-1}(D^*)$ is an unrestricted covering of D^*. Then either $\Lambda \xrightarrow{\omega} D^*$ is isomorphic to $\Lambda_m \to D^*$ for some $m \neq 0$ or it is isomorphic to the universal covering $\widetilde{D}^* \to D^*$. Let ζ be the chart covering Λ, then $F|_\Lambda$ becomes a holomorphic function of ζ on the domain of definition of ζ which is
$$D^* = \{0 < |\zeta| < 1\} \qquad \text{if} \quad \Lambda \cong \Lambda_m \quad \text{for some} \quad m \neq 0$$
$$\widetilde{D}^* = \{\Re\zeta < 0\} \qquad \text{if} \quad \Lambda \cong \widetilde{D}^* \quad (m = 0).$$

If $m \neq 0$ then $F(\zeta)$ has a Laurent expansion

$$F(\zeta) = \sum_{-\infty}^{+\infty} a_h \zeta^h$$

(convergent for $0 < |\zeta| < 1$) and this in terms of z as $\zeta = z^{\frac{1}{m}}$ becomes below on $D^* \subset X$

$$F(z) = \sum_{-\infty}^{+\infty} a_h z^{\frac{h}{m}}$$

(Puiseux expansion of F at a). We have "one" of these Puiseux expansion of F for every component Λ of $\omega^{-1}(D^*)$ of the type Λ_m ($m \neq 0$) (For $m = 1$ this is the usual Laurent expansion).

PROBLEM 2.10. *What sort of expansion or representation we should take for an analytic function $F(\zeta)$ defined on the halfplane $\{\Re\zeta < 0\}$?*

In general one cannot see any sort of expansion of this type but one could restrict the consideration to those function $F(\zeta)$ which are uniformly almost periodic in the strip $-\infty < \Re\zeta < 0$, or have a Dirichlet series expansion

$$F(\zeta) = \sum a_n e^{\lambda_n \zeta} \qquad \lambda_n \in \mathbb{R} \quad \lambda_n \to \infty.$$

This question is completely open.

c) Let $\Sigma_F \xrightarrow{\omega} X$ be as above and let $a \in X$ be a pure ramification point.

DEFINITION 2.11. We say that F present only **algebroid singularities** over a if

(1) the map $\omega^{-1}(D_a^*) \to D_a^*$ is of finite degree. In particular the number of connected component of $\omega^{-1}(D_a^*)$ is finite and each one Λ is of the type Λ_m for some $m \neq 0$.
(2) in the local uniformizing parameter ζ on Λ, $F|_\Lambda$ has only a finite pole i.e.

$$F(\zeta) = \sum_{-N}^{+\infty} a_h \zeta^h \qquad \text{some} \quad N \in \mathbb{Z}.$$

DEFINITION 2.12. An analytic function F over a compact connected complex manifold X of $\dim_{\mathbb{C}} X = 1$ which presents everywhere on X only algebroid singularities is called an **algebraic function** over X.

4. Polar singularities as germs of holomorphic maps

Let $F(\zeta)$ be a holomorphic function defined on the punctured disc $D^* = \{\zeta \in \mathbb{C} \mid 0 < |\zeta| < 1\}$ having a finite pole at the origin:

$$F(\zeta) = \sum_{-N}^{+\infty} a_h \zeta^h \qquad \text{some} \quad N \in \mathbb{Z}.$$

We can consider F as a map of D^* into $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$. If z is the coordinate in $\mathbb{C}$ the map is given by

$$z = F(\zeta).$$

CLAIM 2.13. *The map $z = F(\zeta)$ extends uniquely as a holomorphic map of the unit disc $D = \{\zeta \mid |\zeta| < 1\}$ into $\mathbb{P}^1(\mathbb{C})$*

$$\widetilde{F} \colon D \to \mathbb{P}^1(\mathbb{C}) \qquad \widetilde{F}|_{D^*} = F$$

PROOF. We have

$$F(\zeta) = \zeta^{-N}(a_0 + a_1\zeta + a_2\zeta^2 + \cdots)$$

and we may assume $a_0 \neq 0$ without loss of generality. The holomorphic coordinate w on $\mathbb{P}^1(\mathbb{C})$ around ∞ is $w = \frac{1}{z}$ so that in that coordinate patch the map is given by

$$w = F(\zeta)^{-1} = \zeta^N(a_0 + a_1\zeta + a_2\zeta^2 + \cdots)^{-1} = \zeta^N(b_0 + b_1\zeta + b_2\zeta^2 + \cdots).$$

This is holomorphic also at $\zeta = 0$. ☐

This remarks suggest that when $\dim X = 1$ one should replace the sheaf $\mathcal{O}$ with the sheaf $\widetilde{\mathcal{O}}$ of germs of holomorphic map of X into $\mathbb{P}^1(\mathbb{C})$. The Riemann domain $\widetilde{\Sigma_F}$ of the analytic function defined by a germ of holomorphic map f_{x_0} into $\mathbb{P}^1(\mathbb{C})$ can then constructed as before. Except for the non interesting germ constantly $= \infty$ the Riemann domain $\widetilde{\Sigma}_F$ will be in general larger then the previous one obtained from one of the holomorphic germs of F. On $\widetilde{\Sigma}_F$ the function F is then a holomorphic map in $\mathbb{P}^1(\mathbb{C})$ with $F_{f_{x_0}} = \omega^* f_{x_0}$. We will not make use of this remark but occasionally as $\widetilde{\mathcal{O}}$ looses the ring structure that is present in $\mathcal{O}$.

5. Algebraic functions on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$

a) Let X be compact connected and $\dim_{\mathbb{C}} X = 1$. Let F be an analytic function on X with Riemann domain $\Sigma_F \xrightarrow{\omega} X$.

To say that F is an **algebraic function** means that

(1) $\omega(\Sigma_F) = X \setminus$ at most a finite set A;
(2) each point $a \in X$ is a pure ramification point and there are only finite many points $a_1, \dots, a_k$ in X over which F has a singularity (clearly $X \setminus \omega(\Sigma_F) \subset a_1 \cup \dots \cup a_k$).
(3) at each singular pure ramification point a_i F present only algebroid singularities.

Set $A = a_1 \cup \dots \cup a_k$. If F is algebraic, consider the map

$$\lambda \colon \Sigma_F \setminus \omega^{-1}(A) \to X \setminus A.$$

$X \setminus A$ is still connected and so is $\Sigma_F \setminus \omega^{-1}(A)$. Moreover $(\Sigma_F \setminus \omega^{-1}(A), \lambda, X \setminus A)$ is an unrestricted covering of $X \setminus A$ therefore the number of sheets, i.e. the degree of λ, is constant and equal a certain integer $m \geq 1$. Above any point $z \in X \setminus A$ we have thus exactly m distinct germs of the analytic function F.

b) We now assume that $X = \mathbb{P}^1(\mathbb{C})$. We have the following

Theorem 2.14. *Let z be a non homogeneous coordinate on $\mathbb{P}^1(\mathbb{C})$ (i.e. $z \in \mathbb{C}$ or $z = \infty$ where $w = z^{-1}$ is the local coordinate). Let F be an algebraic function over $\mathbb{P}^1(\mathbb{C})$ with sheet number equal to m and let $A = a_1 \cup \dots \cup a_k$ be the singular set of F in $\mathbb{P}^1(\mathbb{C})$. The m determination of F over any point of $\mathbb{P}^1(\mathbb{C}) \setminus A \setminus \{\infty\}$ are the roots of a polynomial equation of degree m*

$$\Phi(z, u) = \varphi_0(z)u^m + \varphi_1(z)u^{m-1} + \dots + \varphi_m(z) = 0$$

having the following properties

(1) *the coefficients $\varphi_0(z), \dots, \varphi_m(z)$ are polynomials and $\varphi_0(z) \not\equiv 0$;*
(2) *M.C.D.$(\varphi_0(z), \dots, \varphi_m(z)) = 1$;*
(3) *the polynomial $\Phi(z, u)$ is irreducible.*

(Note that (3) implies (2))).

Proof. α. Let $z_0 \in \mathbb{P}^1(\mathbb{C}) \setminus A \setminus \{\infty\}$ and let

$$(f_1(z))_{z_0}, \dots, (f_m(z))_{z_0}$$

be the germ of F above z_0. Consider the equation

$$\Pi(u - f_i(z)) = u^m - s_1(z)u^{m-1} + \dots + (-1)^m s_m(z) = 0. \tag{2.3}$$

The coefficients $s_i(z)$ are the symmetric functions of the germs $f_i(z)$ $1 \leq i \leq m$. Therefore they are holomorphic near z_0. Moreover by analytic continuation along any closed path $\gamma \colon [0,1] \to \mathbb{P}^1(\mathbb{C}) \setminus A \setminus \{\infty\}$ these functions do come back to themselves. Therefore the coefficient $s_i(z)$ of the equation (2.3) are holomorphic on $\mathbb{P}^1(\mathbb{C}) \setminus A \setminus \{\infty\}$.

β. Let $a \in A \cup \infty$ and let z be a local holomorphic coordinate on $\mathbb{P}^1(\mathbb{C})$ centered at a. By assumption, above a, F admits a finite number of Puiseux expansion of type

$$\eta(z) = \sum_{s=-r}^{+\infty} c_s z^{\frac{s}{p}}.$$

Set $\theta(\zeta) = \sum_{-r}^{+\infty} c_s\zeta^s$ and let $\varepsilon = e^{\frac{2i\pi}{p}}$. Consider the p determination corresponding to the considered Puiseux expansion

$$f_1 = \theta(z^{\frac{1}{p}}),\ f_2 = \theta(\varepsilon z^{\frac{1}{p}}),\ \ldots,\ f_p = \theta(\varepsilon^{p-1} z^{\frac{1}{p}}).$$

Consider the symmetric functions

$$\sigma_k = f_1^k + \cdots + f_p^k.$$

We do have

$$\sigma_k(z) = \sum c'_s z^{\frac{s}{p}}\{1 + \varepsilon^s + \cdots + \varepsilon^{s(p-1)}\}$$

where we have set $\theta(\zeta)^k = \sum c'_s\zeta^s$. Now

$$1 + \varepsilon^s + \cdots + \varepsilon^{s(p-1)} = \frac{(\varepsilon^s)^p - 1}{\varepsilon^s - 1} = 0$$

if $\varepsilon^s \neq 1$ i.e. $s \not\equiv 0 \pmod p$. While if $s \equiv 0 \pmod p$ then that sum equals to p. Thus in $\sigma_k(z)$ the factorial powers of z disappears i.e.

$$\sigma_k(z) = \sum_{s=-r}^{+\infty} b_s z^s.$$

This being true for any choice of the Puiseux expansion of type

$$\widetilde{\sigma}_k(z) = \sum_{-N}^{+\infty} a_s z^s.$$

We conclude then that the functions $s_i(z)$ (which are expressible as polynomials in the $\widetilde{\sigma}_k$) are holomorphic at all points of $\mathbb{P}^1(\mathbb{C})$ with finite many exceptions where they present a polar singularity (of finite order).

γ. The following is an easy theorem of Méray: any meromorphic[6] function on $\mathbb{P}^1(\mathbb{C})$ is a rational function of the non homogeneous coordinate z.[7]

We can therefore find polynomials $\varphi_j(z)$ with $\varphi_0(z) \not\equiv 0$ such that

$$s_j(z) = (-1)^j \frac{\varphi_j(z)}{\varphi_o(z)} \qquad M.C.D.(\varphi_0, \varphi_1, \ldots, \varphi_m) = 1.$$

Chasing denominators equation (2.3) became the equation $\Phi(z, u) = 0$.

δ. It remain to show that $\Phi(z, u)$ is irreducible. Let $z_0 \in \mathbb{P}^1(\mathbb{C})\backslash A\backslash\{\infty\}$ and let $f_1(z), \ldots, f_m(z)$ be the m determination of F above z_0. We may assume them holomorphic on a common disc $\{|z - z_0| < \varepsilon\} = D$. By assumption the germs $f_1(z)_{z_0}, \ldots, f_m(z)_{z_0}$ are two by two distinct therefore the sets

$$A_{ij} = \{z \in D \mid f_i(z) = f_j(z)\} \quad i \neq j, \qquad A_{00} = \{z \in D \mid \varphi_0(z) = 0\}$$

are discrete (indeed finite). Set $A = \cup A_{ij}$. Then $A \subset D$. Assume now if possible that $\Phi(z, u)$ is reducible

$$\Phi(z, u) = \Phi_1(z, u)\Phi_2(z, u)$$

then degree in u of $\Phi_i = m_i \geq 1$ and $m_1 + m_2 = m$. Now on Σ_F, $\Phi_1(z, F) \equiv 0$ or $\Phi_2(z, F) \equiv 0$ because Σ_F is connected. This is impossible because for $z \in D \setminus A$ the m values of F above z are distinct while if say $\Phi(z, F) \equiv 0$ these values must satisfy a polynomial equation of degree $< m$. □

[6]A **meromorphic function** $f\colon X \to \mathbb{C}$ is a function such that that is holomorphic on all X except a set of isolated points, which are poles for the function.

[7]Let φ be a meromorphic function on $\mathbb{P}^1(\mathbb{C})$, then φ has a finite number of poles and zeroes that we may assume contained in the affine line $\mathbb{C}$ with affine coordinate z. If $p_1, \ldots, p_n \in$ are the poles with multiplicities $a_1, \ldots, a_n$ and $q_1, \ldots, q_m \in$ are the zeroes with multiplicities $b_1, \ldots, b_n$ then, denoting by

$$f = \frac{(z - p_1)^{a_1} \cdots (z - p_n)^{a_n}}{(z - q_1)^{b_1} \cdots (z - q_m)^{b_m}}\varphi$$

we have that f is meromorphic on $\mathbb{P}^1(\mathbb{C})$, holomorphic and invertible on $\mathbb{C}$. Therefore either f of $1/f$ is holomorphic on $\mathbb{P}^1(\mathbb{C})$ and therefore constant by maximum principle.

c) If we drop the assumption $X = \mathbb{P}^1(\mathbb{C})$ what becomes the previous theorem?

Let us denote by $\mathcal{K}(X)$ the field of meromorphic functions on X. (It can be proved that this field is an algebraic function field of transcendence degree 1). The above theorem becomes:

THEOREM 2.15. *The m determinations of F satisfy an equation of degree m*

$$\Phi(z,u) = u^m + b_1(z)u^{m-1} + \cdots + b_m(z) = 0$$

where $b_i(z) \in \mathcal{K}(X)$. The polynomial $\Phi(z,u)$ is irreducible over $\mathcal{K}(X)$.

PROOF. just as before. □

REMARK 2.16. $\mathcal{K}(\mathbb{P}^1(\mathbb{C}))$ is the quotient field of the ring of polynomials in one variable which is a unique factorization ring. This property is lost in general by the field $\mathcal{K}(X)$ when X is a compact manifold connected of $\dim_{\mathbb{C}} X = 1$.

6. Characterization of algebraic functions over $\mathbb{P}^1(\mathbb{C})$

Consider a polynomial equation

$$\Phi(z,u) = \varphi_0(z)u^m + \varphi_1(z)u^{m-1} + \cdots + \varphi_m(z) = 0.$$

We make on Φ the following assumption

$$\begin{cases} \varphi_0(z) \neq 0 \qquad M.C.D.(\varphi_0, \varphi_1, \ldots, \varphi_m) = 1 \\ \Delta(z) = \text{discriminant of } \Phi(z,u) \text{ w. r. to } u \text{ is } \not\equiv 0 \end{cases} \tag{2.4}$$

the last condition is the necessary and sufficient condition that Φ has no factor of multiplicity ≥ 2.

PROOF. $\Delta(z)$ is the Sylvester resultant of the elimination of u from $\Phi(z,u)$ and $\frac{\partial\Phi}{\partial u}$. If Φ has a factor $A(z,u)$ of multiplicity ≥ 2

$$\Phi = A^2 G \qquad \deg_u A \geq 1$$

then A is a factor of $\frac{\partial\Phi}{\partial u}$ also and thus $\Delta \equiv 0$. Conversely if $\Delta \equiv 0$ Φ and $\frac{\partial\Phi}{\partial u}$ have a common factor A with $\deg_u A \geq 1$ (this by Gauss theorem). Let $\Phi = AG$, $\frac{\partial\Phi}{\partial u} = AL$. We may assume A irreducible. As $\frac{\partial\Phi}{\partial u} = \frac{\partial A}{\partial u}G + A\frac{\partial G}{\partial u} = AL$ we say that A must divide G as A cannot divide $\frac{\partial A}{\partial u}$ thus Φ is divisible by A^2. □

THEOREM 2.17. *Let $\Phi(z,u)$ be as above and satisfy the condition* (2.4). *Let $z_0 \in \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$ be such that*

$$\varphi_0(z_0)\Delta(z_0) \neq 0$$

then $\Phi(z_0,u) = 0$ has m distinct roots $u_1^0, \ldots, u_m^0$. Moreover we can find $\sigma > 0$ end $\varepsilon > 0$ such that

1. *the disc $|u - u_i^0| \leq \varepsilon$ are two by two disjoint;*
2. *for every $z \in |z - z_0| < \sigma$ there exist a unique root $u_i(z)$ inside the disc $|u - u_i^0| < \varepsilon$ which is a holomorphic function of z. This for $1 \leq i \leq m$;*
3. *each function $u_i(z)$ defines an algebraic function U_i over $\mathbb{P}^1(\mathbb{C})$ whose germ satisfy the equation $\Phi(z,u) = 0$;*
4. *the m algebraic functions $U_1, \ldots, U_m$ thus obtained coincide if and only if $\Phi(z,u)$ is an irreducible polynomial.*

PROOF. α. Select first $\varepsilon > 0$ to satisfy condition i). As $\Phi(z_0,u)$ has no zeros on $|u - u_i^0| = \varepsilon$ we have

$$M = \min_{\substack{1 \leq i \leq m \\ |u - u_i^0| = \varepsilon}} |\Phi(z_0,u)| > 0.$$

We can then choose $\sigma > 0$ so small that for $|z - z_0| \leq \sigma$

$$\sup_{\substack{|u - u_i^0| = \varepsilon \\ 1 \leq i \leq m}} |\Phi(z,u) - \Phi(z_0,u)| < \frac{M}{2}.$$

We can consider the logarithmic integral in $|z - z_o| < \sigma$

$$\sigma_i(z) = \frac{1}{2\pi i}\int_{|u-u_i^0|=\varepsilon} \frac{\frac{\partial z}{\partial u}(z,u)}{\Phi(z,u)}du$$

which is well defined as $\Phi(z,u)$ for $|z - z_0| < \sigma$ never vanishes on $|u - u_i^0| = \varepsilon$, because of the above inequality.

Now $\sigma_i(z)$ = number of zeros of $\Phi(z,.) = 0$ inside the circle $|u - u_i^0| < \varepsilon$. It is a continuous function of z and $\sigma_i(z_0) = 1$, thus $\sigma_i(z) = 1$ for $|z - z_0| < \sigma$. Let $u_i(z)$ be the unique root of $\Phi(z,.)$ for $|z - z_0| < \sigma$ inside the circle $|u - u_i^0| < \varepsilon$. We claim that $u_i(z)$ is a holomorphic function of z. Indeed we have

$$u_i(z) = \frac{1}{2\pi i}\int_{|u-u_i^0|=\varepsilon} u\frac{\frac{\partial z}{\partial u}(z,u)}{\Phi(z,u)}du$$

therefore points $i)$ and $ii)$ are established.

β. Let U_i be the analytic function defined by the holomorphic function $u_i(z)$. On Σ_{U_i} the function $\Phi(z, U_i(z))$ will be holomorphic and on $\omega^* u_i$ it vanishes, $\Phi(z, \omega^* u_i(z)) = 0$, by construction. Thus

$$\Phi(z, U_i(z)) \equiv 0.$$

We have to show that U_i is algebraic, and thus for $1 \leq i \leq m$.

γ. We distinguish several cases.

<u>Case 1.</u> (In the vicinity of a finite root of $\Phi(a,u) = 0$ with $a \neq \infty$.) Let $a \neq \infty$ and u_1 be such that $\Phi(a, u_1) = 0$. Since $M.C.D.(\varphi_0, \ldots, \varphi_m) = 1$ the equation $\Phi(a,u) = 0$ is not identically satisfied. Let μ be the multiplicity of that root. If $\mu > 1$ then $\Delta(a) = 0$. In any case we can choose σ sufficiently small so that on the disc $|u - u_i| \leq \varepsilon$ there are no other roots of $\Phi(a,u) = 0$ so that

$$M = \inf_{|u-u_i|=\varepsilon} |\Phi(a,u)| > 0$$

and

$$\sup_{\substack{|u-u_i^0|=\varepsilon \\ 1\leq i\leq m}} |\Phi(z,u) - \Phi(z_0,u)| < \frac{M}{2}.$$

The same argument used before show that

$$\sigma(z) = \frac{1}{2\pi i}\int_{|u-u_i^0|=\varepsilon} \frac{\frac{\partial z}{\partial u}(z,u)}{\Phi(z,u)}du = \mu \qquad |z-a| < \sigma$$

so that the equation $\sigma(z_0, u) = 0$ has μ distinct roots for any $z_0 \neq a$, $|z_0 - a| < \sigma$. Let $v_1(z), \ldots, v_\mu(z)$ be these roots defined near z_0. By the same argument used before show that

$$\sigma(z) = \frac{1}{2\pi i}\int_{|u-u_i^0|=\varepsilon} \frac{\frac{\partial z}{\partial u}(z,u)}{\Phi(z,u)}du \qquad \text{for } |z-a| < \sigma$$

so that the equation $\Phi(z_0, u) = 0$ has μ distinct roots for any $z_0 \neq a$, $|z_0 - a| < \sigma$. Let $\nu_1(z), \ldots, \nu_\mu(z)$ be these roots defined near z_0. By the same argument used before the functions $U_i(z)$ near z_0 appear to be holomorphic.

$$v_1(z) = \sum c_s^{(1)}(z - z_0)^s$$

$$\ldots$$

$$v_\mu(z) = \sum c_s^{(\mu)}(z - z_0)^s.$$

By analytic continuation along the circle $|z - a| = |z_0 - a|$ these μ roots are permuted so that they distribute themselves in a finite number of "cycles" of orders $\nu_1, \nu_2, \ldots, \sum \nu_i = \mu$. The j-th "cycle" on the covering $\Lambda_{\nu_j}(\sigma) \to \{0 < |z - a| < \sigma\}$ will became a uniform holomorphic function and thus will yield a Puiseux expansion:

$$w(z) = \sum_0^\infty c_s(z-a)^{\frac{s}{\nu_j}}$$

which is locking of polar part as for $z \to a$, $w(z) \to u_1$. Therefore use the unicity of the finite root u_1 we have indeed an algebroid singularity.

Case 2. (In the vicinity of an "infinite root" of $\Phi(a,.)$, $a \neq \infty$.) This is the case when at $a \neq \infty$ the equation $\Phi(a,u) = 0$ drops its degree, i.e. when

$$\varphi_0(a) = 0.$$

We make the substitution $u \to w = \frac{1}{u}$ so that (after chasing denominators) the equation $\Phi(z,w)$ becomes

$$\varphi_m(z)w^m + \cdots + \varphi_0(z) = 0.$$

As $\varphi_0(a) = 0$ we went to study the solutions of this equation in the vicinity of the solution $w = 0$. Assume that $\varphi_0(a) = \cdots = \varphi_{\mu-1}(a) = 0$ but $\varphi_\mu(a) \neq 0$ so that the solution $w = 0$ of $\Phi(a,w) = 0$ is of multiplicity μ. Note that $\mu \leq m$ as $M.C.D.(\varphi_0,\ldots,\varphi_m) = 1$. By the previous discussion the solutions of $\Phi(z,w) = 0$ in a small neighborhood $|z-a| < \delta$ near $w = 0$ give a finite number of Puiseux expansion of the sort

$$w = (z-a)^{\frac{q}{\nu_i}}(a_1^{(i)} + a_2^{(i)}(z-a)^{\frac{1}{\nu_i}} + \ldots) \qquad a_i^{(1)} \neq 0$$

$$\sum \nu_i = \mu \qquad q \geq 1.$$

Therefore for $u = \frac{1}{w}$ we get a finite number of Puiseux expansion of the sort

$$u = (z-a)^{\frac{-q}{\nu_i}}(b_1^{(i)} + b_2^{(i)}(z-a)^{\frac{1}{\nu_i}} + \cdots) =$$

$$= \frac{b_1^i}{(z-a)^{\frac{q}{\nu_i}}} + \frac{b_2^i}{(z-a)^{\frac{q-1}{\nu_i}}} + \cdots + \frac{b_q^i}{(z-a)^{\frac{1}{\nu_i}}} + b_{q+1}^{(i)} + \cdots \qquad + b_1^{(i)} \neq 0.$$

Again this show that the singularity we may have are algebroid. In conclusion near a point $a \neq \infty$ we get that the roots of the equation $\Phi(z,u) = 0$ arrange themselves in a finite number of Puiseux expansion of order ν_i with at most a (finite) polar singularity and $\sum \nu_i = m =$ degree of Φ in u.

Case 3. At the point $a = \infty$ we have to use $\zeta = \dfrac{1}{z}$ as local coordinate. The equation $\Phi(z,u) = 0$ becomes after chasing denominators an equation of he same type:

$$\Psi(\zeta,u) = \Psi_0(\zeta)u^m + \Psi_1(z)u^{m-1} + \cdots + \Psi_m(z) = 0$$

with Ψ_i polynomials with $M.C.D.(\Psi_0,\ldots,\Psi_m) = 1$ and and discriminant of Ψ not identically 0. The previous discussion at the point $\zeta = 0$ can be applied and we get in the vicinity of $a = \infty$ again a finite number of Puiseux expansions

$$u = \sum_{s=-N_i}^{+\infty} c_i^{(s)}\zeta^{\frac{s}{\nu_i}} \qquad \sum \nu_i = m$$

therefore each of the analytic functions $U_1,\ldots,U_m$ present all over $\mathbb{P}^1(\mathbb{C})$ only algebroid singularities.

γ. It remains to show that the algebraic functions U_i coincides if and only if Φ is an irreducible polynomial. Assume Φ irreducible. We have to show that the sheet number of U_1 is m, the degree in u of $\Phi(z,u)$. Since on Σ_{U_1}, $\Phi(z,u) = 0$ we see that the sheet number is $l \leq m$. If $l < m$ by the previous Theorem 2.14 U_1 will be defined by an irreducible equation

$$G(z,u) = h_0(z)u^l + h_1(z)u^{l-1} + \cdots + h_l(z) = 0.$$

In a neighborhood of z_0 the polynomials $\Phi(z,u)$ and $G(z,u)$ have a common root, thus their resultant $R(z) \equiv 0$. But then G and Φ have a common factor i.e. G divides Φ as G is irreducible and thus Φ is not irreducible which is a contradiction. Conversely if $U_1 = U_2 = \cdots = U_m = U$ the sheet number of U is m and U satisfy an irreducible equation $G(z,u) = 0$ of degree m in u. By the previous argument G divides Φ and as the degree of Φ is equal to m and $M.C.D.(\varphi_0,\ldots,\varphi_m) = 1$

$$\Phi = Gc \qquad c \neq 0.$$

$\square$

Remark 2.18. An inspection of the previous proof gives without essential modification the following theorem:

THEOREM 2.19. *Let X be a compact connected complex manifold and* $\dim_{\mathbb{C}} X = 1$. *Let $\mathcal{K}(X)$ be the field of meromorphic functions on X and let*

$$\Phi(z,u) = u^m + b_1(z)u^{m-1} + \cdots + b_m(z) = 0$$

be an equation with coefficient $b_i \in \mathcal{K}(X)$ with the property that the discriminant of Φ isn't 0 *in $\mathcal{K}(X)$. The m determination $u_1(z), \ldots, u_m(z)$ of the solution of that equation near a point $z_0 \in X$ which is a regular point of the functions $b_i(z)$ and where the discriminant of Φ is also regular and $\neq 0$ give rise to algebraic functions $U_1, \ldots, U_m$ over X. These functions coincide if and only if the polynomial Φ is irreducible over $\mathcal{K}(X)$.*

CHAPTER 3

Riemann surfaces

1. The Riemann surface of an algebraic function

a) Let X be a compact connected complex manifold of $\dim_{\mathbb{C}} X = 1$ and let U be an algebraic function over X with Riemann domain

$$\Sigma_U \xrightarrow{\omega} X.$$

Then $\omega(\Sigma_U)$ covers X with the exception of finite many points and U presents over X only finite many singular points $a_1, \dots, a_k$. Since a point of X which is not covered by $\omega(\Sigma_U)$ is necessarily a singular point we have, setting $A = a_1 \cup \cdots \cup a_k$, that

$$\Sigma_U \setminus \omega^{-1}(A) \to X \setminus A$$

is an unrestricted covering of $X \setminus A$ with a finite number of sheets. Moreover for each point $a \in A$ we can find a punctured disc

$$D_a^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\} \subset X \setminus A$$

where z is a (convenient) local coordinate centered at a such that

$$\omega^{-1}(D_a^*) = \Lambda^1_{\nu_i}(a) \cup \cdots \cup \Lambda^s_{\nu_s}(a)\,,$$

each connected component $\Lambda^j_{\nu_j}(a)$ of $\omega^{-1}(D_a^*)$ being isomorphic to a punctured disc

$$\Lambda^j_{\nu_j}(a) = \{\zeta \in \mathbb{C} \mid 0 < |\zeta| < 1\}$$

and the map $\omega|_{\Lambda^j}$ being given by

$$z = \zeta^{\nu_j}\,.$$

Moreover

$$\sum \nu_j = m$$

and finally $U|_{\Lambda^j_{\nu_j}(a)} = U(\zeta)$ is a holomorphic function of ζ with a Laurent expansion of the form

$$U(\zeta) = \sum_{-N}^{\infty} c_s \zeta^s$$

convergent in $\Lambda^j_{\nu_j}(a)$. For each $\Lambda^j_{\nu_j}(a)$ consider its natural inclusion in the unit disc

$$\Lambda^j_{\nu_j}(a) \hookrightarrow D^j(a) = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}.$$

We attach the disc $D^j(a)$ to Σ_U by the inclusion $\sigma_j \colon \Lambda^j_{\nu_j}(a) \hookrightarrow D^j(a)$, and we do this for all $\Lambda^j_{\nu_j}(a)$ any $a \in A$. We thus obtain the following space

$$\widetilde{\Sigma}_U = \Sigma_U \bigcup_{\substack{\sigma_j(a) \\ \forall j,\ \forall a}} D^j(a)\,.$$

The following properties are of not to difficult verification.

(1) $\widetilde{\Sigma}_U$ is a complex manifold connected of complex dimension 1 with Hausdorff topology.
(2) $\widetilde{\Sigma}_U$ is also compact (as X is compact).
(3) There exist a natural map, holomorphic,

$$\widetilde{\omega} \colon \widetilde{\Sigma}_U \to X$$

which is surjective and $\widetilde{\omega}|_{\Sigma_U} = \omega$.

(4) The function $U\colon \Sigma_U \to \mathbb{C}$ extends to a holomorphic map $\widetilde{U}\colon \widetilde{\Sigma}_U \to \mathbb{P}^1\mathbb{C}$ which is thus a meromorphic function on the manifold $\widetilde{\Sigma}_U$.

(5) The points of $\widetilde{\Sigma}_U$ correspond one to one with all possible Puiseux expansion

$$u(z) = \sum_{-N}^{\infty} c_s (z-a)^{\frac{1}{s}}$$

of the function U.

Definition 3.1. We call the surface $\widetilde{\Sigma}_U$ with its map

$$\widetilde{\omega}\colon \widetilde{\Sigma}_U \to X$$

the **Riemann surface** of the algebraic function U over X.

[1] The Riemann surface of the algebraic function U should not be confused with the Riemann domain of the same analytic function.

(b) In general let X and Y be compact connected complex manifold of complex dimension 1. Let

$$\lambda\colon Y \to X$$

be a non constant holomorphic map. As every non constant holomorphic map is open it follows that $\lambda(Y)$ being open and closed in X, it must coincide with X i.e. λ is surjective.

Definition 3.2. The data (Y, λ, X) will be called an **abstract Riemann surface** over X.

Riemann (second) existence problem is the following: given an abstract Riemann surface $Y \xrightarrow{\lambda} X$ is it isomorphic to the Riemann surface of an algebraic function over X? In other words we ask if there exists an algebraic function U over X such that, if we consider the Riemann surface $\widetilde{\Sigma}_U \xrightarrow{\widetilde{\omega}} X$ of U, we can find an isomorphism

$$\mu\colon Y \xrightarrow{\sim} \widetilde{\Sigma}_U$$

such that

$$\begin{tikzcd} Y \arrow[rr, "\sim"] \arrow[dr, "\lambda"'] & & \widetilde{\Sigma}_U \arrow[dl, "\widetilde{\omega}"] \\ & X & \end{tikzcd}$$

is commutative. The answer to this problem is affermative. It will be a consequence of Riemann-Roch theorem. Another question which is harder answer is this.

Question 3.3. *Given X what is the the class of complex manifolds Y one can encounter as Riemann surface over X?*

We will show that there is no restriction if $X = \mathbb{P}^1(\mathbb{C})$ on the topological or complex structure of Y.

2. Effective construction of the Riemann surface of an algebraic function over $\mathbb{P}^1(\mathbb{C})$. The hyperelliptic case

(a) We first make the following remark.

Remark 3.4. Given two (abstract) Riemann surfaces $Y_1 \xrightarrow{\lambda_1} X$, $Y_2 \xrightarrow{\lambda_2} X$ we say that they are isomorphic if we can find an isomorphism $\mu\colon Y_1 \xrightarrow{\sim} Y_2$ such that

$$\begin{tikzcd} Y_1 \arrow[rr, "\sim"] \arrow[dr, "\lambda_1"'] & & Y_2 \arrow[dl, "\lambda_2"] \\ & X & \end{tikzcd}$$

commutes.

[1] Following Bourbaki, a dangerous bend is a passage in the text that is designed to forewarn the reader against serious errors.

We will consider Riemann surfaces up to isomorphism. In particular if $r, s \in \mathcal{K}(X)$, $r \neq 0$, are meromorphic functions over X, one can prove that if U is an algebraic function over X, $rU + s$ is algebraic over X and U and $rU + s$ have isomorphic Riemann surfaces.

(b) Let us now take $X = \mathbb{P}^1(\mathbb{C})$ and let U be algebraic over X.

Case of sheet number $= 1$. By the theorem of the previous lecture

$$U = r(z) = \frac{p(z)}{a(z)} \qquad p, q \text{ polynomials} \quad (p, q) = 1$$

i.e. U is a rational function. Then Σ_U, because of the previous remark, is isomorphic to the Riemann surface of a constant function over $\mathbb{P}^1(\mathbb{C})$. Conclusion: $\widetilde{\Sigma}_U \xrightarrow{\omega} \mathbb{P}^1(\mathbb{C})$ is isomorphic to $\mathbb{P}^1(\mathbb{C}) \xrightarrow{id} \mathbb{P}^1(\mathbb{C})$.

Case of sheet number $= 2$. U is a solution of a second order equation

$$a(z)u^2 + b(z)u + c(z) = 0 \qquad a(z) \neq 0 \quad (a, b, c) = 1$$

which is irreducible. This last condition is equivalent to the condition

$$\Delta(z) = b^2 - 4ac$$

is not a square of a polynomial. This because (since the characteristic is $\neq 2$) we can write the above equation in the form

$$(2au - b)^2 = b^2 - 4ac.$$

Replacing U by $2aU - b$ it is enough to consider equations to the simple form

$$u^2 = P(z)$$

where $P(z)$ is a polynomial in z which is not a square. If $P(z)$ has a multiple factor $A(z) \not\equiv 0$, then

$$P(z) = A^2(z)Q(z)$$

and we can replace U by $\frac{U}{A(z)}$. We may thus as well assume that $P(z)$ has no multiple factors. Let us thus consider the algebraic function defined by

$$u^2 = \prod_{j=1}^{l} (z - \alpha_j) \qquad \alpha_i \neq \alpha_j \quad \text{if } i \neq j.$$

Now $\Delta(z) = 4\prod_{j=1}^{p}(z - \alpha_j)$ thus each point $z = \alpha_j$ is a simple root of $\Delta(z)$ and at each one of these points u has a ramification of order 2

$$u = \sqrt{z - \alpha_j}\, w(z)$$

where $w(z) = \sqrt{\prod_{i \neq j}(z - \alpha_j)}$ is a root near $z = \alpha_j$. We have to inspect the situation at $z = \infty$ where $\zeta = \frac{1}{z}$ is the local coordinate. The equation there becomes

$$u^2 \zeta^l = \prod_{i=1}^{l} (1 - \alpha_j \zeta).$$

Claim 3.5. *If* $l \equiv 0 \pmod 2$ *there is no ramification at* ∞. *If* $l \equiv 1 \pmod 2$ *there is ramification at* ∞.

Proof. If $l = 2k$ replacing u by $u\zeta^k$ we see that $u\zeta^k$ has no ramification near $\zeta = 0$ as it satisfies the equation $\nu^2 = \prod_{j=1}^{l}(1 - \alpha_j \zeta)$ thus the same is true for u. If $l = 2k + 1$ replacing u by $u\zeta^{k+1}$ we see that this last is defined by the equation

$$\nu^2 = \zeta \prod_{j=1}^{l} (1 - \alpha_j \zeta)$$

this shows a ramification point of order 2 at $\zeta = 0$ thus the same is true for u. □

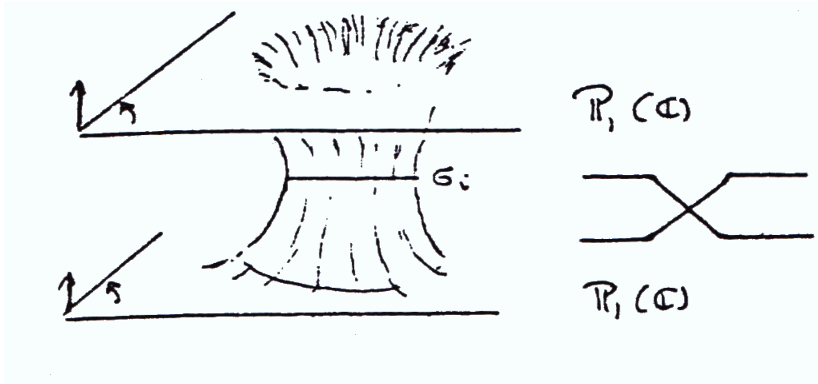


FIGURE 1. A 2-sheeted complex surface over $\mathbb{P}^1(\mathbb{C})$.

Conclusion: if the non homogeneous coordinate z on $\mathbb{P}^1(\mathbb{C})$ is chosen conveniently any Riemann surface with sheet number 2 is isomorphic to the Riemann surface of the algebraic function u defined by an equation of the form

$$u^2 = \prod_{j=1}^{2g+2} (z - \alpha_j) \qquad \alpha_i \neq \alpha_j \text{ if } i \neq j. \tag{3.1}$$

c) Effective construction of the Riemann surface of (3.1)

Let $\sigma_i \colon [0,1] \to \mathbb{P}^1(\mathbb{C})$, $\sigma_i(0) = \alpha_{2i+1}$ and $\sigma_i(1) = \alpha_{2i}$, $1 \leq i \leq g+1$ be arcs whose images are two by two disjoint. Since $\mathbb{P}^1(\mathbb{C})$ is oriented we can talk about the right and left side of σ_i.

Take two copies of $\mathbb{P}^1(\mathbb{C})$ and cut them along the arcs σ_i, $1 \leq i \leq g+1$, identify:

- the left side of σ_i of the first copy with the right side of σ_i on the second copy;
- the right side of σ_i on the first copy with the left side of σ_i on the second copy.

We obtain in this way a 2-sheeted complex surface over $\mathbb{P}^1(\mathbb{C})$ on which the algebraic function u is well defined as a meromorphic function (Figure 1). If we want to study the topological model we may assume the σ_i all on straight segment on the real axis. Reflecting the first copy of $\mathbb{P}^1(\mathbb{C})$ along the real axis we obtain a surface homeomorphic to the previous one. This is homeomorphic to two copies of S^2 joined by $g+1$ "tubes". This is homeomorphic to the surface of a donut with g holes or to the topological sum of g tori.

DEFINITION 3.6. The number g is called the **genus**.

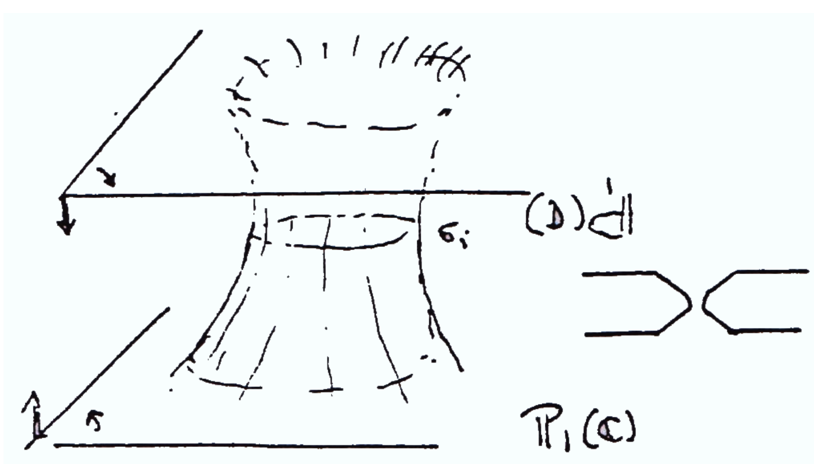


FIGURE 2. Topological model of a 2-sheeted complex surface over $\mathbb{P}^1(\mathbb{C})$.

The surfaces studied above are called **hyperelliptic**. There are hyperelliptic Riemann surfaces of any genus $g = 0, 1, 2, \ldots$.

EXAMPLE 3.7. $u^2 = z$ has a Riemann surface of genus $0 \cong \mathbb{P}^1(\mathbb{C})$. (set for $\lambda \in \mathbb{P}^1(\mathbb{C})$, $u = \lambda$ $z = \lambda^2$).

3. Monodromy groups

(a) Let X be a compact complex connected manifold and of $\dim_{\mathbb{C}} X = 1$ and let u be an algebraic function over X. Let

$$A = a_1 \cup \cdots \cup a_k$$

the set of pure ramification points (i.e. the set of points $a \in X$ such that if $\widetilde{\Sigma}_U \xrightarrow{\omega} X$ is the Riemann surface of U over X then one of the component Λ of $\omega^{-1}(D_a^*(\varepsilon))$ is mapped onto $D_a^*(\varepsilon)$ with a map of degree $m > 1$). Set $\Sigma'_U = \widetilde{\Sigma}_U \setminus \omega^{-1}(A)$. Then

$$\Sigma'_U \xrightarrow{\omega} X \setminus A$$

is an unrestricted covering of $X \setminus A$ with sheet number m. Let $\widetilde{X \setminus A} \xrightarrow{\pi} X \setminus A$ be the universal covering manifold of $X \setminus A$. It is a complex manifold and the fundamental group $\pi_1(X \setminus A)$ operates on $\widetilde{X \setminus A}$ by holomorphic automorphisms. The unrestricted covering $\Sigma'_U \xrightarrow{\omega} X \setminus A$ defines (up to conjugacy in $\pi_1(X \setminus A)$) a subgroup $G \subset \pi_1(X \setminus A)$ so that we can identify Σ'_U with $\frac{\widetilde{X \setminus A}}{G}$ i.e. we have the commutative diagram:

$$\begin{tikzcd}
\Sigma'_U \arrow[rr, "\cong"] \arrow[dr, "\omega"'] & & \frac{\widetilde{X \setminus A}}{G} \arrow[dl] \\
& X \setminus A &
\end{tikzcd}$$

If

$$I_m = \frac{\pi_1(X \setminus A)}{G}$$

this is the typical fiber of ω i.e. I_m is a finite set of m elements. As such I_m is a homogeneous space for the group $\pi_1(X \setminus A)$ which operates on I_m i.e. we have a homomorphism

$$\rho \colon \pi_1(X \setminus A) \to Aut(I_m) = \mathcal{S}_m$$

where $\mathcal{S}_m$ is the symmetric group of permutations of m elements. The group

$$\mathcal{M} = \rho(\pi_1(X \setminus A)) \subset \mathcal{S}_m$$

is a finite group which has the property to be transitive (as I_m is a homogeneous space).

DEFINITION 3.8. $\mathcal{M}$ is called the **monodromy group of the algebraic function** U over X.

Let $o \in X \setminus A$ be the base point from which we calculate $\pi_1(X \setminus A)$ and let $I_m = \{u_1, \ldots, u_m\}$ be the m germs of U over o. For any closed path

$$\gamma \colon [0,1] \to X \setminus A \qquad \gamma(0) = \gamma(1) = o$$

then the operation $\rho(\gamma)$ on I_m is the substitution

$$\rho(\gamma) = \begin{pmatrix} u_{i_1} & \cdots & u_{i_m} \\ u_1 & \cdots & u_m \end{pmatrix}$$

which we obtain by analytic continuation along γ of the m germs $(u_1, \ldots, u_m)$.

EXAMPLE 3.9. Let $X = \mathbb{P}^1(\mathbb{C})$. In this case the group $\pi_1(X \setminus A)$ is generated by the m loops $l_1, \ldots, l_m$ that start from $o \in X \setminus A$ turn around the points $a_1, \ldots, a_m$ respectively in the clockwise direction (see Figure 3) and do come back to o.

Note that if the paths follows around o (in the order of the clock) in the order $l_1, \ldots, l_m$ we have

$$l_1 l_2 \cdots l_m = o.$$

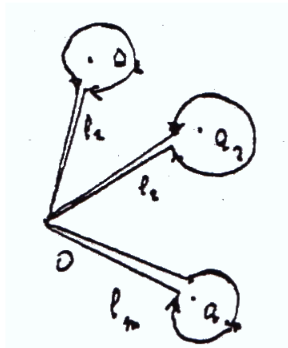


FIGURE 3

Clearly if $S_1 = \rho(l_1), \ldots, S_m = \rho(l_m)$ are known, then the monodromy group is known and we must have (under the above assumption):

$$S_1 \cdot S_2 \cdots S_m = id.$$

b) Let $K = \ker(\rho)$ so that

$$e \to K \to \pi_1(X \setminus A) \xrightarrow{\rho} \mathcal{M} \to e$$

is a short exact sequence of groups. We have

(1) K is a subgroup of finite index in $\pi_1(X \setminus A)$ as $\frac{\pi_1(X\setminus A)}{K} \cong \mathcal{M}$ which is a finite group.
(2) K is an invariant subgroup and indeed

$$K = \bigcap_{\gamma \in \pi_1(X\setminus A)} \gamma G \gamma^{-1}.$$

Indeed if

$$\pi_1(X \setminus A) = G \cup \mu_1 G \cup \mu_2 G \cup \cdots \cup \mu_m G$$

we have that $k \in K$ if and only if

$$k\mu_i G = \mu_i G \qquad \forall 1 \le i \le m$$

thus $\mu_i^{-1} k \mu_i \in G$ for all $1 \le i \le m$. Therefore for any $\gamma \in \pi_1(X \setminus A)$ $\gamma^{-1} k \gamma \in G$.

Set $W = \frac{\widetilde{X\setminus A}}{K}$ so that we have a commutative diagram:

$$\begin{tikzcd} W \arrow[dd, "\pi"] \arrow[dr, "\alpha"] & \\ & \Sigma'_U \arrow[dl, "\omega"] \\ X \setminus A & \end{tikzcd}$$

and

(1) W is a finite covering of $X \setminus A$ with sheet number equal to

$$\text{index of } K \text{ in } \pi_1(X \setminus A) = \# \frac{\pi_1(X \setminus A)}{K} = \# \mathcal{M}.$$

(2) W has a group of automorphism

$$Gal_X(U) = \frac{\pi_1(X \setminus A)}{K} = Aut(W, \pi, X \setminus A) = \mathcal{M}.$$

(3) Σ'_U is obtained by dividing W by the action of $\frac{G}{K}$ (has subgroup of $Gal_X(U)$).

DEFINITION 3.10. The group $Gal_X(U)$ (which is uniquely defined by the analytic function U over X) is called the **Galois group of the analytic function** U over X, the covering $W \xrightarrow{\pi} X \setminus A$ is called the **Galois covering associated to** $\Sigma'_U \xrightarrow{\omega} X \setminus A$.

4. The Galois Riemann surface associated to U

a) Let U be the algebraic function defined over X considered in the previous section. If $\mathcal{K}(X)$ is the field of meromorphic functions over X then U is defined by an irreducible algebraic equation

$$\Phi(z,u) = u^m + b_1(z)u^{m-1} + \cdots + b_m(z) = 0$$

where $b_i(z) \in \mathcal{K}(X)$. Let $u_1, \ldots, u_m$ be the roots of the equation $\Phi(z,u) = 0$ in the algebraic closure $\overline{\mathcal{K}(X)}$ of $\mathcal{K}(X)$ and let $\mathcal{K}(X)(U)$ be the field obtained by adding to $\mathcal{K}(X)$ the algebraic element U defined by $\Phi(z,u) = 0$.

Remark 3.11. $\mathcal{K}(X)(U)$ is isomorphic to the field

$$\frac{\mathcal{K}(X)[U]}{\Phi\mathcal{K}(X)[U]}.$$

Indeed this is the quotient ring of the polynomial ring $\mathcal{K}(X)[U]$ by the ideal $\mathcal{I} = \mathcal{K}(X)[U]\Phi$ (a principal ideal). Every element of $\frac{\mathcal{K}(X)[U]}{\mathcal{I}}$ is a class of polynomials with coefficient in $\mathcal{K}(X)$ in the indeterminate U with respect to the equivalence relation $p, q \in \mathcal{K}(X)[U]$

$$p \sim q \Leftrightarrow \; p - q \in \mathcal{I}.$$

This ring is a field. Indeed if $p(U) \in \mathcal{K}(X)[U]$, $p(U) \neq 0$ in $\mathcal{K}(X)[U]$ means $p(U) \notin \mathcal{I}$ thus $p(U)$ and $\Phi(U)$ have no common factor (as $\Phi(U)$ is irreducible). Therefore the Sylvester resultant[2] of the elimination of U from Φ and p is an element $R \in \mathcal{K}(X)$, $R \neq 0$. But

$$R = A(U)p(U) + B(U)\Phi(U)$$

this means that in $\mathcal{K}(X)(U)$ we have

$$\frac{1}{R}Ap = 1 \qquad \text{i.e } \frac{1}{R}A = p^{-1}.$$

Every element of $\mathcal{K}(X)(U)$ is a rational function in U with coefficients in $\mathcal{K}(X)$. Because of the above remark every element $w \in \mathcal{K}(X)(U)$ can be written in a unique way as

$$w = \alpha_1 + \alpha_2 U + \cdots + \alpha_{m-1}U^{m-1}$$

with $\alpha_i \in \mathcal{K}(X)$.

Now if $\sigma\colon \mathcal{K}(X)(U) \to \overline{\mathcal{K}(X)}$ is a homomorphism we must have $\sigma(U) = u_i$ for some i, as $\sigma(U)$ must go in one of the roots of the equation $\Phi(u,z) = 0$. As the m roots $u_1, \ldots, u_m$ are distinct (because Φ is irreducible) we conclude that there are m homomorphisms

$$\sigma_i\colon \mathcal{K}(X)(U) \to \overline{\mathcal{K}(X)}$$

with

$$\sigma_i(U) = u_i \qquad 1 \leq i \leq m.$$

Definition 3.12. The field $E = \mathcal{K}(X)(u_1, \ldots, u_m)$ generated in $\overline{\mathcal{K}(X)}$ by $\cup\sigma_i(\mathcal{K}(X)(U))$ is called the **splitting field** of the equation $\Phi(u) = \Phi(x,u) = 0$. It is the smallest subfield of $\overline{\mathcal{K}(X)}$ containing $\mathcal{K}(X)$ in which the polynomial $\Phi(u)$ splits into linear factor:

$$\Phi(u) = \prod(u - u_i).$$

Let Γ be the group of automorphisms of E which leave $\mathcal{K}(X)$ pointwise fixed. This group Γ is a finite group isomorphic to a subgroup of permutation of the m letters $u_1, \ldots, u_m$ and has the following properties:

(1) Γ is a finite group (obvious);
(2) $p \in \mathcal{K}(X)$ if and only if $\gamma p = p$ for all $\gamma \in \Gamma$ i.e. $\mathcal{K}(X)$ is the field of fixed elements of Γ.

[2] For the definition and main properties of resultant and discriminant see e.g. [**10, 19**]

This second statement is the core of Galois theory. The proof will be omitted.

b) Let us now go back to the geometrical situation where we had established the commutative diagram:

$$\begin{tikzcd} W \arrow[dd, "\pi"'] \arrow[dr, "\alpha"] & \\ & {\Sigma'}_U \arrow[dl, "\omega"] \\ X \setminus A & \end{tikzcd} \qquad \begin{aligned} &\deg \alpha = \# G/K \\ &\deg \pi = \# \pi_1(X - A)/K = \# \mathcal{M} \\ &\deg \omega = m = \# \pi_1(X - A)/G. \end{aligned}$$

Now we make the following remarks.

Remark 3.13. As $\pi\colon W \to X \setminus A$ is a finite covering we can compactify W into $\widetilde{W} \xrightarrow{\widetilde{\pi}} X$ by the same procedure we used for the construction of Riemann surface to that π extends to a holomorphic map $\widetilde{\pi}$ of the compact manifold $\widetilde{W}$ onto X.

Remark 3.14. Let $o \in X \setminus A$ and let $w_o \in \pi^{-1}(o)$. Let $u_1(z), \dots, u_m(z)$ the m germs of the analytic function U and let us lift them in a neighborhood of w_o. We have m analytic germs, that we will designate by $\widetilde{u}_1(z), \dots, \widetilde{u}_m(z)$ in a neighborhood of w_o. Let $\gamma\colon [0,1] \to W$ be a path from w_o to w_o in W, $\gamma(0) = \gamma(1) = w_o$. By construction $\gamma \in K$ and therefore analytic continuation of $\widetilde{u}_1(z), \dots, \widetilde{u}_m(z)$ along γ brings every germ $\widetilde{u}_i(z)$ into itself. They thus define one valued holomorphic maps

$$\widetilde{u}_i\colon W \to \mathbb{P}^1(\mathbb{C}).$$

Remark 3.15. Since the function U has on X only Puiseux expansion with finite polar part it follows that at each point $w \in \widetilde{W} \setminus W$ the function $\widetilde{u}_i$ extends holomorphically to a holomorphic map into $\mathbb{P}^1(\mathbb{C})$. Thus the functions

$$\widetilde{u}_i\colon \widetilde{W} \to \mathbb{P}^1(\mathbb{C})$$

are well defined meromorphic functions on $\widetilde{W}$.

Remark 3.16. The polynomial $\Phi(z,u)$ on X when lifted $\widetilde{W}$ splits into linear factors

$$\pi * \Phi(z,u) = \prod_{i=1}^{m} (u - \widetilde{u}_i)$$

therefore if $\mathcal{K}(\widetilde{W})$ is the field of all meromorphic functions over $\widetilde{W}$, we have:

$$\mathcal{K}(\widetilde{W}) \supset E = \text{ the splitting field of } \Phi.$$

Remark 3.17. Set on $\widetilde{W}$

$$\Theta = \sum_{i=1}^{m} \lambda_i(z) \widetilde{u}_i \qquad \lambda_i(z) \in \mathcal{K}(X)$$

Since $\mathbb{C} \subset \mathcal{K}(X)$, $\mathcal{K}(X)$ is an infinite field therefore we can find $\lambda_i \in \mathcal{K}(X)$ (for some $\lambda_i \in \mathbb{C}$) such that for all $\gamma \in \mathcal{M} = \frac{\pi_1(X \setminus A)}{K}$, $\gamma \neq id$, we have

$$\Theta(\gamma w) - \Theta(w) \not\equiv 0 \qquad w \in \widetilde{W}.$$

In fact we have to satisfy the linear inequalities $\sum \lambda_i (u_i - u_{\gamma(i)}) \not\equiv 0$ where

$$\gamma = \begin{pmatrix} \gamma(1) & \cdots & \gamma(m) \\ 1 & \cdots & m \end{pmatrix}$$

runs over the finite many substitution of the monodromy groups $\mathcal{M}$, different from the identity. It follows then that:

(1) Θ is an algebraic function over X.
(2) $\widetilde{W} \xrightarrow{\pi} X$ is (up to isomorphism) the Riemann surface of the algebraic function Θ.

REMARK 3.18. $\mathcal{K}(\widetilde{W}) \cong \mathcal{K}(X)(\Theta)$ i.e. every meromorphic function on $\widetilde{W}$ is a rational function in Θ with coefficient in $\mathcal{K}(X)$.

PROOF. Let $v \in \mathcal{K}(\widetilde{W})$ and $\{\gamma_1, \dots, \gamma_s\} = \mathcal{M} = \dfrac{\pi_1(X \setminus A)}{K}$ set

$$v_i = v(\gamma_i w) \qquad \Theta_i = \Theta(\gamma_i w).$$

Let us try to solve the equations

$$\begin{cases} v_1 = \alpha_1 \Theta_1^{s-1}(w) + \alpha_2 \Theta_1^{s-2}(w) + \cdots + \alpha_s \\ v_2 = \alpha_1 \Theta_2^{s-1}(w) + \alpha_2 \Theta_2^{s-2}(w) + \cdots + \alpha_s \\ \qquad \cdots \\ v_s = \alpha_1 \Theta_s^{s-1}(w) + \alpha_2 \Theta_s^{s-2}(w) + \cdots + \alpha_s \end{cases}$$

we get

$$\alpha_r = \frac{\det \begin{pmatrix} \Theta_1^{s-1} & \dots & v_1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \Theta_s^{s-1} & \dots & v_s & \dots & 1 \end{pmatrix}}{\det \begin{pmatrix} \Theta_1^{s-1} & \dots & \Theta_1^{s-r} & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \Theta_s^{s-1} & \dots & \Theta_s^{s-r} & \dots & 1 \end{pmatrix}} \qquad 1 \le r \le s$$

which is a meromorphic function on $\widetilde{W}$. But by its form we have α_r is $\mathcal{M}$-invariant:

$$\alpha_r(\gamma w) = \alpha_r(w).$$

By the argument used in Theorem 2.14 we see that α_r is a meromorphic function on X; $\alpha_r \in \mathcal{K}(X)$ thus

$$v = \sum_{r=1}^{s-1} \alpha_r \Theta^{s-r}.$$

□

REMARK 3.19. $\mathcal{K}(\widetilde{W}) \cong E$. Indeed $\Theta \in E$ as $\Theta = \sum \lambda_i \widetilde{u}_i$ and $\widetilde{u}_i \in E$ therefore $\mathcal{K}(\widetilde{W}) \subset E$. But by Remark 3.16 $E \subset \mathcal{K}(\widetilde{W})$.

Conclusion.

α) Given the Riemann surface $\widetilde{\Sigma}_U \xrightarrow{\widetilde{\omega}} X$ of an analytic function U over X defined by the irreducible equation

$$\Phi(z,u) = u^m + b_1(z) u^{m-1} + \cdots + b_m(z) = 0 \qquad b_i \in \mathcal{K}(X) \tag{3.2}$$

thus defines (up to isomorphism) another Riemann surface

$$\widetilde{W} \xrightarrow{\widetilde{\pi}} X$$

which is a Riemann surface of an analytic algebraic function Θ over X and it is endowed with an automorphism group isomorphic to the monodromy group of U.

β) The field of meromorphic functions $\mathcal{K}(\widetilde{W})$ over $\widetilde{W}$ is isomorphic to the field $\mathcal{K}(X)(\Theta)$ and this is the splitting field for the equation (3.2).

γ) The monodromy group of U appears then as the Galois group of the equation (3.2) and as the group of the automorphism of $\widetilde{W} \to X$.

Note that we have proved in this context directly the theorem of Galois:

THEOREM 3.20. *A meromorphic function $v \in \mathcal{K}(\widetilde{W})$ which is invariant under transformations of the Galois group $\mathcal{M}$:*

$$v(\gamma w) = v(w) \qquad \forall \gamma \in \mathcal{M} \quad w \in \widetilde{W}$$

is a meromorphic function on X

$$v \in \pi^* \mathcal{K}(X)$$

and conversely.

The argument is the same one used in Remark 3.18 or in Theorem 2.14.

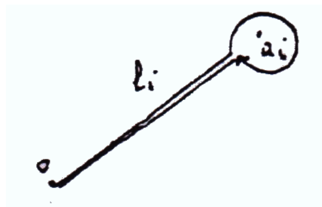


FIGURE 4

5. Effective construction of the Riemann surface of an algebraic function over $\mathbb{P}^1(\mathbb{C})$

a) Let U be an algebraic function over $\mathbb{P}^1(\mathbb{C})$ and let m be its sheet number. Let $A = a_1 \cup \cdots \cup a_k$ be the set of pure ramification points. Let $o \in X \setminus A$ and let

$$\begin{cases} \gamma_i \colon [0,1] \to X \setminus A \qquad \gamma_i(0) = o \quad \gamma_i(1) = a_i \\ 1 \leq i \leq k \end{cases}$$

be two by two disjoint (except for the point o) differentiable paths from o to the points $a_i \in A$. If l_i denote the loop of $\pi_1(o, \mathbb{P}^1(\mathbb{C}) \setminus A)$ obtained by following γ_i to near a_i turning around a_i in the clockwise direction and coming back to o along γ_i (Figure 4), we know that $l_1, \ldots, l_k$ generate $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus A)$ and if the loops follows around o in the clockwise direction we have $l_1 \cdots l_k = id$.

We take m copies of $\mathbb{P}^1(\mathbb{C})$ cut along the paths γ_i and we number them from 1 to m. Let

$$\rho(l_i) = \begin{pmatrix} i_1 & \ldots & i_m \\ 1 & \ldots & m \end{pmatrix}$$

be the substitution of the monodromy group associated to l_i. Identify the left side of γ_i on the on the k-th copy of $\mathbb{P}^1(\mathbb{C})$ with the right hand side of the i_k-th copy of $\mathbb{P}^1(\mathbb{C})$ and do this for $k = 1, \ldots, m$ and for $i = 1, \ldots, k$. We obtain in this way a compact complex manifold Σ_U which is also connected because the monodromy group is transitive. Note that the identification is well defined also around o because $\rho(l_1) \cdots \rho(l_k) = id$, and there is no ramification over o. The manifold Σ_U with its natural projection $\omega \colon \Sigma_U \to \mathbb{P}^1(\mathbb{C})$ is isomorphic to the Riemann surface of the algebraic function U.

EXAMPLE 3.21. Consider the algebraic functions defined by the equation (Figure 5)

$$\Phi(z, u) = u^3 - 3u + 2z = 0$$

on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ where z is the non homogeneous coordinate. This polynomial is irreducible (as it is linear in z) and the discriminant vanishes for $z = \pm 1$ ($\frac{\partial \Phi}{\partial u} = 3(u^2 - 1)$ so where $\Delta(z) = 0$, $u = \pm 1$[3]).

Consider the loops[4]

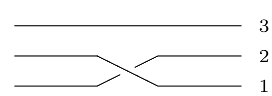
$$l_1 = [-1, 0]$$
$$l_2 = [0, \infty]$$
$$l_3 = [0, 1]$$

and the 3 germs of solutions near $z = 0$ u_1, u_2, u_3 such that

$$\begin{cases} u_1(0) &= & -\sqrt{3} \\ u_2(0) &= & 0 \\ u_3(0) &= & +\sqrt{3} \end{cases}$$

Now when z decreases from 0 to -1 u_1 increases to -1, u_2 decreases to -1 and u_3 increases to 2 as $u^3 - 3u - 2 = (u+1)^2(u-2)$ therefore

$$\rho(l_1) = \begin{pmatrix} u_2 & u_1 & u_3 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

l_1 

[3] Recall that the discriminant of a monic polynomial f of degree n is equal to $n^n f(a_1) \cdots f(a_{n-1})$, where $a_1, \ldots, a_{n-1}$ are the roots of f': in our case $\Delta(z) = 108(z^2 - 1)$.

[4] In this case we have three paths that coincide with the segments $\gamma_1 = [-1, 0], \gamma_2 = [0, +\infty], \gamma_3 = [0, 1]$ and the loops l_1, l_2, l_3 are that loops that we obtain by following γ_i and turning arround the pure ramification points.

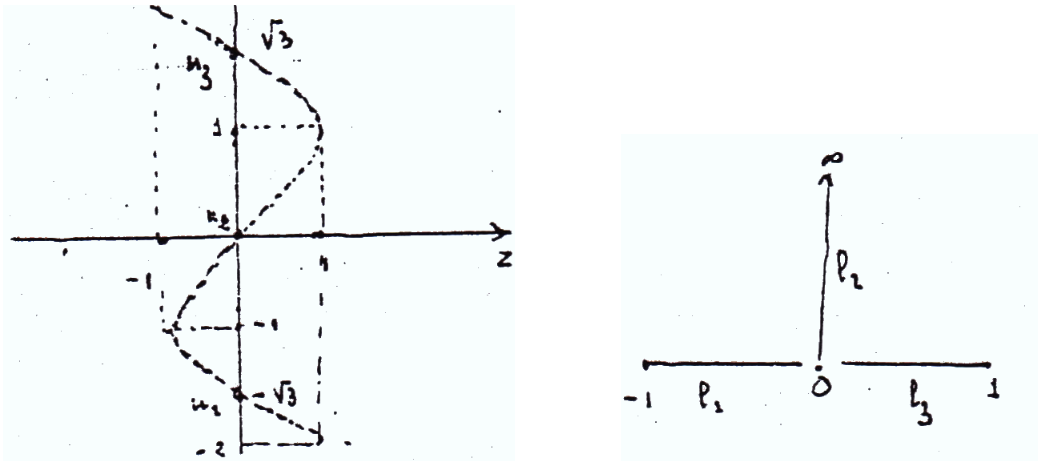
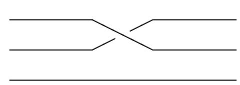


FIGURE 5

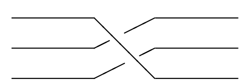
Similarly

$$\rho(l_3) = \begin{pmatrix} u_1 & u_3 & u_2 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

l_3 

thus

$$\rho(l_2) = \rho(l_1)^{-1}\rho(l_3)^{-1} = \begin{pmatrix} u_2 & u_3 & u_1 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

l_2 

6. The construction of the Riemann surface of Lüroth, Clebsch and Clifford

a) We have seen that the Riemann surface of an algebraic function U over a compact connected complex manifold X of $\dim_{\mathbb{C}} X = 1$ is completely determinated by the knowledge of the monodromy group. In particular if $X = \mathbb{P}^1(\mathbb{C})$ and $A = (a_1, \dots, a_k)$ is the set of pure ramification points of U over $\mathbb{P}^1(\mathbb{C})$ then, given $o \in X \setminus A$ the fundamental group $\pi_1(o, X \setminus A)$ is generated by k loops $l_1, \dots, l_k$ going from o to $a_1, \dots, a_k$ and back, with the relation

$$l_1 \cdots l_k = id \tag{3.3}$$

if the loops are token in the proper order. $\pi_1(X \setminus A, o)$ is exactly the group generated by the letters $l_1, \dots, l_k$ satisfying the relation (3.3). The monodromy group $\mathcal{M}$ of the function U is thus given by a group of substitution on m letters (m = number of sheets of U) generated by k substitutions $S_1, \dots, S_k$ where:

$$\begin{cases} S_j = \begin{pmatrix} i_1^{(j)} & \dots & i_m^{(j)} \\ 1 & \dots & m \end{pmatrix} & 1 \le j \le k \\ \\ i_1^{(j)} \dots i_m^{(j)} \text{ a permutation of } & (1, \dots, m) \end{cases}$$

and satisfying the relation

$$S_1 \cdots S_k = id.$$

Moreover the group must be transitive.

The construction of Lüroth-Clebsch-Clifford is based on the following remarks. Let $\mathcal{M}$ be a transitive group of substitutions on m letters generated by k substitutions $S_1, \dots, S_k$ satisfying the condition $S_1 \cdots S_k = id$.

REMARK 3.22. A new system of generators is given by replacing S_i and S_{i+1} by

$$S_i S_{i+1} S_i^{-1} \quad \text{and} \quad S_i$$

in this order. In the monodromy group this amounts to a new choice of loops generating $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus A)$ replacing two consecutive loops l_i, l_{i+1} by the loops $l' = l_i l_{i+1} l_i^{-1}$ and l_i (in this order, see Figure 6).

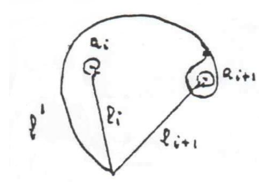


FIGURE 6

REMARK 3.23. A new system of generators is also given by replacing S_i and S_{i+1} by

$$S_{i+1} \quad \text{and} \quad S_{i+1}^{-1}S_iS_{i+1}$$

in this order. For $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus A)$ this amount to replace two consecutive loops l_i, l_{i+1} by the loops l_{i+1} and $l' = l_{i+1}^{-1}l_il_{i+1}$ (in this order, see Figure 7).

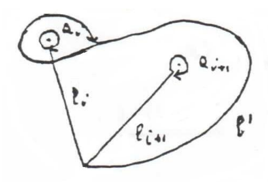


FIGURE 7

THEOREM 3.24 (Lüroth-Clebsch). *Let $\mathcal{M}$ be a transitive group on m letters generated by k substitutions $S_1, \dots, S_k$ related by*

$$S_1 \cdots S_k = id. \tag{3.4}$$

Assume that each substitution S_i is a transposition:

$$S_i = (a, b) \qquad a \neq b \quad a, b \in \{1, \dots, m\}.$$

Then

(1) *k is an even number*
$$k = 2g + 2 + 2(m-2)$$
(2) *by change of generators of the type considered in the above remarks we can find a new set of generators $\Sigma_1, \dots, \Sigma_k$ such that:*
$$\begin{gathered}\Sigma_1 = \Sigma_2 = \cdots = \Sigma_{2g+2} = (1,2)\\ \Sigma_{2g+2+1} = \Sigma_{2g+2+2} = (2,3)\\ \Sigma_{2g+2+3} = \Sigma_{2g+2+4} = (3,4)\\ \cdots\\ \Sigma_{k-1} = \Sigma_k = (m-1, m).\end{gathered}$$

PROOF. (1) That k is even follows by the fact that
$$\prod_{i<j}^{k}(i-j)$$
is left invariant[5] by $S_1 \cdots S_k$. Then k must be even as any S_i changes the sign of that product.
(2) We can first change generators so that the first l substitutions $S_1, \dots, S_l$ operate on the letter 1 and the other $S_{l+1}, \dots, S_k$ do not operate on the letter 1. Clearly l is even.
(3) We can suppose by a new change of generators that (since $S_1 \cdots S_k = id$)
$$S_1 = (1, a) = S_2 \quad S_3 = (1, b) = S_4 \quad \dots \quad S_{l-1} = (1, c) = S_l$$
($a, b, \dots, c$ may or may not be distinct). Remembering the letters we may suppose $a = 2$.

[5]i.e. it is invariant for the left action of the element $S_1 \cdots S_k$

(4) By another change of generators we may assume

$$S_1 = S_2 = \cdots = S_l = (1,2)$$

$$\begin{array}{llll} (1b)(1b)(12)(12) & \text{can} & \text{be replaced} & \text{by} \\ (12)(2b)(2b)(12) & " & " & " \\ (12)(12)(2b)(2b) & & & \end{array}$$

(5) (<u>Lüroth</u>) The k generators can be so chosen that they arrange in m sets G_j:

$$\begin{array}{lccc} G_1) & S_1 = \cdots = S_l = (1,2) & l_1 \equiv 0 & mod\, 2 \\ G_2) & S_{l_1+1} = \cdots = S_{l_1+l_2} = (2,3) & l_2 \equiv 0 & mod\, 2 \\ \ldots & \ldots & \ldots & \ldots \\ G_m) & S_{k-l_m} = \cdots = S_k = (m-1,m) & l_m \equiv 0 & mod\, 2 \end{array}$$

using the transitivity of $\mathcal{M}$ and the above reduction process.

(6) If the set G_i contains for $i < j$ at least four substitution two of these can be shifted into G_{i-1}. Set $i = 2$ to fix the notation

$$\begin{array}{llll} (12)(12)(23)(23)(23)(23) & \text{can} & \text{be replaced} & \text{by} \\ (12)(23)(13)(23)(23)(23) & " & " & " \\ (12)(23)(12)(12)(13)(23) & " & " & " \\ (12)(12)(12)(23)(13)(23) & " & " & " \\ (12)(12)(12)(12)(23)(23) & & & \end{array}$$

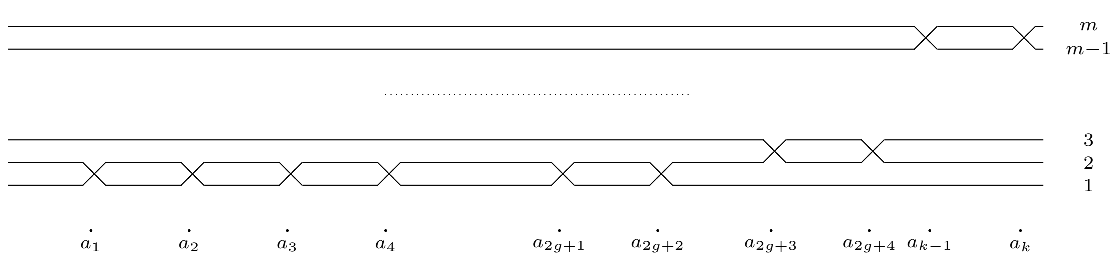
(7) (<u>Clebsch</u>) We may assume

$$\begin{array}{lc} G_1) & S_1 = \cdots = S_{2g+2} = (1,2) \\ G_2) & S_{2g+3} = S_{2g+4} = (2,3) \\ G_3) & S_{2g+5} = S_{2g+6} = (3,4) \\ \ldots & \ldots \\ G_m) & S_{k-1} = S_k = (m-1,m) \end{array}$$

□

REMARK 3.25. For every $k = 2g+2, 2g+2+2, 2g+2+4, \ldots$ the above substitutions Σ define a transitive group on $2, 4, 6, \ldots$ letters with the property $\prod_i^k \Sigma_j = id$. Thus $\mathcal{M}_k$ defines an abstract Riemann surface $\mathcal{R}_k$ with $2, 4, 6, \ldots$ number of sheets.

b) Let now U be an algebraic function on $\mathbb{P}^1(\mathbb{C})$ with the following property. There are k distinct ramification points $a_1, \ldots, a_k$ and the relative substitutions of the monodromy group are all transposition (<u>pure ramification points</u>). If we normalize the substitutions of the monodromy group according to the theorem of Lüroth-Clebsch the Riemann surface of U is obtained by cutting and cross pasting m copies of $\mathbb{P}^1(\mathbb{C})$ according to the indication of the picture[6]:



Let $\mathcal{R}_m$ the surface thus obtained. Topologically we have that, reflecting the m-th sheet along the slit,

$$\begin{array}{ccll} \mathcal{R}_m & = & \mathcal{R}_{m-1} \# S^2 & \text{topological sum with a 2-sphere} \\ \mathcal{R}_{m-1} & = & \mathcal{R}_{m-2} \# S^2 & \\ \ldots & \ldots & \ldots & \\ \mathcal{R}_3 & = & \mathcal{R}_2 \# S^2 & \end{array}$$

[6]see [**4**] section 21.11

thus $\mathcal{R}_m$ is topologically of the same type than $\mathcal{R}_2$ (i.e. homeomorphic). Now $\mathcal{R}_2$ is the Riemann surface of an hyperelliptic algebraic function of genus g that we have already discussed thus we have the following

REMARK 3.26 (Clifford 1887). Under the above assumptions $\mathcal{R}_m$ is homeomorphic to a Riemann surface with 2 sheets and of genus g as indicated by the theorem of Lüroth-Clebsch.

EXERCISE 6.1. Generalize the construction of Lüroth-Clebsch and Clifford replacing $\mathbb{P}^1(\mathbb{C})$ by a compact connected complex manifold X with $\dim_{\mathbb{C}} X = 1$.

7. Even and odd substitution of the monodromy group

Let $S = \begin{pmatrix} i_1 & \dots & i_m \\ 1 & \dots & m \end{pmatrix}$ be a substitution on m letters. We say that S is even or odd if do not or does respectively change the sign of

$$\prod_{i<j}^{m}(i-j).$$

Even substitutions form a subgroup of the symmetric group of m letters, the so called **alternating group**. Given a polynomial on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$,

$$\Phi(z,u) = \varphi_0(z)u^m + \varphi_1(z)u^{m-1} + \cdots + \varphi_m(z)$$

which is irreducible, let us consider a point $a \in \mathbb{C}$ where $\varphi_0(a) \neq 0$ but the discriminant $\Delta(z)$ of a vanishes. The point a is a ramification point (may be not a pure one) of the algebraic function U defined by $\Phi = 0$. To a correspond a substitution S_a of the monodromy group of U.

CLAIM 3.27. *The substitution S_a is an even (odd) substitution if $\Delta(z)$ has in $z = a$ a zero of even (odd) order.*

PROOF. Since $\varphi_0(a) \neq 0$ up to a non vanishing holomorphic function at a we have

$$(\Delta(z))^{\frac{1}{2}} = \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ u_1 & u_2 & \dots & u_m \\ u_1^{m-1} & u_2^{m-1} & \dots & u_m^{m-1} \end{pmatrix} = \prod_{i<j}(u_i - u_j)$$

where $u_1, \dots, u_m$ are the roots of $\Phi(z,u) = 0$ in a small punctured disk centered at a. $\square$

CHAPTER 4

Branches and intersection multiplicity of plane analytic curves

1. Puiseux expansions

Let $\mathcal{O}_0 = \mathbb{C}\{z\}$ the ring of convergent power series near the origin in one variable. This is the ring of germs of holomorphic functions at the origin $0 \in \mathbb{C}$. As $\mathcal{O}_0$ is an integral domain we can consider the quotient field $\mathcal{K}_0$.

PROPOSITION 4.1. *The field $\mathcal{K}_0$ is isomorphic to the field of Laurent expansion near 0 convergent and with a finite polar part:*

$$\mathcal{K}_0 = \left\{ \sum_{s=-N}^{+\infty} c_s z^s, \quad \textit{some } N, \ \limsup \sqrt[s]{|c_s|} < +\infty \right\}.$$

PROOF. Use the remark that if

$$\alpha = z^N(\alpha_0 + \alpha_1 z + \cdots) \qquad \alpha_0 \neq 0$$

is in $\mathcal{O}_0$, then

$$\alpha^{-1} = z^{-N}(\beta_0 + \beta_1 z + \cdots) \qquad \text{with } \beta_0 + \beta_1 z + \cdots \in \mathcal{O}_0.$$

□

Consider now the algebraic closure $\overline{\mathcal{K}}_0$ of $\mathcal{K}_0$.

PROPOSITION 4.2. *The field $\overline{\mathcal{K}}_0$ is isomorphic to the field of Puiseux algebroid expansions*

$$\overline{\mathcal{K}}_0 = \{ \sum_{s=-N}^{+\infty} c_s z^{\frac{s}{\nu}} \quad \textit{some } N, \quad \textit{some } \nu, \quad \limsup \sqrt[s]{|c_s|} < +\infty \}$$

under the obvious operations of sum and multiplication.

Note that $\sum_{-N}^{+\infty} c_s z^{\frac{s}{\nu}} = \sum_{-N}^{+\infty} c_s z^{\frac{sk}{\nu k}}$ for any integer $k > 0$ this shows that sum of two such Puiseux expansion is well defined. No difficulties for the product. One verifies thus that $\{\sum_{-N}^{+\infty} c_s z^{\frac{s}{\nu}} \quad \text{some } N, \quad \text{some } \nu, \quad \limsup \sqrt[s]{|c_s|} < \infty\}$ is a field.

PROOF. If $\alpha = \sum_{-N}^{+\infty} c_s z^{\frac{s}{\nu}}$ is given , we set $\varepsilon = e^{\frac{2\pi i}{\nu}}$ and

$$\alpha_l = \sum_{-N}^{+\infty} c_s z^{\frac{s}{\nu}} \varepsilon^{sl} \qquad 0 \leq l \leq \nu - 1.$$

The elementary symmetric functions $s_i(z)$ of the α_l do not contain fractional power of z thus are in $\mathcal{K}_0$ (cf. Theorem 2.14). Thus α is algebraic over $\mathcal{K}_0$: it satisfies the equation

$$u^\nu + s_1(z)u^{\nu-1} + s_2(z)u^{\nu-2} + \cdots + s_\nu(z) = 0, \qquad s_i(z) \in \mathcal{K}_0.$$

Conversely given an equation over $\mathcal{K}_0$

$$u^m + r_1(z)u^{m-1} + r_2(z)u^{m-2} + \cdots + r_m(z) = 0, \qquad r_i(z) \in \mathcal{K}_0$$

chasing denominators we can put it in the form

$$\varphi^m(z) + \varphi_1(z)u^{m-1} + \cdots + \varphi_m(z) = 0$$

with $\varphi_i(z) \in \mathcal{O}_0$, $\varphi_0(z) \not\equiv 0$, $M.C.D.(\varphi_0, \ldots, \varphi_m) = 1$. We may also assume the equation irreducible or devoid at least of multiple factors. We have seen in Chapter 2 that the m solutions of this equation are of the form

$$\alpha = \sum_{s=-N}^{+\infty} c_s z^{\frac{s}{\nu}}$$

(we have to consider as ν distinct elements the ν expansions $\alpha_l = \sum_{-N}^{+\infty} c_s z^{\frac{s}{\nu}} \varepsilon^{ls}$, $0 \leq l \leq \nu - 1$ $\varepsilon = e^{\frac{2\pi i}{\nu}}$). □

2. Branches of plane analytic curves

a) Let $f(z, u)$ be a holomorphic function defined in a neighborhood U of the origin $(0, 0) \in \mathbb{C}^2$ where z and u are holomorphic coordinates. We assume that:

i: $f(0, 0) = 0$

ii: $f(z, u) \not\equiv 0$ in some connected neighborhood of the origin.

We thus have in a small neighborhood of the origin a Taylor convergent expansion

$$f = f_k(z, u) + f_{k+1}(z, u) + \cdots$$

where f_j are homogeneous polynomials of degree j and $f_k \not\equiv 0$. If we perform a linear change of variables

$$\left\{ \begin{array}{lcl} z & = & \alpha z' + \beta u' \\ u & = & \gamma z' + \delta u' \end{array} \right. \qquad \alpha\delta - \beta\gamma \neq 0 \tag{4.1}$$

the function f becomes as a function of z', u'

$$f(z', u') = f_k(\alpha z' + \beta u', \gamma z' + \delta u') + \cdots .$$

If $f_k(\beta, \delta) \neq 0$ then

$$f(0, u') = u'^k f_k(\beta, \delta) + \cdots \not\equiv 0.$$

Therefore for almost all choices of a linear transformation 4.1 $f(0, u') \not\equiv 0$. We may thus assume, if the coordinates in $\mathbb{C}^2$ are chosen conveniently, that the given function f has the property also

iii: $f(0, u') \not\equiv 0$.

We can thus apply Weierstrass preparation theorem which states that

$$f(z, u) = (u^m + \alpha_1(z) u^{m-1} + \cdots + \alpha_m(z)) e^{Q(z,u)}$$

with $\alpha_i(z)$ holomorphic near 0 and $\alpha_i(0) = 0$ and with $Q(z, u)$ holomorphic in a neighborhood of the origin. In a sufficiently small neighborhood V of the origin the set

$$\{z \in V \mid f(z) = 0\} = \{z \in V \mid u^m + \alpha_1(z) u^{m-1} + \cdots + \alpha_m(z) = 0\}$$

i.e. near the origin the set $\{f = 0\}$ is defined by a polynomial equation

$$u^m + \alpha_1(z) u^{m-1} + \cdots + \alpha_m(z) = 0 \tag{4.2}$$

with $\alpha_i \in \mathbb{C}\{z\}$, $\alpha_i(0) = 0$ $1 \leq i \leq m$.

b) Now we may assume just learned how to solve the equations of type (4.2): all solution have the form

$$u = \sum_{s=i}^{+\infty} c_s z^{\frac{s}{\nu_j}} \qquad \text{(no polar part as the coefficient of } u^m \text{ is 1).}$$

Now it could happen that the polynomial on the left of (4.2) has some multiple factor. If we agree to count each solution of that factor so many time as is its multiplicity we must have $\sum \nu_j = m$. Of course one has to take the attention to consider as different solution the ν_j expansions

$$\left\{ \begin{array}{c} u_l = \sum c_s \varepsilon_j^{ls} z^{\frac{s}{\nu_j}} \\ 0 \leq l \leq \nu_j - 1 \end{array} \right. \qquad \varepsilon_j = e^{\frac{2\pi i}{\nu_j}}$$

We can therefore write in a sufficiently small neighborhood of the origin

$$f(z, u) = e^{Q(z,u)} \prod_j (u - \sum c_s z^{\frac{s}{\nu_j}}) = e^{Q(z,u)} \prod_j (u - P_j(z)) \qquad P_j(z) \in \overline{\mathcal{K}}_0$$

($P_j(z)$ without polar part and vanishing at the origin). For each of the roots of (4.2)

$$u = \sum_{s=1}^{+\infty} c_s z^{\frac{s}{\nu_j}}$$

we can consider the parametric equation of the locus it represents in the form

$$\left\{ \begin{array}{ll} z = & t^{\nu_j} \\ u = & \sum_1^\infty c_s t^s \end{array} \right. \tag{4.3}$$

Note that ν_j roots of the equation (4.2) give one parametric equation (4.3).

c) Conversely let us consider the locus defined near the origin by a set of parametric equations of type

$$\left\{ \begin{array}{lll} z = & t^k & (a_0 + a_1 t + \cdots) \\ u = & t^l & (b_0 + b_1 t + \cdots) \end{array} \right. \qquad \begin{array}{c} a_0 \neq 0,\ k \geq 1 \\ l \geq 1 \end{array} \tag{4.4}$$

so that $z(t) \not\equiv 0$, the series being convergent near $t = 0$. Setting

$$\Theta(t) = t(a_0 + a_1 t + \cdots)^{\frac{1}{k}}$$

and taking Θ as new parametric we get parametric equation of the form

$$\left\{ \begin{array}{lll} z = & \Theta^k & \\ u = & \Theta^l & (c_l + c_{l+1}\Theta + \cdots) \end{array} \right. \qquad l \geq 1$$

This yields a Puiseux expansion

$$u = \sum_1^\infty c_s z^{\frac{s}{k}}$$

and it gives k roots of an algebraic equation of the form

$$w^k + a_0(z) w^{k-1} + \cdots + a_k(z) = 0$$

with $a_i(z) \in \mathcal{O}_0 = \mathbb{C}\{z\}$ and $a_i(0) = 0$.

Remark 4.3. In a Puiseux expansion

$$u = \sum_{s=1}^{+\infty} c_s z^{\frac{s}{k}}$$

we come out with exponents so arranged that $M.C.D.$(all s with $c_s \neq 0$) and k have no common factor[1]. In these condition the corresponding parametric equation

$$\left\{ \begin{array}{ll} z = & \Theta^k \\ u = & \sum c_s \Theta^s \end{array} \right.$$

are said to be written with a **not redundant parametric** Θ.

d) We mention here the following difficult problem:

Problem 4.4. *Given a Puiseux expansion* $u = \sum_1^\infty c_s z^{\frac{s}{k}}$ *can we recognize from its coefficients when it is the root of a global algebraic equation*

$$u^m + s_1(z) u^{m-1} + s_2(z) u^{m-2} + \cdots + s_m(z) = 0, \qquad s_i(z) \in \mathcal{K}(\mathbb{P}^1(\mathbb{C}))?$$

The answer is known for $m = 1$. Then

$$p(z) u = q(z)$$

p, q polynomials $p \not\equiv 0$

$$p = p_0 + p_1 z + \cdots + p_l z^l, \qquad q = q_0 + q_1 z + \cdots + q_k z^k$$

then $u = \sum_{-N}^\infty c_s z^s$ must satisfy the recursive relations

$$p_0 c_{k+1} + p_1 c_k + \cdots + p_l ck + 1 - l = 0$$
$$p_0 c_{k+2} + p_1 c_{k+1} + \cdots + p_l ck + 2 - l = 0$$

this condition is also a sufficient condition for $u = \sum c_s z^s$ to represent an element of $\mathcal{K}(\mathbb{P}^1(\mathbb{C}))$.

[1] Every Puiseux expansion can be written in this form and it is called the **canonical form** of the series.

3. Intersection multiplicity

a) Let $f(z,u), g(z,u)$ be two holomorphic functions defined in a neighborhood U of the origin in $\mathbb{C}^2$, such that

$$f(0,0) = 0 = g(0,0) \qquad f(z,u) \not\equiv 0 \quad g(z,u) \not\equiv 0.$$

By a suitable choice of coordinates we may assume $f(0,u) \not\equiv 0, g(0,u) \not\equiv 0$ so that we can write them in the Weierstrass form

$$\begin{aligned} f(z,u) =& (u^m + \alpha_1(z)u^{m-1} + \cdots + \alpha_m(z))e^{Q(z,u)} \\ g(z,u) =& (u^l + \beta_1(z)u^{l-1} + \cdots + \beta_l(z))e^{H(z,u)} \end{aligned}$$

where Q and H are holomorphic near the origin and where $\alpha_i(z)$ and $\beta_i(z)$ are holomorphic in z and vanish for $z = 0$.

In the field $\mathcal{K}_0$ the pseudopolynomials

$$\begin{aligned} A(z,u) :=& (u^m + \alpha_1(z)u^{m-1} + \cdots + \alpha_m(z)) = \prod_1^m (u - P_j(z)) \\ B(z,u) :=& (u^l + \beta_1(z)u^{l-1} + \cdots + \beta_l(z)) = \prod_1^l (u - Q_j(z)) \end{aligned}$$

split into linear factors and $P_j(z), Q_j(z)$ are Puiseux expansions without polar part and vanishing at the origin. By elimination of u from $A(z,u)$ and $B(z,u)$ we get as the Sylvester resultant[2] $R(f,g) = R(z)$:

$$R(z) = \prod_{\substack{1 \le i \le m \\ 1 \le j \le l}} (P_i(z) - Q_j(z)) \ \in \mathcal{O}_0 = \mathbb{C}\{z\}. \tag{4.5}$$

Now $R(z)$ vanishes for $z = 0$. There are two possibilities:

---

[2]By the theory of elimination we remember that the resultant of $A(z,u)$ and $B(z,u)$ $R(A,B)$ with respect to u is the determinant of the Sylvester matrix:

$$\begin{pmatrix}
1 & \alpha_1(z) & \dots & \dots & \alpha_m(z) & & & & & & & & \\
 & 1 & \alpha_1(z) & \dots & \dots & \dots & \dots & \dots & \alpha_m(z) & & & & \\
 & & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & & \\
 & & & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\
 & & & & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
 & & & & & 1 & \alpha_1(z) & \dots & \dots & \dots & \dots & \dots & \alpha_m(z) \\
1 & \beta_1(z) & \dots & \beta_l(z) & & & & & & & & & \\
 & 1 & \beta_1(z) & \dots & \beta_l(z) & & & & & & & & \\
 & & \dots & \dots & \dots & \dots & \dots & & & & & & \\
 & & & \dots & \dots & \dots & \dots & \dots & & & & & \\
 & & & & \dots & \dots & \dots & \dots & \dots & & & & \\
 & & & & & \dots & \dots & \dots & \dots & \dots & & & \\
 & & & & & & \dots & \dots & \dots & \dots & \dots & & \\
 & & & & & & & \dots & \dots & \dots & \dots & \dots & \\
 & & & & & & & & & 1 & \beta_1(z) & \dots & \beta_l(z)
\end{pmatrix}.$$

The importance of the resultant of two polynomial is that $R(A,B) = 0$ if and only if A and B have a common non constant factor. Now we want to prove that this definition of resultant and the definition 4.5 are equivalent. Indeed, as we have

$$A(z,u) = \prod_1^m (u - P_j(z)) \qquad B(z,u) = \prod_1^l (u - Q_j(z)),$$

for every $i \in \{1, \dots, m\}$,for the property of the resultant:

$$R(A,B) = R((u - P_i(z))h(z), B(z,u)) = R(uh(u + P_i(z)), B(z, u + P_i(z)))) = B(z, P_i(z))R(h(z,u), B(z,u))$$

then

$$R(A,B) = \prod_i B(z, P_i(z)) = \prod_{i,j} (P_i(z) - Q_j(z)).$$

(1) either $R(z) \equiv 0$ and this can happens only when $f(z,u) = 0$ and $g(z,u) = 0$ have a common branch (and thus $f(z,u)$ and $g(z,u)$ have a common factor near the origin of positive degree in u)
(2) or in a sufficiently small neighborhood V of $(0,0) \in \mathbb{C}^2$ $f(z,u) = 0$ and $g(z,u) = 0$ have the origin as an isolated common zero.

Therefore if $f(z,u) = 0$ and $g(z,u) = 0$ have no common branches trough the origin then the origin is an isolated common zero. We set in this case the following definition.

Definition 4.5. $I_0(f,g)$ = **multiplicity of intersection of** $f = 0$ **and** $g = 0$ **at the origin** = **order of the root** $z = 0$ **in** $R(z)$ = minimal integer h such that $R(z) \in \mathfrak{m}^h$, $R(z) \notin \mathfrak{m}^{h+1}$ where $\mathfrak{m} = \mathbb{C}\{z\}(z)$ is the maximal ideal of the local ring $\mathbb{C}\{z\}$.

b) Assume that $f = 0$ and $g = 0$ have only one branch trough the origin so that the pseudopolynomials $A(z,u), B(z,u)$ are irreducible; then those branches have parametric equations

$$c = \begin{cases} z = & t^m \\ u = & \sum_1^\infty b_s t^s \end{cases} \qquad \text{for } f = 0$$

$$r = \begin{cases} z = & t^l \\ u = & \sum_1^\infty c_s t^s \end{cases} \qquad \text{for } g = 0$$

and t is a non redundant parametric in both cases. Then:

$$\begin{aligned} I_0(f,g) &= \text{order in } t \text{ of } g(t^m, \textstyle\sum b_s t^s) \\ &= \text{order in } t \text{ of } f(t^l, \textstyle\sum c_s t^s) \qquad \leftarrow {}^3 \\ &\overset{def}{=} I_0(c,r) \end{aligned}$$

and we recognize that this number is independent of the choice of the coordinates in $\mathbb{C}^2$ and of the non redundant parametric equations of the branches c and r.

[3] Indeed, for the Weierstrass preparation Theorem we have:

$$f(z,u) = (u - P(z))e^{S(z,u)} \qquad g(z,u) = (u - Q(z))e^{T(z,u)}$$

so the order of $g(t^m, \sum b_s t^s)$ in 0 is equal to the order of

$$B(t^m, \sum b_s t^s) = \sum b_s t^s - Q(t^m) = at^N + \dots.$$

Thus we have $O_0(g(t^m, \sum b_s t^s)) = N$. Note that

$$B(z, \sum b_s t^s) = az^{\frac{N}{m}} + \dots.$$

If we consider all determination of the parametrization $\sum_1^{+\infty} b_s \varepsilon_\lambda^s t^s$ where ε_λ is any m-th root of unity we can write

$$B(z, \sum_1^{+\infty} b_s \varepsilon_\lambda^s t^s) = a\varepsilon N_\lambda z^{\frac{N}{m}} + \dots.$$

Hence

$$\prod_{\lambda=1}^{m} B(z, \sum_1^{+\infty} b_s \varepsilon_\lambda^s t^s) = bz^N + \dots$$

and so we have

$$O(\prod_\lambda B(z, \sum_1^{+\infty} b_s \varepsilon_\lambda^s t^s)) = O_0(g(t^m, \sum b_s t^s)).$$

Now we consider all determination of $\sum c_s t^s$ $\sum c_s \varepsilon_\beta^s t^s$ where ε_β is any l-th root of unity, so we can write

$$\prod_\lambda B(z, \sum_1^{+\infty} b_s \varepsilon_\lambda^s t^s) = \prod_{\alpha\ \beta} \left(\sum c_s \varepsilon_\beta^s t^s - \sum b_s \varepsilon_\lambda^s t^s\right).$$

Hence

$$O_0(g(t^m, \sum b_s t^s)) = O(\prod_{\alpha\ \beta} \left(\sum c_s \varepsilon_\beta^s t^s - \sum b_s \varepsilon_\lambda^s t^s\right)) = O(\sum b_s \varepsilon_\lambda^s t^s - \sum c_s \varepsilon_\beta^s t^s) = O_0(f(t^l, \sum c_s t^s))$$

where the last equality is obtained by reversing the role of f and g in the above discussion.

Therefore (distributive law of the intersection number) if $f = 0$ near the origin represents the set of branches $\sum m_j c_j$ and $g = 0$ near the origin represent the set of branches $\sum \mu_s r_s$ (m_j and μ_s integers ≥ 1) then if $c_j \neq r_s$, $\forall j\ \forall s$, we have:

$$I_0(f, g) = I_0(\sum m_j s_j, \sum \mu_s r_s) = \sum m_j \mu_s I_0(c_j, r_s).$$

c) Suppose that

$$f = f_s + f_{s+1} + \cdots \qquad f_s \not\equiv 0 \ (f_j \text{ homogeneous polynomial of degree } j \text{ in } u, z.)$$

$$g = g_t + g_{t+1} + \cdots \qquad g_t \not\equiv 0 \ (g_k \text{ homogeneous polynomial of degree } k \text{ in } u, z.)$$

are the Taylor expansion of f and g near the origin in $\mathbb{C}^2$.

THEOREM 4.6 (Corrado Segre criterion). *Assume that $f_s = 0$ and $g_t = 0$ have no nontrivial common root, then*

$$I_0(f, g) = st.$$

PROOF. It is not restrictive to assume $f_s(0, u) \not\equiv 0$ $g_t(0, u) \not\equiv 0$ and that f and g are polynomials of respective degrees s and t as provided by the Weierstrass preparation theorem. Thus

$$\begin{aligned} f(z, u) &= u^s + (\alpha_1(z) + \alpha_2(z) + \cdots)u^{s-1} + (\beta_2(z) + \cdots)u^{s-2} + \cdots + (\gamma_s(z) + \cdots) \\ &= u^s + \alpha_1(z)u^{s-1} + \beta_2(z)u^{s-2} + \cdots + \gamma_s(z) + \text{ higher order terms} \end{aligned}$$

(we must have $\beta_1(z) \equiv 0, \ldots, \gamma_1(z) \equiv 0, \ldots, \gamma_{s-1}(z) \equiv 0$) and thus

$$f_s(z, u) = u^s + \alpha_1(z)u^{s-1} + \beta_2(z)u^{s-2} + \cdots + \gamma_s(z).$$

Similarly

$$g(z, u) = u^t + \sigma_1(z)u^{t-1} + \tau_2(z)u^{t-2} + \cdots + \mu_t(z) + \text{ higher order terms}$$

and thus

$$g_t(z, u) = u^t + \sigma_1(z)u^{t-1} + \tau_2(z)u^{t-2} + \cdots + \mu_t(z)$$

(we must have that similarly the coefficient of u^{t-j} vanishes at the origin of order $\geq j$). Then[4]

$$R(f, g) = R(f_s, g_t) + \text{ higher order terms } = cz^{st} + \text{ higher order terms}$$

and $c \neq 0$ become of the assumption. □

4. The formula of Caccioppoli

This formula enable us to calculate intersection multiplicity $I_0(f, g)$ by means of an integral. It is given by the following.

THEOREM 4.8 (Caccioppoli [**5**]). *For $\varepsilon > 0$ $\sigma > 0$ generic and sufficiently small the surface*

$$\Gamma = \{(z, u) \in V \mid \ |f| = \varepsilon, \ |g| = \sigma\}$$

[4]The first equality is a consequence of the multilinearity of the determinant, the second one is true becouse the following theorem holds:

THEOREM 4.7. *Let*

$$F_n = A_n(z) + A_{n-1}(z)x + \cdots + A_0(z)x^n,$$

$$G_m = B_m(z) + B_{m-1}(z)x + \cdots + B_0(z)n^m,$$

where A_i, B_i are homogeneous polynomials of degree i in z, and $A_0, B_0 \neq 0$. If $R(z)$ is the resultant of F and G with respect to x, then either $R = 0$ or R is homogeneous of degree mn.

PROOF. [**19**] Theorem 10.9 page 30. □

is differentiable[5] *and compact and we have*[6]

$$I_0(f,g) = \frac{1}{(2\pi i)^2}\int_\Gamma \frac{df\wedge dg}{fg}. \qquad \left(\begin{array}{c} \textit{generalized Cauchy formula} \\ \dfrac{1}{(2\pi i)^2}\displaystyle\int_{|z_1|=\varepsilon,|z_2|=\sigma}\frac{dz_1\wedge dz_2}{z_1 z_2} = 1\end{array}\right)$$

5. Multiplicity of intersection as linking number

It is enough to define the multiplicity of intersection for two distinct branches

$$c = \begin{cases} z = & t^m \\ u = & \sum_1^{+\infty} c_s t^s\end{cases} \qquad \gamma = \begin{cases} z = & t^\mu \\ u = & \sum_1^{+\infty} d_s t^s\end{cases}$$

Let $S^3(\varepsilon) = \{(z,u)\in\mathbb{C}^2 \mid |z|^2+|u|^2=\varepsilon^2\}$ be a small sphere of radius $\varepsilon>0$. Then

$$c\cap S^3(\varepsilon) \text{ defines an oriented knot } l_c \text{ on } S^3(\varepsilon)$$
$$\gamma\cap S^3(\varepsilon) \text{ defines an oriented knot } l_\gamma \text{ on } S^3(\varepsilon)$$

and

$$l_c\cap l_\gamma = \emptyset$$

as two distinct branches have only the origin in common in a small neighborhood of the origin. Then the **linking number** $\mathcal{L}(l_c,l_\gamma)$ is defined. It is an integer obtained as follows. Consider $l_c=\delta E$ as a boundary of a 2 chain E and set

$$\mathcal{L}(l_c,l_\gamma) = \text{number of intersection of } E \text{ with } l_\gamma$$

one has $\mathcal{L}(l_c,l_\gamma)=\mathcal{L}(l_\gamma,l_c)$.

THEOREM 4.9. *One has*

$$I_0(c,\gamma) = \mathcal{L}(l_c,l_\gamma).$$

REMARK 4.10. We can take the stereographic projection of $\mathcal{S}^3(\varepsilon)$ on $\mathbb{R}^3$. Send a current of unit density along l_c (in its direction) this produces a magnetic field $\overrightarrow{\mathbb{H}}(p)$ at each point $p\notin l_c$. The potential of this field (up to a constant $c_0\neq 0$ dependent of the units of measure) is given by the solid angle Ω_p from which l_c is seen from p. The solid angle Ω_p is a multivalued function and increases by 4π if we go once around l_c in the proper direction. Thus the work of $\overrightarrow{\mathbb{H}}(p)$ along l_γ is given by

$$\int_{l_\gamma} d\Omega_p = c_0 4\pi\mathcal{L}(l_c,l_\gamma) \qquad \text{(Gauss integral.)}$$

EXERCISE 5.1.

(1) Relate formula of Caccioppoli with the Gauss integral.
(2) Consider on $S^3(1)$ the Hopf fibering

$$S^3(1)\to\mathbb{P}^1(\mathbb{C})$$

wich associate to each point $a\in S^3(1)$ the punctured line $a\mathbb{C}^*$ in $\frac{\mathbb{C}^2\setminus\{0\}}{\mathbb{C}^*}=\mathbb{P}^1(\mathbb{C})$. Consider the stereographic projection of $S^3(1)$ on $\mathbb{R}^3$. Then the circles that appear as

---

[5]We have to show that $J=J((f,g)) = f_xg_y - f_yg_x\neq 0$ where f and g without common factors and $f(0)=g(0)=0$. Let $h(x,y)$ an irreducible factor of f and assume that $J=0$. Case 1 If $h(x,y)=x$ (x divide h), we can write $f=x^n\varphi$ where x doesn't divide φ. One has

$$0 = nx^{n-1}\varphi g + x^n\varphi_x g_y - x^n f_y g_x$$

thus x divide g_y and, since $g(0)=0$, x divide g. Case 2 If h is an irreducible factor of f but n doesn't divide h, we can consider a parametrization of h $(\alpha(t),\beta(t))$. We can consider the functions

$$\varphi(t,z) = f(\alpha(t),\beta(t)+z) \qquad \Psi(t,z) = g(\alpha(t),\beta(t)+z).$$

Note that φ and Ψ are without common factor and z divide φ. Since

$$\varphi_t = f_x\alpha' + f_y\beta' \quad \varphi_z = f_y$$
$$\Psi_t = g_x\alpha' + g_y\beta' \quad \Psi_z = g_y$$

we have

$$\varphi_t\Psi_z - \varphi_z\Psi_t = \alpha' J.$$

For Case 1 we have the desidered result.

[6]A proof of this formula will be given later (footnote 7).

fibers in the Hopf fibering go into circles in $\mathbb{R}^3$ (or straight lines) that pass trought two antipodal point of the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Prove then that any two of the circles of the Hopf fibering have linking number 1.

(3) On the basis of point (2) prove the theorem that states

$$I_0(c, \gamma) = \mathcal{L}(l_c, l_\gamma).$$

(4) Establish the formula of Caccioppoli by means of the Cauchy integral.

6. Severi's definition

From the formula of Caccioppoli we deduce the following fact. Let $\alpha, \beta \in \mathbb{C}$ be sufficiently small and generic then the equations

$$f(z) - \alpha = 0 \qquad g(z) - \beta = 0$$

have in V exactly $I_0(f, g)$ common solutions

$$p_1(\alpha, \beta), \ldots, p_{I_0(f,g)}(\alpha, \beta)$$

which are simple intersections of $f - \alpha = 0$ and $g - \beta = 0$ (i.e. the two analytic curves intersect at p_j transversely: $(df \wedge dg)_{p_j} \neq 0$)), therefore

Theorem 4.11. *$f = 0$ $g = 0$ have multiplicity of intersection $I_0(f, g)$ at the origin if and only if given a neighborhood V of the origin we can find $\varepsilon > 0$ such that for $|\alpha| \leq \varepsilon$, $|\beta| \leq \varepsilon$ (α, β) outside a set of measure 0 we have that*

$$\#\{\{f - \alpha = 0\} \cap \{g - \beta = 0\}\} = I_0(f, g)$$

and at any point $p \in \{f - \alpha = 0\} \cap \{g - \beta = 0\}$

$$(df \wedge dg)_p \neq 0.$$

Proof. [7] This by virtue of Sard theorem and the fact that the integrand in Cacciopoli's formula outside $\{f - \alpha = 0\} \cap \{g - \beta = 0\}$ is a closed 2 form. Indeed for α outside a set of measure 0 $f = \alpha$ is a non singular submanifold of V. If β outside a set of measure 0 we have then that $g = \beta$ is a non critical value for $g_{|f=\alpha}$. Thus, there, $df \wedge dg \neq 0$. At each one of these points the intersection multiplicity is one by Cauchy formula. But the total integral of Caccioppoli is the sum of the corresponding integral at the points of intersection thus

$$I_0(f, g) = \#\{\{f - \alpha = 0\} \cap \{g - \beta = 0\}\}.$$

□

[7] We can proove Severi's formula without Cacciopoli's formula. Indeed, we can choose α and β such that for every point $p \in \{f - \alpha = 0\} \cap \{g - \beta = 0\}$ $(df \wedge dg)_p \neq 0$, this is possible becouse for Sard theorem the set of the critical point has measure 0. We can suppose, up to a change of coordinates, that $f(0, y) \not\equiv 0$ and $g(0, y) \not\equiv 0$. We define

$$F(x, y) = f(x, y) - \alpha \qquad G(x, y) = g(x, y) - \beta$$

and we consider the resultant of F and G with respect to y, thus we have:

$$R = R(F, G) = R(x, \alpha, \beta)|_{\substack{\alpha = 0 \\ \beta = 0}} = R(x).$$

If we consider a closed curve Γ such that $\{f-\alpha = 0\} \cap \{g-\beta = 0\} \cap \Gamma = \emptyset$ and Γ sorround $\{f-\alpha = 0\} \cap \{g-\beta = 0\}$, thus for the theorem of logarithmic indicator and becouse every intersection of F and G are simple, we have:

$$\#\{\{f - \alpha = 0\} \cap \{g - \beta = 0\}\} = \int_{\zeta \in \Gamma} \frac{R'(F, G)}{R(F, G)} d\zeta = \int_{\zeta \in \Gamma} \frac{R'(x)}{R(x)} = I_0(f, g).$$

xx

We can now prove the formula of Caccioppoli, but we need the following definition:

Definition 4.12. Let $f \in \mathcal{O}(U)$, we say that λ is the **local degree** of f if f define a λ-sheeted analytic cover on U.

Let, now, f and g as in the Theorem 4.8. Note that the local degree λ of the map (f, g) is equal to the cardinality of $\{f - \alpha = 0\} \cap \{g - \beta = 0\}$. Thus for Severi's formula:

$$I_0(f, g) = \#\{\{f - \alpha = 0\} \cap \{g - \beta = 0\}\} = \lambda = \frac{1}{(2\pi i)^2} \int_{(f^{-1}, g^{-1}\Gamma} \frac{dz_1 \wedge dz_2}{z_1 z_2} = \frac{1}{(2\pi i)^2} \int_\Gamma \frac{df \wedge dg}{fg}.$$

7. Intersection multiplicity as "Hilbert function"

a) Given in a neighborhood of the origin V in $\mathbb{C}^2$ the analytic functions $f(z_1, z_2)$, $g(z_1, z_2)$ with

$$f(0,0) = 0 = g(0,0) \qquad f(z_1, z_2) \not\equiv 0,\ g(z_1, z_2) \not\equiv 0$$

and f and g without common factors in $\mathcal{O}_0 = \mathbb{C}\{z_1, z_2\}$, then by **Hilbert nullstellensatz** [**8**, Theorem 7, page 97] if $\mathfrak{m} = \mathbb{C}\{z_1, z_2\}(z_1, z_2)$ is the maximal ideal of $\mathcal{O}_0$ we have

$$\mathfrak{m}^h \subset \mathcal{O}_0(f, g) \subset \mathfrak{m}$$

for some $h \geq 1$. Hence

$$\delta(f, g) = \dim_{\mathbb{C}} \frac{\mathcal{O}_0}{\mathcal{O}_0(f, g)} < +\infty.$$

THEOREM 4.13. *We have*

$$\delta(f, g) = I_0(f, g).$$

The proof is based on the following remarks.

LEMMA 4.14. *If $g = g_1 g_2$ in $\mathcal{O}_0$ (g_1, g_2 non units) then*

$$\delta(f, g_1 g_2) = \delta(f, g_1) + \delta(f, g_2).$$

PROOF. Set $\overline{\mathcal{O}} = \frac{\mathcal{O}}{\mathcal{O}(f)}$ then

$$\delta(f, g_1 g_2) = \dim_{\mathbb{C}} \frac{\overline{\mathcal{O}}}{\overline{\mathcal{O}}(g_1 g_2)}.$$

Now we have the exact sequence

$$0 \to \frac{\overline{\mathcal{O}}(g_2)}{\overline{\mathcal{O}}(g_1 g_2)} \to \frac{\overline{\mathcal{O}}}{\overline{\mathcal{O}}(g_1 g_2)} \to \frac{\overline{\mathcal{O}}}{\overline{\mathcal{O}}(g_2)} \to 0$$

thus

$$\dim_{\mathbb{C}} \frac{\overline{\mathcal{O}}}{\overline{\mathcal{O}}(g_1 g_2)} = \dim_{\mathbb{C}} \frac{\overline{\mathcal{O}}}{\overline{\mathcal{O}}(g_1)} + \dim_{\mathbb{C}} \frac{\overline{\mathcal{O}}(g_2)}{\overline{\mathcal{O}}(g_1 g_2)}. \tag{4.6}$$

Now g_2 is not a zero divisor in $\frac{\mathcal{O}}{\mathcal{O}(f)} = \overline{\mathcal{O}}$ as one verifies directly (f, g_2 have no common factors). Therefore

$$\frac{\overline{\mathcal{O}}}{\overline{\mathcal{O}}(g_1)} \cong \frac{\overline{\mathcal{O}}(g_2)}{\overline{\mathcal{O}}(g_1 g_2)} \tag{4.7}$$

is an isomorphism defined by multiplication by g_2. Then (4.6) and (4.7) yield the desired formula. □

COROLLARY 4.15. *If $f = \prod f_i^{m_i}$, $g = \prod g_j^{\mu_j}$ are the decomposition of f and g into irreducible factors in $\mathcal{O}_0$ then*

$$\delta(f, g) = \sum m_i \mu_j \delta(f_i, g_j).$$

From this formula and the distributive law for multiplicity of intersection one realizes that it is enough to prove the theorem for f and g irreducible in $\mathcal{O}_0$. Let then f be irreducible and let

$$\begin{cases} z_1 = & t^\nu \\ z_2 = & \sum_1^{+\infty} c_s t^s \end{cases}$$

be the parametric equations of the branch defined by $f = 0$ in terms of a non redundant parameter t then

$$\overline{\mathcal{O}} = \frac{\mathcal{O}}{\mathcal{O}(f)} \cong \mathbb{C}\{t^\nu, \sum c_s t^s\} = A.$$

Also set

$$g(t^\nu, \sum c_s t^s) = t^{I_0(f,g)} w(t)$$

with $w(0) \neq 0$ then the theorem follows from the

LEMMA 4.16.
$$\dim_{\mathbb{C}} \frac{A}{t^{I_0(f,g)} w(t) A} = I_0(f,g).$$

PROOF. The proof is omitted[8]. □

b) Complements In general set $\mathcal{O}_0 = \mathbb{C}\{z_1, \dots, z_n\}$ the ring of convergent power series in n variables. Let $\mathcal{I} = \mathcal{O}(f_1, \dots, f_l)$ be an ideal in $\mathcal{O}_0$. We set
$$\delta(\mathcal{I}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_0}{\mathcal{I}} \qquad \text{(can be } +\infty).$$
Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_0$ and set
$$\delta_k(\mathcal{I}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_0}{\mathcal{I} + \mathfrak{m}^{k+1}}$$
then
$$\delta_k(\mathcal{I}) < +\infty \qquad \left(\leq \binom{n+k}{k} \right)$$
Set
$$\nu_l(\mathcal{I}) = \dim \frac{\mathfrak{m}^l}{\mathfrak{m}^{l+1}} - \dim \frac{\mathcal{I} \cap \mathfrak{m}^l}{\mathcal{I} \cap \mathfrak{m}^{l+1}}$$
for $l \geq 1$.

PROPOSITION 4.17. **a:** $\delta_k(\mathcal{I}) = 1 + \nu_1(\mathcal{I})) + \cdots + \nu_k(\mathcal{I})$
b: $\nu_k(\mathcal{I})) = 0 \iff \mathfrak{m}^k \in \mathcal{I}$
c: $\nu_k(\mathcal{I}) = 0 \Rightarrow \nu_{k+l}(\mathcal{I}) = 0 \quad \forall l \geq 0$
d: $\delta(\mathcal{I}) = \lim_{k \to +\infty} \delta_k(\mathcal{I})$.

PROPOSITION 4.18. **a:** $\delta(\mathcal{I}) \leq s \iff \mathfrak{m}^s \subset \mathcal{I}$
b: $\mathfrak{m}^s \subset \mathcal{I} \Rightarrow \delta(\mathcal{I}) = \delta_s(\mathcal{I})$
c: $\delta(\mathcal{I}) > s \iff \delta_s(\mathcal{I}) > s$.

Set $\Omega(s) = \{(\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n \mid \sum \sigma_i \leq s\}$, set $A(\mathcal{I}, s) = (u^\beta_\gamma)$ where $\beta \in \Omega(s)$ and $\gamma \in \{1, \dots, l\} \times \Omega(s)$ by $u^\beta_{(i\ \sigma)}$ =coefficient of z^σ in $z^\beta f_i$.

PROPOSITION 4.19 (Bochnak-Łojasiewicz criterion). *In the above notation $\delta_s(\mathcal{I}) > s$ if and only if* $\operatorname{rk} A(\mathcal{I}, s) < \binom{n+s}{s} - s$.

PROPOSITION 4.20. *Let $A(\mathcal{I})$ be the algebra generated by $f_1, \dots, f_l$. If $\delta(\mathcal{I}) = r < +\infty$ and $l_1(z), \dots, l_r(z)$ are elements of $\mathcal{O}_0$ whose images spam $\frac{\mathcal{O}}{\mathcal{I}}$ then for all $g(z) \in \mathcal{O}_0$ we have an expression*
$$g = \varphi_1 l_1 + \cdots + \varphi_r l_r \qquad \textit{with } \varphi_i \in A(\mathcal{I})$$
(i.e. $\mathcal{O}$ is a finitely generated $A(\mathcal{I})$ module).

[8] A proof can be found in Chapter 8 of [**13**]

CHAPTER 5

Bezout theorem

1. Bezout theorem in $\mathbb{P}^2(\mathbb{C})$

DEFINITION 5.1. Let x_0, x_1, x_2 be homogeneous coordinates in $\mathbb{P}^2(\mathbb{C})$. Let $f(x_0, x_1, x_2)$ be an irreducible homogeneous polynomial of degree m. The set

$$C = \{(x_0, x_1, x_2) \in \mathbb{P}^2(\mathbb{C}) \mid f(x_0, x_1, x_2) = 0\}$$

is called an **irreducible curve**. An irreducible curve determines the corresponding irreducible polynomial up to a constant factor (use the Hilbert Nullstellensatz). A **positive cycle** is a linear combination with positive integer coefficients of irreducible curves

$$C = \sum m_i C_i$$

where the sum is finite, i.e. is an element of the free abelian monoid generated by the irreducible curves. We call **degree of** C the degree of f, m, if C is irreducible and we set by definition

$$\text{degree } C = \sum m_i \text{ degree of } C_i.$$

Cycles positive of degree k are thus in one to one correspondence with the projective space of dimension $\binom{2+k}{2} - 1$ whose points represent homogeneous polynomials of degree k, non identically zero, up to multiplication by a non zero constant factor.

THEOREM 5.2 (Bezout). *Let C and Γ be two positive cycles in $\mathbb{P}^2(\mathbb{C})$ without common irreducible components then*

$$I(C, \Gamma) = \sum_{a \in \text{ supp } C \cap \text{ supp } \Gamma} I_a(C, \Gamma)$$

is well defined and we have

$$I(C, \Gamma) = \text{ degree of } C \cdot \text{ degree of } \Gamma.$$

PROOF. The fact that $I(C, \Gamma)$ is well defined follows from the previous lecture. Also it follows that that number is invariant by homotopy. Let f, g be polynomials corresponding to C and Γ. We can homotopy f and g into product of linear homogeneous factors (avoiding that in the homotopy f and g acquire common factors of positive degree). Therefore we can assume

$$f = \prod_1^m l_i \qquad g = \prod_1^l \lambda_i \qquad m = \deg C \;\; l = \deg \Gamma.$$

Now

$$I(C, \Gamma) = I(\sum \{l_i = 0\}, \sum \{\lambda_j = 0\}) = \sum_{i,j} I(\{l_i = 0\}, \{\lambda_j = 0\}) = ml$$

as the intersection multiplicity of two distinct projective lines is one. □

2. Bezout theorem in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$

Let (x_0, x_1), (y_0, y_1) be homogeneous coordinates on the the first respectively the second copy of $\mathbb{P}^1(\mathbb{C})$. Instead of the graded ring of homogeneous polynomials in x_0, x_1, x_2 as we have done for $\mathbb{P}^2$ we consider here the bigraded ring $\mathbb{C}[x_0, x_1; y_0, y_1]$ of polynomials homogeneous in x_0, x_1 and homogeneous in y_0, y_1. Repeating the same consideration as before we can define positive cycles. Each positive cycles has a bidegree (m_1, m_2). For instance an irreducible curve of bidegree (α, β) will be the set of the zeros on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ of an irreducible polynomial $f(x_0, x_1; y_0, y_1)$ bihomogeneous of degree α and β. With the same argument as before we have the following

THEOREM 5.3 (Bezout). *Let C and Γ be two positive cycles in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ without common components then*

$$I(C,\Gamma)=\sum_{a\in\ supp\ C\cap\ supp\ \Gamma} I_a(C,\Gamma)$$

is well defined and if

$$\textit{bidegree of } C=(m_1,m_2)$$
$$\textit{bidegree of } \Gamma=(\mu_1,\mu_2)$$

then

$$I(C,\Gamma)=m_1\mu_2+m_2\mu_1.$$

3. Chasles theorem

DEFINITION 5.4. By a **reduced algebraic correspondence** in $\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ we means the set

$$\Gamma=\{(x_0,x_1)\times(y_0,y_1)\mid f(x_0,x_1;y_0,y_1)=0\}$$

where $f\in\mathbb{C}_0[x_0,x_1;y_0,y_1]$ has no multiple components. We say that the correspondence is **non degenerate** if f has no factor in x_0,x_1 alone or in y_0,y_1 alone.

Let Γ be a reduced non degenerate correspondence of bidegree (α,β). This means that to a point $(y_0^{(0)},y_1^{(0)})$ in the second copy of $\mathbb{P}^1(\mathbb{C})$ correspond α points on the first copy of $\mathbb{P}^1(\mathbb{C})$ which are distinct if $(y_0^{(0)},y_1^{(0)})$ avoids finite many positions. Similarly we can interpret β. Note now that the diagonal $\Delta\subset\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ is an irreducible correspondence of bidegree $(1,1)$ defined by

$$f=x_1y_0-x_0y_1.$$

$I(\Gamma,\Delta)$ is the number of fixed points of the correspondence Γ (counted with proper multiplicity).

THEOREM 5.5. *If Γ is reduced non degenerate of bidegree (α,β) then, if $\Delta\not\subset\Gamma$,*

$$I(\Gamma,\Delta)=\textit{ number of fixed points }=\alpha+\beta.$$

4. Poncelet polygons

a) If we set

$$\begin{cases} x_0 &= u_0^2\\ x_1 &= u_0u_1\\ x_2 &= u_1^2\end{cases}$$

then we define a holomorphic one to one map

$$\mathbb{P}^1(\mathbb{C})\to\mathbb{P}^2(\mathbb{C}),\qquad (u_0,u_1)\in\mathbb{P}^1(\mathbb{C}),\quad (x_0,x_1,x_2)\in\mathbb{P}^2(\mathbb{C}),$$

whose image is the irreducible curve of degree 2 (conic)

$$x_1^2-x_0x_1=0. \tag{5.1}$$

Given a conic

$$\sum_0^3 a_{ij}x_ix_j=0\qquad \det(a_{ij})\neq 0\text{ (non degenerate conic)} \tag{5.2}$$

then the theory of reduction to canonical form of quadratic form shows that there exist a transformation of $PL(2,\mathbb{C})=\frac{GL(3,\mathbb{C})}{\mathbb{C}*}$ that transform (5.2) into (5.1). Therefore $PL(2,\mathbb{C})$ operate transitively on non degenerate conics; each one of them is a biholomorphic image of $\mathbb{P}^1(\mathbb{C})$.

b) Let C_1,Γ_1 be two non degenerate conics having 4 distinct points in common. Then one can show that they also have 4 distinct tangent in common (the cross ratio of the 4 points on C_1 equals the cross ratio of the 4 tangent in the proper order). If we take 2 of the common points into the cyclic points we may without loss of generality assume that C and Γ are two cycles. Let $p_0\in\mathbb{C}$ consider from p_0 one of the two tangent to Γ, call it r_1, and let p_1 be the point where r_1 intersect C again. Start from p_1 with the tangent line r_2 to Γ, different from r_1, to obtain the further intersection point p_2 with C. Start from p_2 etc....(see figure 1). Consider the

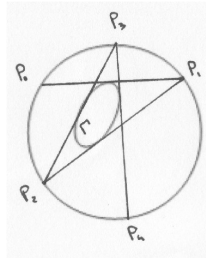


FIGURE 1

correspondence

$$p_0 \to p_n.$$

This correspondence is given by a reduced non degenerate correspondence Λ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ (where $\mathbb{P}^1(\mathbb{C})$ is identified with C). We have that

(1) Λ is invariant by reflection along the diagonal;
(2) Λ is of indices $(2, 2)$.

By Chasles theorem

$$I(\Delta, \Lambda) = 4 \quad \text{if } \Delta \not\subset \Lambda.$$

<u>Case 1</u>. $n = 2k$, then 4 closed polygons (i.e. with $p_n = p_0$) can be obtained taking p_k in one of the four points of $C \cap \Gamma$. The polygon is then $p_0 p_1 \dots p_k p_{k-1} \dots p_0$.

<u>Case 2</u>. $n = 2k + 1$ then 4 closed polygons can be obtained taking p_k on one of the 4 points where the tangent to C is also tangent to Γ. The polygon is then $p_0 p_1 \dots p_k p_k p_{k-1} \dots p_0$.(see figure 2).

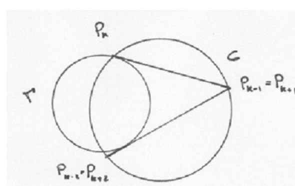


FIGURE 2

<u>Conclusion</u> (Poncelet theorem). Either there are only the 4 degenerate closed polygons $p_0 \dots p_n$ described above or, if there is a further one then there are infinite many starting from any point p_0 we get $p_n = p_0$.(see figure 3).

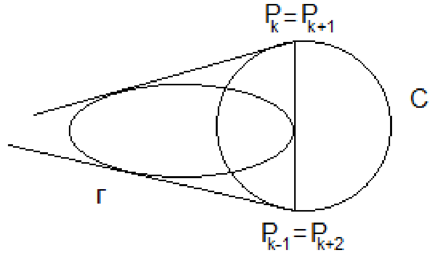


FIGURE 3

EXAMPLE 5.6. Consider the 2 cycles Γ, C inscribed and circumscribed to a triangle $p_0 p_1 p_2$ then from any point p_0' of C there is an inscribed triangle in C circumscribed to Γ.

5. Generalization of the intersection multiplicity

a) Let V_1 V_2 be analytic sets defined in a neighborhood U of the origin $0 \in \mathbb{C}^n$. Assume that

(1) $\dim_0 V_1 + \dim_0 V_2 = n$
(2) $V_1 \cap V_2$ has 0 as an isolated point of intersection.

Set $S(\varepsilon) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum |z_i|^2 = \varepsilon\}$ and consider on $S(\varepsilon)$ for ε small the two knots (oriented)

$$l_{V_1} = V_1 \cap S(\varepsilon) \qquad \text{and} \qquad l_{V_2} = V_2 \cap S(\varepsilon).$$

Define

$$I_0(V_1, V_2) = \mathcal{L}(l_{V_1}, l_{V_2}) = \text{ the linking number of } l_{V_1} \text{ and } l_{V_2}.$$

If V_1 and V_2 are linear subspace we have

$$I_0(V_1, V_2) = 1.$$

Extend the definition by postulating distributive law to positive cycles.

b) Let V_1 and V_2 be algebraic subvarieties of $\mathbb{P}^n(\mathbb{C})$ (or positive cycles) of pure dimension[1] d and $n-d$ respectively. Then

$$V_1 \cong m_1 \mathbb{P}^d(\mathbb{C}) \qquad \text{(homology)}$$

$$V_2 \cong m_2 \mathbb{P}^{n-d}(\mathbb{C}) \qquad \text{(homology)}$$

where m_1 and m_2 are called the **orders of degrees** of V_1 and V_2 respectively. Set[2]

$$I(V_1, V_2) = \sum_{a \in V_1 \cap V_2} I_a(V_1, V_2)$$

Then one has the general Bezout theorem:

$$I(V_1, V_2) = m_1 m_2.$$

c) Let V_1, V_2 be algebraic subvarieties of $\mathbb{P}^n(\mathbb{C})$ (or positive cycles) of pure dimension d and δ respectively. Assume that

$$d + \delta \geq n.$$

Then $V_1 \cap V_2$ is an algebraic variety Z (or cycle) whose irreducible components have all dimension $\geq d + \delta - n$. Assume that Z is pure dimensional of dimension $d + \delta - n$, then

$$Z \cong \deg(Z) \mathbb{P}^{d+\delta-n}(\mathbb{C}) \qquad \text{(homology)}$$

where $\deg(Z)$ is the order of degree of Z. From the general Bezout theorem there follows that, under the specified assumptions

$$\deg(Z) = \deg(V_1) \deg(V_2).$$

d)

In particular let $F(z_1, \dots, z_n)$ be a holomorphic function defined in a neighborhood of the origin in $\mathbb{C}^n$ with $F(0, 0, \dots, 0) = 0$ but $F \not\equiv 0$. Let

$$\gamma = \{z_\alpha = P_\alpha(t) \qquad \alpha = 1, \dots, n$$

be an irreducible branch trough the origin ($P_\alpha(0) = 0 \;\; \alpha = 1, \dots n$) given by parametric equations in the non redundant parameter t. Assume that $\gamma \cap \{F = 0\}$ has the origin as an isolated point, then one has

$$I_0(\gamma, \{F = 0\}) = \text{ order of } F(P_1(t), \dots, P_n(t))$$

where $F = 0$ denotes the cycle represented by the function F.

e) Let V_1 V_2 be two analytic subvarieties defined in a neighborhood U of the origin in $\mathbb{C}^n$ such that

(1) $\dim_0 V_1 + \dim_0 V_2 = n$
(2) $V_1 \cap V_2 = \{0\}$ in U.

---

[1] An algebraic manifold X is said to be **pure dimensionale** if at avery point $p \in X$ we have $\dim_x X = \dim X$.
[2] Here it is assumed that $V_1 \cap V_2$ is a finite set.

Let $\mathcal{I}(V_1)$ and $\mathcal{I}(V_2)$ be the ideals in $\mathcal{O}_0 = \mathbb{C}\{z_1, \dots, z_n\}$ of holomorphic germs at the origin vanish on V_1 and V_2, then

$$\mathcal{I}(V_1) + \mathcal{I}(V_2)$$

is a zero dimensional ideal in $\mathcal{O}_0 = \mathbb{C}\{z_1, \dots, z_n\}$ and thus by Hilbert Nullstellensatz we must have

$$\mathfrak{m}^h \subset \mathcal{I}(V_1) + \mathcal{I}(V_2) \subset \mathfrak{m}$$

for some $h > 0$. Therefore

$$\delta(\mathcal{I}(V_1), \mathcal{I}(V_2)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_0}{\mathcal{I}(V_1) + \mathcal{I}(V_2)} < +\infty.$$

The question arises to decide if we have

$$\delta(\mathcal{I}(V_1), \mathcal{I}(V_2)) = I_0(V_1, V_2).$$

The answer to this question is NO.

Example 5.7. (due to Groebner) In $\mathbb{C}$. V_1 is the algebraic variety defined by parametric equation

$$\begin{cases} z_1 &= \rho\lambda^4 \\ z_2 &= \rho\lambda^3 \\ z_3 &= \rho\lambda \\ z_4 &= \rho \end{cases} \qquad (\text{ no } \lambda^2) \qquad \dim_0(V_1) = 2$$

so that

$$\mathcal{I}(V_1) = \mathcal{O}_0(z_1^2 z_3 - z_2^2, z_1 z_4 - z_2 z_3, z_1 z_3^2 - z_2^2 z_4, z_2 z_4^2 - z_3).$$

V_2 is the plane $\{z_1 = z_4 = 0\}$ i.e.

$$\mathcal{I}(V_2) = \mathcal{O}_0(z_1, z_4).$$

Then

$$\mathcal{I}(V_1) + \mathcal{I}(V_2) = \mathcal{O}_0(z_1, z_4, z_2 z_3, z_2^3, z_3^3)$$

and

$$\delta(\mathcal{I}(V_1) + \mathcal{I}(V_2)) = 5$$

while by Bezout theorem since V_2 has order 4, we have $I_0(V_1, V_2) = 4$.

Remark 5.8. The coincidence of the Hilbert defect δ with the multiplicity of intersection is assumed if

$$\mathcal{I}(V_1) = \mathcal{O}_0(f_1, \dots, f_l) \qquad \text{and } \dim_0 V_1 = n - l$$
$$\mathcal{I}(V_2) = \mathcal{O}_0(g_1, \dots, g_h) \qquad \text{and } \dim_0 V_2 = n - h$$
$$l + h = n$$

i.e. for the so called **complete intersections**.

Remark 5.9. If we wish an algebraic definition of the intersection multiplicity in general one has to write it as a Euler number by the "formula of the Tor" of Serre [**?**].

Remark 5.10. If the surrounding space is not a manifold one must expect fractional values of the intersection multiplicity.

Example 5.11. Consider in $\mathbb{P}^3(\mathbb{C})$ the conic

$$V = \{z_1^2 + z_2^2 + z_3^2 = 0\} \qquad (z_0, z_1, z_2, z_3 \text{ homogeneous coordinates in } \mathbb{P}^3(\mathbb{C}))$$

and let l_1, l_2 two generators, then

$$I_0(l_1, l_2) = \frac{1}{2}.$$

Indeed $2l_1$ is a plane section so that

$$I(2l_1, 2l_2) = 2$$

as two plane sections have 2 points in common thus

$$4I(l_1, l_2) = 2 \qquad \text{i.e.} \quad I(l_1, l_2) = \frac{1}{2}.$$

Let $\mathbb{C}^3$ be the space of the non homogeneous coordinates z_1, z_2, z_3 and consider $V \cap S(2)$ i.e.

$$\begin{cases} |z_1|^2 + |z_2|^2 + |z_3|^2 = 2 \\ z_1^2 + z_2^2 + z_3^2 = 0 \end{cases}$$

Now set $z_j = x_j + iy_j$ thus

$$\sum x_j^2 + \sum y_j^2 = 2$$
$$\sum x_j^2 - \sum y_j^2 = 0$$
$$\sum x_j y_j = 0$$

i.e. $V \cap S(2)$ can be identified to the set of couples of 2 vectors $\vec{x} = (x_1, x_2, x_3)$ $\vec{y} = (y_1, y_2, y_3)$ of length one and orthogonal in $\mathbb{R}^3$. Therefore $V \cap S(2) \cong SO(3) = \mathbb{P}^3(\mathbb{R})$. The linking number of the two knots defined by l_1 and l_2 on $V \cap S(2)$ is indeed equal $\frac{1}{2}$.

6. Topology of complex compact connected manifolds of complex dimension one

a) Every compact connected complex manifold of complex dimension one is a differentiable (hence topological) oriented compact manifold of real dimension 2 (surface). From topology we borrow the following

Theorem 5.12. *Every topological oriented compact connected surface X of dimension 2 is homeomorphic to the connected sum of S^2 (the 2-sphere) with a finite number g of tori $T_i = T = \dfrac{\mathbb{R}^2}{\mathbb{Z}^4}$*

$$X = S^2 \# T_1 \# \cdots \# T_g$$

b) $g = g(X)$ is a topological invariant (the genus of X) and it has the following properties

(1) $g(X) = \frac{1}{2} b_1(X) = \frac{1}{2} \dim_{\mathbb{R}} H^1(X, \mathbb{R})$

(2) $2g - 2$ is the Euler characterstic of X; if X is triangulated with α_0 vertices, α_1 edges and α_2 triangles:

$$2 - 2g = \alpha_0 - \alpha_1 + \alpha_2 = \dim H^0(X, \mathbb{R}) - \dim H^1(X, \mathbb{R}) + \dim H^2(X, \mathbb{R}).$$

(3) Let $\gamma_i \colon S^1 \to X$, $1 \le i \le l$, be a set of homeomorphisms of the 1-sphere S^1 into X such that

$$\gamma_i(S^1) \cap \gamma_j(S^1) = \emptyset \qquad i \ne j. \tag{5.3}$$

Let us consider all set of homeomorphism $\{\gamma_i\}_{1 \le i \le l}$ of this sort and such that

$$X \setminus \cup \gamma_i(S^1) \tag{5.4}$$

is connected, then

$$l \le g(X)$$

and the maximum value $g(X)$ is attained.

c) $H^1(X, \mathbb{Z})$ is generated by $2g$ 1-cycles

$$a_i \colon S^1 \to X \qquad \text{(homeomorphism for } 1 \le i \le g\text{)}$$
$$b_i \colon S^1 \to X \qquad \text{(homeomorphism for } 1 \le i \le g\text{)}$$

with the property

$$H^1(X, \mathbb{Z}) = \bigoplus_1^g \mathbb{Z} a_i + \bigoplus_1^g \mathbb{Z} b_i$$
$$I(a_i, a_j) = 0 \qquad I(b_i, b_j) = 0$$
$$I(a_i, b_j) = -I(b_j, a_i) = \delta_{ij} \quad \text{(Kronecker delta)}$$

so that

$$\begin{pmatrix} I(a_i, a_j) & I(a_i, b_j) \\ I(b_i, a_j) & I(b_i, b_j) \end{pmatrix} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

d) We may assume that in the previous statement all cycles a_i and b_j start and end at a same point $0 \in X$. Then $\pi_1(X, 0)$ is generated by $a_1, \ldots, a_g, b_1, \ldots, b_g$ and by a suitable ordering the group $\pi_1(X, 0)$ is the free group generated by these letters subject to the relation

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = id.$$

e) One can also assume that the cycles a_i b_j for all i, j have only the point 0 as a common point. Then $X \setminus (\cup a_i \cup b_j)$ is homeomorphic to a polygon P_{4g} with $4g$ sides a_i b_i a_i^{-1} b_i^{-1} with $1 \leq i \leq g$ so that $X = P_{4g}/$(identification of the sides a_i a_i^{-1} b_i b_i^{-1} in the corresponding order).

THEOREM 5.13 (Harnack). *We call a compact complex manifold X with $\dim_{\mathbb{C}} X = 1$ have real structure if on X an antiholomorphic map $\tau\colon X \to X$ is given with $\tau^2 = id$. Then*

(1) *every connected component of the fixed point set of τ is homeomorphic to an analytic imbedding $\alpha\colon S^1 \to X$;*
(2) *the number $N(X, \tau)$ of the connected components of the fixed points set is bounded by*

$$N(X, \tau) \leq g + 1;$$

(3) *for every value of $g \geq 0$ there exist X with real structure τ and $N(X, \tau) = g + 1$ (Harnack curves);*
(4) *every Harnack curves is isomorphic to the "doubling" of a plane region in $\mathbb{C}$ bounded by $g + 1$ cycles.*

PROOF. (1) Let $a \in X$ be a fixed point of τ and z a local coordinate centered at a, then in a small neighborhood of a τ is given by an equation

$$z' = \overline{g(z)} \qquad g(0) = 0$$

with $g(z)$ holomorphic. As $\tau^2 = id$ we must have

$$\overline{z} = g(\overline{g(z)})$$

so that we have

$$\frac{dg}{dz}(0)\overline{\frac{dg}{dz}(0)} = 1.$$

Hence $w = g(z)$ can be taken as coordinate and the equation of τ become

$$w' = \overline{w}.$$

Thus now α is obvious as the fixed point set must consists of the analytic compact manifolds.

(2) If there were $g(x) + 2$ connected components $C_1, C_2, \ldots, C_{g+2}$ in the fixed set of τ then $X \setminus \cup_1^{h+1} C_i$ for some $h \leq g$ is disconnected in 2 components A, B and C_{g+2} belongs to one of them say A. But τ exchanges A with B thus C_{g+2} could not be in the fixed set.
(3) The Riemann surface of

$$u^2 = -\prod_{i=1}^{2g+2} (u - i)$$

has the required property.
(4) Since inversion by reciprocal radii is antiholomorphic it follows that the doubling of a plane region in $\mathbb{C}$ bounded by $g + 1$ circles gives a complex compact manifold with a real structure of the type of Harnack. The fact that every Harnack curve is of this type follows from the remarks:
   (a) the $g + 1$ component of the fixed set of τ decompose X into two connected components;
   (b) each of these component is isomorphic (by a theorem in uniformization theory) to a plane region in $\mathbb{C}$ bounded by $g + 1$ circles.

□

7. Riemann-Hurwitz-Zeuthen formula

Let X and Y be compact connected complex manifold of complex dimension 1. Let $f\colon X \to Y$ be a holomorphic non constant map. Then

(1) f is open and surjective $f(X) = Y$;
(2) X is a n-sheeted ramified covering of Y with a finite set of branch points on Y;
(3) at every point $x_0 \in X$ $y_0 = f(x_0) \in Y$ local coordinates can be chosen ζ and z so that in a connected small neighborhood U of x_0 $f|_U$ is given by equation
$$z = \zeta^{\nu(x_0)}$$
where $\nu(x_0)$ is an integer ≥ 1 depending only on x_0 and f.
(4) for all $x \in X$ $\nu(x)$ is defined and $\nu(x) > 1$ only at finite many points of X.

THEOREM 5.14. *Let $g(X)$ and $g(Y)$ denote the genus of X and Y respectively. We have*
$$2g(X) - 2 = n(2g(Y) - 2) + \sum_{x\in X}(\nu(x) - 1)$$

PROOF. Choose a triangulation of Y with the property that the vertices contains all points $y = f(x)$ with $\nu(x) > 1$. Let $\alpha_0, \alpha_1, \alpha_2$ be the number of vertices, edges and triangles of this triangulation. By f we can "lift" this triangulation to a triangulation of X and let $\beta_0, \beta_1, \beta_2$ be the number of vertices, edges and triangle of this triangulation of X. Then
$$\beta_2 = n\alpha_2$$
$$\beta_1 = n\alpha_1$$
$$\beta_0 = n\alpha_0 - \sum_{x\in X}(\nu(x) - 1).$$
Thus
$$\begin{aligned} 2g(X) - 2 = -(\beta_0 - \eta_1 + \beta_2) &= -\left(n(\alpha_0 - \alpha_1 + \alpha_2) - \sum_{x\in X}(\nu(x) - 1)\right) \\ &= n(2g(Y) - 2) + \sum_{x\in X}(\nu(x) - 1). \end{aligned}$$
□

<u>Consequences:</u>

(1) $\sum_{x\in X}(\nu(x) - 1) \equiv 0 \pmod 2$ i.e. the number of branch points on X is even;
(2) if $Y = \mathbb{P}^1(\mathbb{C})$ then (Riemann-Hurwitz)
$$2g(X) - 2 = \sum_{x\in X}(\nu(x) - 1) - 2n.$$
(3) (theorem of Weber)
   (a) $g(X) \leq g(Y)$ for all f holomorphic non constant;
   (b) if $g(X) = g(Y)$ there are only 3 possibilities:
      (i) $g(X) = g(Y) = 0$ then $\sum_{x\in X}(\nu(x) - 1) = 2(n - 1)$;
      (ii) $g(X) = g(Y) = 1$ then $\sum_{x\in X}(\nu(x) - 1) = 0$ (i.e. no ramification);
      (iii) $g(X) = g(Y) > 1$ then $n = 1$ i.e. $f\colon X \xrightarrow{\sim} Y$ is an isomorphism.

PROOF. If $g(X) = 0$ then $n(g(Y) - 1) + \frac{1}{2}\sum_{x\in X}(\nu(x) - 1) = -1$ which is possible only if $g(Y) = 0$ and $\sum_{x\in X}(\nu(x) - 1) = 2n - 2$. If $g(X) \geq 1$ then
$$g(Y) - 1 = \frac{1}{n}\{g(X) - 1 - \frac{1}{2}\sum_{x\in X}(\nu(x) - 1)\} \leq \frac{1}{n}(g(X) - 1)$$
therefore $g(Y) \leq g(X)$. If $g(X) = g(Y)$ is 0 or 1 then i and ii follows directly from the formula. If $g = g(X) = g(Y) > 1$ then
$$(n - 1)(g - 1) + \frac{1}{2}\sum_{x\in X}(\nu(x) - 1) = 0$$
thus $n = 1$ (and $\sum_{x\in X}(\nu(x) - 1) = 0$). □

In particular if $g(X) \geq 2$ every non constant holomorphic map $f\colon X \to X$ is an automorphism.

CHAPTER 6

Meromorphic functions on complex manifolds

1. Preliminaries

a) The Levi form.

Definition 6.1. Let Ω be an open subset of $\mathbb{C}^*$, Let $\partial\Omega$ be its boundary $\partial\Omega = \overline{\Omega} \setminus \Omega$. We say that $\partial\Omega$ is **smooth** if for all $x_0 \in \partial\Omega$ we can find a neighborhood $U(x_0)$ and a C^∞ function

$$\varphi\colon U(x_0) \to \mathbb{R}$$

such that

$$\Omega \cap U(x_0) = \{z \in U(x_0) \mid \varphi(z) < 0\} \qquad d\varphi(x_0) \neq 0.$$

Given an open set Ω in a complex manifold X of pure dimension n and with a smooth boundary $\partial\Omega$ we can always find a global function

$$\Phi\colon X \to \mathbb{R}$$

such that

$$\Omega = \{z \in X \mid \Phi(z) < 0\} \qquad d\Phi(z) \neq 0 \quad \forall z \in \partial\Omega.$$

This is done by selecting a covering $\mathcal{U} = \{U_i\}_{i\in J}$ of X and by considering in each U_i a defining function, for $\partial\Omega \cap U_i$, φ_i $\left(\varphi_i = \begin{cases} +1 & \text{if } U_i \cap \Omega = \emptyset \\ -1 & \text{if } U_i \subset \Omega \end{cases}\right)$. On $U_i \cap U_j$ we have $\varphi_i = h_{ij}\varphi_j$ with h_{ij} C^∞ and $h_{ij} > 0$. Then we can find C^∞ functions $k_i\colon U_i \to \mathbb{R}$ such that $\log h_{ij} = k_j - k_i$, hence $h_{ij} = \frac{e^{k_i}}{e^{k_j}}$. Set $h_i = e^{k_i}$ we have then $\varphi_i h_i = \varphi_j h_j = \Phi$ and Φ has the required property. To show that $\{k_i\}$ exist we choose a partition of unit $\{\rho_i\}_{i\in J}$ subordinate to $\mathcal{U}$ and set

$$k_j = \sum \rho_i \log h_{ij}.$$

Then

$$k_j - k_i = \sum \rho_s \log h_{sj} - \sum \rho_s \log h_{si} = \sum \rho_s \log \frac{h_{sj}}{h_{si}} = \sum \rho_s \log h_{ij} \qquad (\text{as } h_{is}h_{sj} = h_{ij}).$$

b) Given on an open set Ω of $\mathbb{C}^n$ a C^∞ function $\varphi\colon \Omega \to \mathbb{R}$, at any point $a \in \Omega$ we can consider the Taylor expansion of φ

$$\varphi(z) = \varphi(a) + \sum \partial_\alpha\varphi(a)(z_\alpha - a_\alpha) + \sum \partial_{\overline{\alpha}}\varphi(a)(\overline{z_\alpha} - \overline{a_\alpha}) + \frac{1}{2}\sum \partial_{\alpha\beta}\varphi(a)(z_\alpha - a_\alpha)(z_\beta - a_\beta) +$$

$$+\frac{1}{2}\sum \partial_{\overline{\alpha}\overline{\beta}}\varphi(a)(\overline{z_\alpha} - \overline{a_\alpha})(\overline{z_\beta} - \overline{a_\beta}) + \sum \partial_{\alpha\overline{\beta}}\varphi(a)(z_\alpha - a_\alpha)(\overline{z_\beta} - \overline{a_\beta}) + \mathcal{O}(||z-a||^3).$$

Because φ is real valued, the quadratic form

$$\mathcal{L}(\varphi)_a(v) = \sum \partial_{\alpha\overline{\beta}}\varphi(a)v_\alpha v_\beta$$

is hermitian. It is called the **Levi form** of φ at a. A biholomorphic change of coordinates near a acts on $\mathcal{L}(\varphi)_a$ with the linear change of variables

$$v \to J(a)v$$

where $J(a)$ is the Jacobian matrix of that change of coordinates. In particular the number of positive and negative eigenvalue of $\mathcal{L}(\varphi)_a$ is independent of the choice of local holomoprhic coordinates.

REMARK 6.2. If $(d\varphi)_a \neq 0$ we can perform a change of holomorphic coordinates centered at a in which the new coordinate z_1 is

$$\sum \partial_\alpha \varphi(a)(z_\alpha - a_\alpha) + \frac{1}{2}\sum \partial_{\alpha\beta}\varphi(a)(z_\alpha - a_\alpha)(z_\beta - a_\beta).$$

Then φ in those new coordinates tokes the form

$$\varphi(z_1) = \varphi(a) + 2\mathbb{R}ez_1 + \mathcal{L}(\varphi)_a(z) + o(||z||^3).$$

c) We apply the previous considerations to the defining function of a smooth boundary point $a \in \partial\Omega$, $\Omega \subset \mathbb{C}^n$. We may choose the coordinates with a at the origin. Then the real tangent plane to $\partial\Omega$ at 0 is

$$\sum \partial_\alpha \varphi(0) z_\alpha + \sum \partial_{\overline{\alpha}} \varphi(0)\overline{z}_\alpha = 0$$

and it contains the $(n-1)$ complex plane

$$\sum \partial_\alpha \varphi(0) z_\alpha = 0$$

which is called the **analytic tangent plane** to $\partial\Omega$ at a, usually denoted by $T_a(\Omega)$. Consider the Levi-form of φ restricted to $T_a(\Omega)$

$$\mathcal{L}(\varphi)|_{T_a(\Omega)} = \begin{cases} \sum \partial_{\alpha\overline{\beta}}\varphi(0)v_\alpha \overline{v}_\beta \\ \sum \partial_\alpha \varphi(0) v_\alpha = 0 \end{cases}$$

We obtain in this way a hermitian form in $n-1$ variables. We claim that the number of positive and negative eigenvalues of $\mathcal{L}(\varphi)|_{T_a(\Omega)}$ is independent both from the choice of local holomorphic coordinates in $\mathbb{C}^n$ and of the choice of the defining function φ for the boundary of Ω near a. The first part of the statement follows from the previous remark, the second from the fact that if φ and ψ are defining function for $\partial\Omega$ near $0 \in \partial\Omega$ we must have

$$\varphi = h\psi \qquad h \in C^\infty \text{ near } a \quad h(0) > 0.$$

then a direct calculation shows that:

$$\mathcal{L}(\varphi)|_{T_0(\partial\Omega)} = h(0)\mathcal{L}(\psi)|_{T_a(\partial\Omega)}$$

d) Using φ in the form of the remark above, we then derive the following geometric interpretation of the number $p(0)$ of positive and $q(0)$ of negative eigenvalues of $\mathcal{L}(\varphi)|_{T_0(\partial\Omega)}$. Clearly we have

$$p(a) + q(a) \leq n - 1.$$

There is an analytic disc of dimension $p = p(0)$

$$\tau \colon D^p \to \mathbb{C}$$

$D^p = \{t \in \mathbb{C}^p \mid \sum |t_i|^2 < 1\}$, τ biholomorphic, such that

$$\begin{cases} \tau(0) = a \\ \tau(D^p) \setminus \{a\} \subset \overline{\Omega}^c \end{cases} \tag{6.1}$$

Analogously there is an analytic disc of dimension $q = q(0)$

$$\tau \colon D^q \to \mathbb{C}$$

$D^q = \{t \in \mathbb{C}^q \mid \sum |t_i|^2 < 1\}$, τ biholomorphic, such that

$$\begin{cases} \tau(0) = a \\ \tau(D^q) \setminus \{a\} \subset \Omega \end{cases} .$$

e) The field of rational function of an algebraic variety. Let $\mathbb{P}^n(\mathbb{C})$ be the projective space of n dimensions and let $z_0, \ldots, z_n$ denote the homogeneous coordinates there. Let $\mathbb{C}_0[z_0, \ldots, z_n]$ be the graded ring of homogeneous polynomials in these coordinates and let $J \subset \mathbb{C}_0[z_0, \ldots, z_n]$ be a prime ideal. We set

$$V = \{z \in \mathbb{P}^n(\mathbb{C}) \mid f(z) = 0 \; \forall f \in J \text{ homogeneous } \}$$

and

$$A(V) = \frac{\mathbb{C}_0[z_0, \ldots, z_n]}{J}.$$

The ring $A(V)$ is called the **coordinate ring on the algebraic variety** V. $A(V)$ is an integral domain as J is a prime ideal, therefore one can consider the quotient field

$$Q(V) = \{\frac{p}{q} \mid p, q \in A(V) \quad q \neq 0,\ \deg p = \deg q\}$$

where of course $\frac{p}{q} = \frac{p'}{q'}$ if and only if $pq' - qp' = 0$. Let us suppose that $z_0 \notin J$ (this can be achieved by a suitable surjective transformation of the coordinates), then

$$\mathbb{C}^n = \{z \in \mathbb{P}^n(\mathbb{C}) \mid z_0 \neq 0\}$$

$$V_1 = V \cap \mathbb{C}^n = \{z \in \mathbb{C}^n \mid f(1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}) = 0\ \forall f \in J\}$$

and it is defined by a prime ideal J_1 in the ring $\mathbb{C}[\zeta_1, \dots, \zeta_n]$ where $\zeta_i = \frac{z_i}{z_0}$ are the non homogeneous coordinates covering $\mathbb{C}^n$. If we consider the ring

$$A_1(V_1) = \frac{\mathbb{C}[\zeta_1, \dots, \zeta_n]}{J_1}$$

and its quotient field

$$Q_1(V_1) = \{\frac{p}{q} \mid p, q \in A_1(V) \quad q \neq 0,\ \deg p = \deg q\}$$

we realize that we have an isomorphism

$$Q(V) \to Q_1(V_1)$$

given by

$$\frac{p(z_0, \dots, z_n)}{q(z_0, \dots, z_n)} \mapsto \frac{p(1, \zeta_1, \dots, \zeta_n)}{q(1, \zeta_1, \dots, \zeta_n)}.$$

Therefore for question regarding the field $Q(V)$ is unessential if we work with $V \subset \mathbb{P}^n$ or with $V_1 \subset \mathbb{C}^n$.

Theorem 6.3. *Transcendence degree of* $Q(V)$ = *transcendence degree of* $Q_1(V) = \dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V_1$.

Proof. Since $z_0 \notin J$, $\{z_0 = 0\}$ is nowhere dense in V so that

$$\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} V_1.$$

Also as $Q(V) \cong Q_1(V_1)$ they have the same transcendence degree. Finally if d is the transcendence degree of $Q_1(V_1)$ we may assume that $\zeta_1, \dots, \zeta_d$ are algebraically independent and that $\zeta_{d+1}, \dots, \zeta_n$ depends algebraically upon $\zeta_1, \dots, \zeta_d$. Also, replacing $\zeta_{d+1}, \dots, \zeta_n$ by suitable linear combination over $\mathbb{C}$ we may assume[1] $Q_1(V_1) = \mathbb{C}(\zeta_1, \dots, \zeta_d, \zeta_{d+1})$. We then consider the minimal equations of ζ_{d+1} over $\mathbb{C}(\zeta_1, \dots, \zeta_d)$

$$p_i(\zeta_1, \dots, \zeta_d, \zeta_{d+1}) = 0$$

and we can assume p_i to be a polynomial in all variables devoid of factors in $\zeta_1, \dots, \zeta_d$ only. Moreover we will have $\zeta_{d+j} = \frac{p_j(\zeta_1, \dots, \zeta_{d+1})}{q(\zeta_1, \dots, \zeta_j)}$ for $2 \leq j \leq n - d$ with p_j, q polynomials and $q \neq 0$. Consider the projection $\pi\colon \mathbb{C}^n \to \mathbb{C}^d$ where $\mathbb{C}^d$ is the space of the coordinates $\zeta_1, \dots, \zeta_d$. One then find that outside the algebraic variety $W \subset \mathbb{C}^d$ given by the zeros of the coefficients of the highest power of ζ_{d+1} in p, $q = 0$, and the zeros of the discriminant of p. With respect to ζ_{d+1} we get that

$$\pi|_{\pi^{-1}(\mathbb{C}^d \setminus W) \cap V_1}$$

has finite fibers and is a local isomorphism of V_1 on $\mathbb{C}^d$. Since $\dim_{\mathbb{C}} V_1 = \dim_{\mathbb{C}}(V_1, a)$ for any regular point $a \in V_1$ we obtain thus that

$$\dim_{\mathbb{C}} V_1 = d = \text{transcendence degree of } Q_1(V_1).$$

$\square$

[1] By primitive element theorem.

f) Shilov boundary of a compact polycilinder. Let

$$\overline{P} = \{z \in \mathbb{C} \mid |z_i| \le r_i,\ 1 \le i \le n\}$$

be a compact polycilinder of polyradius $(r_1, \dots, r_n)$, $r_i > 0$. Let f be any holomorphic function defined on a neighborhood of $\overline{P}$, then

$$\sup_{\overline{P}} |f| = \sup_{\substack{|z_1| = r_1 \\ \dots \\ |z_n| = r_n}} |f|$$

as it follows from the maximum principle in one variable applied n times. Set

$$\mathcal{S}(P) = \{z \in \mathbb{C}^n \mid |z_i| = r_i\ , \quad 1 \le i \le n\}$$

we thus have

(1) for all f holomorphic near $\overline{P}$, $\sup_{\overline{P}} |f| = \sup_{\mathcal{S}(P)} |f|$;
(2) $\mathcal{S}(P)$ is a minimal closed subset with property i).

Definition 6.4. We call $\mathcal{S}(P)$ the Shilov boundary of $\overline{P}$.

g)

Lemma 6.5 (Schwarz Lemma). *Let f be holomorphic in a neighborhood of $\overline{P}$ and suppose that $f \in \mathfrak{m}_0^h$ i.e. f vanishes at the origin with all derivatives of order $\le h-1$. Then we have for all $z \in P$*

$$|f(z)| \le \frac{||z||^h}{||r||^h} \sup_{\mathcal{S}(P)} |f| \qquad ||z|| = \sup_{1 \le i \le n} |z_i|.$$

Proof. Set $z = \lambda a$, with $||a|| = 1$, then $\frac{f(\lambda a)}{\lambda^h}$ is holomorphic for $||\lambda a|| \le ||r||$ i.e. $|\lambda| \le ||r||$. The maximum principle gives:

$$\left| \frac{f(\lambda a)}{\lambda^h} \right| \le \frac{\sup_{\overline{P}} |f|}{||r||^h}$$

i.e.

$$|f(z)| \le \sup_{S(P)} |f| \frac{||z||^h}{||r||^h}$$

as $||z|| = ||\lambda a|| = |\lambda|$. □

2. Meromorphic functions on complex manifolds

a) Let X be a complex manifold and let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on X for every open set $U \subset X$, it is defined by the space $\mathcal{H}(U)$ and the natural restriction maps. The space $\mathcal{H}(U)$ is a ring. Let $\mathcal{D}(U)$ be subset of $\mathcal{H}(U)$ of divisors of zeros i.e. $\mathcal{D}(U)$ is the set of those holomorphic functions on U vanishing on some connected component of U. Let $Q(U)$ be the quotient ring of $\mathcal{H}(U)$ with respect to $D(U)$ i.e. $Q(U)$ is the quotient $\dfrac{f}{g}$ with $f \in \mathcal{H}(U)$, $g \in \mathcal{H}(U) \setminus D(U)$ with identifications

$$\frac{f}{g} = \frac{f'}{g'} \qquad \text{if and only if } fg' = f'g.$$

If $V \subset U$ is an inclusion of open sets, the restriction map $r_V^U \colon \mathcal{H}(U) \to \mathcal{H}(V)$ sends $\mathcal{H}(U) \setminus D(U)$ into $\mathcal{H}(V) \setminus D(V)$ and thus induces a homomorphism of rings

$$r_V^U \colon Q(U) \to Q(V).$$

We obtain in this way a presheaf. The corresponding sheaf $\mathcal{M}$ is called the **sheaf of germs of meromorphic functions** on X. The ring

$$\mathcal{K}(X) = \Gamma(X, \mathcal{M})$$

is called the **ring of meromorphic functions** on X. Note that

$$Q(X) \subset \mathcal{K}(X)$$

but $Q(X)$ may be actually smaller than $\mathcal{K}(X)$.

EXAMPLE 6.6. Take $X = \mathbb{P}^1(\mathbb{C})$ the Riemann's sphere. Then $\mathcal{H}(X) = \mathbb{C}$ thus $Q(X) = \mathbb{C}$ while $\mathcal{K}(X)$ is isomorphic to the field of all rational functions in one variable t, $\mathcal{K}(X) \cong \mathbb{C}(t)$.

REMARK 6.7. If X is connected then $\mathcal{K}(X)$ and $Q(X)$ are fields.

In the sequel we will always assume that X is a connected manifold.

b) Meromorphic functions and holomorphic line bundles

DEFINITION 6.8. Let X be a complex manifold, by a **holomorphic line bundle** on X we means a triple, (F, π, X) where F is a complex manifold, $\pi\colon F \to X$ a holomorphic surjective maps such that

(1) π is of maximal rank;
(2) for all $x \in X$, $\pi^{-1}(x) \cong \mathbb{C}$ in such a way that:
   (a) the map
$$F \times_X F \to F$$
   given by $(u, v) \to u + v$ is holomorphic;
   (b) the map
$$\mathbb{C} \times F \to F$$
   given by $(\lambda, v) \to \lambda v$ is holomorphic.

Given two holomorphic line bundles (F, π, X), (E, ω, X) over X a **morphism** (or bundle map) is a holomorphic map $f\colon F \to E$ such that

(1) $\pi = \omega \circ f$;
(2) for every $x \in X$ the induced map $f_x\colon \pi^{-1}(x) \to \omega^{-1}(x)$ is $\mathbb{C}$-linear.

A holomorphic line bundle (F, π, X) is said to be **trivial** if it is isomorphic to the bundle $(X \times \mathbb{C}, pr_X, X)$

Every holomorphic line bundle is locally trivial (as it follows from the implicit function theorem). Therefore there exist an open covering $\mathcal{U} = \{U_i\}$ of X and biholomorphic maps
$$\Phi_i\colon \pi^{-1}(U_i) \to U_i \times \mathbb{C}$$
such that $\pi \circ \Phi_i^{-1}(x, y) = x$, $(x, y) \in U_i \times \mathbb{C}$. On $U_i \cap U_j$ we have two trivializations of F and thus
$$\Phi_i \circ \Phi_j^{-1}(x, v) = (x, g_{ij} v)$$
with
$$g_{ij}\colon U_i \cap U_j \to \mathbb{C}^* \tag{6.2}$$
holomorphic and non zero. In $U_i \cap U_j \cap U_k$ we must have
$$g_{ij} g_{jk} = g_{ik} \tag{6.3}$$

DEFINITION 6.9. The collection of $\{g_{ij}\}$ are called **transition functions** of F (relative to the local trivializations Φ_i).

Conversely given on an open covering $\mathcal{U} = \{U_i\}$ a system of transition functions (6.2) satisfying the consistency conditions (6.3) one can construct the holomorphic line bundle with local trivializations on the sets U_i having the given system as a system of transition functions. Two holomorphic line bundles given on the same covering $\mathcal{U} = \{U_i\}$ with transition functions $\{g_{ij}\}$, $\{f_{ij}\}$ are isomorphic if and only if there exist holomorphic maps $\lambda_i\colon U_i \to \mathbb{C}^*$ such that
$$g_{ij} = \lambda_i f_{ij} \lambda_j^{-1} \qquad \text{on } U_i \cap U_j.$$

Given a holomorphic line bundle (F, π, X) we can consider the space of holomorphic sections
$$\Gamma(X, F) = \{s\colon X \to F \mid s \text{ holomorphic }, \ \pi \circ s = id_X\}.$$
In terms of local trivializations of F on the covering $\mathcal{U} = \{U_i\}$, a holomorphic section is given by a collection
$$s_i\colon U_i \to \mathbb{C}$$
of holomorphic functions such that
$$s_i = g_{ij} s_j \qquad \text{on } U_i \cap U_j$$

Given two holomorphic sections $s_0 = \{s_{0i}\}$ and $s_1 = \{s_{1i}\}$ of the bundle F, if s_0 is not identically zero on any open set, we can construct a meromorphic function on X $\frac{s_1}{s_0}$ given locally by $\left\{\frac{s_{1i}}{s_{0i}}\right\}$. The following proposition shows that we obtain in this way all elements of $\mathcal{K}(X)$.

Proposition 6.10. *Every meromorphic function m on a complex manifold X is the quotient of two holomorphic sections of an appropriate holomorphic line bundle on X.*

Proof. At every point $x \in X$ we can find a neighborhood V such that

$$m|_V = \frac{p}{q} \qquad p \in \mathcal{K}(V),\ q \in \mathcal{K}(V) \setminus D(V).$$

Since the ring $\mathcal{O}_x$ is a unique factorization domain, if we take V sufficiently small we may assume that the germs p_x and q_x of p and q at x are coprime. But if p_x and q_x are coprime and if V is sufficiently small, then also the germs p_y and q_y of p and q at any point $y \in V$ are coprime[2]. Let $\{V(x_i)\}_{i\in I}$ be a covering of X with such neighborhood. Then on $V(x_i)$

$$m|_{V(x_i)} = \frac{p_i}{q_i} \qquad p_i \in \mathcal{K}(V(x_i))\ q_i \in \mathcal{K}(V(x_i)) \setminus D(V(x_i)).$$

and on $V(x_i) \cap V(x_j)$

$$\frac{p_i}{q_i} = \frac{p_j}{q_j} \quad \text{i.e. } p_i q_j = p_j q_i.$$

By the Euclid Lemma p_i and p_j must divide p_i i.e. $p_i = g_{ij} p_j$ with g_{ij} a unit in $\mathcal{H}(V_i) \cap \mathcal{H}(V_j)$, this means that g_{ij} on $V(x_i) \cap V(x_j)$ is holomorphic and never zero. It follows then that we also have

$$q_i = g_{ij} q_j.$$

$$g_{ij} g_{jk} g_{ki} = 1 \quad \text{on} \quad V(x_i) \cap V(x_j) \cap V(x_k).$$

This shows that the collection $\{g_{ij}\}$ is a set of transition functions of a holomorphic line bundle F over X, the collection $\{p_i\}$ gives a holomorphic section s_1 of F and the collection $\{q_i\}$ gives a holomorphic section s_0 of F with s_0 not identically zero on any open set. We have thus proved that $m = \frac{s_1}{s_0}$ as required. ∎

c) Let (F, π, X) be a holomorphic line bundle on X given on a covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X by transition functions $\{g_{ij}\}$. One can consider the "l-th tensor power" of F, (F^l, π_l, X) which is given by the transition functions $\{g_{ij}^l\}$. We can consider the graded ring

$$\mathcal{J}(X, F) = \bigoplus_{l=0}^{\infty} \Gamma(X, F^l)$$

of the holomorphic sections of the different tensor powers of F (F^0 is the trivial bundle). Note that if $s \in \Gamma(X, F^l)$ and $t \in \Gamma(X, F^m)$ then $st \in \Gamma(X, F^{l+m})$. If X is connected, as we always assume, then $\mathcal{J}(X, F)$ is an integral domain and we can consider the field of quotients

$$Q(X, F) = \{\frac{s_1}{s_0} \mid s_1, s_0 \in \Gamma(X, F^l) \text{ for some } l, s_0 \neq 0\}.$$

We have

$$Q(X, F) \subset \mathcal{K}(X)$$

in particular $Q(X) = Q(X, \text{trivial bundle})$.

Theorem 6.11. *For every holomorphic line bundle F the field $Q(X, F)$ is algebraically (integrally?) closed in $\mathcal{K}(X)$ (X connected).*

[2]The germs p_x and q_x are coprime thus $R(p, q) \neq 0$ in x. Let $x_1, \dots, x_n$ a coordinate system centered in x. The resultant R is a continuous function in this coordinates thus there exist a neighborhood V of x such that $R(p, q)|_V \neq 0$.

PROOF. Let $h \in \mathcal{K}(X)$ be algebraic over $Q(X,F)$ i.e. h satisfies an equation

$$h^\nu + k_1 h^{\nu-1} + \cdots + k_\nu \equiv 0$$

where $k_i \in Q(X,F)$. Let $k_i = \frac{s_i}{t_i}$ with $s_i, t_i \in \Gamma(X, F^{l_i})$, multiplying the above equation by $\prod_{i=1}^{\nu} t_i$ we obtain an equation

$$\sigma_0 h^\nu + \sigma_1 h^{\nu-1} + \cdots + \sigma_\nu \equiv 0$$

where $\sigma_i \in \Gamma(X, F^l)$ for a suitable l ($l = \sum_{i=1}^{\nu} l_i$) and where $\sigma_0 \not\equiv 0$. After multiplication by $\sigma_0^{\nu-1}$ the above equation can be written as follows:

$$(\sigma_0 h)^\nu + \sigma_1 (\sigma_0 h)^{\nu-1} + \cdots + \sigma_0^{\nu-1}\sigma_\nu \equiv 0.$$

At each point $x \in X$ $\sigma_0 h$ satisfies an equation with holomorphic coefficient and with the coefficient of the highest power equal one. This shows that $\sigma_0 h$ is meromorphic at x and integral over $\mathcal{O}_x$, since $\mathcal{O}_x$ is integrally closed $\sigma_0 h = \tau$ must be holomorphic at x. Hence $\sigma_0 \in \Gamma(X, F^l)$ and also $\sigma_0 h \in \Gamma(X, F^l)$, thus

$$h = \frac{\tau}{\sigma_0} = \frac{\sigma_0 h}{\sigma_0} \in Q(X,F).$$

□

3. Pseudoconcave manifolds

DEFINITION 6.12. A connected complex manifold X is called **pseudoconcave** if we can find a non empty open subset $Y \subset X$ with the following properties

(1) Y is relative compact in X: $Y \Subset X$;
(2) $\partial Y = \overline{Y} \setminus Y$ is smooth and the Levi form of ∂Y restricted to the analytic tangent plane has at least one negative eigenvalue at each point of ∂Y (as usual the defining function for ∂Y is chosen so that is < 0 on Y and > 0 outside of Y).

In particular for any point $z_0 \in \partial Y$ there is an analytic disc of dimension ≥ 1 which is tangent at z_0 to ∂Y and is contained, except z_0, in Y.

EXAMPLE 6.13. (1) Every compact connected manifold is pseudoconcave (take $Y = X$ then $\partial Y = \emptyset$ thus condition ii is void).

(2) Let Z be a compact connected manifold of $\dim_{\mathbb{C}} Z \geq 2$. Let $\{a_1, \dots, a_m\}$ be a finite subset of Z. Then $X = Z \setminus \{a_1, \dots, a_m\}$ is pseudoconcave (take for Y the complement of a set of disjoint coordinate balls centered at the points a_i).

(3) Not every pseudoconcave manifold is compactificable (i.e. isomorphic to an open subset of a compact manifold). For instance if we take $\mathbb{P}^2(\mathbb{C}) \setminus \{0\} \supset \mathbb{C}^2 \setminus \{0\}$ and if z_1, z_2 are the holomorphic coordinates on $\mathbb{C}^2$, we can consider the exterior form

$$\varphi_\varepsilon = dz_1 \wedge dz_2 + \varepsilon \partial\overline{\partial} \log(|z_1|^2 + |z_2|^2) \quad \text{with } \varepsilon \neq 0$$

and define a function f to be holomorphic if it satisfies the differential equation

$$df \wedge \varphi_\varepsilon = 0.$$

In this way we define a complex structure on $\mathbb{C}^2 \setminus \{0\}$ (which agrees with the natural one if $\varepsilon = 0$). One can show that this complex structure can be extended to $\mathbb{P}^2(\mathbb{C}) \setminus \{0\}$, that if $\varepsilon \neq 0$ is small gives to $\mathbb{P}^2(\mathbb{C}) \setminus \{0\}$ a pseudoconcave structure and that it is not compactificable.

REMARK 6.14. Every holomorphic function on a pseudoconcave manifold is constant.

PROOF. In fact let f be holomorphic non constant on X and let $z_0 \in \overline{Y}$ such that $|f(z_0)| = \sup_{\overline{Y}} |f|$. By the maximum modulus principle, $z_0 \in \partial Y$. If D is a 1-dimensional disc tangent to ∂Y at z_0 and except to z_0 contained in Y then $|f|_D$ has a maximum on an interior point of D. Thus f is constant on D and there is an interior point $z_1 \in \overline{Y}$ such that $|f(z_1)| = |f(z_0)| = \sup_{\overline{Y}} |f|$. This is a contradiction. □

In particular a pseudoconcave manifold (not reduced at a single point) cannot be isomorphic to any local closed submanifold of numerical space $\mathbb{C}^N$ (otherwise there will be a polynomial on $\mathbb{C}^N$ inducing on X a non constant holomorphic function). More generically we can prove the following

THEOREM 6.15. *For any holomorphic line bundle F on a pseudoconcave manifold X we have*

$$\dim_{\mathbb{C}} \Gamma(X, F) < \infty.$$

We will deduce this theorem from the following useful

LEMMA 6.16. *Let F be a holomorphic line bundle over a pseudoconcave manifold X there exist a finite number of points $a_1, \ldots, a_k$ in X and an integer $h = h(F)$ such that if $s \in \Gamma(X, F)$ vanishes at each point a_i of order $\geq h$ then $s \equiv 0$.*

PROOF. Let Y be as in the definition of pseudoconcave manifolds. For every point $x \in \overline{Y}$ we can choose a coordinate polycilinder P_x, coordinates p_i,$i = 1, \ldots, n$, with center x and of radius r_x such that

(1) $F|_{\overline{P}_x}$ is trivial;

(2) $\mathcal{S}(\overline{P}_x) = \{y \in U \mid |p_i(y) - p_i(x)| = r_x,\ 1 \leq i \leq n\} \subset Y$

where U is the coordinate patch on which p_i are coordinates. This is possible in view of the pseudoconcavity of Y[3]. Let P'_x be the concentric polycilinder to P_x with radius $r_x e^{-1}$. We can select a finite number of points $a_1, \ldots, a_k$ such that

(3) $\cup P'_{a_i} \supset \overline{Y}$.

Let F be given by transition functions

$$f_{ij} \colon \overline{P}_{a_i} \cap \overline{P}_{a_j} \to \mathbb{C}^*$$

and set

$$||F|| = \sup_{i,j} \sup_{\overline{P}_{a_i} \cap \overline{P}_{a_j}} |f_{ij}| = e^{\mu}.$$

Note that since $f_{ij} = f_{ji}^{-1}$ we must have $\mu \geq 0$. Now choose h integer with $h > \mu$, for instance $h = \{\mu\} + 1$ where $\{\mu\}$ denotes the integral part of μ. Let $s \in \Gamma(X, F)$ vanishing at the point a_i of order $\geq h$. The section s is given by holomorphic functions $s_i \colon \overline{P}_{a_i} \to \mathbb{C}$. We set

$$M = \sup_i \sup_{\overline{P}_{a_i}} |s_i|.$$

There exist a point $z_0 \in \mathcal{S}(\overline{P}_{a_{i_0}})$, for some a_{i_0}, such that

$$|s_{i_0}(z_0)| = M$$

(indeed $\mathcal{S}(\overline{P}_{a_i}))$ is the Shilov boundary of $\overline{P}_{a_i}$). Since $z_0 \in Y$ there exist a $P'_{a_{j_0}}$ containing z_0. Certainly $j_0 \neq i_0$ and we will have

$$s_{i_0}(z_0) = f_{i_0 j_0}(z_0) s_{j_0}(z_0).$$

Therefore

$$M = |s_{i_0}(z_0)| = |f_{i_0 j_0}(z_0)||s_{j_0}(z_0)| \leq ||F|||s_{j_0}(z_0)|.$$

By Schwarz's Lemma

$$|s_{j_0}(z_0)| \leq M \frac{||z_0||^h}{r^h_{a_{j_0}}}$$

where

$$||z_0|| = \sup |p_i(z_0) - p_i(a_{j_0})| \leq r_{a_{j_0}} e^{-1}.$$

The functions p_i is being the coordinates on $P_{a_{j_0}}$. Hence

$$M \leq ||F|| M e^{-h} = e^{\mu - h} M.$$

But $\mu - h \leq 0$. Thus $M = 0$. Hence also $s \equiv 0$. □

[3]It is clear that we can find P_x with this property if $x \in Y$. If $x \in \partial Y$ for definition of pseudoconcave manifold the Levi form of ∂Y restricted to the analytical tangent plane has at least one negative eigenvalue thus, as we can see in 6.1, we can find a disc D_x of dimension ≥ 1 that is contained in Y. Then we will choose $P_x = D_x$.

THEOREM 6.15. The natural map

$$\Gamma(X,F) \to \coprod_{i=1}^{k} \frac{\mathcal{O}_{a_i}}{\mathfrak{m}_{a_i}^h},$$

which associates to each section $s \in \Gamma(X,F)$ the Taylor expansion of s up to order $h-1$ at each point a_i, is an injective map by the previous Lemma. The right-hand-space is a finite dimensional vector space over $\mathbb{C}$ (dimension $\leq k\binom{n+h}{h}$). □

REMARK 6.17. Let X be pseudoconcave and $Y \subset X$ as in the definition. Then $\Gamma(X,F) \to \Gamma(Y,F)$ is injective. Using Hartogs theorem and the pseudoconcavity of Y we can construct an open neighborhood $\widetilde{Y}$ of $\overline{Y}$ such that the restriction map $r^{\widetilde{Y}}_{Y}\colon \Gamma(\widetilde{Y},F) \to \Gamma(Y,F)$ is an isomorphism. Now $r^{\widetilde{Y}}_{Y}$ is a compact map for the Fréchet topology of $\Gamma(\widetilde{Y},F)$ and $\Gamma(Y,F)$. Thus the Fréchet space $\Gamma(Y,F)$ is locally compact and therefore finite dimensional. Thus would give a more direct proof of Theorem 6.15. However the previous proof has the merit to give an estimate for dimension of $\Gamma(X,F)$ which will be useful in the sequel.

4. Analytic and algebraic dependence of meromorphic functions

DEFINITION 6.18. Let X be a connected complex manifold. Let $f_1,\dots,f_k \in \mathcal{K}(X)$. We say that these meromorphic functions are **analytically dependent** if

$$df_1 \wedge \cdots \wedge df_k \equiv 0$$

wherever this is defined.

In other words $f_1,\dots,f_k$ are analytically dependent if at any point where each of these functions is holomorphic the Jacobian $\frac{\partial(f_1\dots f_k)}{\partial(z_1,\dots,z_n)}$ with respect to a system $z_1,\dots,z_n$ of local holomorphic coordinates, has rank $< k$.

DEFINITION 6.19. The meromorphic functions $f_1,\dots,f_k$ are said **algebraically dependent** if there exists a non identically zero polynomial $p(x_1,\dots,x_k)$ in k variables and complex coefficients such that

$$p(f_1,\dots,f_k) \equiv 0$$

wherever it is defined.

CLAIM 6.20. *Algebraic dependence implies analytic dependence.*

PROOF. In fact if $k > n = \dim_{\mathbb{C}} X$ there is nothing to prove. Assume $k \leq n$. Without loss of generality we may assume that $f_1,\dots,f_{k-1}$ are algebraically independent. Let $p(x_1,\dots,x_k)$ be a polynomial $\not\equiv 0$ of minimal degree in z_k such that

$$p(f_1,\dots,f_k) \equiv 0$$

differentiating this identity we get

$$\sum \frac{\partial p}{\partial x_i}(f)df_i \equiv 0.$$

But $\frac{\partial p}{\partial x_k}(f) \not\equiv 0$, thus we get a non trivial linear relation between the differentials df_i on an open dense subset of X. This implies that $df_1 \wedge \cdots \wedge df_k \equiv 0$ wherever defined on X. □

The converse of this statement (except for $k=1$) is not true in general. For instance the functions $f_s(z) = e^{z^s}$, $s = 1,2,3,\dots$ in $\mathcal{K}(\mathbb{C})$ are all algebraically independent[4]

[4]Let p a generic polynomial in n variables and let $f_1,\dots,f_n$ n functions of the forms $f_i(z) = f_{s_i}(z)$, thus we have:

$$p(f_1,\dots,f_n) = \sum_I a_I e^{z^I}$$

where $I = (i_1,\dots,i_n)$ and $z^I = \sum_{j=1}^n i_j z^j$. We can consider, with respect to the lexicografical order, the multindex:

$$H = \max I \mid a_I \neq 0.$$

If

$$p(f_1,\dots,f_n) = 0$$

while any two of them are analytically dependent. The converse is however true for pseudoconcave manifolds; we have in fact the following

THEOREM 6.21. *Let X be a pseudoconcave manifold. If $f_1, \ldots, f_k, f \in \mathcal{K}(X)$ are analytically dependent then they are also algebraically dependent (i.e. on pseudoconcave manifolds analytic dependence = algebraic dependence).*

PROOF. It is not restrictive to assume that $f_1, \ldots, f_k$ are analytically independent. Otherwise replace $f_1, \ldots, f_k$ by a maximal subset of $\{f_1, \ldots, f_k, f\}$. There exist a holomorphic line bundle F on X and holomorphic sections $s_i \in \Gamma(X, F)$ $0 \leq i \leq k$ with $s_0 \not\equiv 0$ such that

$$f_i = \frac{s_i}{s_0}.$$

Indeed for each f_i there exist a holomorphic line bundle F_i and holomorphic sections $t_0^{(i)}, t_1^{(i)}$ with $t_0^{(i)} \not\equiv 0$ such that $f_i = \frac{t_1^{(i)}}{t_0^{(i)}}$. Taking $F = F_1 \cdots F_k$ then $s_0 = \prod_{i=1}^{k} t_0^{(i)} \in \Gamma(X, F)$ and $s_0 \not\equiv 0$. Moreover $s_i = t_0^{(i)} \cdots t_1^{(i)} \cdots t_0^{(i)} \in \Gamma(X, F)$ and we have $f_i = \frac{s_i}{s_0}$. We can choose a covering of $\overline{Y}$ by coordinate concentric polycilinders $P_{a_i} \supset P'_{a_i}$, $1 \leq i \leq N$, as in the Lemma 6.16 such that

(1) $F|_{\overline{P}_{a_i}}$ is trivial;
(2) $\mathcal{S}(\overline{P}_{a_i}) \subset Y$;
(3) $\cup P'_{a_i} \supset \overline{Y}$;
(4) at each point a_i the functions $f_1, \ldots, f_k$ are holomorphic and $f_1 - f_1(a_i) = \zeta_1^{(i)}, \ldots, f_k - f_k(a_i) = \zeta_k^{(i)}$ can be taken among a set of local holomorphic coordinates[5].

This can be done by small translation in their coordinate patches of the polycilinders $P_{a_i} \supset P'_{a_1}$ as conditions (1), (2), (3) are not affected by these translations and as the set of points where condition iv) cannot be satisfied has an open dense complement. Also there exist a holomorphic line bundle G on X and holomorphic sections $\sigma_0, \sigma_1 \in \Gamma(X, G)$ with $\sigma_0 \not\equiv 0$ such that

$$f = \frac{\sigma_1}{\sigma_0}.$$

We may assume that:

(5) $G|_{\overline{P}_{a_i}}$ is trivial;
(6) f is holomorphic at each point a_i.

As in the Lemma 6.16 we define

$$||F^k|| = e^{k\mu} \qquad \text{(where } F^k = F \cdots F \text{ , } k \text{ times.)}$$
$$||G|| = e^{\omega}.$$

Consider a generic polynomial in $(k+1)$ variables of degree r in each one of the variables $x_1, \ldots, x_k$ and of degree s in x_{k+1}.

$$P(x_1, \ldots, x_{k+1}) = \sum c_{\alpha_1 \ldots \alpha_k \alpha_{k+1}} x_1^{\alpha_1} \cdots x_k^{\alpha_k} x_{k+1}^{\alpha_{k+1}}$$

where $0 \leq \alpha_i \leq r$ for $1 \leq i \leq k$ and $0 \leq \alpha_{k+1} \leq s$. Let

$$\pi(x_0, \ldots, x_k, y_0, y_1) = x_0^{kr} y_0^s P(\frac{x_1}{x_0}, \ldots, \frac{x_k}{x_0}, \frac{y_1}{y_0})$$

---

then

$$\sum_I a_I e^{z_I - z_H} = a_H + \sum_{I<H} a_I e^{z^I - z^H} = 0$$

for all $z \in \mathbb{C}$. Thus we can consider $z \in \mathbb{R}$ and we have:

$$0 = a_H + \sum_{I<H} a_I e^{z^I - z^H} \overset{z \to \infty}{\longrightarrow} a_H$$

this shows that $a_H = 0$.

[5]Indeed the set of the point that satisfy this condition is an open non empty set and the group of automorphism of the polycolinder acts transitively on each P_{a_i}. Indeed $\mathrm{Aut}(P) = \mathrm{Aut}(D)^n$, where P is an n dimensional polycilinder and D is a disc, and the group of automorphism of the disc acts transitively on D (see [**11**]). Thus if the points a_i don't satisfy this condition acting with an automorphism of P_{a_i} we can change the center without changing the polycilinder.

be the corresponding homogeneous polynomial. These polynomials form a vector space $W(r,s)$ over $\mathbb{C}$ of dimension $(r+1)^k(s+1)$. Now note that

$$\pi(s_0, s_1, \dots, s_k, \sigma_0, \sigma_1) \in \Gamma(X, F^{kr}G^s)$$

so we can define in this way a linear map

$$\varepsilon\colon W(r,s) \to \Gamma(X, F^{kr}G^s).$$

The theorem will be proved if we show that $\ker \varepsilon \neq 0$. For this we will estimate $\dim_{\mathbb{C}} Im\varepsilon$. Let h be the smallest integer $> kr\mu + s\omega$. The map which associates to $\pi(s_0, \dots, s_k, \sigma_0, \sigma_1)$ the Taylor expansion up to order $h-1$ of the function $P(f_1, \dots, f_k, f)$ at each point a_i gives a linear map:

$$Im\varepsilon \to \prod_{i=1}^{N} \frac{\mathbb{C}\{\zeta_1^{(i)}, \dots, \zeta_k^{(i)}\}}{\mathfrak{m}_{a_i}^h},$$

where $\mathfrak{m}_{a_i}$ is the maximal ideal of the local ring $\mathbb{C}\{\zeta_1^{(i)}, \dots, \zeta_k^{(i)}\}$ of convergent power series in the variables $\zeta^{(i)}$. By Lemma 6.16 this map is injective. Now the target space has a dimension

$$\delta = N\binom{[kr\mu + s\omega] + 1 + k}{k} =$$

$$= N\left(\frac{[kr\mu + s\omega] + 1 + k}{k}\right)\left(\frac{[kr\mu + s\omega] + 1 + k}{k-1}\right)\cdots\left(\frac{[kr\mu + s\omega] + 1 + k}{1}\right)$$

$$\leq N([kr\mu + s\omega] + 1 + k)^k \leq Nk^k\mu^k r^k + \text{ lower order terms in } r.$$

If we select s such that

$$s + 1 > Nk^k\mu^k,$$

then if r is sufficiently large, we get

$$\dim_{\mathbb{C}} W(r,s) > \dim_{\mathbb{C}} Im\varepsilon$$

and therefore $\ker \varepsilon \neq 0$. □

5. Algebraic fields of meromorphic functions

a)

DEFINITION 6.22. By an **algebraic field of transcendence degree** d we means a finite algebraic extension of the field $\mathbb{C}(t_1, \dots, t_d)$ of all rational functions in d variables.

Since the field $\mathbb{C}$ is of characteristic zero, any extension of this kind is primitive so is of the form $\mathbb{C}(t_1, \dots, t_d, \theta)$ with θ algebraic over $\mathbb{C}(t_1, \dots, t_d)$. Chasing denominators are dividing off any factor in the variables $t_1, \dots, t_d$ only we may assume that P is a polynomial in all the variables and that is irreducible. If V is the algebraic variety defined in $\mathbb{C}^{d+1}$ where $t_1, \dots, t_d, t$ are coordinates by the equation

$$P(t_1, \dots, t_d, t) = 0.$$

Then V is an irreducible variety and the field $\mathbb{C}(t_1, \dots, t_d, \theta)$ is isomorphic to the field of rational functions on V. Moreover $d = \dim_{\mathbb{C}} V$. We want to prove the following:

THEOREM 6.23. *On a pseudoconcave manifold X of complex dimension n, the field $\mathcal{K}(X)$ of all meromorphic functions is an algebraic field of transcendence degree $d \leq n$.*

That the transcendence degree of $\mathcal{K}(X)$ cannot exceed $\dim_{\mathbb{C}} X$ follows already from the fact that on pseudoconcave manifolds algebraic and analytic dependence are the same. The remaining part of the theorem is a consequence of the following

PROPOSITION 6.24. *Let X be a pseudoconcave manifold and let $f_1, \dots, f_k \in \mathcal{K}(X)$ be algebraically independent. There exists an integer $\nu = \nu(f_1, \dots, f_k)$ such that any $f \in \mathcal{K}(X)$ which is algebraically dependent on $f_1, \dots, f_k$ satisfies a non trivial equation over $\mathbb{C}(f_1, \dots, f_k)$ of degree $\leq \nu$.*

PROOF. We follows the proof of the Theorem 6.21. First we find a holomorphic line boundle F and holomorphic sections $s_0 \not\equiv 0, s_1, \dots, s_k$ of F such that $f_i = \frac{s_i}{s_0}$, $1 \leq i \leq k$,. Secondly we find coordinate polycilinders $P_{a_i} \supset P'_{a_1}$, $1 \leq i \leq N$, such that

(1) F restricted to a neighborhood of $\overline{P}_{a_i}$ is trivial;
(2) $\mathcal{S}(\overline{P}_{a_i}) \subset Y$
(3) $\cup P'_{a_1} \supset \overline{Y}$
(4) at each point a_i, $f_1 - f_1(a_i) = \zeta_1^{(i)}, \dots, f_k - f_k(a_i) = \zeta_k^{(i)}$ are holomorphic and can be taken among a set of local holomorphic coordinates.

Thirdly, since the conditions i) ii) iii) iv) remain valid by small translations of P_{a_i} within its coordinate patch, we may determine for each a_i a small closed neighborhood $V(a_i)$ so that no matter how we translate the center a_i of P_{a_i} on a point of $V(a_i)$ the above four conditions remain valid. Let Q_i be the union of the translations of P_{a_i} just considered and let us compute $||F||$ with respect to the covering $\{Q_i\}_{1\leq i\leq N}$. Finally, from the proof of Theorem 6.21 we realize that there exist a holomorphic line bundle G and two holomorphic sections $\sigma_0 \not\equiv 0, \sigma_1$ of G such that $f = \frac{\sigma_1}{\sigma_0}$ and satisfying the following condition

(5) $G|_{Q_i}$ is trivial.

We set
$$||F^k|| = ||F||^k = e^{k\mu}$$
and we choose
$$\nu + 1 > Nk^k\mu^k.$$
Then ν depends only from $f_1, \dots, f_k$ but not from f. We also define with respect to the covering $\{Q_i\}$
$$||G|| = e^{\omega}$$
and we choose the centers a_i of P_{a_i} in $V(a_i)$ so that at a_i also f is holomorphic , $1 \leq i \leq N$. We can now proceed as in the proof of Theorem 6.21 and we realize that if r is sufficiently large f satisfies a non trivial equation over $\mathbb{C}(f_1, \dots, f_k)$ of degree $\leq \nu$. □

THEOREM 6.23. Let $f_1, \dots, f_k$ be a maximal set of algebraically independent meromorphic functions. Let $f \in \mathcal{K}(X)$ be so chosen that is degree α over $\mathbb{C}(f_1, \dots, f_k)$ is maximal. This is possible by virtue of Proposition 6.24. We claim that
$$\mathcal{K}(X) = \mathbb{C}(f_1, \dots, f_k, f).$$
Clearly $\mathbb{C}(f_1, \dots, f_k, f) \subset \mathcal{K}(X)$. Let $h \in \mathcal{K}(X)$, we can find $\Theta \in \mathcal{K}(X)$ such that
$$\mathbb{C}(f_1, \dots, f_k, f, h) = \mathbb{C}(f_1, \dots, f_k, \Theta).$$
Then
$$\alpha \geq [\mathbb{C}(f_1, \dots, f_k, \Theta) : \mathbb{C}(f_1, \dots, f_k)] =$$
$$= [\mathbb{C}(f_1, \dots, f_k, \Theta) : \mathbb{C}(f_1, \dots, f_k, f)] \cdot [\mathbb{C}(f_1, \dots, f_k, f) : \mathbb{C}(f_1, \dots, f_k)].$$
But the second factor of this product equals α therefore the first factor equals 1. This means that $h \in \mathbb{C}(f_1, \dots, f_k, f)$ and thus our contention is proved. □

b) As an application of the previous theorem we can prove the following.

THEOREM 6.25. *Let X be a pseudoconcave manifold and let $\tau\colon X \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic map of rank $n = \dim_{\mathbb{C}} X$ at some point of X. Then $Im\tau$ is contained in an irreducible algebraic variety Y of the same dimension than X.*

PROOF. Let Y be the smallest algebraic subvariety of $\mathbb{P}^N(\mathbb{C})$ containing $\tau(X)$. Certainly Y exists and is irreducible, it is defined by the homogeneous prime ideal
$$\mathcal{P}_Y = \{p \in \mathbb{C}[z_0, \dots z_N] \mid p \circ \tau = 0\}$$
where $\mathbb{C}[z_0, \dots z_N]$denotes the graded ring of homogeneous polynomials on $\mathbb{P}^N(\mathbb{C})$. Let $\mathcal{R}(Y)$ be the field of rational functions on Y. Any element $f \in \mathcal{R}(Y)$ is represent as a quotient of two homogeneous polynomials of the same degree $f = \frac{p}{q}$ with $q \notin \mathcal{P}_Y$. If $f = \frac{p}{q} = \frac{p'}{q'}$ then $pq' - p'q \in \mathcal{P}_Y$. This shows that $f \circ \tau$ is a well defined meromorphic function on X. We have therefore defined a, necessarily injective, homomorphism
$$\tau^*\colon \mathcal{R}(Y) \to \mathcal{K}(X).$$

Now

$$\dim_{\mathbb{C}} Y = \text{trans degree of } \mathcal{R}(Y) \leq \text{trans degree of } \mathcal{K}(X) \leq \dim_{\mathbb{C}} X.$$

But $\dim_{\mathbb{C}} Y \geq \dim_{\mathbb{C}} \tau(X) = \dim_{\mathbb{C}}(X)$ by the assumption about the rank of the map τ. Consequently $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X$. □

In particular *every connected complex compact submanifold of* $\mathbb{P}^N(\mathbb{C})$ *is a projective algebraic variety* (Chow theorem).[6]

c)

THEOREM 6.26. *Let* $X \subset \mathbb{P}^N(\mathbb{C})$ *be an irreducible algebraic submanifold of* $\mathbb{P}^N(\mathbb{C})$*. Let* $\mathcal{R}(X)$ *the field of rational functions on* X *and* $\mathcal{K}(X)$ *the field of meromorphic functions on* X*. We have*

$$\mathcal{R}(X) \cong \mathcal{K}(X).$$

PROOF. (1) Consider on $\mathbb{P}^N(\mathbb{C})$ the bundle of hyperplane sections. This is given as follows. Consider the covering of $\mathbb{P}^N(\mathbb{C})$:

$$U_i = \{z = (z_0, \ldots, z_N) \in \mathbb{P}^N(\mathbb{C}) \mid z_i \neq 0\}.$$

On $U_i \cap U_j$ $g_{ij} = \frac{z_j}{z_i}$ is holomorphic and $\neq 0$. We have on $U_i \cap U_j \cap U_k \neq \emptyset$

$$g_{ij} g_{jk} = g_{ik}$$

therefore it defines a holomorphic line bundle F on $\mathbb{P}^N(\mathbb{C})$. If $s_i \colon U_i \to \mathbb{C}$, $' \leq i \leq n$ represents a holomorphic section we have

$$s_i = \frac{z_j}{z_i} s_j \qquad \text{on } U_i \cap U_j$$

i.e.

$$z_i s_i = z_j s_j = \varphi(z_0, z_1, \ldots, z_N)$$

is a holomorphic homogeneous function of $z_0, \ldots, z_N$ of degree 1 i.e.

$$\varphi(z_0, \ldots, z_N) = \sum_{j=0}^{N} a_j z_j \qquad a_j \in \mathbb{C}.$$

We thus have $\Gamma(\mathbb{P}^N(\mathbb{C}), F) \cong$ space of homogeneous polynomials of degree 1 in $z_0, \ldots, z_N$.

(2) More generally we have $\Gamma(\mathbb{P}^N(\mathbb{C}), F^l) \cong$ space of homogeneous polynomials of degree l in $z_0, \ldots, z_N$ so that

$$\mathcal{A}(\mathbb{P}^N(\mathbb{C}), F) \cong \mathbb{C}[z_0, \ldots, z_N]$$

where $\mathbb{C}[z_0, \ldots, z_N]$ denotes the graded ring of homogeneous polynomials in the variables $[z_0, \ldots, z_N]$. Therefore

(a) $Q(\mathbb{P}^N(\mathbb{C}), F)$ is a subfield of $\mathcal{K}(\mathbb{P}^N(\mathbb{C}))$;
(b) transcendence degree of $Q(\mathbb{P}^N(\mathbb{C}), F) = N =$ transcendence degree of $\mathcal{K}(\mathbb{P}^N(\mathbb{C}))$;
(c) as $Q(\mathbb{P}^N(\mathbb{C}), F)$ is algebraically closed in $\mathcal{K}(\mathbb{P}^N(\mathbb{C}))$ we must have (Theorem 6.21) $Q(\mathbb{P}^N(\mathbb{C}), F) = \mathcal{K}(\mathbb{P}^N(\mathbb{C}))$.

But

$$Q(\mathbb{P}^N(\mathbb{C}), F) = \mathcal{R}(\mathbb{P}^N(\mathbb{C}))$$

hence

$$\mathcal{R}(\mathbb{P}^N(\mathbb{C})) = Q(\mathbb{P}^N(\mathbb{C}), F) = \mathcal{K}(\mathbb{P}^N(\mathbb{C}))$$

and the theorem is proved if $X = \mathbb{P}^N(\mathbb{C})$, (theorem of Hurwitz).

---

[6]The theorem of Chow is a consequence of Theorem 6.25 and connectedness theorem for complex algebraic variety [**16**], i.e. *If* U *is a non-empty Zariski open subset of a complex irreducible quasiprojective variety* Y*, then* U *is connected and dense in the complex topology.* Now assume that $X \subset Y$ where X is a compact complex manifold of dimension n and Y an irreducible projective variety of dimension n. Let $U \subset Y$ be the Zariski open subset of smooth points, then U is a connected complex manifold of dimension n and $X \cap U$ is a closed submanifold of the same dimension. According to connectedness theorem we have $U = U \cap X$ and then $Y = \overline{U} = \overline{U \cap X} \subset X$.

(3) In the general case for a projective submanifold $X \subset \mathbb{P}^N(\mathbb{C})$ we have by means of the same argument, replacing F by $F|_X$ that

$$Q(X, F|_X) = \mathcal{K}(X).$$

Clearly $\mathcal{R}(X) \subset \mathcal{K}(X) = Q(X, F|_X)$. We have to show that

$$\mathcal{R}(X) = Q(X, F|_X).$$

(4) Let $\mathcal{P}(X) \subset \mathbb{C}_0[z_0, \ldots, z_N]$ be the homogeneous ideal defining X in $\mathbb{P}^N(\mathbb{C})$. We may assume without loss of generality that $z_0 \notin \mathcal{P}(X)$. Then $x_i|_X = s_i \in \Gamma(X, F|_X)$ and

$$\mathcal{R}(X) = \mathbb{C}\left(\frac{x_1}{x_0}|_X, \ldots, \frac{x_N}{x_0}|_X\right) = \mathbb{C}\left(\frac{s_1}{s_0}, \ldots, \frac{s_1}{s_0}\right).$$

Let $\mathcal{K}(X) = \mathcal{R}(X)(\Theta)$ with $\Theta = \frac{\sigma_1}{\sigma_0}$, σ_1, σ_0 in $\Gamma(X, F|_X^l)$ (for some l) and $\sigma_0 \not\equiv 0$. This by virtue of the previous remark. Consider the maps

$$\tau_2\colon X \to \mathbb{P}^{N+2}(\mathbb{C})$$
$$\tau_1\colon X \to \mathbb{P}^{N+1}(\mathbb{C})$$

defined by

$$\begin{aligned} x &\xrightarrow{\tau_2} (s_0^l, s_1 s_0^{l-1}, \ldots, s_N s_0^{l-1}, \sigma_0, \sigma_1) = (z_0, \ldots, z_N, z_{N+1}, z_{N+2}) \\ x &\xrightarrow{\tau_1} (s_0^l, s_1 s_0^{l-1}, \ldots, s_N s_0^{l-1}, \sigma_0) = (z_0, \ldots, z_N, z_{N+1}). \end{aligned}$$

Then we realize that both τ_1 and τ_2 are holomorphic everywhere and of maximal rank. Therefore by the previous theorem

$$\tau_1(X) = Y_1 \qquad \tau_2(X) = Y_2$$

are algebraic manifolds of dimension N in $\mathbb{P}^{N+1}(\mathbb{C})$ and $\mathbb{P}^{N+2}(\mathbb{C})$ respectively. If $\mathcal{P}(Y_1)$, $\mathcal{P}(Y_2)$, denote the corresponding homogeneous ideals we have that $z_0 \notin \mathcal{P}(Y_1)$ and $z_0 \notin \mathcal{P}(Y_2)$. Let

$$\mathbb{C}^{l+2} = \{z = (z_0, \ldots, z_{N+2}) \in \mathbb{P}^{N+2}(\mathbb{C}) \mid z_0 \neq 0\}$$
$$\mathbb{C}^{l+1} = \{z = (z_0, \ldots, z_{N+2}) \in \mathbb{P}^{N+1}(\mathbb{C}) \mid z_0 \neq 0\}$$
$$\mathbb{C}^{l} = \{z = (z_0, \ldots, z_{N+2}) \in \mathbb{P}^{N}(\mathbb{C}) \mid z_0 \neq 0\}$$

where $y_i = \frac{z_i}{z_0}$ are taken as holomorphic coordinates. Set

$$Z = X \cap \mathbb{C}^l, \quad Z_1 = Y_1 \cap \mathbb{C}^{l+1}, \quad Z_2 = Y_2 \cap \mathbb{C}^{l+2}$$

and let

$$\pi_2\colon \mathbb{C}^{l+2} \to \mathbb{C}^{l+1}$$
$$\pi_1\colon \mathbb{C}^{l+1} \to \mathbb{C}^{l}$$

be the natural projections. We have one to one surjective holomorphic maps

$$\pi_2\colon Z_2 \to Z_1$$
$$\pi_1\colon Z_1 \to Z.$$

We have

$$\mathcal{R}(X) \cong \mathcal{R}(Z) \subset \mathcal{R}(Z_1) \cong \mathcal{R}(Y_1) \subset \mathcal{R}(Z_2) \cong \mathcal{R}(Y_2) = \mathcal{K}(X).$$

(5) The desired equality $\mathcal{R}(X) \cong \mathcal{K}(X)$ then is a consequence of the following

LEMMA 6.27. *Let $\pi\colon \mathbb{C}^{N+1} \to \mathbb{C}^N$ be a linear surjective map. Let $Z_1 \subset \mathbb{C}^{N+1}$, $Z \subset \mathbb{C}^N$ be irreducible algebraic manifolds of the same dimension. Suppose that $\pi(Z_1) = Z$ and that*

$$\pi|_{Z_1}\colon Z_1 \to Z$$

is surjective and one to one. (It will appear from he proof that it is enough to assume that $\pi(Z_1)$ is an open and dense set in Z and that on an open and dense subset of $\pi(Z_1)$ the fiber of the map $\pi|_{Z_1}$ consists of a single point.) Then $\mathcal{R}(Z_1) \cong \mathcal{R}(Z)$

PROOF. Let $x_1, \dots, x_N$ be the coordinates in $\mathbb{C}^N$ and let $x_1, \dots, x_N, y$ be the coordinates in $\mathbb{C}^{N+1}$ so that

$$\pi(x_1, \dots, x_N, y) = (x_1, \dots, x_N).$$

We have

$$\pi^* \mathcal{R}(Z) \subset \mathcal{R}(Z_1)$$

and $\mathcal{R}(Z)$ and $\mathcal{R}(Z_1)$ have the same transcendence degree. Therefore y is algebraic over $\mathcal{R}(Z)$. Let

$$y^\mu + a_1(x) y^{\mu-1} + \cdots + a_\mu(x) = 0 \qquad a_i(x) \in \mathcal{R}(Z) \tag{6.4}$$

its minimal equation. Let $\mathcal{P}(Z_1) \subset \mathbb{C}[x_1, \dots, x_N, y]$ be the ideal of Z_1 and let $f_{alpha}(x, y)$ be a set of generators for $\mathcal{P}(Z_1)$, $1 \leq \alpha \leq k$. Set $a_i(x) = \frac{b_i(x)}{b_0(x)}$ with b_i polynomials and $b_0(x) \notin \mathcal{P}(Z)$ the ideal of Z. There exist $c_\alpha(x)$ polynomials with $c_\alpha(x) \notin \mathcal{P}(Z)$ such that

$$\begin{cases} c_\alpha(x) f_\alpha(x, y) \equiv A_\alpha(x, y)(b_0(x) y^\mu + b_1(x) y^{\mu-1} + \cdots + b_\mu(x)) \quad mod\, \mathcal{P}(Z) \\ \qquad\qquad 1 \leq \alpha \leq k. \end{cases}$$

This shows that if $x \in Z$, $b_(x) \neq 0$, and $\prod c_\alpha(x) \neq 0$, and

$$\Delta(x) = \text{discr}_y\ (b_0(x) y^\mu + b_1(x) y^{\mu-1} + \cdots + b_\mu(x)) \neq 0$$

above x $\pi^{-1}(x) \cap Z$ consist of μ distinct points. By assumption $\mu = 1$ i.e. $y = \frac{b_1(x)}{b_0(x)} \in \mathcal{R}(Z)$. Hence $\mathcal{R}(Z) = \mathcal{R}(Z_1)$. □

□

CHAPTER 7

Cech cohomology

1. Cech cohomology with values in a sheaf

a) Let X be a topological space and let $\mathcal{F}$ be a sheaf of abelian groups over X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X. We define

$$\mathcal{C}^q(\mathcal{U}, \mathcal{F}) = \prod_{i_0,\dots,i_q} \Gamma(U_{i_0,\dots,i_q}, \mathcal{F}) \qquad \text{for } q \geq 0$$

where $(i_0, \dots, i_q) \in I^{q+1}$ and where, for simplicity of notation, we set

$$U_{i_0,\dots,i_q} = U_{i_0} \cap \dots \cap U_{i_q}.$$

An element $f \in \mathcal{C}^q(\mathcal{U}, \mathcal{F})$ is thus given by a collection $\{f_{i_0,\dots,i_q}\}$ where $f_{i_0,\dots,i_q} \in \Gamma(U_{i_0,\dots,i_q}, \mathcal{F})$. Obviously $\mathcal{C}^q(\mathcal{U}, \mathcal{F})$ is an abelian group. We define

$$\delta_q \colon \mathcal{C}^q(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{q+1}(\mathcal{U}, \mathcal{F})$$

by the formula

$$(\delta_q f)_{i_0,\dots,i_q,i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j r^{U_{i_0,\dots,\widehat{i_j},\dots,i_{q+1}}}_{U_{i_0,\dots,i_{q+1}}} f_{i_0,\dots,\widehat{i_j},\dots,i_{q+1}}$$

where r^U_V for $V \subset U$ denote the restriction map

$$r^U_V \colon \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}).$$

One verifies that

$$\delta_{q+1} \circ \delta_q = 0$$

so that we obtain a sequence of abelian groups and homomorphisms

$$\mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_0} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_1} \mathcal{C}^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_2} \dots$$

which is a complex (of cochain, i.e. with differential operator of degree $+1$). Its cohomology in dimension q will be denoted by

$$H_q(\mathcal{U}, \mathcal{F}) = \frac{\ker \delta_q}{\operatorname{Im} \delta_{q-1}} \qquad \delta_{-1} = 0.$$

Note that since $\mathcal{F}$ is a sheaf we must have

$$H^0(\mathcal{U}, \mathcal{F}) = \ker \delta_0 = \Gamma(X, \mathcal{F}) = \mathcal{F}(X).$$

So that the above complex can be completed with the augmentation map

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \xrightarrow{\varepsilon} \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_0} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_1} \dots$$

(and this sequence is exact on $\Gamma(X, \mathcal{F})$ and on $\mathcal{C}^0(\mathcal{U}, \mathcal{F})$).

Definition 7.1. The groups $H^q(\mathcal{U}, \mathcal{F})$ are called the **Cech cohomology groups** of the covering $\mathcal{U}$ with values in $\mathcal{F}$.

b)

Definition 7.2. A **simplicial complex** is a collection of a set K and a subset $\Phi(K)$ of its finite parts such that, if the **simplex** $(k_{i_0}, \dots, k_{i_q}) \in \Phi(K)$ then also all its faces $(k_{i_0}, \dots, \widehat{k_{i_j}}, \dots, k_{i_q}) \in \Phi(K)$ for all j such that $0 \leq j \leq q$.

Let, in $\mathbb{R}^K$, $e(k)$ denote the unit versor with all components zero except the k-th one which is equal 1. For every $(k_{i_0}, \ldots, k_{i_q}) \in \Phi(K)$ let $\sigma_{i_0,\ldots,i_q}$ be the q-simplex convex envelope of $e(k_{i_o}), \ldots, e(k_{i_q})$. We set

$$|K| = \cup_{q=0}^{\infty} \cup_{(k_{i_0},\ldots,k_{i_q})\in\Phi(K)} \sigma_{i_0,\ldots,i_q}$$

this is a collection of simplices closed and such that if a simplex is in $|K|$ all its faces are also in $|K|$.

Definition 7.3. We call $|K|$ the **geometric realization** of the simplicial complex $(K, \Phi(K))$.

In general a geometric q-simplex in an euclidean space $\mathbb{R}^K$ is the convex envelope of $q+1$ linearly independent points. A polyhedron P is a collection of $0,1,2,\ldots$-simplices in $\mathbb{R}^K$ such that if a simplex is in P then also all its faces are in P. The geometric realization of a simplicial complex is a polyhedron. Given a topological space X and an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X we can construct a simplicial complex $\omega(\mathcal{U})$, called the **nerve of the covering** $\mathcal{U}$ as follows. The 0-simplices or vertices of $\omega(\mathcal{U})$ are the elements $i \in I$ (or $U_i \in \mathcal{U}$). The q-simplices $(i_o, \ldots, i_q) \in I^{q+1}$ (or $U_{i_o\ldots,i_q}$) are those $(q+1)$-tuples $(i_o, \ldots, i_q)$ such that $U_{i_0} \cap \cdots \cap U_{i_q} \neq \emptyset$. Clearly if $(i_o, \ldots, i_q) \in \Phi(\omega(\mathcal{U}))$, i.e. $U_{i_0} \cap \cdots \cap U_{i_q} \neq \emptyset$, then $(i_0, \ldots, \widehat{i_j}, \ldots, i_q) \in \Phi(\omega(\mathcal{U}))$ as $U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_q} \neq \emptyset$ as $\emptyset \neq U_{i_0} \cap \cdots \cap U_{i_q} \subset U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_q}$. If a sheaf $\mathcal{F}$ is given on X then we obtain a law that associates to each q-simplex of $\omega(\mathcal{U})$ $U_{i_0,\ldots,i_q} \in \Phi(\omega(\mathcal{U}))$ an abelian group $\Gamma(U_{i_0,\ldots,i_q}, \mathcal{F})$:

$$U_{i_0,\ldots,i_q} \mapsto \Gamma(U_{i_0,\ldots,i_q}, \mathcal{F}).$$

If $U_{i_0,\ldots,\widehat{i_j},\ldots,i_q}$ is a face of $U_{i_o,\ldots,i_q}$; $U_{i_0,\ldots,\widehat{i_j},\ldots,i_q} = \delta_j(U_{i_0,\ldots,i_q})$ we obtain a natural map

$${}^{+}\delta_j(\mathcal{F})\colon \Gamma(U_{i_0,\ldots,\widehat{i_j},\ldots,i_q}, \mathcal{F}) \to \Gamma(U_{i_o,\ldots,i_q}, \mathcal{F})$$

and these maps are compatible with repeated face operations. We obtain therefore a system on $\omega(\mathcal{U})$ of **local coefficients** and we can construct the cochains groups of $\omega(\mathcal{U})$ based on these local coefficients

$$\mathcal{C}^q(\omega(\mathcal{U}), \mathcal{F}) = \{\sum_{U_{i_0,\ldots,i_q}\neq\emptyset} f_{i_0,\ldots,i_q} U_{i_0,\ldots,i_q}\}, \qquad f_{i_0,\ldots,i_q} \in \Gamma(U_{i_0,\ldots,i_q}, \mathcal{F}),$$

and the cohomology operator becomes the operator δ_q defined before. It is clear from this construction that the formula of δ_q has been inverted and why $\delta_{q+1} \circ \delta_q = 0$. In conclusion the cochain group

$$\mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_0} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_1} \mathcal{C}^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_2} \ldots$$

is nothing else but the cochain group of the nerve $\omega(\mathcal{U})$ of $\mathcal{U}$ with the local coefficient system provided by the assignments

$$U_{i_0,\ldots,i_q} \mapsto \Gamma(U_{i_0,\ldots,i_q}, \mathcal{F}).$$

c) Let now $\mathcal{V} = \{V_j\}_{j\in J}$ be another covering of the space X which is a refinement of the

covering $\mathcal{U}$: $(\mathcal{V} \prec \mathcal{U})$. By this we mean that we can find a refinement function $\tau\colon J \to I$ such that $V_j \subset V_{\tau(j)}$ for all $j \in J$. Clearly $\mathcal{V}$ can be a refinement of $\mathcal{U}$ in many different ways. We will write

$$\mathcal{V} \prec_\tau \mathcal{U}$$

if we need to make explicit the refinement function. If $\omega(\mathcal{U})$ and $\omega(\mathcal{V})$ are the nerves of $\mathcal{U}$ and $\mathcal{V}$ respectively and if $\mathcal{V} \prec_\tau \mathcal{U}$ then we have a natural map

$$\tau_\omega\colon \omega(\mathcal{V}) \to \omega(\mathcal{U});$$

this map is simplicial, i.e. if $V_{j_0,\ldots,j_q} \neq \emptyset$ then $V_{\tau(j_0),\ldots,\tau(j_q)} \neq \emptyset$ (obvious as $V_{j_0,\ldots,j_q} \subset V_{\tau(j_0),\ldots,\tau(j_q)}$). Moreover if $\mathcal{V} \prec_\tau \mathcal{U}$ and $\mathcal{V} \prec_\sigma \mathcal{U}$ then the maps τ_ω and σ_ω are $\omega(\mathcal{U})$-near in the sense that $\tau_\omega(j)$ and $\sigma_\omega(j)$ both lay in the same 1-simplex $U_{\tau(j)} \cap U_{\sigma(j)}$ of $\omega(\mathcal{U})$.[1] Therefore τ_ω and σ_ω are

[1] A map $\varphi\colon \omega(\mathcal{V}) \to \omega(\mathcal{U})$ is **simplicial** if

(1) for each vertex v of $\omega(\mathcal{V})$, $\varphi(v)$ is a vertex of $\omega(\mathcal{U})$
(2) for each $q+1$-simplex $(i_0, \ldots, i_q)$ of $\omega(\mathcal{V})$, the vertices $\varphi(i_0), \ldots, \varphi(i_q)$ all lie in some closed simplex of $\omega(\mathcal{U})$
(3) for each $(s) = (i_0, \ldots, i_q) \in \omega(sV)$ and $p = \sum_{j0}^{q} a_j i_j \in (s)$ $\varphi(p) = \sum_{j=0}^{q} a_j \varphi(i_j)$.

homotopic simplicial maps of $\omega(\mathcal{V})$ into $\omega(\mathcal{U})$. Given $\mathcal{V} \prec_\tau \mathcal{U}$ then a complex homomorphism

$$\tau^*\colon \mathcal{C}^*(\mathcal{U},\mathcal{F}) \to \mathcal{C}^*(\mathcal{V},\mathcal{F})$$

is defined by

$$(\tau^* f)_{j_0,\dots,j_q} = r^{U_{\tau(j_0),\dots,\tau(j_q)}}_{V_{j_0,\dots,j_q}} f$$

and therefore we get a homomorphism of the corresponding cohomology groups

$$\tau^{\mathcal{U}}_{\mathcal{V}}\colon H^*(\mathcal{U},\mathcal{F}) \to H^*(\mathcal{V},\mathcal{F}).$$

Since two refinement functions $\mathcal{V} \prec_\tau \mathcal{U}$, $\mathcal{V} \prec_\sigma \mathcal{U}$ give homotopic maps of $\omega(\mathcal{V})$ into $\omega(\mathcal{U})$ then the corresponding associated maps in cohomology coincide i.e.

$$\tau^{\mathcal{U}}_{\mathcal{V}} = \sigma^{\mathcal{U}}_{\mathcal{V}}$$

i.e. the map on the level of cohomology depends only on the coverings $\mathcal{U}$ and $\mathcal{V}$ and not on the refinement function.[2] We then define

$$H^q(X,\mathcal{F}) = \lim_{\to\mathcal{U}} H^q(\mathcal{U},\mathcal{F}) \qquad q > 0$$

as the Chech cohomology group of X with values in $\mathcal{F}$.

2. Homomorphism of sheaves

a) Given on X two sheaves of abelian groups represented by their corresponding stacks, $(\mathcal{F},\pi,X)$ and $(\mathcal{G},\omega,X)$, a homomorphism of $\mathcal{F}$ into $\mathcal{G}$ is a continuous map $\varphi\colon \mathcal{F} \to \mathcal{G}$ such that

(1) $\omega \circ \varphi = \pi$
(2) for every $x \in X$ $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is a group homomorphism.

We say that another simplicial map $\psi\colon \omega(\mathcal{V}) \to \omega(sU)$ is a **simplicial approximation** to φ if for each vertex $v \in \omega(\mathcal{V})$ $\varphi(St(v)) \subset St(\psi(\mathcal{V}))$ where

$$St(v) = \bigcup_{\substack{(s) \in \omega(\mathcal{V}) \\ v \in (s)}} (s).$$

THEOREM 7.4. *Suppose $\psi\colon \omega(\mathcal{V}) \to \omega(\mathcal{U})$ is a simplicial approximation of φ. Then, for any $p \in \omega(\mathcal{V})$ $\varphi(p)$ and $\psi(p)$ lie in a common closed simplex of $\omega(sU)$.*

PROOF. [**17**] theorem 2 page 92. □

THEOREM 7.5. *Let ψ a simplicial approximation to φ then there exists a homotopy between ψ and Φ.*

PROOF. [**17**] theorem 3 page 92. □

Thus we say that τ_ω and σ_ω are two simplicial maps and σ_ω is a simplicial approximation to τ_ω.

[2]To prove this fact it is sufficies to show that for $f \in Z^q(\mathcal{U},\mathcal{F})$

$$\{r^{U_{\tau(j_0),\dots,\tau(j_q)}}_{V_{j_0,\dots,j_q}} f\} - \{r^{U_{\sigma(j_0),\dots,\sigma(j_q)}}_{V_{j_0,\dots,j_q}} f\} \in \delta C^{q-1}(\mathcal{V}).$$

Put

$$k_{\nu_1,\dots,\nu_q} = \sum_{t=1}^{q} (-1)^{t-1} r_W f_{\tau(\nu_1),\dots,\tau(\nu_t),\sigma(\nu_t),\dots,\sigma(\nu_q)} \quad W = \bigcap_{t=1}^{q} V_{\nu_t}.$$

By an easy calculation we see that

$$r^{U_{\tau(j_0),\dots,\tau(j_q)}}_{V_{j_0,\dots,j_q}} f - r^{U_{\sigma(j_0),\dots,\sigma(j_q)}}_{V_{j_0,\dots,j_q}} f = (dk)_{j_0,\dots,j_q}.$$

Note that $k_{\nu_1,\dots,\nu_q}$ is note necessarily skew-simmetric in the indices so we put

$$\tilde{k}_{\nu_1,\dots,\nu_q} = \frac{1}{q!} \sum sign \begin{pmatrix} \nu_1 & \dots & \nu_q \\ \mu_1 & \dots & \mu_q \end{pmatrix} k_{\mu_1,\dots,\mu_q}$$

where the summation is taken over all permutation of $\nu_1,\dots,\nu_q$. Since $\{(d\tilde{k})_{\nu_1,\dots,\nu_q}\}$ is skew-symmetric then $\{(d\tilde{k})_{\nu_1,\dots,\nu_q}\} \in \delta C^{q-1}(\mathcal{V})$ and as before, by an easy calculation, we see that

$$\{r^{U_{\tau(j_0),\dots,\tau(j_q)}}_{V_{j_0,\dots,j_q}} f\} - \{r^{U_{\sigma(j_0),\dots,\sigma(j_q)}}_{V_{j_0,\dots,j_q}} f\} = \{(d\tilde{k})_{j_1,\dots,j_q}\} \in dC^{q-1}(\mathcal{V}).$$

In particular if φ is injective then $\mathcal{F}$ can be considered as a subsheaf of $\mathcal{G}$ and we can then define the quotient sheaf $\frac{\mathcal{G}}{\mathcal{F}}$ as the quotient of the space $\mathcal{G}$ by the equivalence relation

$$g_x \sim g'_x \text{ if and only if } g_x - g'_x \in \varphi(\mathcal{F}_x).$$

One has $(\frac{\mathcal{G}}{\mathcal{F}})_x = \frac{\mathcal{G}_x}{\mathcal{F}_x}$.

b)

DEFINITION 7.6. A sequence of sheaves and homomorphisms

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0 \tag{7.1}$$

is called **exact** if and only if for every $x \in X$

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$

is an exact sequence.

In particular if $\mathcal{F}$ is a subsheaf of $\mathcal{G}$ we have the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \frac{\mathcal{G}}{\mathcal{F}} \longrightarrow 0.$$

Given a sheaf homomorphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ this induces a homomorphism

$$\varphi^*_{\mathcal{U}}\colon H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{U}, \mathcal{G}) \qquad \text{for all } q \geq 0$$

and thus at the limit a homomorphism

$$\varphi^*\colon H^q(X, \mathcal{F}) \to H^q(X, \mathcal{G}) \qquad \text{for all } q \geq 0.$$

c) One has the following important theorem of Serre [**14**]:

THEOREM 7.7 (Serre). *Given an exact sequence* (7.1) *of sheaves on a paracompact space X, then one has an exact cohomology sequence*

$$0 \longrightarrow H^0(X, \mathcal{F}) \xrightarrow{\alpha^*} H^0(X, \mathcal{G}) \xrightarrow{\beta^*} H^0(X, \mathcal{H}) \xrightarrow{\delta} H^1(X, \mathcal{F}) \xrightarrow{\alpha^*} H^1(X, \mathcal{G}) \xrightarrow{\beta^*} H^1(X, \mathcal{H}) \xrightarrow{\delta} \dots$$

3. Theorem of Leray

a) Cech cohomology is of no practical use unless one can consider on the topological space X a system $\{\mathcal{U}^\alpha\}_{\alpha \in A}$ of coverings which has the following properties (with respect to a given sheaf $\mathcal{F}$ of abelian groups on X):

(1) $\{\mathcal{U}^\alpha\}_{\alpha \in A}$ is cofinal to all coverings of X (i.e. given a covering $\mathcal{U}$ of X we can find an $\mathcal{U}^\alpha \prec \mathcal{U}$)
(2) for any $\alpha \in A$, any $U_{i_0,\dots,i_q} \neq \emptyset$ with $U_i \in \mathcal{U}^\alpha$ we have $H^r(U_{i_0,\dots,i_q}, \mathcal{F}) = 0$ for all $r > 0$.

One has in fact the following fundamental theorem

THEOREM 7.8 (Leray). [3]
Let $\mathcal{F}$ be a sheaf of abelian groups on X and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X such that

$$H^r(U_{i_0,\dots,i_p}, \mathcal{F}) = 0 \quad \forall r > 0 \; \forall i_0, \dots, i_p$$

then the natural map

$$H^q(\mathcal{U}, \mathcal{F}) \to H^q(X, \mathcal{F})$$

is an isomorphism for all $q > 0$.

REMARK 7.9. If $q = 0$ we have already seen that, without any assumption on the covering $\mathcal{U}$, we have for any sheaf $\mathcal{F}$

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$$

[3] For the proof of this theorem see [**6**] Corollary at page 213.

REMARK 7.10. Let $q = 1$. We claim that without any assumption on the refinement of covering $\mathcal{V} \prec_\tau \mathcal{U}$ we have, for any sheaf $\mathcal{F}$ of abelian groups that

$$\tau^* \colon H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{V}, \mathcal{F})$$

is injective (and thus at the limit $H^1(\mathcal{U}, \mathcal{F}) \to H^1(X, \mathcal{F})$ is injective). Indeed let $\mathcal{V} = \{V_j\}_{j \in J}$, $\mathcal{U} = \{U_a\}_{a \in A}$, let $f \in \mathcal{Z}(\mathcal{U}, \mathcal{F}) = \ker\{\mathcal{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \mathcal{C}^2(\mathcal{U}, \mathcal{F})\}$ i.e. $f = \{f_{ab}\}$, $f_{ab} \in \Gamma(U_a \cap U_b, \mathcal{F})$, with

$$f_{ab} + f_{bc} + f_{ca} = 0 \qquad \text{on } U_a \cap U_b \cap U_c.$$

Suppose $\tau^*(f)$ is a boundary

$$\tau^*(f) \in \operatorname{Im}\{\mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \mathcal{C}^1(\mathcal{U}, \mathcal{F})\}$$

i.e.

$$f_{\tau(j)\tau(k)} = g_k - g_j \quad \text{ on } V_j \cap V_k,\ g_j \in \Gamma(V_j, \mathcal{F})$$

thus for any $a \in A$

$$f_{a\tau(k)} - f_{a\tau(j)} = g_k - g_j \quad \text{ on } U_a \cap V_j \cap V_k$$

i.e.

$$\varphi_a = g_j - f_{a\tau(j)} = g_k - f_{a\tau(k)}$$

is well defined on $\Gamma(U_a, \mathcal{F})$. We claim that

$$f_{ab} = \varphi_a - \varphi_b \qquad \text{on } U_a \cap U_b \ \forall a, b \in A.$$

Indeed

$$f_{ab}|_{U_a \cap U_b \cap V_j} = f_{a\tau(j)} - f_{b\tau(j)} = (f_{a\tau(j)} - g_j) - (f_{b\tau(j)} - g_j) = \varphi_a - \varphi_b \qquad \forall j.$$

REMARK 7.11. Let $q = 1$ and $\mathcal{V} \prec \mathcal{U}$. If we assume that

$$H^1(\mathcal{V}_{|U_a}, \mathcal{F}) = 0$$

then

$$H^1(\mathcal{U}, \mathcal{F}) \to H^1(\mathcal{V}, \mathcal{F})$$

is surjective. Thus at the limit, if $H^1(U_a, \mathcal{F}) = 0$ then $H^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\sim} H^1(X, \mathcal{F})$. Indeed let $g_{ij} \in Z^1(\mathcal{U}, \mathcal{F})$

$$g_{ij} + g_{jk} + g_{ki} = 0 \qquad \text{on } V_i \cap V_j \cap V_k$$

n $U_a \cap \mathcal{V}$ we have

$$g_{ij}|_{U_a \cap V_i \cap V_j} = h_{aj} - h_{ai}$$
$$g_{ij}|_{U_b \cap V_i \cap V_j} = h_{bj} - h_{bi}$$

hence

$$h_{ab} = h_{aj} - h_{bj} = h_{ai} - h_{bi} \in \Gamma(U_a \cap U_b, \mathcal{F}).$$

Claim

$$h_{ab} + h_{bc} + h_{ca} = 0 \qquad \text{on } U_a \cap U_b \cap U_c$$
$$h_{ab} + h_{bc} + h_{ca}|_{V_j} = h_{aj} - h_{bj} + h_{bj} - h_{cj} + h_{cj} - h_{aj} = 0$$

this for all j thus the conclusion. To show that $\{h_{\tau(i)\tau(j)} - g_{ij}\}$ is a coboundary. Now

$$h_{\tau(i)\tau(j)} - g_{ij} = h_{\tau(i)j} - h_{\tau(j)j} - (h_{\tau(i)j} - h_{\tau(i)i}) = h_{\tau(i)i} - h_{\tau(j)j}$$

but $h_{\tau(i)i} \in \Gamma(V_i, \mathcal{F})$ $\forall i$ thus the second claim also proved.

REMARK 7.12. In general Leray theorem[4] can be more precisely stated as follows: if

$$H^1(U_{i_o}, \mathcal{F}) = H^1(U_{i_o,i_1}, \mathcal{F}) = \cdots = H^1(U_{i_o \ldots, i_{q-1}}, \mathcal{F}) = 0$$
$$H^2(U_{i_o}, \mathcal{F}) = H^2(U_{i_o,i_1}, \mathcal{F}) = \cdots = H^2(U_{i_o \ldots, i_{q-2}}, \mathcal{F}) = 0$$
$$\cdots\cdots\cdots$$
$$H^{q-1}(U_{i_0}, \mathcal{F}) = 0$$

for $q - 1 > 0$ and for all $i_0, \ldots, i_q - 1$, then $H^q(\mathcal{U}, \mathcal{F}) \to H^q(X, \mathcal{F})$ is injective. If

$$H^1(U_{i_0}, \mathcal{F}) = H^1(U_{i_0,i_1}, \mathcal{F}) = \cdots = H^1(U_{i_0 \ldots, i_q}, \mathcal{F}) = 0$$

[4]See [**6**] Theorem 5.4.1 page 212

$$H^2(U_{i_0},\mathcal{F}) = H^2(U_{i_0,i_1},\mathcal{F}) = \cdots = H^2(U_{i_0,\dots,i_{q-1}},\mathcal{F}) = 0$$
$$\cdots\cdots\cdots$$
$$H^q(U_{i_0},\mathcal{F}) = 0$$

for all $i_0,\dots,i_q$ then

$$H^q(\mathcal{U},\mathcal{F}) \xrightarrow{\sim} H^q(X,\mathcal{F}).$$

b)

LEMMA 7.13. *Let X be a paracompact contractible topological space (i.e. such that there exist a continuous map $F\colon X\times[0,1]\to X$ with $F(x,0)=x$ and $F(x,1)=x_0\in X$, where x_0 is a fixed point of X). Then*

$$H^j(X,\mathcal{G}) = 0 \qquad \forall j>0$$

for any constant sheaf of abelian groups $\mathcal{G}$ on X.

REMARK 7.14. In the applications we have in view it is enough to have Lemma 7.13 when X is an open convex subset of some $\mathbb{R}^n$.

Let K be a polyhedron union of open simplices $\{\sigma_\alpha\}_{\alpha\in A}$. For every vertex k of K we denote by $U(k)$ the star of k that is the union of the simplices σ_α which have a vertex in k. Then $\mathcal{U}=\{U(k)\}_{k \text{ is a vertex of } K}$ is an open covering of K which (by virtue of the Lemma 7.13) is a Leray covering. Therefore for any constant sheaf $\mathcal{G}$ (for instance $\mathcal{G}=\mathbb{Z}$) we have

$$H^i(\mathcal{U},\mathcal{G}) \cong H^i(K,\mathcal{G}) \quad \forall i\geq 0.$$

But $|\omega(\mathcal{U})|\cong K$ as one easily verify. Thus $H^i(\omega(\mathcal{U}),\mathcal{G})$ is the simplicial cohomology of K with coefficients in $\mathcal{G}$ and we have

$$H^i(\omega(\mathcal{U}),\mathcal{G}) = H^i(\mathcal{U},\mathcal{G}) = H^i(X,\mathcal{G}).$$

THEOREM 7.15. *The simplicial cohomology of a polyhedron K with values in an abelian group G is isomorphic to the Cech cohomology of K with values in the sheaf $\mathcal{G}$ of locally constant section of G . Hence it is independent of the triangulation of K.*

In particular one can take $K=\partial\sigma_{n+1}$ where

$$\sigma_{n+1} = \{t_1\dots t_{n+1}\in\mathbb{R}^{n+1} \mid \sum t_i = 1, t\geq 0\}.$$

Then K is homeomorphic to S^n the n-sphere and we have

$$H^j(S^n,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad j=0,n \\ 0 & \text{if} \quad j\neq 0,n \end{cases}$$

It follows that we cannot produce a homomorphism

$$\tau\colon\mathbb{R}^m\to\mathbb{R}^n \qquad \text{if } m\neq n$$

as this will produce a homomorphism of their Alexandroff compactification

$$\tilde{\tau}\colon \mathbb{R}^m\cup\{\infty\} = S^m \to \mathbb{R}^n\cup\{\infty\} = S^n$$

(theorem of invariance of dimension).

c) Let X be a complex space and let (E,π,X) be a holomorphic vector bundle over X. Let $\mathcal{O}(E)$ be the sheaf of germs of holomorphic sections of E. If the fiber of E is $\mathbb{C}^r$ locally $\mathcal{O}(E)\cong\mathcal{O}^r$. Let $U\subset X$ be an open set of holomorphy in X then

THEOREM 7.16 (H. Cartan, J.P. Serre). *For all $j>0$ we have*

$$H^j(U,\mathcal{O}(E)) = 0.$$

Now the intersection of two open set of holomorphy is also an open set of holomorphy therefore if $\mathcal{N}=\{$ open set of holomorphy of $X\}$ we have that any covering $\mathcal{U}=\{U_i\}_{i\in I}$ with $\mathcal{U}\subset\mathcal{N}$ is a Leray covering. Hence

$$H^i(\mathcal{U},\mathcal{O}(E)) \cong H^i(X,\mathcal{O}(E)) \qquad \forall i\geq 0.$$

Note that coordinate balls and coordinate polycilinders are in $\mathcal{N}$ so that the set of $\mathcal{U}\subset\mathcal{N}$ is cofinal to the set of all covering of X.

4. Acyclic resolutions

Let $\mathcal{F}$ be a sheaf of abelian groups on a paracompact topological space X.

DEFINITION 7.17. By a **resolution** of $\mathcal{F}$ we mean an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_2 \xrightarrow{d_2} \dots \tag{7.2}$$

The resolution is called **acyclic** if

$$H^q(X, \mathcal{F}_j) = 0 \qquad \forall q > 0 \, \forall j \geq 0.$$

THEOREM 7.18 (De Rham). *If* (7.2) *is an acyclic resolution of $\mathcal{F}$ then the Cech cohomology of X with values in $\mathcal{F}$ is naturally isomorphic to the cohomology of the complex*

$$\Gamma(X, \mathcal{F}_0) \xrightarrow{d_0^*} \Gamma(X, \mathcal{F}_1) \xrightarrow{d_1^*} \Gamma(X, \mathcal{F}_2) \xrightarrow{d_2^*} \dots$$

i.e.

$$H^q(X, \mathcal{F}) \cong \frac{\ker\{\Gamma(X, \mathcal{F}_q) \xrightarrow{d_q^*} \Gamma(X, \mathcal{F}_q + 1)\}}{\mathrm{Im}\{\Gamma(X, \mathcal{F}_{q-1}) \xrightarrow{d_{q-1}^*} \Gamma(X, \mathcal{F}_q)\}}.$$

PROOF. [5] □

EXAMPLE 7.19. A sheaf $\mathcal{G}$ of abelian groups is called **flabby**[6] if for every open set $U \subset X$ we have that the restriction map

$$r_U^X : \Gamma(X, \mathcal{G}) \to \Gamma(U, \mathcal{G})$$

is surjective. For a flabby sheaf $\mathcal{G}$ one has that

$$H^q(X, \mathcal{G}) = 0 \qquad \forall q > 0.$$

Any sheaf of abelian groups $\mathcal{F}$ can be considered as a subsheaf of a flabby sheaf $\mathcal{C}^0$ that is the sheaf of arbitrary (not necessarily continuous) sections of $\mathcal{F}$. Making use of this fact one realizes that any sheaf $\mathcal{F}$ admits a flabby (thus acyclic) resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \mathcal{C}^2 \longrightarrow \dots.$$

---

[5] We divide the proof of this theorem in two steps.

a) For the exactness of (7.2) we have that

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d_0} d_0 \longrightarrow 0$$

is exact. Hence we get the exact sequence of cohomology groups

$$0 \xrightarrow{H} {}^0(\mathcal{F}) \longrightarrow H^0(\mathcal{F}_0) \xrightarrow{d_0^*} H^0(d\mathcal{F}_0) \longrightarrow H^1(\mathcal{F}) \longrightarrow H^1(\mathcal{F}_0) \xrightarrow{d_0^*} \dots \xrightarrow{H} {}^{q-1}(\mathcal{F}_0) \xrightarrow{d_0^*} H^{q-1}(d\mathcal{F}_0) \longrightarrow H^q(\mathcal{F}) \longrightarrow H^q(\mathcal{F}_0) \xrightarrow{d_0^*} \dots.$$

Since (7.2) is acyclic $H^q(sF_0) = 0$ for $q \geq 1$. therefore we obtain the exact sequences

$$0 \longrightarrow H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{F}_0) \xrightarrow{d_0^*} H^0(d_0\mathcal{F}_0) \longrightarrow H^1(\mathcal{F}) \longrightarrow 0$$

$$0 \longrightarrow H^{q-1}(d_0\mathcal{F}_0) \longrightarrow H^q(\mathcal{F}) \longrightarrow 0 \quad q = 2, 3, 4, \dots.$$

Hence

$$H^1(\mathcal{F}) \cong \frac{H^0(d_0\mathcal{F}_0)}{d_0^* H^0(\mathcal{F}_0)} \quad H^q(\mathcal{F}) \cong H^{q-1}(d\mathcal{F}_0) \quad q \geq 2.$$

b) From the exact sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d_0} d_0 \longrightarrow 0$$

and

$$0 \longrightarrow d_{q-1}\mathcal{F}^{q-1} \longrightarrow \mathcal{F}_q \xrightarrow{d_q} d_q\mathcal{F}_q \longrightarrow 0$$

since (7.2) is acyclic and for the point a) we have:

$$H^q(\mathcal{F}) \cong H^{q-1}(d_0\mathcal{F}_0) \cong H^{q-2}(d_1\mathcal{F}_1) \cong \dots \cong H^1(d_{q-2}\mathcal{F}^{q-2}) \cong \frac{H^0(d_{q-1}\mathcal{F}^{q-1})}{d_{q-1}^* H^0(\mathcal{F}^{q-1})} = \frac{\ker\{\Gamma(X, \mathcal{F}_q) \xrightarrow{d_q^*} \Gamma(X, \mathcal{F}_q + 1)\}}{\mathrm{Im}\{\Gamma(X, \mathcal{F}_{q-1}) \xrightarrow{d_{q-1}^*} \Gamma(X, \mathcal{F}_q)\}}$$

[6] Also called **flasque** in the literature.

EXAMPLE 7.20. A sheaf $\mathcal{G}$ is called **soft** if for any closed subset $A \subset X$ the restriction map

$$r_A^X : \Gamma(X, \mathcal{G}) \to \Gamma(A, \mathcal{G})$$

is surjective. For a soft sheaf one has again

$$H^q(X, \mathcal{G}) = 0 \qquad \forall q > 0.$$

Therefore any soft resolution of a sheaf $\mathcal{F}$ is an acyclic resolution.

EXAMPLE 7.21. Let X be a differentiable manifold of $\dim_{\mathbb{R}} X = n$ and let $\mathcal{A}^r$ denote the sheaf of germs of C^∞ differential forms of degree r. Let

$$d\colon \mathcal{A}^r \to \mathcal{A}^{r+1}$$

be the sheaf homomorphism induced by exterior differentiation. The kernel of the homomorphism

$$d\colon \mathcal{A}^0 \to \mathcal{A}^1$$

is the sheaf of germs of constant functions $\mathbb{R}$. We thus have a sequence of sheaf homomorphisms

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^n \xrightarrow{d} 0. \tag{7.3}$$

As $d \circ d = 0$ this is a complex of shaves. Moreover the sequence is an exact sequence of sheaves (Poincaré Lemma). As the sheaves $\mathcal{A}^i$ are soft[7] the (7.3) is a soft resolution of the constant sheaf $\mathbb{R}$. De Rham Theorem then states that

$$H^q(X, \mathbb{R}) \cong \frac{\ker\{\Gamma(X, \mathcal{A}^q(X)) \xrightarrow{d_q^*} \Gamma(X, \mathcal{A}^{q+1}(X))\}}{\operatorname{Im}\{\Gamma(X, \mathcal{A}^{q-1}(X)) \xrightarrow{d_{q-1}^*} \Gamma(X, \mathcal{A}^q(X))\}} = \frac{\text{closed } q\text{-forms}}{\text{exact } q\text{-forms}} \; \forall q \geq 0.$$

In particular $H^q(X, \mathbb{R}) = 0$ if $q > n$.

EXAMPLE 7.22. Let X be a complex manifold of complex dimension n. Let $\mathcal{A}^{r,s}$ be the sheaf of germs of C^∞ complex valued forms of type r, s i.e. forms φ that in a system of local coordinates z have an expression of type

$$\varphi = \sum_{\substack{\alpha_0 < \cdots < \alpha_r \\ \beta_0 < \cdots < \beta_s}} a_{\alpha_0, \ldots, \alpha_r, \beta_0, \ldots, \beta_s}(z) dz_{\alpha_0} \wedge \cdots \wedge dz_{\alpha_r} \wedge d\overline{z}_{\beta_0} \wedge \cdots \wedge d\overline{z}_{\beta_s}.$$

Let $\overline{\partial}$ be the operator of exterior differentiation with respect to antiholomorphic coordinates. This gives sheaf homomorphism of type:

$$\overline{\partial}\colon \mathcal{A}^{r,s} \to \mathcal{A}^{r,s+1}.$$

We obtain in this way a sequence of sheaves and homomorphisms

$$0 \longrightarrow \Omega^r \longrightarrow \mathcal{A}^{r,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{r,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \mathcal{A}^{r,n} \longrightarrow 0$$

where $\Omega^r = \ker\{\mathcal{A}^{r,0} \to \mathcal{A}^{r,1}\}$ is the sheaf of germs of holomorphic r-forms. As $\overline{\partial} \circ \overline{\partial} = 0$ and as we have the analog of Poincaré lemma (Dolbeault lemma) we obtain a soft resolution [8] of the

[7] This is a consequence of the partition of unity.

[8] The sheaves $\mathcal{A}^{r,s}$ are soft sheaves because the sheaf $\mathcal{A}$ is fine i.e. for any locally finite open covering $\mathcal{U} = \{U_i\}$ there is a family of homomorphism $\{h_i : \mathcal{A} \to \mathcal{A}\}$ such that:

(1) *supp* $h_i \subset U_i$
(2) $\sum_i h_i = id.$

Indeed let $\mathcal{U} = \{U_i\}$ be a locally finite open covering of X and ρ_i a partition of unity subordinate to $\mathcal{U}$. For $s = f_p \in \mathcal{A}_p$ define

$$h_i s = (\rho_i f)_p$$

and this definition is indipendent from the choice of f. Let $\mathcal{U} = \{h_i s; g, U\} = \{g_q \mid q \in U\} \subset \mathcal{A}$ be a neighborhood of $h_i s$, since $g_p = h_i s = (\rho_i f)_p$, there is a neighborhood $V \subset U$ of p such that $g|_V = (\rho_i f)|_V$. Hence for $t \in \mathcal{U}(s; f, V)$, $q = \omega(t) \in V$,

$$h_i s = h_i f_q = (\rho_i f)_q = g_q \in \mathcal{U}(h_i s; g, U).$$

Thus h_i is continuous, hence a homomorphism. Clearly (1) (2) are satisfied. Note that module of fine sheaf is a fine sheaf, hence $\mathcal{A}^{r,s}$ are fine.

sheaf Ω^r. In particular for $r=0$ we have a soft resolution of $\mathcal{O}$ and , by De Rham Theorem

$$H^q(X,\mathcal{O}) \cong \frac{\ker\{\Gamma(X,\mathcal{A}^{0,q}(X)) \xrightarrow{d_q^*} \Gamma(X,\mathcal{A}^{0,q+1}(X))\}}{\operatorname{Im}\{\Gamma(X,\mathcal{A}^{q-1}(X)) \xrightarrow{d_{0,q-1}^*} \Gamma(X,\mathcal{A}^{0,q}(X))\}} = \frac{\overline{\partial}\text{-closed } q\text{-forms}}{\overline{\partial}\text{-exact } q\text{-forms}} \quad \forall q \geq 0.$$

We thus have $H^q(X,\mathcal{O}) = 0$ if $q > n$.

Example 7.23. More generally: let $E \xrightarrow{\pi} X$ be a holomorphic vector bundle over X given over a covering $\mathcal{U} = \{U_i\}$ of X by transitions functions

$$g_{ij} \colon U_i \cap U_j \to GL(r,\mathbb{C}) \quad \text{(holomorphic)}$$

$$g_{ij}g_{jk} = g_{ik} \quad \text{on } U_i \cap U_j \cap U_k.$$

By an (r,s)-form with values in E we means a collection $\varphi_{U_i}^{r,s}$ of forms of type r,s on the U_i such that

$$\varphi_{U_i}^{r,s} = g_{ij}\varphi_{U_j}^{r,s}.$$

Clearly we have

$$\overline{\partial}\varphi_{U_i}^{r,s} = g_{ij}\overline{\partial}\varphi_{U_j}^{r,s}$$

(as g_{ij} is holomorphic). Hence we can consider the sheaves $\mathcal{O}^{r,s}(E)$ of germs of C^∞ form of type r,s with values in E. We have thus an exact sequence of sheaves

$$0 \longrightarrow \Omega^r(E) \longrightarrow \mathcal{A}^{r,0}(E) \xrightarrow{\overline{\partial}} \mathcal{A}^{r,1}(E) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{r,n}(E) \longrightarrow 0$$

which is a soft resolution of the sheaf $\Omega^r(E)$ of germs of holomorphic r-forms with values in E.

5. Complementary remarks

a) In the construction of Cech cohomology groups we have made use of open coverings $\mathcal{U} = \{U_i\}_{i\in I}$ of the base topological space X. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X by open sets $U_i \Subset X$. Let $\overline{\mathcal{U}} = \{\overline{U}_i\}_{i\in I}$ the corresponding covering of X by the compact sets $\overline{U}_i$. As those compact coverings are cofinal to the set of all coverings we can define as before $\overline{H}^q(\overline{\mathcal{U}},\mathcal{F})$ replacing the local system

$$U_{i_o,\dots,i_q} \mapsto \Gamma(U_{i_o,\dots,i_q},\mathcal{F})$$

by the system

$$\overline{U}_{i_o,\dots,i_q} \mapsto \Gamma(\overline{U}_{i_o,\dots,i_q},\mathcal{F}) \quad (\overline{U}_{i_0,\dots,i_q} = \overline{U}_{i_0} \cap \cdots \cap \overline{U}_{i_q})$$

and then define

$$\overline{H}^q(X,\mathcal{F}) = \varinjlim\nolimits_{\mathcal{U}} \overline{H}^q(\mathcal{U},\mathcal{F}).$$

If the space X is paracompact and locally compact, for every open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X we can find another covering $\mathcal{V} = \{V_i\}_{i\in I}$ (with the same set of indices) such that

$$\forall i \in I \quad V_i \Subset U_i.$$

Hence

$$\mathcal{V} = \{V_i\}_{i\in I} \prec \overline{\mathcal{V}} = \{\overline{V}_i\}_{i\in I} \prec \mathcal{U} = \{U_i\}_{i\in I} \prec \overline{\mathcal{U}} = \{\overline{U}_i\}_{i\in I}$$

and therefore natural maps

$$\overline{H}^q(\overline{\mathcal{U}},\mathcal{F}) \xrightarrow{\alpha} H^q(\mathcal{U},\mathcal{F}) \xrightarrow{\beta} \overline{H}^q(\overline{\mathcal{V}},\mathcal{F}).$$

At the limit we obtain then

$$\overline{H}^q(X,\mathcal{F}) \xrightarrow{\alpha} H^q(X,\mathcal{F}) \xrightarrow{\beta} \overline{H}^q(X,\mathcal{F}).$$

and $\beta \circ \alpha = id$. Thus α is surjective and β injective i.e. we must have

$$\overline{H}^q(X,\mathcal{F}) \cong H^q(X,\mathcal{F}).$$

b) This remark sometimes produces simplifications. For instance, to establish the conclusions of Lemma 7.13 it is enough to have a lemma of the following type:

Lemma 7.24. *Let Ω be a convex open bounded subset of $\mathbb{R}^n$ and $\mathcal{G}$ constant sheaf. Then $H^q(\overline{\Omega},\mathcal{G}) = 0$ for $q > 0$.*

This is easier to prove as $\overline{\Omega}$ is compact and one can take as a model of $\overline{\Omega}$ the unit cube $\overline{Q} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid 0 \le x_i \le 1 \text{ for } 1 \le i \le n\}$. Given a polyhedron K one has to take as natural covering the following

$$\overline{U}_K = \frac{2}{3} \text{ closed star of } K$$

then $\overline{\mathcal{U}} = \{\overline{U}_k\}_{k \in K}$ has again a nerve $\cong K$.

c) Let Ω be open and bounded in $\mathbb{C}^n$. Then

$$H^n(\overline{\Omega}, \mathcal{O}) = 0.$$

Indeed by De Rham Theorem it is enough to show that any $(0, n)$-form

$$\Psi = f(z) d\bar{z}_1 \dots d\bar{z}_n$$

is well defined and C^∞ in some neighborhood U of $\overline{\Omega}$ is $\overline{\partial}$-exact. Now, up to restricting U, we may assume that $f \in C^\infty(\overline{U})$ so that

$$u(z) = \int_\Omega \frac{f(\zeta)}{|z - \zeta|^{2n-1}} dv \quad (dv = \text{euclidean volume element})$$

is well defined and C^∞. We have $\Delta u = f$ on $\overline{\Omega}$ (up to a universal constant $= (2\pi)^{2n}$ to multiply f). Hence as $\Delta = \sum \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\alpha}$ we can set (neglecting the constant)

$$\varphi = \sum (-1)^\alpha \frac{\partial u}{\partial z_\alpha} d\overline{z_1} \wedge \dots \wedge \widehat{d\overline{z_\alpha}} \wedge \dots \wedge d\overline{z_n}$$

and we get

$$\overline{\partial} \varphi = \Psi.$$

The same is true for any open set, i.e. $H^n(\Omega, \mathcal{O}) = 0$, but harder to prove.

CHAPTER 8

Finiteness theorem

1. Preliminaries on Fréchet spaces

a) Let E be a vector space over $\mathbb{C}$. A topology on E is compatible with the vector space structure if

$$\begin{array}{rcl} E \times E & \to & E \\ (u, v) & \mapsto & (u + v) \end{array} \tag{8.1}$$

$$\begin{array}{rcl} \mathbb{C} \times E & \to & E \\ (\lambda, v) & \mapsto & (\lambda v) \end{array} \tag{8.2}$$

are continuous maps.

If $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ is a fundamental system of neighborhoods of the origin in E then for any $v \in E$ $\{v + U_\alpha\}_{\alpha \in A}$ is a fundamental system of neighborhoods of v. Thus the topology on E is determined by the datum of a fundamental system of neighborhoods of the origin. In general we assume that E is locally convex i.e. a fundamental system of neighborhoods $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of the origin can be found that consists of open convex sets U_α such that $\lambda U_\alpha \subset U_\alpha$ if $|\lambda| \leq 1$. If we set

$$p_{U_\alpha}(v) = \inf\{|\lambda| > 0 \mid v \in \lambda U_\alpha\}$$

so that

$$U_\alpha = \{v \in E \mid p_{U_\alpha}(v) < 1\}$$

we have:

$$p_{U_\alpha}(u + v) \leq p_{U_\alpha}(u) + p_{U_\alpha}(v) \tag{8.3}$$

$$p_{U_\alpha}(\lambda v) = |\lambda| p_{U_\alpha}(v) \tag{8.4}$$

A function $p\colon E \to \mathbb{R}^+$ verifying (8.3) (8.4) is called a seminorm. A locally convex topological vector space E has thus a topology that can be defined by a set of seminorms. Any set of seminorms on a topological vector space defines on it a locally convex topology.

b) Let E be a locally convex topological vector space in which the topology can be defined by a countable set $\{p_n\}_{n \in \mathbb{N}}$ of seminorms. Then the topology of E can also be defined by the invariant distance

$$d(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{p_n(x - y)}{1 + p_n(x - y)}.$$

If E is complete (i.e. every Cauchy sequence converges) then E is also called a **Fréchet space**. We will usually assume that the space E has a Hausdorff topology. A Fréchet space is a Baire space i.e. the intersection of countably many open dense sets is not empty.

Definition 8.1. A subset S of E is called of the **first category** if S is contained in a countable union of closed sets $\{F_n\}_{n \in \mathbb{Z}}$

$$S \subset \cup F_n$$

such that $E \setminus F_n$ is open and dense. A subset S of E is of the **second category** if it is not of the first category .

Theorem 8.2. *Let $u\colon E \to F$ be a continuous linear map between Fréchet spaces. If $u(E)$ is of second category then $u(E) = F$ and u is an open map.*

In particular if u is surjective then u must be open as any Fréchet space is a Baire space i.e. is of second category.

COROLLARY 8.3. *If $u(E)$ is of finite codimension in F then $u(E)$ is closed in F.*

PROOF. Let $w_1, \ldots, w_r \in F$ be linearly independent and such that

$$u(E) + \mathbb{C}w_1 + \cdots + \mathbb{C}w_r = F.$$

Consider the map

$$w \colon E \times \mathbb{C}^r \ toF$$

where

$$w(e, \lambda_1, \ldots, \lambda_r) = u(e) + \lambda_1 w_1 + \cdots + \lambda_r w_r.$$

Then w is linear and continuous and surjective thus w is open i.e., $E = w^{-1}w(E)$, we must have that $w(E)$ is closed. □

COROLLARY 8.4. *Let E be a Fréchet space and F a closed subspace of E then $G = \frac{E}{F}$ is a Fréchet space (with the quotient topology).*

PROOF. The topology of G is defined by the sequence of seminorms

$$p_n(g) = \inf_{\lambda(v)=g} p_n(v)$$

where $\lambda \colon E \to \frac{E}{F} = G$ is the natural map and where $\{p_n(g)\}$ describe a sequence of seminorms defining the topology of E. As E is of second category so is G of second category. Let $\widehat{G}$ be the completion of G and let

$$\mu \colon G \to \widehat{G}$$

the natural imbedding of G in its completion. Clearly μ is continuous and has an image of second category. Hence μ is surjective i.e. μ is a homomorphism. This shows that G is complete hence a Fréchet space. □

REMARK 8.5. Fréchet spaces have the very remarkable property that closed subspaces and quotient by closed subspaces are shell Fréchet spaces.

c)

DEFINITION 8.6. Let E,F be Fréchet space and let $\mu \colon E \to F$ be a continuous linear map. We say that μ is a **compact map** if we can find a neighborhood U of the origin in E such that $\overline{\mu(U)}$ is compact in F.

(Note that the continuity f μ is a consequence of this property).

THEOREM 8.7 (F. Riesz and L. Schwartz). *Let μ, ν be continuous linear maps from the Fréchet space E into the Fréchet space F. Assume that μ is surjective and ν is compact. Then*

$$\mu + \nu \colon E \to F$$

has an image of finite codimension (hence closed).

EXERCISE 1.1. If E is a locally compact Fréchet space then E is finite dimensional. Indeed by assumption there exist a neighborhood U of the origin in E such that $\overline{U}$ is compact. Thus the identity Id_E is a compact map. We can apply the above theorem with $E = F$, $\mu = Id_E$, $\nu = -Id_E$. Then $\mu + \nu = 0$ hence $F = E$ is finite dimensional.

The theorem above is essentially a consequence of the fact stated in this exercise.

2. Cech cohomology with values in a locally free sheaf $\mathcal{O}(E)$

a) Let E be a holomorphic vector bundle over the complex manifold X. If r is the dimension of the fiber of E then for any open set $U \subset X$ where $E|_U \cong U \times \mathbb{C}^r$ we have

$$\Gamma(U, \mathcal{O}(E)) = \mathcal{H}(U)^r$$

where $\mathcal{H}(U)$ denote the space of holomorphic functions on U. Now

(1) $\Gamma(U, \mathcal{O}(E))$ with the topology of uniform convergence on compact sets has the structure of a Fréchet space. The Banach open mapping theorem ensures that the Fréchet space structure on $\Gamma(U, \mathcal{O}(E))$ is independent of the trivialization $E|_U \cong U \times \mathbb{C}^r$ chosen for E over U.

(2) If $V \subset U$ is open and a subset of U the natural restriction map

$$r_V^U \colon \Gamma(U, \mathcal{O}(E)) \to \Gamma(V, \mathcal{O}(E))$$

is continuous.

(3) If V is $\Subset U$ then r_V^U is also a compact map.

The proof of this last statement (3) follows from the Vitali's theorem.

b) Let $\mathcal{W} = \{W_j\}_{j \in J}$ be a countable covering of X by coordinate patches W_j on which $E|_{W_i}$ is trivial. Then the groups

$$C^q(\mathcal{W}, \mathcal{O}(E)) = \prod \Gamma(W_{j_0, \dots j_q}, \mathcal{O}(E))$$

as countable product of Fréchet spaces have the structure of Fréchet space. Since the restriction map are continuous we have that the complex

$$C^0(\mathcal{W}, \mathcal{O}(E)) \xrightarrow{\delta} C^1(\mathcal{W}, \mathcal{O}(E)) \xrightarrow{\delta} C^2(\mathcal{W}, \mathcal{O}(E)) \xrightarrow{\delta} \dots$$

is a topological complex of Fréchet space and continuous maps.

3. Finiteness theorem

THEOREM 8.8. *Let A, B be open subset of a complex manifold X, and let E be a holomorphic vector bundle on X. Assume that*

(1) $B \Subset A$

(2) $r_B^{A*} \colon H^q(A, \mathcal{O}(E)) \to H^q(B, \mathcal{O}(E))$ *is surjective*

then

$$\dim_{\mathbb{C}} H^q(B, \mathcal{O}(E)) < \infty$$

PROOF. (1) Choose a countable covering $\mathcal{U} = \{U_i\}_{i \in I}$ of A by open set of holomorphy contained in coordinate patches on which E is trivial. Because by assumption (1) we may assume $\overline{B} \cap U_i \neq \emptyset$ for only finite many U_i, say for $1 \leq i \leq k$. We may consider for each U_i $1 \leq i \leq k$ an open subset $\widehat{U_i} \Subset U_i$ so that $B \subset \bigcup_{i=1}^k \widehat{U_i}$. Now we choose a countable covering $\mathcal{V} = \{V_j\}_{j \in J}$ of B by open set of holomorphy with the condition that

$$\mathcal{V} \prec \{\widehat{U_i} \cap B\}_{1 \leq i \leq k}.$$

Let for each V_j be $\tau(j) \in \{1, \dots, k\}$ be so chosen that

$$V_j \subset \widehat{U}_{\tau(j)}$$

(2) Consider now the cochain groups

$$\Gamma(U_{i_0, \dots, i_q}, \mathcal{O}(E)) = \prod \Gamma(U_{i_0, \dots, i_q}, \mathcal{O}(E)) \text{ on } A$$

$$\Gamma(V_{i_0, \dots, i_q}, \mathcal{O}(E)) = \prod \Gamma(V_{j_0, \dots, j_q}, \mathcal{O}(E)) \text{ on } B$$

and the restriction map

$$\tau^* \colon C^q(\mathcal{U}, \mathcal{O}(E)) \to C^q(\mathcal{V}, \mathcal{O}(E))$$

given by τ i.e.

$$(\tau^* f)_{j_0, \dots, j_q} = f_{\tau(j_0), \dots, \tau(j_q)}|_{V_{j_0, \dots, j_q}} \ \forall f \in C^q(\mathcal{U}, \mathcal{O}(E)).$$

Now
$$V_{j_0,\dots,j_q} \subset \widehat{U}_{\tau(j_0),\dots,\tau(j_q)} \Subset U_{\tau(j_0),\dots,\tau(j_q)}.$$
Therefore, with loose but obvious notations, the set
$$\left\{ f \in C^q(\mathcal{U}, \mathcal{O}(E)) \mid \sup_{\overline{\widehat{U}}_{\alpha_0,\dots,\alpha_q}} ||f_{\alpha_0,\dots,\alpha_q}|| < 1, 1 \le \alpha_0 \le \cdots \le \alpha_q \le k \right\}$$
is a neighborhood of the origin in $C^q(\mathcal{U}, \mathcal{O}(E))$ and has relatively compact image in $C^q(\mathcal{V}, \mathcal{O}(E))$. Therefore τ^* is a compact map.

(3) Let Z^q denote the cocycles. we consider the map
$$w \colon Z^q(\mathcal{U}, \mathcal{O}(E)) \oplus C^{q-1}(\mathcal{U}, \mathcal{O}(E)) \to Z^q(\mathcal{U}, \mathcal{O}(E))$$
given by
$$w(\alpha \oplus \beta) = \tau^*\alpha + \delta\beta.$$
By Leray theorem (and Cartan Serre's theorem) we have that
$$H^q(\mathcal{U}, \mathcal{O}(E)) = H^q(A, \mathcal{O}(E))$$
$$H^q(\mathcal{V}, \mathcal{O}(E)) = H^q(B, \mathcal{O}(E))$$
and by assumption the map $r_B^{A^*}$, which is induced by τ^*, is surjective. This means that w is also a surjective map. Now $Z^q(\mathcal{U}, \mathcal{O}(E))$, $Z^q(\mathcal{V}, \mathcal{O}(E))$ and $C^{q-1}(\mathcal{U}, \mathcal{O}(E))$ are Fréchet space, w is surjective and τ^* is compact. Therefore $\delta = w - \tau^*$ has an image in $Z^q(\mathcal{U}, \mathcal{O}(E))$ of finite codimension i.e. $\delta C^{q-1}(\mathcal{U}, \mathcal{O}(E))$ is of finite codimension in $Z^q(\mathcal{V}, \mathcal{O}(E))$. This means that
$$\dim_{\mathbb{C}} H^q(\mathcal{V}, \mathcal{O}(E)) = \dim_{\mathbb{C}} \frac{Z^q(\mathcal{U}, \mathcal{O}(E))}{\delta C^{q-1}(\mathcal{V}, \mathcal{O}(E))} < \infty.$$
Hence by Leray's theorem
$$\dim_{\mathbb{C}} H^q(B, \mathcal{O}(E)) = H^q(\mathcal{V}, \mathcal{O}(E)) < \infty.$$
□

Corollary 8.9. *If X is a compact manifold we can take $B = A = X$ therefore for a compact manifold X and any holomorphic vector bundle E over X we have*
$$\dim_{\mathbb{C}} H^q(X, \mathcal{O}(E)) < \infty \quad \forall q \ge 0.$$

In particular for X compact and $q = 0$ we find another proof of Theorem 6.15.

4. Strongly pseudoconvex domains

a) Let X be a complex manifold and let $B \Subset X$ have a smooth boundary. We may assume that there exist a C^∞ function $\Phi \colon X \to \mathbb{R}$ with
$$B = \{x \in X \mid \Phi(x) < 0\} \qquad d\Phi(x) \ne 0 \; \forall x \in \partial B = \overline{B} \setminus B.$$

Definition 8.10. We will say that B is **strongly pseudoconvex** if for all $z_0 \in \partial B$
$$\mathcal{L}(\Phi)_{z_0}(u) = \begin{cases} \sum \frac{\partial \Phi}{\partial z_\alpha \partial \overline{z_\beta}}(z_0) u_\alpha \overline{u}_\beta \\ \sum \frac{\Phi}{\partial z_\alpha}(z_0) u_\alpha = 0 \end{cases}$$
is positive definite.

This condition as we have seen is independent from the choice of the function Φ.

Remark 8.11. Replacing Φ by $\Psi = e^{C\Phi} - 1$ where $c > 0$ is sufficient large, we may assume that for all $z_0 \in \partial B$
$$\sum \frac{\partial^2 \Psi}{\partial z_\alpha \partial \overline{z_\beta}}(z_0) u_\alpha \overline{u}_\beta > 0$$
(without restriction to analytic tangent plane to ∂B at z_0). Indeed
$$\partial\overline{\partial}\Psi = e^{C\Phi} C\{\partial\overline{\partial}\Phi + C|\partial\Phi|^2\}.$$

REMARK 8.12. If ∂B is strongly pseudoconvex (and smooth) at z_0 then by a suitable choice of local holomorphic coordinate at z_0 we may assume that, in this coordinates patch, $B \cap U$ has a strongly elementary convex boundary at z_0.

REMARK 8.13. Open convex set Ω of $\mathbb{C}^n$ are domain of holomorphy, in particular we have

$$H^q(\Omega, \mathcal{O}(E)) \qquad \forall q > 0$$

for all vector bundle E.

b)

THEOREM 8.14. *Let B be a strongly pseudoconvex subset of the complex manifold X.Let E be any holomorphic vector bundle on X then*

$$\dim_{\mathbb{C}} H^q(B, \mathcal{O}(E)) < \infty \quad \forall q > 0.$$

PROOF. Step 1 (Mayer-Vietoris sequence) Let X be a topological space and $X = X_1 \cup X_2$ with X_1 and X_2 open sets. Let $\mathcal{F}$ be any sheaf of abelian groups on X then one has the following exact sequence

$$0 \to H^0(X, \mathcal{F}) \xrightarrow{\alpha} H^0(X_1, \mathcal{F}) \oplus H^0(X_2, \mathcal{F}) \xrightarrow{\beta} H^0(X_1 \cap X_2, \mathcal{F}) \longrightarrow$$
$$\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X_1, \mathcal{F}) \oplus H^1(X_2, \mathcal{F}) \longrightarrow H^1(X_1 \cap X_2, \mathcal{F}) \longrightarrow \cdots$$

PROOF. Let

$$0 \to \mathcal{F} \to C^0 \to C^1 \to C^2 \to \ldots$$

be a flabby resolution of $\mathcal{F}$. We have for every $q \geq 0$ exact sequence

$$0 \to \Gamma(X, C^q) \xrightarrow{\alpha} \Gamma(X_1, C^q) \oplus \Gamma(X_2, C^q) \xrightarrow{\beta} \Gamma(X_1 \cap X_2, C^q) \to 0$$

where $\alpha(a) = a|_{X_1} \oplus a|_{X_2}$ and $\beta(a \oplus b) = a|_{X_1 \cap X_2} - b|_{X_1 \cap X_2}$. This lead to a short exact sequence of complexes (taking the direct sum over q). The corresponding cohomology sequence is the sequence above. □

Step 2

LEMMA 8.15 (Brumps Lemma). *Let $B \Subset X$ and strongly pseudoconvex. There exist (arbitrarily fine) finite coverings $\mathcal{U} = \{U_i\}_{1 \leq i \leq t}$ of ∂B and for each $\mathcal{U}$ a sequence of open sets B_j with smooth strongly pseudoconvex boundaries $0 \leq j \leq t$ such that*

1. $B = B_0 \subset B_1 \cdots \subset B_t$
2. $B_0 \Subset B_t$
3. $B_i \setminus B_{i-1} \Subset U_i$ $1 \leq i \leq t$
4. $H^r(B_i \cap U_J, \mathcal{O}(E)) = 0$ *for all i, j $r > 0$ and for all holomorphic vector bundle E over X.*

PROOF. Let $B = \{x \in X \mid \Phi(x) < 0\}$ $d\Phi(x) \neq 0$ for all $x \in \partial B$ and $\mathcal{L}(\Phi)_x > 0$ for all $x \in \partial B$. We may assume that in any open neighborhood U of ∂B we do have $d\Phi \neq 0$ and $\mathcal{L}(\Phi) > 0$. Let $\mathcal{U} = \{U_i\}_{1 \leq i \leq t}$ be a covering of ∂B with coordinate balls $U_i \subset U$ sufficiently small that $U_i \cap B$ is elementary convex with strongly convex boundary ∂B. Select C^∞ function $\rho_i \geq 0$ on X $1 \leq i \leq t$ with

1. *$supp\rho_i \Subset U_i$*
2. $\sum \rho_i(x) > 0$ $\forall x \in \partial B$.

Choose $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$ successively > 0 and sufficiently small so that

$$\Phi_1 = \Phi - \varepsilon_1 \rho_1, \Phi_2 = \Phi - \varepsilon_2 \rho_2, \ldots, \Phi_t = \Phi - \varepsilon_t \rho_t$$

have the following properties

1. $\mathcal{L}(\Phi_i) > 0$ for all $x \in U$
2. setting $B_0 = B$
$$B_i = B \cup \{x \in U \mid \Phi_i(x) < 0\}$$
∂B_i is smooth.

(3) $B_i \cap U_j$ is elementary convex in the coordinate patch U_i and thus an open set of holomorphy.

Then the four stated properties are verified as

$$B = B_0 \subset B_1 \cdots \subset B_t \text{ because } \Phi = \Phi_0 \geq \Phi_1 \geq \ldots \Phi_t$$

$$B_0 \Subset B_t \text{ because } \Phi_0 > \Phi_t \text{ on } \partial B$$

$$B_i \setminus B_{i-1} \Subset U_i \text{ as } \Phi_i - \Phi_{i-1} = \varepsilon_i \rho_i \text{ has support in } U_i$$

$$H^r(B_i \cap U_J, \mathcal{O}(E)) = 0 \text{ for } r > 0 \text{ as } B_i \cap U_i \text{ is a domain of holomorphy.}$$

□

Step 3 There exists $A \subset X$ open with $B \Subset A$ and

$$H^r(A, \mathcal{O}(E)) \to H^r(B, \mathcal{O}(E))$$

surjective for every $r > 0$. Take $A = B_t$ in the previous step. It is sufficient to show that

$$H^r(B_i, \mathcal{O}(E)) \to H^r(B_{i-1}, \mathcal{O}(E))$$

is surjective for $1 \leq i \leq t$. Now $B_i = B_{i-1} \cup U_i \cap B_i$ and by the Mayer-Vietoris sequence we get the exact sequence

$$H^r(B_i, \mathcal{O}(E)) \to H^r(B_{i-1}, \mathcal{O}(E)) \oplus H^r(U_i \cap B_i, \mathcal{O}(E)) \to H^r(U_i \cap B_{i-1}, \mathcal{O}(E)).$$

Since for $r > 0$ $H^r(U_i \cap B_i, \mathcal{O}(E)) = 0 = H^r(U_i \cap B_{i-1}, \mathcal{O}(E))$ we get the conclusion.

Step 4 it is enough how to apply the finiteness theorem of the previous section. □

c) Levi problem solution is given by the following theorem.

THEOREM 8.16. *Let $B \Subset X$ be strongly pseudoconvex. For any divergent sequence $\{x_\nu \subset B\}$ we can find a holomorphic function f on B such that*

$$\sup |f(x_\nu)| = +\infty.$$

PROOF. As $B \Subset X$ it is not restrictive to assume that the sequence x_ν converges to a boundary point $z_0 \in \partial B$. Let

$$B = \{x \in X \mid \Phi(x) < 0\}$$

with $\mathcal{L}(\Phi) > 0$ in a neighborhood U of B and $d\Phi \neq 0$ in the same neighborhood. With a suitable choice of coordinates $z_1, \ldots, z_n$ near z_0 we can write

$$\Phi(z) = \Re f(z) + \mathcal{L}(\Phi)_{z_0}(z) + O(||z^3||)$$

with f holomorphic in a neighborhood $V(z_0)$ of z_0. If $V(z_0)$ is sufficiently small

$$\{z \in V(z_0) \mid f(z) = 0\} \cap B = \emptyset.$$

We can choose $\rho \geq 0$, *supp*$\rho \Subset V(z_0)$, $\rho(z_0) > 0$ and $\varepsilon > 0$ so small that satisfy $\Phi_1 = \Phi - \varepsilon\rho$ $d\Phi_1 \neq 0$ on U $\mathcal{L}(\Phi_1) > 0$ on U and

$$A = B \cup \{x \in U \mid \Phi_1(x) < 0\}$$

has a smooth strongly pseudoconvex boundary. Write $A = B \cup A \cap V$ and apply the Mayer-Vietoris sequence to $\mathcal{O}$

$$0 \to H^0(A, \mathcal{O}) \to H^0(B, \mathcal{O}) \oplus H^0(A \cap V, \mathcal{O}) \to H^0(B \cap V, \mathcal{O}) \xrightarrow{\delta} H^1(A, \mathcal{O}) \to \ldots$$

By the previous theorem

$$r = \dim_{\mathbb{C}} H^1(A, \mathcal{O}) < \infty.$$

Consider on $B \cap V$ the functions

$$\frac{1}{f}, \frac{1}{f^2}, \ldots, \frac{1}{f^{r+1}}.$$

These are holomorphic on $B \cap V$ and there must be a set of constant $c_1, c_2, \ldots, c_{r+1}$ not all 0 such that

$$\delta(c_1 \frac{1}{f} + c_2 \frac{1}{f^2} + \cdots + c_{r+1} \frac{1}{f^{r+1}}) = 0.$$

Thus there exist holomorphic functions $h_B \subset H^0(B, \mathcal{O})$ and $h_{A\cap V} \in H^0(A \cap V, \mathcal{O})$ such that

$$\sum_{i=1}^{r+1} c_i \frac{1}{f^i} h_{A\cap V} = h_B = m.$$

Hence m is a meromorphic function on A, holomorphic on B and with principal part at z_0 equal to $\sum_{i=1}^{r+1} c_i \frac{1}{f^i}$. Thus $f = m|_A = h_B$ gives

$$\lim |h_B(x_\nu)| = \infty.$$

□

APPENDIX A

Kodaira dimension.

In this chapter we want to treat another numerical invariant of complex manifold: the Kodaria dimension.

Let X be a smooth compact manifold. We know that there are two dimensions that we can consider on X: its dimension on $\mathbb{C}$, that we suppose equal to n, and its algebraic dimension $a(X)$ which is less than or equal to the first. However, as we seen in the previous chapters, these two dimensions are the same if X is an algebraic manifold and for this reason the Kodaira dimension is an important tool in algebraic geometry.

Let $K \to X$ the line canonical bundle on X. Let $K = \Omega^n_X$ and consider its tensor powers $K^{\otimes i}$ each of which is still a line bundle on X. Let $\Gamma(X, K^{\otimes i})$ the set of the sections of the bundle $K^{\otimes i}$ and note that there is a canonical pairing:

$$\Gamma(X, K^{\otimes i}) \bigotimes \Gamma(X, K^{\otimes j}) \to \Gamma(X, K^{\otimes i+j}).$$

Thus the set:

$$R(X) = \bigoplus_{i \geq 0} \Gamma(X, K^{\otimes i}).$$

is a ring called **the canonical ring** of the manifold X. Let

$$Q(X, K) = \{\frac{s_0}{s_1} \mid s_0, s_1 \in \Gamma(X, K^{\otimes i}), s_0 \neq 0\}$$

the field of quotient of the ring $R(X)$. As we prove in Theorem 6.11, $Q(X, K)$ is a algebrically closed subflied of $\mathcal{K}(X)$.

Definition A.1. We define the **Kodaira dimension** $kod(X)$ of the manifold X as follows:

(1) if $\Gamma(X, K^{\otimes i}) = 0$ for all $i \geq 0$ then $kod(X) = -\infty$;
(2) otherwise $kod(X)$ is the trescendence degree on $\mathbb{C}$ of the field $Q(X, K)$.

Remark A.2. Clearly we have

$$kod(X) \geq a(X) \geq \dim_{\mathbb{C}} X = n.$$

Example A.3. Let $S \subset \mathbb{P}^3$ be a surface of degree d so we have:

$$kod(S) = \begin{cases} -\infty & \text{if} \quad d \leq 3 \\ 0 & \text{if} \quad d = 4 \\ 2 & \text{if} \quad d \geq 5 \end{cases}$$

Definition A.4. The manifold X is of **general type** if its Kodaira dimension is maximal:

$$kod(X) = \dim_{\mathbb{C}} X.$$

Note that if $kod(X) = 0$ then $Q(X, K) = \mathbb{C}$ so we have that $\dim \Gamma(X, K^{\otimes i}) = 0, 1$ for all i and, if X is of general type, then the field of meromorphic function on X coincides with the field $Q(X, K)$. Thus for a manifold of general type each meromorphic function is quotient of two sections of an appropriate tensor product of the canonical bundle. We give, now, a result that will be useful later.

Proposition A.5. *The filed $Q(X, K)$ is finitely generated on $\mathbb{C}$.Moreover let $f_1, \ldots, f_k \in Q(X, K)$ such that*

$$Q(X, K) = \mathbb{C}(f_1, \ldots, f_k)$$

then there exist $N \gg 0$ and $s_0, s_1, \ldots, s_k \in \Gamma(X, K^{\otimes N})$ such that

$$f_i = \frac{s_i}{s_0} \text{ for all } i = 1, \ldots, k.$$

PROOF. Let $f_1, \dots, f_k \in Q(X, K)$, so each $f_i = \frac{s_{i1}}{s_{i0}}$ where $s_{i1}, s_{i0} \in \Gamma(X, K^{\otimes n_i})$ for some n_i. Thus we define

$$s_0 = s_{10} \cdots s_{k0} \qquad s_i = s_{10} \cdots s_{i1} \cdots s_{k0} \quad i = 1, \dots, k$$

and $N = \sum_i n_i$.

Let $f_1, \dots, f_k \in Q(X, K)$ a maximal set of algebraically indipendent rational section. We have to show that there exists α such that each $f \in Q(X, K)$ that is algebrically dependent from $f_1, \dots, f_k$ satisfy an equation of degree $\leq \alpha$. This fact implies the statement indeed: let $h \in Q(X, K)$, we can find $\Theta \in Q(X, K)$ such that

$$\mathbb{C}(f_1, \dots, f_k, f, h) = \mathbb{C}(f_1, \dots, f_k, \Theta).$$

Then

$$\alpha \geq [\mathbb{C}(f_1, \dots, f_k, \Theta) : \mathbb{C}(f_1, \dots, f_k)] =$$
$$= [\mathbb{C}(f_1, \dots, f_k, \Theta) : \mathbb{C}(f_1, \dots, f_k, f)] \cdot [\mathbb{C}(f_1, \dots, f_k, f) : \mathbb{C}(f_1, \dots, f_k)].$$

But the second factor of this product equals α therefore the first factor equals 1. Thus:

$$Q(X, K) = \mathbb{C}(f_1, \dots, f_k, f).$$

To show the existence of α we follows the proof of the Theorem 6.21: we can find coordinate polycilinders $P_{a_i} \supset P'_{a_i}$, $1 \leq i \leq n$, such that:

(1) $K^{\otimes N}|_{\overline{P}_{a_i}}$ is trivial;
(2) $\mathcal{S}(\overline{P}_{a_i}) \subset Y$;
(3) $\cup P'_{a_i} \supset \overline{Y}$;
(4) at each point a_i the functions $f_1, \dots, f_k$ are holomorphic and $f_1 - f_1(a_i) = \zeta_1^{(i)}, \dots, f_k - f_k(a_i) = \zeta_k^{(i)}$ can be taken among a set of local holomorphic coordinates.

Let $\overline{N}$ such that $f = \frac{\sigma_0}{\sigma_1}$ with $\sigma_0, \sigma_1 \in K^{\otimes \overline{N}}$, we can suppose that:

(5) $K^{\otimes \overline{N}}|_{\overline{P}_{a_i}}$ is trivial;
(6) f is holomorphic in each a_i.

We can consider $||K|| = e^{\mu}$ thus $||K^{\otimes N}|| = e^{\mu N}$ and $||K^{\otimes \overline{N}}|| = e^{\mu \overline{N}}$. Let p a generic polynomial in $k+1$ variables of degree r in $x_1, \dots, x_k$ and of degree α in x_{k+1}:

$$P(x_1, \dots, x_{k+1}) = \sum c_{i_1 \dots i_k i_{k+1}} x_1^{i_1} \cdots x_k^{i_k} x_{k+1}^{i_{k+1}} \quad 1 \leq \imath_j \leq r \;\; j = 1, \dots, k \quad 1 \leq i_{k+1} \leq \alpha$$

and its corresponding homogeneous polynomial:

$$\pi(x_0, \dots, x_k, y_0, y_1) = x_0^{kr} y_0^{\alpha} P(\frac{x_1}{x_0}, \dots, \frac{x_k}{x_0}, \frac{y_1}{y_0}).$$

So we have

$$\pi(s_0, s_1, \dots, s_k, \sigma_0, \sigma_1) \in \Gamma(X, K^{rN + \alpha \overline{N}})$$

thus, as in Theorem 6.21, if α is such that:

$$\alpha + 1 > nk^k \mu^k$$

and r is sufficiently large, f satisfy an equation of degree $\leq \alpha$. Note that the condition $\alpha + 1 > nk^k \mu^k$ depends only from the functions $f_1, \dots, f_k$ and doesn't depend from f thus this prove the statement for all $f \in Q(X, K)$. □

We now want to give an alternative definition of the Kodaira dimension. Thus we define the $i-$**canonical map** as follows:

$$\begin{array}{rcl} \Phi_i \colon X \setminus \{\text{base point of } X\} & \to & \mathbb{P}(\Gamma(X, K^{\otimes i}))^* \\ p & \mapsto & (s_1(p), \dots, s_i(p)) \end{array}$$

where $s_1, \dots, s_i$ is a basis for $\Gamma(X, K^{\otimes i})$. So we can define the Kodaira dimension of X:

$$Kod(X) = \begin{cases} -\infty & \text{if} \quad \Gamma(X, K^{\otimes i} = 0) \; \forall i \\ \max_i a(\Phi_i(X)) & \end{cases}$$

where $a(\Phi_i(X))$ is the algebraic dimension of $\Phi_i(X)$.

THEOREM A.6. *The two definitions of Kodaira dimension of the manifold X are equivalent.*

Proof. Let $\mathcal{K}(\Phi_i(X))$ the field of meromorphic functions of the manifold $\Phi_i(X)$, thus clearly we have that for all i

$$\mathcal{K}(\Phi_i(X)) \subset Q(X, K).$$

As we see in Proposition A.5 if $i > N$ then $f_1, \ldots, f_k \in \mathcal{K}(\Phi_i(X))$ thus

$$\mathcal{K}(\Phi_i(X)) = Q(X, K).$$

$\square$

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