ON THE T^1-LIFTING THEOREM

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This paper addresses two questions in the theory of functors of Artin rings that have raised some interest in recent years: smoothness as a consequence of the T^1-lifting condition and estimates on the dimension of the hull induced by estimates on the dimension of the space of curvilinear obstructions.

Let k be a field, Art_k the category of Artinian local algebras over k with residue field k, and let $F : Art_k \to Sets$ be a covariant functor with $F(k) = \{pt\}$ (following Schlessinger [Sch], we call such an F a functor of Artin rings). Following [Kaw1], we will say that F is a deformation functor if it satisfies Schlessinger's conditions (H1), (H2) and (H3) (that is, if it admits a hull) and if moreover we are given a k-vector space T^2_F which is an obstruction space for F (see definition 0.1). Recall that Schlessinger's conditions (H1) and (H2) imply that $F(k[\epsilon])$ is a vector space, which we call T^1_F (where as usual $k[\epsilon] = k[\epsilon]/\epsilon^2$).

One can then introduce the T^1-lifting property for F (see definition 1.1) and define in T^2_F the subset of curvilinear obstructions (see definition 2.1).

The main result of this paper is the following theorem:

Theorem A. *Let F be a deformation functor. If F satisfies the T^1-lifting condition and* $\operatorname{char} k = 0$, *then F is smooth.*

In the paper [Ran1] Ran introduced the T^1-lifting condition for the functor Def_X of infinitesimal deformations of a compact complex manifold X and proved that it implies smoothness. In [Kaw1] Kawamata extended the definition of T^1-lifting to an arbitrary functor, and proved theorem A under the additional assumption that F be prorepresentable.

In the first section of this paper we prove theorem A, and show how this leads to improvements of some of Kawamata's applications in [Kaw1].

In the second section of this paper we give a new and shorter proof of Ran-Kawamata's estimate on the dimension of the hull of a functor in terms of the curvilinear obstructions. We also point out that it is necessary to assume that the ground field is algebraically closed; in particular, we give an example of a prorepresentable functor over $\mathbb{R}$ for which the estimate does not hold.

This paper is a slightly modified version of [FM2]. We recently received an erratum [Kaw3] for [Kaw1] from Kawamata, written after receiving [FM2]. There he modifies the definition of the T^1-lifting property. This definition depends on the particular deformation problem at hand and not only on the functor. It is clear that the new definition implies the old one, therefore our result is stronger.

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0. Preliminaries and notation

We recall some basic notions about small extensions. A small extension of A by M is a short exact sequence

$$e: \qquad 0 \longrightarrow M \longrightarrow B \xrightarrow{f} A \longrightarrow 0$$

where f is a morphism in Art_k and the B-module M is a k-vector space; one can also say that $f: B \to A$ is a small extension. Small extensions of A by M up to isomorphism form a vector space $Ex(A,M)$; given a morphism $\phi: A' \to A$ in Art_k and a linear map $\psi: M \to M'$ there are induced linear maps $\phi^*: Ex(A,M) \to Ex(A',M)$ and $\psi_*: Ex(A,M) \to Ex(A,M')$.

Recall that if F satisfies (H1) and (H2), a small extension as above induces an action of $T^1_F \otimes M$ on $F(B)$, transitive on the fibres of $F(B) \to F(A)$. When M is one-dimensional we will talk of the action of T^1_F on $F(B)$, assuming that an isomorphism $M \to k$ as been fixed.

Definition 0.1. (cf [Kaw2], [FM1]) Let F be a functor of Artin rings, and let T^2_F be a k-vector space. We say that T^2_F is an *obstruction space* for F if for every small extension e as above we are given a map $ob_e: F(A) \to T^2_F \otimes M$ such that
(i) the image of $F(B) \to F(A)$ is $ob_e^{-1}(0)$;
(ii) ob_e is functorial.

Remark 0.2. Let T^2_F be an obstruction space for F. Fix $A \in Art_k$, $a \in F(A)$, M a finite dimensional k-vector space. Then the map $Ex(A,M) \to T^2_F \otimes M$ defined by $e \mapsto ob_e(a)$ is linear.

Let $\widehat{Art_k}$ be the category of complete local k-algebras R such that $R/\mathbf{m}_R^n \in Art_k$ for every $n \in \mathbb{N}$. One can define the notion of small extensions in $\widehat{Art_k}$ in the obvious way. For such an R, define $h_R(A) = Hom(R,A)$. Let T^2_R be the vector space $Ex(R,k)$; T^2_R can be computed as $(I/\mathbf{m}I)^\vee$, where $R = P/I$, $P = k[[x_1,\dots,x_n]]$, and I is an ideal contained in $\mathbf{m}_P^2$. There is a canonical small extension

$$u_R: \qquad 0 \longrightarrow T^{2\vee}_R \longrightarrow P/\mathbf{m}I \longrightarrow R \longrightarrow 0$$

which is universal, in that every small extension of R can be obtained from u_R by a unique pushforward.

Lemma 0.3. *T^2_R is naturally an obstruction space for h_R; every other obstruction space contains T^2_R canonically as a vector subspace.*

Proof. The obstruction map is defined by requiring that $ob_e(a) = g$ where e is a small extension as above, $a \in h_R(A)$ and $g \in Hom(T^2_R, M)$ is defined to be such that $a^*e = g_*u_R$. The rest of the proof is an elementary computation, see e.g. [FM1] §5. □

1. Proof of theorem A.

Definition 1.1. ([Ran1], [Kaw1]) Let $A_n = k[t]/(t^{n+1})$, $B_n = k[x,y]/(x^{n+1}, y^2)$ and let $B_n \to A_n$ be the map defined by $x \mapsto t, y \mapsto 0$. A functor of Artin rings F has the T^1*–lifting property* if, for every $n \in \mathbb{N}$, the natural map

$$F(B_{n+1}) \to F(B_n) \times_{F(A_n)} F(A_{n+1})$$

is surjective.

Proof of theorem A.. Arguing as in [Kaw1] it is sufficient to prove that for every integer $m \geq 1$ $F(A_{m+1}) \to F(A_m)$ is surjective. Fix such an m, and assume that $m \geq 2$.

In addition to A_n and B_n, we need to consider the following auxiliary k-algebras:

$$C_n = k[x,y]/(x^{n+1}, x^n y, y^2), V_n = k[t,s]/(t^{n+1}, t^2 s, s^2), A'_n = k[t,s]/(t^{n+1}, ts, s^2).$$

In order to reduce notation to a minimum, the following convention will be in force. All algebras we consider are quotients of either $A = k[t,s]$ or $B = k[x,y]$; a morphism between these rings and the induced morphism between any two quotients will be denoted by the same letter. Let $i : A \to A$ (resp. $j : B \to B$) be the identity, and let $f : A \to B$ (resp. $g : B \to A$, resp. $q : A \to A$) be defined by $f(t,s) = (x+y, x^m)$ (resp. $g(x,y) = (t,0)$, resp. $q(t,s) = (t, t^m)$).

As an example, the cartesian diagram

$$\begin{array}{ccc} C_n & \xrightarrow{g} & A_n \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle i} \\ B_{n-1} & \xrightarrow{g} & A_{n-1} \end{array}$$

induces a canonical isomorphism between C_n and $B_{n-1} \times_{A_{n-1}} A_n$ for all $n \geq 1$; in particular the T^1-lifting condition can be rephrased by saying that, given a $c \in F(C_n)$, there exists $b \in F(B_n)$ having the same projections to B_{n-1} and A_n.

If F satisfies (H4), then the T^1-lifting condition becomes that $F(B_n) \to F(C_n)$ is onto for every $n \in \mathbb{N}$. The cartesian diagram (in characteristic zero)

$$\begin{array}{ccc} A_{m+1} & \xrightarrow{i} & A_m \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f} \\ B_m & \xrightarrow{j} & C_m \end{array}$$

together with condition (H1) immediately implies that $j : F(A_{m+1}) \to F(A_m)$ is surjective and the theorem; in fact, this is essentially Kawamata's proof.

The general case requires a longer argument. Fix $a \in F(A_m)$, and let $c' = f(a) \in F(C_m)$. Let $b \in F(B_m)$ be chosen such that $c = j(b) \in F(C_m)$ has the same projection as c' to both $F(A_m)$ and $F(B_{m-1})$ (such a b exists by the T^1-lifting assumption). In particular note that $g(c) = a \in F(A_m)$.

Consider the two cartesian diagrams

$$\begin{array}{ccc} V_{m+1} & \xrightarrow{i} & V_m \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f} \\ B_m & \xrightarrow{j} & C_m \end{array} \qquad\qquad \begin{array}{ccc} V_{m+1} & \xrightarrow{i} & V_m \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q} \\ A_{m+1} & \xrightarrow{i} & A_m. \end{array}$$

Note that the first diagram is cartesian because the characteristic is zero.

We will prove that there exists $v \in F(V_m)$ such that $f(v) = c$ and $q(v) = a$; by condition (H1) applied to the first diagram, v lifts to $v' \in F(V_{m+1})$; considering the second diagram, we will deduce that a lifts to $F(A_{m+1})$.

To do this, note that $q : V_m \to A_m$ factors as

$$V_m \xrightarrow{i} A'_m \xrightarrow{f} C_m \xrightarrow{g} A_m.$$

The morphisms $i : A'_m \to A_m$ and $A'_m \to k[\epsilon]$ (induced by $t \mapsto 0$, $s \mapsto \epsilon$) define an isomorphism $A'_m \to k[\epsilon] \times_k A_m$; by condition (H2), the map $F(A'_m) \to t_F \times F(A_m)$ is an isomorphism. We will in the following omit this isomorphism and identify $F(A'_m)$ with $t_F \times F(A_m)$. Note that $q : F(A'_m) \to F(A_m)$ is precisely the t_F action on $F(A_m)$ induced by $i : A_m \to A_{m-1}$, as it is easy to verify.

Claim 1. *There exists $w \in t_F$ such that $(w \oplus a) \in F(A'_m)$ satisfies $f(w \oplus a) = c \in F(C_m)$ and $q(w \oplus a) = a \in F(A_m)$.*

Claim 2. *Let $a \in F(A_m)$ and $w \in t_F$ be such that $q(w \oplus a) = a \in F(A_m)$. Then there exists $v \in F(V_m)$ such that $i(v) = (w \oplus a) \in F(A'_m)$.*

Let v be the lifting of c obtained by combining the two claims. Then $q(v) = a \in F(A_m)$, and the proof is complete. □

Proof of claim 1. As $j(c) = j(c') \in F(B_{m-1})$, there exists $w \in t_F$ that $w \cdot c' = c$, via the natural action of t_F on $F(C_m)$ induced by $j : C_m \to B_{m-1}$.

We will now prove that $(w \oplus a)$ satisfies claim 1. Let $C_m[\epsilon] = C_m \times_k k[\epsilon]$, and identify $F(C_m[\epsilon])$ with $t_F \times F(C_m)$; then the action is induced by the map $\overline{j} : C_m[\epsilon] \to C_m$ given by $\overline{j}(x, y, \epsilon) = (x, y, x^m)$. By assumption $\overline{j}(w \oplus c') = c$. Note now that $f : A'_m \to C_m$ factors as

$$A'_m \xrightarrow{\overline{f}} C_m[\epsilon] \xrightarrow{\overline{j}} C_m,$$

where $\overline{f}(t, s) = (x + y, \epsilon)$. It is easy to verify that $\overline{f}(w \oplus a) = (w \oplus c')$, hence $f(w \oplus a) = c$.

Since $q : A'_m \to A_m$ factors as $A'_m \xrightarrow{f} C_m \xrightarrow{g} A_m$, $g(c) = a$ implies that $q(w \oplus a) = a$.

□

Proof of claim 2. Let e be the small extension

$$0 \longrightarrow k \xrightarrow{t^{m+1}} A_{m+1} \xrightarrow{i} A_m \longrightarrow 0;$$

we will prove that the small extension

$$0 \longrightarrow k \xrightarrow{ts} V_m \xrightarrow{i} A'_m \longrightarrow 0,$$

is exactly $q^*e - p^*e$. From this and 0.2 it follows that $(w \oplus a) \in F(A'_m)$ is unobstructed. In order to see this consider the universal small extension u of A'_m

$$u: \qquad 0 \longrightarrow \langle t^{m+1}, ts, s^2 \rangle \longrightarrow k[t,s]/(t^{m+2}, t^2 s, ts^2, s^3) \xrightarrow{i} A'_m \longrightarrow 0$$

and the following commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle t^{m+1}, ts, s^2 \rangle & \longrightarrow & k[t,s]/(t^{m+2}, t^2 s, ts^2, s^3) & \xrightarrow{i} & A'_m & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow i & & \downarrow i & & \\ 0 & \longrightarrow & k & \xrightarrow{t^{m+1}} & A_{m+1} & \xrightarrow{i} & A_m & \longrightarrow & 0 \end{array}$$

where $\phi(t^{m+1}) = 1$, $\phi(ts) = \phi(s^2) = 0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle t^{m+1}, ts, s^2 \rangle & \longrightarrow & k[t,s]/(t^{m+2}, t^2 s, ts^2, s^3) & \xrightarrow{i} & A'_m & \longrightarrow & 0 \\ & & \downarrow \psi & & \downarrow q & & \downarrow q & & \\ 0 & \longrightarrow & k & \xrightarrow{t^{m+1}} & A_{m+1} & \xrightarrow{i} & A_m & \longrightarrow & 0 \end{array}$$

where $\psi(t^{m+1}) = \psi(ts) = 1$, $\psi(s^2) = 0$. Hence $p^*e = \phi_*u$, $q^*e = \psi_*u$ and therefore $q^*e - p^*e = (\psi - \phi)_*u$ is the bottom row of the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle t^{m+1}, ts, s^2\rangle & \longrightarrow & k[t,s]/(t^{m+2}, t^2s, ts^2, s^3) & \xrightarrow{i} & A'_m & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \psi-\phi} & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle \|} & & \\ 0 & \longrightarrow & k & \xrightarrow{ts} & k[t,s]/(t^{m+1}, t^2s, s^2) & \xrightarrow{i} & A'_m & \longrightarrow & 0. \end{array}$$

Note that here we have used that $m \geq 2$; in fact, if $m = 1$, we would get $k[t,s]/(t^3, t^2s, ts - s^2)$ which is not isomorphic to V_1. □

Remark 1.2. In the above proof the case $m = 1$ is not included. One can prove that this is not needed to ensure smoothness, as in [FM1], lemma 5.6. One can also prove the case $m = 1$ directly using the simpler argument for functors satisfying also (H4), which applies in this case in view of condition (H2). Actually we find Ran's proof in [Ran1] difficult to understand, as he does not assume (H4) (that is, that $H^0(X, T_X) = 0$) and deals extensively only with the $m = 1$ case.

In the proof of theorem A we never used Schlessinger condition (H3). So we have in fact proven the following, stronger result.

Theorem A'. *Let F be a functor of Artin rings, and assume that F satisfies (H1) and (H2), has an obstruction space T^2_F and the T^1-lifting property. If* $\operatorname{char} k = 0$ *then $F(A_{m+1}) \to F(A_m)$ is surjective for every $m \geq 1$.*

Theorem A' implies in fact that such an F is unobstructed: this is proven in [FM1], corollary 6.15.

Corollary 1.3. In [Kaw1] one can remove the assumption F simple from theorem 3 and $H^0(X, T_X) = 0$ from theorem 4.

2. Dimension of local moduli

In the paper [Ran2] Ran gave an estimate on the dimension of some Hilbert schemes using deformation theory arguments. In [Kaw2] Kawamata raised some objections to Ran's proof and gave a new proof.

Definition 2.1. Let F be a deformation-type functor, and let $v \in T^2_F$. We say that v is a *curvilinear obstruction* if $v \otimes t^n$ is in the image of ob_{ϕ_n} for some $n \in \mathbb{N}$, where $\phi_n : k[t]/t^{n+1} \to k[t]/t^n$ is the obvious map.

Theorem 2.2. *Let F be a deformation functor, and let T^{2c}_F be the vector subspace of T^2_F generated by curvilinear obstructions. If k is algebraically closed, then the dimension of the hull of F is $\geq \dim T^1_F - \dim T^{2c}_F$.*

Example 2.3. If k is not algebraically closed then theorem 2.2 may fail. Assume for instance that $k = \mathbb{R}$, and let $R = k[x,y]/(x^3, y^3, x^2+y^2)$. Then $x^3, y^3 \in T^{2\vee}_R$ are in the kernel of every curvilinear obstruction; hence $\dim R = 0$, $\dim T^1_R = 2$, $\dim T^{2c}_R = 1$. Similar examples can be constructed whenever the base field k is not algebraically closed.

Lemma 2.4. *Let k be an algebraically closed field. Let $R \in \widehat{Art_k}$, and $g \in \mathbf{m}_R$ a non-nilpotent element. Then there is a local homomorphism $\psi : R \to k[[t]]$ such that $\psi(g) \neq 0$.*

Proof. As $g = 0$ is a proper subset of $Spec R$, there is a curve in $Spec R$ not contained in it. For a detailed proof, see [FM1], Lemma 5.4. □

Proposition 2.5. *Let $R \in \widehat{Art_k}$, and assume $k = \overline{k}$. Let T^{2c}_R be the vector subspace of T^2_R generated by curvilinear obstructions. Then $\dim R \geq \dim T^1_R - \dim T^{2c}_R$.*

Proof. Write $R = P/I$, with P a power series algebra and $I \subset \mathbf{m}_P^2$. Let $d = \dim T^2_R = \dim(I/\mathbf{m}I)$. By Nakayama's lemma, I can be generated by d elements, say $I = (f_1, \ldots, f_d)$. As k is algebraically closed, by repeated application of lemma 2.4 (and possibly reordering the f_i's) we can assume that there exists an $h \leq d$ such that the following holds:
1) $f_i \notin \sqrt{(f_1, \ldots, f_{i-1})}$ for $i \leq h$;
2) for all $i \leq h$ there exists a morphism $\phi_i : P \to k[[t]]$ in $\widehat{Art_k}$ such that $\phi_i(f_i) \neq 0$, $\phi_i(f_j) = 0$ if $j < i$ and $\deg \phi_i(f_j) \geq \deg \phi_i(f_i)$ if $j > i$;
3) $I \subset \sqrt{(f_1, \ldots, f_h)}$.
Condition 3) implies that $\dim R = \dim P/(f_1, \ldots, f_h)$, hence it is enough to prove $h \leq \dim T^{2c}_R$. For $i = 1, \ldots, h$, let $v_i \in T^{2c}_R$ be the obstruction induced by ϕ_i. Then, since $f_1, \ldots, f_d$ are linearly independent in $I/\mathbf{m}I$, the elements $v_1, \ldots, v_h$ are also linearly independent in $(I/\mathbf{m}I)^\vee$, as the associated matrix contains a triangular $h \times h$ minor with nonzero elements on the diagonal. □

Proof of theorem 2.2. Let $h_R \to F$ be a smooth morphism (that is, R is a hull for F). Then T^2_F is naturally an obstruction space for h_R, hence by 0.3 it contains T^2_R as a vector subspace and $T^{2c}_F = T^{2c}_R$. The theorem follows from proposition 2.5. □

Remark 2.6. Kawamata's proof of theorem 2.2 also uses the assumption $k = \overline{k}$, although this is not explicitly stated. In [Kaw2], Kawamata considers functors on Art_Λ, where Λ is a local complete noetherian k-algebra. Our proof can be easily adapted to this more general situation, with minor changes.

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