PROVE IT YOURSELF THE BAKER-CAMPBELL-HAUSDORFF FORMULA

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1. Notation and set-up

Let R be a unitary associative $\mathbb{Q}$-algebra and $I\subset R$ a nilpotent ideal. Denote by $1+I=\{1+a\mid a\in I\}\subset R$; notice that every element of $1+I$ is invertible and $(1+a)^{-1}=1+(-a+a^2+\cdots)\in 1+I$. Denote by $\operatorname{End}(R)$ the associative $\mathbb{Q}$-algebra of endomorphisms of R, considered as a vector space over $\mathbb{Q}$.

Define the **exponential**

$$e\colon I\to 1+I\subset R,\qquad e^a=\sum_{n\geq 0}\frac{a^n}{n!},$$

and the **logarithm**

$$\log\colon 1+I\to I,\qquad \log(1+a)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{a^n}{n}.$$

We assume already proved that exponential and logarithm are one the inverse of the other, i.e. for every $a,b\in I$ we have

$$\log(e^a)=a,\qquad e^{\log(1+b)}=1+b.$$

2. The Hausdorff formula

Given $a\in R$ denote

$$\operatorname{ad}a\colon R\to R,\qquad (\operatorname{ad}a)(x)=[a,x]=ax-xa.$$

Exercise 2.1. Prove that for every $a,b\in R$ and $n\geq 0$ we have

$$(\operatorname{ad}a)^n b=\sum_{i=0}^{n}(-1)^i\binom{n}{i}a^{n-i}ba^i=\sum_{i=0}^{n}\binom{n}{i}a^{n-i}b(-a)^i.$$

Deduce that if $a\in I$ then $\operatorname{ad}a$ is nilpotent in $\operatorname{End}(R)$ and therefore thre are defined the invertible operators

$$e^{\operatorname{ad}a}=\sum_{n\geq 0}\frac{(\operatorname{ad}a)^n}{n!}\in\operatorname{End}(R),$$

$$\frac{e^{\operatorname{ad}a}-I}{\operatorname{ad}a}=\sum_{n\geq 0}\frac{(\operatorname{ad}a)^n}{(n+1)!}\in\operatorname{End}(R),$$

$$\frac{\operatorname{ad}a}{e^{\operatorname{ad}a}-I}=\left(\frac{e^{\operatorname{ad}a}-I}{\operatorname{ad}a}\right)^{-1}=\sum_{n\geq 0}\frac{B_n}{n!}(\operatorname{ad}a)^n\in\operatorname{End}(R),$$

where B_n are the Bernoulli numbers.

Exercise 2.2. In the notation above prove that:

(1) For every $a\in I$ and $b\in R$

$$e^{\operatorname{ad}a}b:=\sum_{n\geq 0}\frac{(\operatorname{ad}a)^n}{n!}b=e^abe^{-a}.$$

(2) For every $a\in I$ and $b\in R$ we have $ab=ba$ if and only if $e^ab=be^a$.
(3) For every $a,b\in I$ we have $e^ab=be^a$ if and only if $e^ae^b=e^be^a$.
(4) Given $a,b\in I$ such that $ab=ba$, then

$$e^{a+b}=e^ae^b=e^be^a,\qquad \log((1+a)(1+b))=\log(1+a)+\log(1+b).$$

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Let t be a variable and denote by $d\colon R[t]\to R[t]$, $d(a)=a'$, the derivation operator:

$$(\sum a_n t^n)'=\sum n a_n t^{n-1}.$$

Multiplication on the left give an injective morphism of $\mathbb{Q}$-algebras

$$\phi\colon R[t]\to \operatorname{End}(R[t]),\qquad \phi(a)b=ab.$$

Prove that:

$$\phi(a')=[d,\phi(a)],\qquad \forall\, a\in R[t],$$

$$\phi(e^a)=e^{\phi(a)},\qquad \phi((e^a)')=de^{\phi(a)}-e^{\phi(a)}d,\qquad \phi((e^a)'e^{-a})=d-e^{\operatorname{ad}\phi(a)}d,\qquad \forall\, a\in I[t],$$

and deduce from the injectivity of ϕ that

$$(e^a)'e^{-a}=\frac{e^{\operatorname{ad} a}-1}{\operatorname{ad} a}(a').$$

Now, let $a,b\in I$ and define

$$Z=\log(e^{ta}e^b)\in I[t].$$

Prove that

$$\frac{e^{\operatorname{ad} Z}-1}{\operatorname{ad} Z}(Z')=(e^Z)'e^{-Z}=a.$$

and therefore

Therefore $Z=Z(t)$ is the solution of the Cauchy problem

$$Z'=\frac{\operatorname{ad} Z}{e^{\operatorname{ad} Z}-1}(a)=\sum_{n\geq 0}\frac{B_n}{n!}(\operatorname{ad} Z)^n(a),\qquad Z(0)=Z_0=b,$$

where the B_n's are the Bernoulli numbers ($\sum \frac{B_n}{n!}t^n=\frac{t}{e^t-1}$).

Theorem 2.3. *Given $a,b\in I$ we have*

$$e^ae^b=e^{a\bullet b},\quad \textit{where}\quad a\bullet b=\sum_{n\geq 0}Z_n,$$

and

$$Z_0=b,\qquad Z_{r+1}=\frac{1}{r+1}\sum_{m\geq 0}\frac{B_m}{m!}\sum_{i_1+\cdots+i_m=r}(\operatorname{ad} Z_{i_1})(\operatorname{ad} Z_{i_2})\cdots(\operatorname{ad} Z_{i_m})a.$$

Proof. Exercise. Hint $\log(e^{ta}e^b)=Z=Z_0+tZ_1+\cdots+t^nZ_n+\cdots$. □

The first terms of the above series are

$$a\bullet b=a+b+\frac{1}{2}[a,b]+\frac{1}{12}[a,[a,b]]+\frac{1}{12}[b,[a,b]]+\cdots$$

Since $(e^ae^b)e^c=e^a(e^be^c)$ the product $I\times I\xrightarrow{\bullet}I$ is associative. If L is a Lie subalgebra of I and $a,b\in L$, then $a\bullet b\in L$ and $a\bullet b-a-b$ belongs to the Lie ideal generated by $[a,b]$.

The formula of Theorem 2.3 allows to define for every nilpotent Lie algebra L a map

$$L\times L\to L,\qquad (a,b)\mapsto a\bullet b$$

commuting with morphisms of Lie algebras. Notice that if $[a,b]=0$ then $a\bullet b=a+b$ and then $a\bullet(-a)=0$.

3. Tree summation formula for BCH product

Recall that a tree is called a *rooted tree* if one vertex has been designated the *root*. Every rooted tree has a natural structure of directed tree such that, for every vertex u, there exists a unique directed path from u to the root. We shall write $u\to v$ if the vertex v belongs to the directed path from u to the root. A *leaf* is a vertex without incoming edges. A vertex is called *internal* if it is not a leaf.

From now on, we consider only planar binary rooted trees; we denote by $\mathcal{B}$ the set of finite planar binary rooted trees with the root at the top and the leaves at the bottom (i.e., every directed path moves upward); binary means that every internal vertex has exactly two incoming edges.

We also write

$$\mathcal{B}=\bigcup_{n>0}\mathcal{B}_n,$$

where $\mathcal{B}_n$ is the set of planar binary rooted trees with n leaves and, for every $\Gamma\in\mathcal{B}$, we denote by $L(\Gamma)$ the set of leaves of Γ. The planarity of the tree gives, for every internal vertex v, a total ordering of the edges ending on v, from the leftmost to the rightmost.

Definition 3.1. A *rightmost branch* of a planar binary rooted tree $\Gamma \in \mathcal{B}$ is a maximal connected subgraph $\Omega \subset \Gamma$, with at least two vertices and with the property that every edge of Ω is a rightmost edge of Γ.

Definition 3.2. A *local rightmost leaf* is a leaf lying on a rightmost branch. Given an internal vertex v, we call $m(v)$ the leaf lying on the rightmost branch containing v. We also denote by $d(v)$ the distance between v and $m(v)$.

Definition 3.3. A *subroot* is the vertex of a rightmost branch which is nearest to the root. The set of subroots of a finite planar rooted tree Γ will be denoted by $R(\Gamma)$.

Let R be a (non associative) algebra over $\mathbb{Q}$ and $\Gamma \in \mathcal{B}$. Labelling the leaves of Γ with elements of R, we can associate the product element in R obtained by the usual operadic rules, i.e., we perform the product of R at every internal vertex in the order arising from the planar structure of the directed tree. Given any map $f : L(\Gamma) \to R$ (the labelling), we denote by $Z_\Gamma(f) \in R$ the corresponding product element.

Definition 3.4. Given two leaves l_1 and l_2 in $\Gamma \in \mathcal{B}$, we say $l_1 \preceq l_2$ if $l_1 = l_2$ or there exists a subroot $v \in R(\Gamma)$ such that $l_2 = m(v)$ and $l_1 \to v$.

Definition 3.5. For every poset $(A, \leq)$, we denote

$$\mathcal{B}(A) = \{(\Gamma, f) \,|\, \Gamma \in \mathcal{B},\, f : (L(\Gamma), \preceq) \to (A, \leq), f \text{ monotone } \}$$

In a similar way we define $\mathcal{B}_n(A)$, for every $n > 0$.

Definition 3.6. Let $b_n = B_n/n!$, or equivalently $\sum b_n t^n = \frac{t}{e^t - 1}$. Given a poset A and $(\Gamma, f) \in \mathcal{B}(A)$, let us define

$$c_{(\Gamma,f)} := \prod_{v \in R(\Gamma)} \frac{b_{d(v)}}{t(v)} \,,$$

where for every subroot $v \in R(\Gamma)$, we have

$$t(v) = \text{ number of leaves } u \in L(\Gamma) \text{ such that } u \to v \text{ and } f(u) = f(m(v)).$$

Theorem 3.7. *Let L be a nilpotent Lie algebra, then for every $a, b \in L$ we have*

$$a \bullet b = \sum_{(\Gamma,f) \in \mathcal{B}(b \leq a)} c_{(\Gamma,f)} Z_\Gamma(f). \tag{1}$$

Proof. Left as exercise. Hint: let $\mathcal{A} \subset \mathcal{B}(b \leq a)$ be the subset of trees having every local rightmost leave labelled by a; then we have

$$\sum_{(\Gamma,f) \in \mathcal{B}(b \leq a)} c_{(\Gamma,f)} Z_\Gamma(f) = b + \sum_{(\Gamma,f) \in \mathcal{A}} c_{(\Gamma,f)} Z_\Gamma(f).$$

Now use Theorem 2.3 and induction on the number of subroots. □

References: details about Hausdorff formula and its proof can be found in the book of B.C. Hall: *Lie Groups, Lie Algebras, and representations. An elementary introduction.* A detailed proof of Theorem 3.7 will appear in a forthcoming paper by D. Iacono and M. Manetti.