Kodaira-Spencer formality of products of complex manifolds

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Abstract We shall say that a complex manifold X is *Kodaira-Spencer formal* if its Kodaira-Spencer differential graded Lie algebra $A^{0,*}_X(\Theta_X)$ is formal; if this happen, then the deformation theory of X is completely determined by the graded Lie algebra $H^*(X,\Theta_X)$ and the base space of the semiuniversal deformation is a quadratic singularity. Determine when a complex manifold is Kodaira-Spencer formal is generally difficult and we actually know only a limited class of cases where this happen. Among such examples we have Riemann surfaces, projective spaces, holomorphic Poisson manifolds with surjective anchor map $H^*(X,\Omega^1_X)\to H^*(X,\Theta_X)$ [4] and every compact Kähler manifold with trivial or torsion canonical bundle, see [9] and references therein. In this short note we investigate the behavior of this property under finite products. Let X,Y be compact complex manifolds; we prove that whenever X and Y are Kähler, then $X\times Y$ is Kodaira-Spencer formal if and only if the same holds for X and Y (Corollary 2). A revisit of a classical example by Douady shows that the above result fails if the Kähler assumption is dropped.

1 Review of differential graded (Lie) algebras and formality

In this section every vector space and tensor product is intended over a fixed field $\mathbb{K}$ of characteristic 0. In rational homotopy theory, an important role is played by the notion of formality of a differential graded algebra [2, p. 260]. A similar role in deformation theory is played by the notion of formality of a differential graded Lie algebra [5, p. 52].

Definition 1. A DG-algebra (short for differential graded commutative algebra) is the data of a $\mathbb{Z}$-graded vector space $A=\oplus_{n\in\mathbb{Z}}A^n$, equipped with a differential

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$d\colon A^n \to A^{n+1}$, $d^2 = 0$, and a product

$$A^n \times A^m \to A^{n+m}, \qquad (a,b) \mapsto ab,$$

which satisfy the following conditions:

1. (associativity) $(ab)c = a(bc)$,
2. (graded commutativity) $ab = (-1)^{\deg(a)\deg(b)} ba$,
3. (graded Leibniz) $d(ab) = d(a)b + (-1)^{\deg(a)} a d(b)$.

In particular every DG-algebra is also a cochain complex and its cohomology inherits a structure of graded commutative algebra. A morphism of DG-algebras is simply a morphism of graded algebras commuting with differentials. A DG-algebra A is called unitary if there exists a unit $1 \in A^0$ such that $1a = a$ for every $a \in A$.

Typical examples of DG-algebras are the de Rham complex $A_X^{*,*}$ and the Dolbeault complex $A_X^{0,*}$ of a holomorphic manifold X, equipped with the usual wedge product of differential forms.

Definition 2. A morphism $f\colon A \to B$ of DG-algebras is called a quasi-isomorphism if it is a quasi-isomorphism of the underlying cochain complexes. Two DG-algebras are said to be quasi-isomorphic if they are equivalent under the equivalence relation generated by quasi-isomorphisms.

A DG-algebra A is called formal if it is quasi-isomorphic to its cohomology algebra $H^*(A)$.

Example 1 (The Iwasawa DG-algebra). Probably the simplest example of non formal DG-algebra is the Iwasawa algebra: consider the vector space V with basis e_1, e_2, e_3 and the unique differential on the exterior algebra $R = \oplus_i R^i$, $R^i := \bigwedge^i V$ such that

$$de_1 = de_2 = 0, \qquad de_3 = -e_1 \wedge e_2\ .$$

According to Leibniz rule we have

$$d(e_1 \wedge e_2) = d(e_2 \wedge e_3) = d(e_1 \wedge e_3) = d(e_1 \wedge e_2 \wedge e_3) = 0$$

and there exists an obvious injective morphism $j\colon H^*(R) \hookrightarrow R$ of cochain complexes whose image is the graded vector subspace spanned by the six linearly independent vectors $1, e_1, e_2, e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_3$; however j is not a morphism of algebras.

Whenever $\mathbb{K} = \mathbb{R}$ the algebra R can be identified with the algebra of right-invariant differential forms on the Lie group of real matrices of type

$$\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix},$$

by setting $e_1 = dx_1$, $e_2 = dx_2$ and $e_3 = dx_3 - x_1 dx_2$. The non formality of R may be easily checked, as in [7], by computing the triple Massey products; here we obtain again this result as a consequence of Proposition 2.

Definition 3. A DG-Lie algebra (short for differential graded Lie algebra) is the data of a $\mathbb{Z}$-graded vector space $L = \oplus_{n\in\mathbb{Z}} L^n$, equipped with a differential $d\colon L^n \to L^{n+1}$, $d^2 = 0$, and a bracket

$$L^n \times L^m \to L^{n+m}, \qquad (a,b) \mapsto [a,b],$$

which satisfy the following conditions:

1. (graded anti commutativity) $[a,b] = -(-1)^{\deg(a)\deg(b)}[b,a]$;
2. (graded Leibniz) $d[a,b] = [da,b] + (-1)^{\deg(a)}[a,db]$;
3. (graded Jacobi) $[[a,b],c] = [a,[b,c]] - (-1)^{\deg(a)\deg(b)}[b,[a,c]]$.

As above, every DG-Lie algebra is also a cochain complex and its cohomology inherits a structure of graded Lie algebra. A morphism of DG-Lie algebras is simply a morphism of graded Lie algebras commuting with differentials.

Example 2. The Kodaira-Spencer DG-Lie algebra KS_X of a complex manifold X is defined as the Dolbeault complex $A_X^{0,*}(\Theta_X)$ of the holomorphic tangent sheaf equipped with the natural extension of the usual bracket on smooth sections of Θ_X, see e.g. [6].

If L is a DG-Lie algebra and A is a DG-algebra, then the tensor product $L\otimes A$ has a natural structure of DG-Lie algebra, where:

$$d(x\otimes a) = dx\otimes a + (-1)^{\deg(x)} x\otimes da, \quad [x\otimes a, y\otimes b] = (-1)^{\deg(a)\deg(y)}[x,y]\otimes ab\,.$$

Let's denote by **Art** the category of Artin local $\mathbb{K}$-algebras with residue field $\mathbb{K}$ and by **Set** the category of sets. Unless otherwise specified, for every $A \in$ **Art** we shall denote by $\mathfrak{m}_A$ its maximal ideal. Every DG-Lie algebra L gives a functor

$$\mathrm{MC}_L\colon \mathbf{Art} \to \mathbf{Set}, \qquad \mathrm{MC}_L(A) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; dx + \frac{1}{2}[x,x] = 0 \right\}$$

The equation $dx + [x,x]/2 = 0$ is called the Maurer-Cartan equation and MC_L is called the Maurer-Cartan functor associated to L. Two elements $x, y \in \mathrm{MC}_L(A)$ are said to be gauge equivalent if there exists $a \in L^0 \otimes \mathfrak{m}_A$ such that

$$y = e^a * x := x + \sum_{n=0}^{\infty} \frac{[a,-]^n}{(n+1)!}([a,x] - da).$$

Then we define the functor $\mathrm{Def}_L\colon$ **Art** $\to$ **Set** defined as (we refer to [5, 12, 13] for details):

$$\mathrm{Def}_L(A) = \frac{\mathrm{MC}_L(A)}{\text{gauge equivalence}}\,.$$

The projection $\mathrm{MC}_L \to \mathrm{Def}_L$ is a formally smooth natural tranformation: this means that, given a surjective morphism $A \xrightarrow{\alpha} B$ in the category **Art**, an element $x \in$

$\mathrm{MC}_L(B)$ can be lifted to $\mathrm{MC}_L(A)$ if and only if its equivalence class $[x] \in \mathrm{Def}_L(B)$ can be lifted to $\mathrm{Def}_L(A)$.

In this paper we shall need several times the following results (for a proof see e.g. Theorem 5.71 of [13]). A morphism of DG-Lie algebras $f\colon L \to M$ is called a quasi-isomorphism if the induced map in cohomology $f\colon H^*(L) \to H^*(M)$ is an isomorphism of graded Lie algebras.

Theorem 1 (Schlessinger-Stasheff [18]). *Let $L \to M$ be a morphism of differential graded Lie algebras. Assume that:*

1. *$H^0(L) \to H^0(M)$ is surjective,*
2. *$H^1(L) \to H^1(M)$ is bijective,*
3. *$H^2(L) \to H^2(M)$ is injective.*

Then the induced natural transformation $\mathrm{Def}_L \to \mathrm{Def}_M$ is an isomorphism of functors.

Corollary 1. *Let $L \to M$ be a quasi-isomorphism of differential graded Lie algebras. Then the induced natural transformation $\mathrm{Def}_L \to \mathrm{Def}_M$ is an isomorphism of functors.*

The notion of formality extends immediately to differential graded Lie algebras. A DG-Lie algebra L is called formal if it is connected to the graded Lie algebra $H^*(L)$ by a finite chain of quasi-isomorphisms of DG-Lie algebras.

As a first application of Theorem 1 we have therefore that for a formal DG-Lie algebra L the functor Def_L is determined by the graded Lie algebra structure on $H^*(L)$.

Proposition 1. *If a differential graded Lie algebra L is formal, then the two maps*

$$\mathrm{Def}_L(\mathbb{K}[t]/(t^3)) \to \mathrm{Def}_L(\mathbb{K}[t]/(t^2)), \qquad \mathrm{Def}_L(\mathbb{K}[t]/(t^n)) \to \mathrm{Def}_L(\mathbb{K}[t]/(t^2))$$

have the same image for every $n \geq 3$.

Proof. We may assume that L is a graded Lie algebra and therefore its Maurer-Cartan equation becomes $[x,x]=0$, $x \in L^1$. Therefore $tx_1 \in \mathrm{Def}_L(\mathbb{K}[t]/(t^2))$ lifts to $\mathrm{Def}_L(\mathbb{K}[t]/(t^3))$ if and only if there exists $x_2 \in L^1$ such that

$$t^2[x_1,x_1] \equiv [tx_1+t^2x_2, tx_1+t^2x_2] \equiv 0 \pmod{t^3} \iff [x_1,x_1]=0$$

and $[x_1,x_1]=0$ implies that $tx_1 \in \mathrm{Def}_H(\mathbb{K}[t]/(t^n))$ for every $n \geq 3$. □

An example of non formal DG-Lie algebra is provided by the next proposition.

Proposition 2. *Let $\mathfrak{n}_3(\mathbb{K})$ be the Lie algebra of strictly upper triangular 3×3 matrices and let R the Iwasawa DG-algebra defined above. Then:*

1. *the differential graded Lie algebra $\mathfrak{n}_3(\mathbb{K}) \otimes R$ is formal and the functor $\mathrm{Def}_{\mathfrak{n}_3(\mathbb{K})\otimes R}$ is smooth;*

2. *the differential graded Lie algebra* $\mathfrak{sl}_2(\mathbb{K})\otimes R$ *is not formal and the functor* $\mathrm{Def}_{\mathfrak{sl}_2(\mathbb{K})\otimes R}$ *is not smooth.*

Proof. Let's denote by $C\subset R$ the DG-vector subspace spanned by $e_3, e_1\wedge e_2$ and by $I\subset\mathfrak{n}_3(\mathbb{K})$ the Lie ideal of matrices of type

$$\begin{pmatrix}0&0&t\\0&0&0\\0&0&0\end{pmatrix},\qquad t\in\mathbb{K}\,.$$

Since $I=[\mathfrak{n}_3(\mathbb{K}),\mathfrak{n}_3(\mathbb{K})]$ and $[I,\mathfrak{n}_3(\mathbb{K})]=0$, the subcomplex $I\otimes C$ is an acyclic Lie ideal of $\mathfrak{n}_3(\mathbb{K})\otimes R$. The formality of $\mathfrak{n}_3(\mathbb{K})\otimes R$ is now an immediate consequence of the easy facts that, the projection

$$\pi\colon\ \mathfrak{n}_3(\mathbb{K})\otimes R\to\frac{\mathfrak{n}_3(\mathbb{K})\otimes R}{I\otimes C}$$

is a quasi-isomorphism and

$$\pi\circ(\mathrm{Id}\otimes j)\colon\ \mathfrak{n}_3(\mathbb{K})\otimes H^*(R)\to\frac{\mathfrak{n}_3(\mathbb{K})\otimes R}{I\otimes C}$$

is a morphism of differential graded Lie algebras. The smoothness of $\mathrm{Def}_{\mathfrak{n}_3(\mathbb{K})\otimes R}$ follows from the fact that the Maurer-Cartan equation in $H^*(\mathfrak{n}_3(\mathbb{K})\otimes R)=\mathfrak{n}_3(\mathbb{K})\otimes H^*(R)$ is trivial.

Next, we shall use Proposition 1 in order to prove that $M=\mathfrak{sl}_2(\mathbb{K})\otimes R$ is not formal. More precisely we shall prove that there exists an element in $\mathrm{MC}_M(\mathbb{K}[t]/(t^2))$ which lifts to $\mathrm{MC}_M(\mathbb{K}[t]/(t^3))$ but does not lift to $\mathrm{MC}_M(\mathbb{K}[t]/(t^4))$. Denote by u,v,h the standard basis of $\mathfrak{sl}_2(\mathbb{K})$:

$$[u,v]=h,\qquad [h,u]=2u,\qquad [h,v]=-2v,$$

and consider the element $\xi=ue_1t+ve_2t-he_3t^2\in\mathrm{MC}_M(\mathbb{K}[t]/(t^3))\subset M^1\otimes\mathbb{K}[t]/(t^3)$. A generic element of $M^1\otimes\mathbb{K}[t]/(t^4)$ lifting $ue_1t+ve_2t\in\mathrm{MC}_M(\mathbb{K}[t]/(t^2))$ may be written as

$$\eta=ue_1t+ve_2t+(ae_1+be_2+ce_3)t^2+(\alpha e_1+\beta e_2+\gamma e_3)t^3,\quad a,b,c,\alpha,\beta,\gamma\in\mathfrak{sl}_2(\mathbb{K})\,.$$

Assume that η satisfies the Maurer-Cartan equation. Since

$$d\eta=ce_1\wedge e_2t^2+\gamma e_1\wedge e_2t^3,\qquad \frac{1}{2}[\eta,\eta]=he_1\wedge e_2t^2+(\cdots)t^3$$

we must have $c=-h$; therefore the coefficient of $e_1\wedge e_3t^3$ in $\frac{1}{2}[\eta,\eta]$ is equal to $[u,c]=[u,-h]=[h,u]=2u\neq 0$ and this gives a contradiction. □

Lemma 1. *Let* L,M *be DG-Lie algebras and* B *a DG-algebra:*

1. *if* L *and* B *are formal, then* $L\otimes B$ *is a formal DG-Lie algebra;*

2. *if B is unitary, $H^*(B)\neq 0$ and $L\otimes B$ is a formal, then also L is formal;*
3. *the DG-Lie algebra $L\times M$ is formal if and only if L and M are formal.*

Proof. The first item is clear, while the second and the third are exactly Corollaries 3.5 and 3.6 of [14]. □

2 Deformations of products of compact complex manifolds

From now on we work over the field $\mathbb{C}$ of complex numbers; every complex manifold is assumed compact and connected.

By a general and extremely fruitful principle, introduced by Schlessinger-Stasheff [18], Deligne [1], Drinfeld and developed by many others, over a field of characteristic 0, every "reasonable" deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic DG-Lie algebras giving the same deformation theory.

For instance, deformations of a compact complex manifold X are controlled by the quasi-isomorphism class of the Kodaira-Spencer differential graded Lie algebra $KS_X=A^{0,*}_X(\Theta_X)$ of differential forms valued in the holomorphic tangent sheaf [6, 17]. This means that the functor $\operatorname{Def}_X\colon\mathbf{Art}\to\mathbf{Set}$ of infinitesimal deformations of X is isomorphic to the functors $\operatorname{Def}_{KS_X}$.

Here we must pay attention to the fact that the corresponding cohomology graded Lie algebra $H^*(A^{0,*}_X(\Theta_X))=H^*(X,\Theta_X)$ is not a complete invariant under quasi-isomorphisms and, in general, its knowledge is not sufficient to determine the deformation theory of X, although $H^1(X,\Theta_X)$ is the space of first order deformations, $H^2(X,\Theta_X)$ is an obstruction space and the quadratic bracket

$$q\colon H^1(X,\Theta_X)\to H^2(X,\Theta_X),\qquad q(\xi)=\frac{1}{2}[\xi,\xi],$$

is the obstruction to lifting a first order deformation of X up to second order. In particular the vanishing of the bracket on $H^*(X,\Theta_X)$ does not imply that X is unobstructed.

Whenever the Kodaira-Spencer algebra KS_X is formal, the deformations of X are determined by the graded Lie algebra $H^*(X,\Theta_X)$ and the base space of the Kuranishi family is analytically isomorphic to the germ at 0 of the nullcone of the quadratic map q.

As noticed above, in general the Kodaira-Spencer algebra is not formal, even for projective manifolds. For example, Vakil proved [19, Thm. 1.1] that for every analytic singularity $(U,0)$ defined over $\mathbb{Z}$ there exists a complex surface S with very ample canonical bundle such that its local moduli space is analytically isomorphic to the germ at 0 of $U\times\mathbb{C}^n$ for some integer $n\geq 0$. Choosing $U=\{(x,y)\in\mathbb{C}^2\mid xy(x-y)=0\}$ and taking S as above, the Kodaira-Spencer algebra of S cannot be formal. As a warning against possible mistakes, we note that such a surface S is obstructed although the bracket on $H^*(S,\Theta_S)$ is trivial.

Consider now two compact connected complex manifolds X,Y; given two deformations $X_A \to \operatorname{Spec}(A)$, $Y_A \to \operatorname{Spec}(A)$, of X,Y over the same basis, their fibred product

$$X_A \times_{\operatorname{Spec}(A)} Y_A \to \operatorname{Spec}(A)$$

is a deformation of the product $X \times Y$. Therefore it is well defined a natural transformation of functors

$$\alpha\colon \operatorname{Def}_X \times \operatorname{Def}_Y \to \operatorname{Def}_{X\times Y}\,.$$

It is easy to describe α in terms of morphisms of differential graded Lie algebras: denote by $p\colon X\times Y \to X$ and $q\colon X\times Y\to Y$ the projections; since

$$p_* p * \Theta_X = \Theta_X \otimes p_*\mathscr{O}_{X\times Y} = \Theta_X, \qquad q_* q * \Theta_Y = \Theta_Y \otimes q_*\mathscr{O}_{X\times Y} = \Theta_Y$$

and $\Theta_{X\times Y} = p^*\Theta_X \oplus q^*\Theta_Y$, we may define two natural injective morphisms of differential graded Lie algebras

$$p^*\colon KS_X \to KS_{X\times Y}, \qquad q^*\colon KS_Y \to KS_{X\times Y}.$$

Since $[p^*\eta, q^*\mu] = 0$ for every $\eta \in KS_X$, $\mu \in KS_Y$, we get a morphism of differential graded Lie algebras

$$p^* \times q^*\colon KS_X \times KS_Y \to KS_{X\times Y} \tag{1}$$

inducing α at the level of associated deformation functors.

Lemma 2. *Assume X,Y compact and connected. Then the morphism α is an isomorphism if and only if*

$$H^0(X,\Theta_X)\otimes H^1(Y,\mathscr{O}_Y) = H^1(X,\mathscr{O}_X)\otimes H^0(Y,\Theta_Y) = 0\,.$$

Proof. By Künneth formula ([8, Thm. 6.7.8], [10, Thm. 14]) we have:

$$\begin{aligned} H^i(X\times Y, \Theta_{X\times Y}) &= H^i(X\times Y, p^*\Theta_X)\oplus H^i(X\times Y, q^*\Theta_Y),\\ H^i(X\times Y, p^*\Theta_X) &= \bigoplus_j H^j(X,\Theta_X)\otimes H^{i-j}(Y,\mathscr{O}_Y),\\ H^i(X\times Y, q^*\Theta_Y) &= \bigoplus_j H^j(X,\mathscr{O}_X)\otimes H^{i-j}(Y,\Theta_Y). \end{aligned} \tag{2}$$

The morphism $p^*\colon KS_X \to KS_{X\times Y}$ is injective in cohomology and the image of $H^i(X,\Theta_X)$ is the subspace $H^i(X,\Theta_X)\otimes H^0(Y,\mathscr{O}_Y) \subset H^i(X\times Y, p^*\Theta_X)$; similarly for the morphism q^*. Thus, $H^0(KS_{X\times Y}) = H^0(KS_Y)\oplus H^0(KS_Y)$,

$$\begin{aligned} &H^1(KS_{X\times Y}) =\\ &\quad = H^1(KS_X)\oplus H^1(KS_Y)\oplus (H^0(X,\Theta_X)\otimes H^1(Y,\mathscr{O}_Y))\oplus(H^1(X,\mathscr{O}_X)\otimes H^0(Y,\Theta_Y)) \end{aligned}$$

and we have an injective map $H^2(KS_X)\oplus H^2(KS_Y) \to H^2(KS_{X\times Y})$.

If α is an isomorphism then, looking at first order deformations, we have

$$H^0(X,\Theta_X)\otimes H^1(Y,\mathscr{O}_Y) = H^1(X,\mathscr{O}_X)\otimes H^0(Y,\Theta_Y) = 0\,.$$

Conversely, it is sufficient to apply Theorem 1 to the DG-Lie morphism $p^*\times q^*$. ☐

The assumption of Lemma 2 is satisfied in most cases; for instance, a theorem of Matsumura [15] implies that $H^0(X,\Theta_X)=0$ for every compact manifold of general type X. If $H^1(X,\mathscr{O}_X)\otimes H^0(Y,\Theta_Y)\neq 0$, then it is easy to describe deformations of $X\times Y$ that are not a product. Assume that X is a Kähler manifold, then $b_1(X)\neq 0$ and there exists at least one surjective homomorphism $\pi_1(X)\xrightarrow{g}\mathbb{Z}$. Since $H^0(Y,\Theta_Y)\neq 0$, there exists at least a nontrivial one parameter subgroup $\{\theta_t\}\subset \mathrm{Aut}(Y)$, $t\in\mathbb{C}$, of holomorphic automorphisms of Y. Therefore we get a family of representations

$$\rho_t\colon \pi_1(X)\to \mathrm{Aut}(Y),\qquad \rho_t(\gamma)=\theta_t^{g(\gamma)},\qquad t\in\mathbb{C}$$

inducing a family of locally trivial analytic Y-bundles over X. Moreover, Kodaira and Spencer proved that projective spaces $\mathbb{P}^n$ and complex tori $(\mathbb{C}^q/\Gamma)$ have unobstructed deformations, while the product $(\mathbb{C}^q/\Gamma)\times\mathbb{P}^n$ has obstructed deformations for every $q\geq 2$ and every $n\geq 1$ [11, page 436]. This was the first example of obstructed manifold.

Let's denote by $B_X^*=\{\phi\in A_X^{0,*}\mid \partial\phi=0\}$ the DG-algebra of antiholomorphic differential forms on a complex manifold X. In the above setup we can define two morphisms

$$h_1\colon KS_X\otimes B_Y^*\to KS_{X\times Y},\qquad h_1(\phi\otimes\eta)=p^*(\phi)\wedge q^*(\eta),$$

$$h_2\colon B_X^*\otimes KS_Y\to KS_{X\times Y},\qquad h_2(\phi\otimes\eta)=p^*(\phi)\wedge q^*(\eta).$$

It is straightforward to check that h_1,h_2 are morphisms of differential graded Lie algebras and that the image of h_1 commutes with the image of h_2. This implies that the morphism (1) extends naturally to a morphism of differential graded Lie agebras

$$h\colon (KS_X\otimes B_Y^*)\times(B_X^*\otimes KS_Y)\to KS_{X\times Y} \tag{3}$$

Theorem 2. *For every pair of compact connected Kähler manifolds X,Y the morphism (3) is an injective quasi-isomorphism of differential graded Lie algebras. In particular, considering $H^*(X,\mathscr{O}_X)$ and $H^*(Y,\mathscr{O}_Y)$ as graded commutative algebras (with the usual cup product), there exists an isomorphism of functors*

$$\mathrm{Def}_{X\times Y}\cong \mathrm{Def}_{KS_X\otimes H^*(Y,\mathscr{O}_Y)}\times \mathrm{Def}_{KS_Y\otimes H^*(X,\mathscr{O}_X)}\,.$$

Proof. If X is compact Kähler, the $\partial\overline{\partial}$-lemma implies that $B_X^i\subset A_X^{0,i}$ is a set of representative for the Dolbeault cohomology group $H^i(X,\mathscr{O}_X)$ and therefore B_X^* is isomorphic to $H^*(X,\mathscr{O}_X)$ as a DG-algebra. Now, the formulas (2) imply immediately that the morphism (3) is a quasi-isomorphism. ☐

Corollary 2. *Let X,Y be compact Kähler manifolds. Then $KS_{X\times Y}$ is a formal DG-Lie algebra if and only if KS_X and KS_Y are formal.*

Proof. Immediate consequence of Lemma 1 and Theorem 2. ☐

3 A DG-Lie revisitation of an example by Douady

We want to prove, by a deeper study of a classical example by Douady [3, p. 18] that Corollary 2 fails without the Kähler assumption. The non Kähler manifold involved in this example is the Iwasawa manifold X, defined as the quotient of the group of complex matrices of type

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}$$

by the right action of the cocompact subgroup of matrices with coefficients in the Gauss integers. By a (non trivial) result by Nakamura [16, p. 96] (cf. also [6, Lemma 6.5]), the morphism of DG-algebras

$$j\colon R \to A_X^{0,*}, \qquad j(e_1) = d\overline{z_1}, \quad j(e_2) = d\overline{z_2}, \quad j(e_3) = d\overline{z_3} - \overline{z_1}d\overline{z_2},$$

is a quasi-isomorphism. Being X parallelizable the morphism of DG-Lie algebras $H^0(X,\Theta_X)\otimes R \to A_X^{0,*}(\Theta_X)$ is a quasi-isomorphism; in view of the isomorphism of Lie algebras $\mathfrak{n}_3(\mathbb{C}) \simeq H^0(X,\Theta_X)$:

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mapsto a\frac{\partial}{\partial z_1} + b\left(\frac{\partial}{\partial z_2} + z_1\frac{\partial}{\partial z_3}\right) + c\frac{\partial}{\partial z_3}\,.$$

we get that the Kodaira-Spencer algebra of the Iwasawa manifold X is quasi-isomorphic to the formal DG-Lie algebra $\mathfrak{n}_3(\mathbb{C})\otimes R$.

Consider now $Y = \mathbb{P}^1$, then $H^*(Y,\Theta_Y) = H^0(Y,\Theta_Y) \simeq \mathfrak{sl}_2(\mathbb{C})$ and therefore the Kodaira-Spencer algebra KS_Y is quasi-isomorphic to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Since every differential form in the image of j is antiholomorphic, as above we can define a morphism of DG-Lie algebras

$$(KS_X \otimes B_Y^*) \times (R \otimes KS_Y) \to KS_{X\times Y} \tag{4}$$

which, by Künneth formula is a quasi-isomorphism. Thus the Kodaira-Spencer algebra of $X\times Y$ is quasi-isomorphic to $(\mathfrak{n}_3(\mathbb{C})\otimes R)\times(\mathfrak{sl}_2(\mathbb{C})\otimes R)$.

Since $\mathfrak{sl}_2(\mathbb{C})\otimes R$ is not formal, by Lemma 1, also the Kodaira-Spencer algebra of $X\times Y$ is not formal. It is possible to prove, using the above results, that the base space of the Kuranishi family of $X\times Y$ is isomorphic to $(\mathbb{C}^6\times U, 0)$, where $U\subset\mathbb{C}^6$ is a cone defined by six homogeneous polynomials of degree 3.

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