# DEFORMATIONS OF COMPLEX MANIFOLDS AND HOLOMORPHIC MAPS

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## Lecture 1. Differential graded lie algebras and deformation functors

Unless otherwise specified, every vector space is considered over a fixed field $\mathbb{K}$ of characteristic 0; by the symbol $\otimes$ we mean the tensor product $\otimes_{\mathbb{K}}$ over the field $\mathbb{K}$.

We denote by $\mathbf{G}$ the category of $\mathbb{Z}$-graded $\mathbb{K}$-vector spaces. The objects of $\mathbf{G}$ are the $\mathbb{K}$-vector spaces $V$ endowed with a $\mathbb{Z}$-graded direct sum decomposition $V = \oplus_{i\in\mathbb{Z}} V_i$. The elements of $V_i$ are called homogeneous of degree $i$. The morphisms in $\mathbf{G}$ are the degree-preserving linear maps.

If $V = \oplus_{n\in\mathbb{Z}} V_n \in \mathbf{G}$ we write $\deg(a; V) = i \in \mathbb{Z}$ if $a \in V_i$; if there is no possibility of confusion about $V$ we simply denote $\overline{a} = \deg(a; V)$.

Given two graded vector spaces $V, W \in \mathbf{G}$ we denote by $\mathrm{Hom}^n(V, W)$ the vector space of $\mathbb{K}$-linear maps $f\colon V \to W$ such that $f(V_i) \subset W_{i+n}$ for every $i \in \mathbb{Z}$. Observe that $\mathrm{Hom}^0(V, W) = \mathrm{Hom}_{\mathbf{G}}(V, W)$ is the space of morphisms in the category $\mathbf{G}$.

The tensor product, $\otimes\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, and the *internal Hom*, $\mathrm{Hom}^*\colon \mathbf{G}^{op} \times \mathbf{G} \to \mathbf{G}$, are defined in the following way: given $V, W \in \mathbf{G}$, we set

$$V \otimes W = \bigoplus_{i\in\mathbb{Z}} (V \otimes W)_i, \quad \text{where} \quad (V \otimes W)_i = \bigoplus_{j\in\mathbb{Z}} V_j \otimes W_{i-j},$$

$$\mathrm{Hom}^*(V, W) = \bigoplus_n \mathrm{Hom}^n_{\mathbb{K}}(V, W).$$

**Definition 1.1.** If $V, W \in \mathbf{G}$, the *twist map* $\mathtt{tw}\colon V \otimes W \to W \otimes V$ is the linear map defined by the rule $\mathtt{tw}(v \otimes w) = (-1)^{\overline{v}\,\overline{w}} w \otimes v$, for every pair of homogeneous elements $v \in V$, $w \in W$.

Unless otherwise specified we shall use the *Koszul signs convention*. This means that we choose as natural isomorphism between $V \otimes W$ and $W \otimes V$ the twist map $\mathtt{tw}$ and we make every commutation rule compatible with $\mathtt{tw}$. More informally, to "get the signs right", whenever an "object of degree $d$ passes on the other side of an object of degree $h$, a sign $(-1)^{dh}$ must be inserted". As an example, if $f, g \in \mathrm{Hom}^*(V, W)$, then $f \otimes g \in \mathrm{Hom}^*(V \otimes V, W \otimes W)$ is defined by the rule $(f \otimes g)(u \otimes v) = (-1)^{\overline{g}\,\overline{u}} f(u) \otimes g(v)$.

We denote by $\mathbf{DG}$ the category of $\mathbb{Z}$-graded differential $\mathbb{K}$-vector spaces (also called complexes of vector spaces). The objects of $\mathbf{DG}$ are the pairs $(V, d)$ where $V = \oplus V_i$ is an object of $\mathbf{G}$ and $d\colon V \to V$ is a linear map, called *differential* such that $d(V_i) \subset V_{i+1}$ and $d^2 = d \circ d = 0$. The morphisms in $\mathbf{DG}$ are the degree-preserving linear maps commuting with the differentials.

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For simplicity we will often consider **G** as the full subcategory of **DG** whose objects are the complexes with trivial differential.

Given $(V,d)$ in **DG** we denote as usual by $Z^*(V)=\ker d$ the space of cocycles, by $B^*(V)=d(V)$ the space of coboundaries and by $H^*(V)=Z^*(V)/B^*(V)$ the cohomology of $V$. A morphism in **DG** is called a *quasi-isomorphism* if it induces an isomorphism in homology. A differential graded vector space $(V,d)$ is called *acyclic* if $H^*(V)=0$.

If $(V,d),(W,\delta)\in \mathbf{DG}$ then also $(V\otimes W, d\otimes Id+Id\otimes\delta)\in\mathbf{DG}$; according to Koszul signs convention, since $\delta\in \operatorname{Hom}^1_{\mathbb{K}}(W,W)$, we have $(Id\otimes\delta)(v\otimes w)=(-1)^{\overline{v}}v\otimes\delta(w)$. Notice also that the definition of the differential on tensor products commutes with twist maps, i.e.

$$\mathtt{tw}\circ(d\otimes Id+Id\otimes\delta)=(\delta\otimes Id+Id\otimes d)\circ\mathtt{tw}\colon V\otimes W\to W\otimes V.$$

There exists also a natural differential $\rho$ on $\operatorname{Hom}^*(V,W)$ given by the formula

$$\rho f=\delta\circ f-(-1)^{\overline{f}}f\circ d,$$

$$(\rho f)v=\delta(fv)-(-1)^{\overline{f}}f(dv),\qquad \text{for every } v\in V.$$

The Kunneth's formulas assert that the natural maps

$$H^*(V)\otimes H^*(W)\to H^*(V\otimes W),\qquad H^*(\operatorname{Hom}^*(V,W))\to \operatorname{Hom}^*(H^*(V),H^*(W))\ ,$$

are isomorphisms of graded vector spaces. In particular if $W$ is acyclic then also $V\otimes W$, $\operatorname{Hom}^*(V,W)$ and $\operatorname{Hom}^*(W,V)$ are acyclic.

The fiber product of two morphisms $B\xrightarrow{f}D$ and $C\xrightarrow{h}D$ in the category **DG** is defined as the complex

$$C\times_D B=\bigoplus_n (C\times_D B)_n,\qquad (C\times_D B)_n=\{(c,b)\in C_n\times B_n\,|\,h(c)=f(b)\},$$

with differential $d(c,b)=(dc,db)$. We point out that if $f$ is a surjective quasi-isomorphism, then also the projection $C\times_D B\to C$ is a surjective quasi-isomorphism.

**Definition 1.2.** A graded (associative, $\mathbb{Z}$-commutative) algebra is a graded vector space $A=\oplus A_i\in\mathbf{G}$ endowed with a product $A_i\times A_j\to A_{i+j}$ satisfying the properties:

(1) $a(bc)=(ab)c$.
(2) $a(b+c)=ab+ac$, $(a+b)c=ac+bc$.
(3) (Koszul signs convention) $ab=(-1)^{\overline{a}\,\overline{b}}ba$ for $a,b$ homogeneous.

The algebra $A$ is unitary if there exists $1\in A_0$ such that $1a=a1=a$ for every $a\in A$.

Let $A$ be a graded algebra, then $A_0$ is a commutative $\mathbb{K}$-algebra in the usual sense; conversely every commutative $\mathbb{K}$-algebra can be considered as a graded algebra concentrated in degree 0. If $I\subset A$ is a homogeneous left (resp.: right) ideal then $I$ is also a right (resp.: left) ideal and the quotient $A/I$ has a natural structure of graded algebra.

**Example 1.3.** The exterior algebra $A=\bigwedge^* V$ of a $\mathbb{K}$-vector space $V$, endowed with wedge product, is a graded algebra with $A_i=\bigwedge^i V$.

**Example 1.4** (Polynomial algebras)**.** Given a set $\{x_i\}$, $i\in I$, of homogeneous indeterminates of integral degree $\overline{x_i}\in\mathbb{Z}$ we can consider the graded algebra $\mathbb{K}[\{x_i\}]$. As a $\mathbb{K}$-vector space $\mathbb{K}[\{x_i\}]$ is generated by monomials in the indeterminates $x_i$ subjected to the relations $x_ix_j=(-1)^{\overline{x_i}\,\overline{x_j}}x_jx_i$. In a similar way it is defined $A[\{x_i\}]$ for every graded algebra $A$.

Notice that the exterior algebras are exactly the polynomial algebras where every indeterminate has degree $+1$.

**Definition 1.5.** A *dg-algebra* (differential graded algebra) is the data of a graded algebra $A$ and a $\mathbb{K}$-linear map $s\colon A\to A$, called *differential*, with the properties:

(1) $s(A_n)\subset A_{n+1}$, $s^2=0$;
(2) (graded Leibnitz rule) $s(ab)=s(a)b+(-1)^{\overline{a}}as(b)$.

A morphism of dg-algebras is a morphism of graded algebras commuting with differentials; the category of dg-algebras is denoted by **DGA**.

**Example 1.6** (Koszul algebras)**.** Let $V$ be a vector space and consider the graded algebra

$$A=\bigoplus_{i\le 0}A_i,\qquad A_i=\bigwedge^{-i}V\,,$$

with the wedge product as a multiplication map. Given a linear map $f\colon V\to\mathbb{K}$, we may define a differential $s\colon A_i\to A_{i+1}$

$$s=f\lrcorner\colon \bigwedge^{-i}V\to\bigwedge^{-i+1}V\,,\qquad i<0\,,$$

where the contraction operator $\lrcorner$ is defined by the formula

$$f\lrcorner(v_1\wedge\cdots\wedge v_h)=\sum_{j=1}^{h}(-1)^{j-1}f(v_j)v_1\wedge\cdots\wedge\widehat{v_j}\wedge\cdots\wedge v_h.$$

Leibnitz rule implies that on a polynomial algebra $\mathbb{K}[\{x_i\}]$, every differential $s$ is uniquely determined by the values $s(x_i)$.

**Example 1.7.** Let $t,dt$ be indeterminates of degrees $\overline{t}=0$, $\overline{dt}=1$; on the polynomial algebra $\mathbb{K}[t,dt]=\mathbb{K}[t]\oplus\mathbb{K}[t]dt$ there exists an obvious differential $d$ such that $d(t)=dt$, $d(dt)=0$. Since $\mathbb{K}$ has characteristic 0, we have $H^*(\mathbb{K}[t,dt])=H^0(\mathbb{K}[t,dt])=\mathbb{K}$. More generally if $(A,s)$ is a dg-algebra then $A[t,dt]=A\otimes\mathbb{K}[t,dt]$ is a dg-algebra, with differential

$$s(a\otimes p(t))=s(a)\otimes p(t)+(-1)^{\overline{a}}a\otimes p'(t)dt,\quad s(a\otimes q(t)dt)=s(a)\otimes q(t)dt.$$

**Definition 1.8.** A *differential graded Lie algebra* (DGLA for short) is the data of a differential graded vector space $(L,d)$ together a with bilinear map $[-,-]\colon L\times L\to L$ (called bracket) of degree 0 such that:

(1) (graded skewsymmetry) $[a,b]=-(-1)^{\deg(a)\deg(b)}[b,a]$.
(2) (graded Jacobi identity) $[a,[b,c]]=[[a,b],c]+(-1)^{\deg(a)\deg(b)}[b,[a,c]]$.
(3) (graded Leibniz rule) $d[a,b]=[da,b]+(-1)^{\deg(a)}[a,db]$.

The Leibniz rule implies in particular that the bracket of a DGLA $L$ induces a structure of graded Lie algebra on its cohomology $H^*(L)=\oplus_i H^i(L)$.

**Example 1.9.** Given a differential graded vector space $(V,\overline{\partial})$, the space $\operatorname{Hom}^*(V,V)$, with the bracket

$$[f,g]=fg-(-1)^{\deg(f)\deg(g)}gf$$

and the differential

$$df=[\overline{\partial},f]=\overline{\partial}f-(-1)^{\deg(f)}f\overline{\partial}$$

is a differential graded Lie algebra. The natural map

$$H^*(\operatorname{Hom}^*(V,V))\xrightarrow{\simeq}\operatorname{Hom}^*(H^*(V),H^*(V)).$$

is an isomorphism of graded Lie algebras

**Example 1.10.** Given a differential graded Lie algebra $L$ and a commutative $\mathbb{K}$-algebra $\mathfrak{m}$ there exists a natural structure of DGLA in the tensor product $L\otimes\mathfrak{m}$ given by

$$d(x\otimes r)=dx\otimes r,\qquad [x\otimes r,y\otimes s]=[x,y]\otimes rs.$$

If $\mathfrak{m}$ is nilpotent (for example if $\mathfrak{m}$ is the maximal ideal of a local artinian $\mathbb{K}$-algebra), then the DGLA $L\otimes\mathfrak{m}$ is nilpotent; under this assumption, for every $a\in L^0\otimes\mathfrak{m}$ the operator

$$\operatorname{ad}a\colon L\otimes\mathfrak{m}\to L\otimes\mathfrak{m},\qquad \operatorname{ad}a(b)=[a,b]\,,$$

is a nilpotent derivation and

$$e^{\operatorname{ad}a}=\sum_{n=0}^{+\infty}\frac{(\operatorname{ad}a)^n}{n!}\colon L\otimes\mathfrak{m}\to L\otimes\mathfrak{m}$$

is an automorphism of the differential graded Lie algebra $L\otimes\mathfrak{m}$.

In order to introduce the basic ideas of the use of DGLAs in deformation theory, we begin with an example where technical difficulties are reduced at minimum [16]. Consider a finite complex of vector spaces

$$(V,\overline{\partial}):\qquad 0\longrightarrow V^0\xrightarrow{\overline{\partial}}V^1\xrightarrow{\overline{\partial}}\cdots\xrightarrow{\overline{\partial}}V^n\longrightarrow 0.$$

Given a local artinian $\mathbb{K}$-algebra $A$ with maximal ideal $\mathfrak{m}_A$ and residue field $\mathbb{K}$, we define a deformation of $(V,\overline{\partial})$ over $A$ as a complex of $A$-modules of the form

$$0\longrightarrow V^0\otimes A\xrightarrow{\overline{\partial}_A}V^1\otimes A\xrightarrow{\overline{\partial}_A}\cdots\xrightarrow{\overline{\partial}_A}V^n\otimes A\longrightarrow 0,$$

such that its residue modulo $\mathfrak{m}_A$ gives the complex $(V,\overline{\partial})$. By base change $\operatorname{Hom}_A(V^i\otimes A,V^j\otimes A)=\operatorname{Hom}(V^i,V^j\otimes A)$ and, since $A$ is a finite dimensional vector space over $\mathbb{K}$, we have $\operatorname{Hom}(V^i,V^j\otimes A)=\operatorname{Hom}(V^i,V^j)\otimes A$. Since, as a $\mathbb{K}$ vector space, $A=\mathbb{K}\oplus\mathfrak{m}_A$, the above condition are equivalent to say that

$$\overline{\partial}_A=\overline{\partial}+\xi,\qquad\text{where}\quad \xi\in\operatorname{Hom}^1(V,V)\otimes\mathfrak{m}_A.$$

The "integrability" condition $\overline{\partial}_A^2=0$ becomes

$$0=(\overline{\partial}+\xi)^2=\overline{\partial}\xi+\xi\overline{\partial}+\xi^2=d\xi+\frac{1}{2}[\xi,\xi],$$

where $d$ and $[\ ,\ ]$ are the differential and the bracket on the differential graded Lie algebra $\operatorname{Hom}^*(V,V)\otimes\mathfrak{m}_A$ (Example 1.10). Two deformations $\overline{\partial}_A,\overline{\partial}_A'$ are isomorphic if there exists a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & V^0\otimes A & \xrightarrow{\overline{\partial}_A} & V^1\otimes A & \xrightarrow{\overline{\partial}_A} & \cdots & \xrightarrow{\overline{\partial}_A} & V^n\otimes A & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\phi_0} & & \downarrow{\scriptstyle\phi_1} & & & & \downarrow{\scriptstyle\phi_n} & & \\
0 & \longrightarrow & V^0\otimes A & \xrightarrow{\overline{\partial}_A'} & V^1\otimes A & \xrightarrow{\overline{\partial}_A'} & \cdots & \xrightarrow{\overline{\partial}_A'} & V^n\otimes A & \longrightarrow & 0
\end{array}$$

such that every $\phi_i$ is an isomorphism of $A$-modules whose specialization to the residue field is the identity. Therefore we can write $\phi:=\sum_i\phi_i=Id+\eta$, where $\eta\in\operatorname{Hom}^0(V,V)\otimes\mathfrak{m}_A$ and, since $\mathbb{K}$ is assumed of characteristic 0 we can take the logarithm and write $\phi=e^a$ for some $a\in\operatorname{Hom}^0(V,V)\otimes\mathfrak{m}_A$. The commutativity of the diagram is therefore given by the equation $\overline{\partial}_A'=e^a\circ\overline{\partial}_A\circ e^{-a}$. Writing $\overline{\partial}_A=\overline{\partial}+\xi$, $\overline{\partial}_A'=\overline{\partial}+\xi'$ and using the relation $e^a\circ b\circ e^{-a}=e^{\operatorname{ad}a}(b)$ we get

$$\xi'=e^{\operatorname{ad}a}(\overline{\partial}+\xi)-\overline{\partial}=\xi+\frac{e^{\operatorname{ad}a}-1}{\operatorname{ad}a}([a,\xi]+[a,\overline{\partial}])=\xi+\sum_{n=0}^{\infty}\frac{(\operatorname{ad}a)^n}{(n+1)!}([a,\xi]-da).$$

In particular, both the integrability condition and isomorphism are entirely written in terms of the DGLA structure of $\mathrm{Hom}^*(V,V)\otimes\mathfrak{m}_A$. This leads to the following general construction.

Denote by **Art** the category of local artinian $\mathbb{K}$-algebras with residue field $\mathbb{K}$ and by **Set** the category of sets (we ignore all the set-theoretic problems, for example by restricting to some universe). Unless otherwise specified, for every objects $A\in\mathbf{Art}$ we denote by $\mathfrak{m}_A$ its maximal ideal. Given a differential graded Lie algebra $L$ we define a covariant functor $\mathrm{MC}_L\colon\mathbf{Art}\to\mathbf{Set}$,

$$\mathrm{MC}_L(A)=\left\{x\in L^1\otimes\mathfrak{m}_A \mid dx+\frac{1}{2}[x,x]=0\right\}$$

The equation $dx+[x,x]/2=0$ is called the *Maurer-Cartan* equation and $\mathrm{MC}_L$ is called the Maurer-Cartan functor associated with $L$.
Two elements $x,y\in L\otimes\mathfrak{m}_A$ are said to be *gauge equivalent* if there exists $a\in L^0\otimes\mathfrak{m}_A$ such that

$$y=e^a*x:=x+\sum_{n=0}^{\infty}\frac{(\mathrm{ad}\,a)^n}{(n+1)!}([a,x]-da).$$

The operator $*$ is called *gauge action*; in fact we have $e^a*(e^b*x)=e^{a\bullet b}*x$, where $\bullet$ is the Baker-Campbell-Hausdorff product in the nilpotent Lie algebra $L^0\otimes\mathfrak{m}_A$, and then $*$ is an action of the exponential group $\exp(L^0\otimes\mathfrak{m}_A)$ on the graded vector space $L\otimes\mathfrak{m}_A$.
It is not difficult to see that the set of solutions of the Maurer-Cartan equation is stable under the gauge action and then it makes sense to consider the functor $\mathrm{Def}_L\colon\mathbf{Art}\to\mathbf{Set}$ defined as

$$\mathrm{Def}_L(A)=\frac{\mathrm{MC}_L(A)}{\text{gauge equivalence}}.$$

*Remark* 1.11. Given a surjective morphism $A\xrightarrow{\alpha}B$ in the category **Art**, an element $x\in\mathrm{MC}_L(B)$ can be lifted to $\mathrm{MC}_L(A)$ if and only if its equivalence class $[x]\in\mathrm{Def}_L(B)$ can be lifted to $\mathrm{Def}_L(A)$. In fact if $[x]$ lifts to $\mathrm{Def}_L(A)$ then there exists $y\in\mathrm{MC}_L(A)$ and $b\in L^0\otimes\mathfrak{m}_B$ such that $\alpha(y)=e^b*x$. It is therefore sufficient to lift $b$ to an element $a\in L^0\otimes\mathfrak{m}_A$ and consider $x'=e^{-a}*y$.

The above computation shows that the functor of infinitesimal deformations of a complex $(V,\overline{\partial})$ is isomorphic to $\mathrm{Def}_L$, where $L$ is the differential graded Lie algebra $\mathrm{Hom}^*(V,V)$.

The utility of this approach relies on the following result, sometimes called *basic theorem of deformation theory.*

**Theorem 1.12** (Schlessinger-Stasheff, Deligne, Goldman-Millson)**.** *Let $f\colon L\to M$ be a morphism of differential graded Lie algebras (i.e. $f$ commutes with differential and brackets). Then $f$ induces a natural transformation of functors* $\mathrm{Def}_L\to\mathrm{Def}_M$*. Moreover, if:*

(1) $f\colon H^0(L)\to H^0(M)$ *is surjective;*
(2) $f\colon H^1(L)\to H^1(M)$ *is bijective;*
(3) $f\colon H^2(L)\to H^2(M)$ *is injective;*

*then* $\mathrm{Def}_L\to\mathrm{Def}_M$ *is an isomorphism.*

*Proof.* See e.g. [14]. □

**Definition 1.13.** On the category of differential graded Lie algebras consider the equivalence relation generated by: $L\sim M$ *if there exists a quasiisomorphism* $L\to M$. We shall say that two DGLAs are *quasiisomorphic* if they are equivalent under this relation.

**Example 1.14.** Denote by $\mathbb{K}[t,dt]$ the differential graded algebra of polynomial differential forms over the affine line and for every DGLA $L$ denote $L[t,dt] = L\otimes\mathbb{K}[t,dt]$. As a graded vector space $L[t,dt]$ is generated by elements of the form $aq(t)+bp(t)dt$, for $p,q\in\mathbb{K}[t]$ and $a,b\in L$. The differential and the bracket on $L[t,dt]$ are

$$d(aq(t)+bp(t)dt) = (da)q(t) + (-1)^{\deg(a)}aq(t)'dt + (db)p(t)dt,$$

$$[aq(t), ch(t)] = [a,c]q(t)h(t), \quad [aq(t), ch(t)dt] = [a,c]q(t)h(t)dt.$$

For every $s\in\mathbb{K}$, the evaluation morphism

$$e_s\colon L[t,dt]\to L, \quad e_s(aq(t)+bp(t)dt) = q(s)a$$

is a quasiisomorphism of differential graded Lie algebras.

**Corollary 1.15.** *If $L, M$ are quasiisomorphic DGLAs, then there exists an isomorphism of functors $\operatorname{Def}_L\simeq\operatorname{Def}_M$.*

**Definition 1.16.** A differential graded Lie algebra $L$ is called *formal* if it is quasiisomorphic, to its cohomology graded Lie algebra $H^*(L)$.

**Lemma 1.17.** *For every differential graded vector space $(V,\overline{\partial})$, the differential graded Lie algebra $\operatorname{Hom}^*(V,V)$ is formal.*

*Proof.* For every index $i$ we choose a vector subspace $H^i\subset Z^i(V)$ such that the projection $H^i\to H^i(V)$ is bijective. The graded vector space $H=\oplus H^i$ is a quasiisomorphic subcomplex of $V$. The subspace $K=\{f\in\operatorname{Hom}^*(V,V)\mid f(H)\subset H\}$ is a differential graded Lie subalgebra and there exists a commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\alpha} & \operatorname{Hom}^*(V,V) & \longrightarrow & \operatorname{Hom}^*(H,V/H) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle Id} & & \\
0 & \longrightarrow & \operatorname{Hom}^*(H,H) & \longrightarrow & \operatorname{Hom}^*(H,V) & \longrightarrow & \operatorname{Hom}^*(H,V/H) & \longrightarrow & 0.
\end{array}$$

The maps $\alpha$ and $\beta$ are morphisms of differential graded Lie algebras. Since $\operatorname{Hom}^*(H,V/H)$ is acyclic and $\gamma$ is a quasiisomorphism, it follows that also $\alpha$ and $\beta$ are quasiisomorphisms. □

A generic deformation of $(V,\overline{\partial})$ over $\mathbb{K}[[t]]$ is a differential of the form $\tilde{\partial}=\overline{\partial}+tx_1+t^2x_2+\cdots$, where $x_i\in\operatorname{Hom}^1(V,V)$ for every $i$. Taking the series expansion of the integrability condition $[\tilde{\partial},\tilde{\partial}]=0$ we get an infinite number of equations

$$\begin{array}{ll}
1) & [\overline{\partial},x_1] = dx_1 = 0\\
2) & [x_1,x_1] = -2[\overline{\partial},x_2] = -2dx_2\\
\vdots & \quad\vdots\\
n) & \sum_{i=1}^{n-1}[x_i,x_{n-i}] = -2[\overline{\partial},x_n] = -2dx_n
\end{array}$$

The first equation tell us that $\overline{\partial}+tx_1$ is a deformation over $\mathbb{K}[t]/(t^2)$ of $\overline{\partial}$ if and only if $\overline{\partial}x_1+x_1\overline{\partial}=0$. The second equation tell us that $\overline{\partial}+tx_1$ extends to a deformation over $\mathbb{K}[[t]]$ only if the morphism of complexes $x_1\circ x_1$ is homotopically equivalent to 0.

Vice versa, the existence of $x_1,x_2$ satisfying equations 1) and 2) is also sufficient to ensure that $\overline{\partial}+tx_1$ extends to a deformation over $\mathbb{K}[[t]]$. According to Lemma 1.17, the proof of this fact follows immediately from the following proposition.

**Proposition 1.18.** *If a differential graded Lie algebra $L$ is formal, then the two maps*

$$\operatorname{Def}_L(\mathbb{K}[t]/(t^3)) \to \operatorname{Def}_L(\mathbb{K}[t]/(t^2))$$

$$\operatorname{Def}_L(\mathbb{K}[[t]]) := \lim_{\leftarrow n} \operatorname{Def}_L(\mathbb{K}[t]/(t^n)) \to \operatorname{Def}_L(\mathbb{K}[t]/(t^2))$$

*have the same image.*

*Proof.* According to Corollary 1.15 we may assume that $L$ is a graded Lie algebra and therefore its Maurer-Cartan equation becomes $[x,x]=0$, $x \in L^1$. Therefore $tx_1 \in \operatorname{Def}_L(\mathbb{K}[t]/(t^2))$ lifts to $\operatorname{Def}_L(\mathbb{K}[t]/(t^3))$ if and only if there exists $x_2 \in L^1$ such that

$$t^2[x_1,x_1] \equiv [tx_1 + t^2x_2, tx_1 + t^2x_2] \equiv 0 \pmod{t^3} \iff [x_1,x_1]=0$$

and $[x_1,x_1]=0$ implies that $tx_1 \in \operatorname{Def}_H(\mathbb{K}[t]/(t^n))$ for every $n \geq 3$. □

**Definition 1.19** ([17])**.** A covariant functor $F\colon \mathbf{Art} \to \mathbf{Set}$ is called *smooth* if for every surjective morphism $A \to B$ in **Art**, the map $F(A) \to F(B)$ is surjective.

**Corollary 1.20.** *If a DGLA $L$ is quasiisomorphic to a DGLA with trivial bracket, then* $\operatorname{Def}_L$ *is smooth.*

*Proof.* Immediate consequence of Corollary 1.15. □

## Lecture 2. Deformations of complex manifolds

Unless otherwise specified, every complex manifold is assumed compact and connected. For every complex manifold $X$ we denote by:

- $\Theta_X$ the holomorphic tangent sheaf of $X$.
- $\mathcal{A}_X^{p,q}$ the sheaf of differentiable $(p,q)$-forms of $X$. More generally if $\mathcal{E}$ is locally free sheaf of $\mathcal{O}_X$-modules we denote by $\mathcal{A}_X^{p,q}(\mathcal{E}) \simeq \mathcal{A}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{E}$ the sheaf of $(p,q)$-forms of $X$ with values in $\mathcal{E}$ and by $A_X^{p,q}(\mathcal{E}) = \Gamma(X, \mathcal{A}_X^{p,q}(\mathcal{E}))$ the space of its global sections.

**Definition 2.1.** Let $(B,b_0)$ be germ of complex spaces. A *deformation* $X \xrightarrow{i} \mathcal{X} \xrightarrow{f} (B,b_0)$ of a compact complex manifold $X$ over $(B,b_0)$ is a pair of holomorphic maps

$$X \xrightarrow{i} \mathcal{X} \xrightarrow{f} B$$

such that:

(1) $fi(X) = b_0$.
(2) There exists an open neighbourhood $b_0 \in U \subset B$ such that $f\colon f^{-1}(U) \to U$ is a proper flat holomorphic map.
(3) $i\colon X \to f^{-1}(b_0)$ is an isomorphism of complex manifolds.

$\mathcal{X}$ is called the total space of the deformation and $(B,b_0)$ the base germ space.

**Definition 2.2.** Two deformations of $X$ over the same base

$$X \xrightarrow{i} \mathcal{X} \xrightarrow{f} (B,b_0), \qquad X \xrightarrow{j} \mathcal{X}' \xrightarrow{g} (B,b_0)$$

are isomorphic if there exists an open neighbourhood $b_0 \in U \subset B$, and a commutative diagram of holomorphic maps

$$\begin{array}{ccc} X & \xrightarrow{i} & f^{-1}(U) \\ {\scriptstyle j}\downarrow & \swarrow & \downarrow{\scriptstyle f} \\ g^{-1}(U) & \xrightarrow[g]{} & U \end{array}$$

with the diagonal arrow a holomorphic isomorphism.

For every pointed complex manifold $(B, b_0)$ we denote by $\mathrm{Def}_X(B, b_0)$ the set of isomorphism classes of deformations of $X$ with base $(B, b_0)$. It is clear from the definition that if $b_0 \in U \subset B$ is open, then $\mathrm{Def}_X(B, b_0) = \mathrm{Def}_X(U, b_0)$. If $(B, b_0)$ is the Spec of a local artinian $\mathbb{C}$-algebra $A$, then we will denote

$$\mathrm{Def}_X(A) = \mathrm{Def}_X(B, b_0).$$

Notice that every element of $\mathrm{Def}_X(A)$ can be interpreted as a morphism of sheaves of algebras $\mathcal{O}_A \to \mathcal{O}_X$ such that $\mathcal{O}_A$ is flat over $A$ and $\mathcal{O}_A \otimes_A \mathbb{C} \to \mathcal{O}_X$ is an isomorphism. Define the functor

$$\mathrm{Def}_X \colon \mathbf{Art} \to \mathbf{Set}$$

of infinitesimal deformations of $X$ by setting $\mathrm{Def}_X(A)$ as the set of isomorphism classes of deformations of $X$ over $A$. This functor is isomorphic to the deformation functor associated to the *Kodaira-Spencer* differential graded Lie algebra of $X$, that is

$$KS_X = A_X^{0,*}(\Theta_X) = \oplus_i A_X^{0,i}(\Theta_X).$$

The differential on $KS_X$ is the Dolbeault differential, while the bracket is defined in local coordinates as the $\overline{\Omega}^*$-bilinear extension of the standard bracket on $\mathcal{A}_X^{0,0}(\Theta_X)$ ($\overline{\Omega}^*$ is the sheaf of antiholomorphic differential forms). By Dolbeault theorem we have $H^i(A_X^{0,*}(\Theta_X)) = H^i(X, \Theta_X)$ for every $i$. The isomorphism $\mathrm{Def}_{KS_X} \to \mathrm{Def}_X$ is obtained by thinking, via Lie derivation, the elements of $A_X^{0,i}(\Theta_X)$ as derivations of degree $i$ of the sheaf of graded algebras $\oplus_i \mathcal{A}_X^{0,i}$. More precisely, with every $x \in \mathrm{MC}_{KS_X}(A)$ we associate the deformation

$$\mathcal{O}_A(x) = \ker(\mathcal{A}_X^{0,0} \otimes A \xrightarrow{\overline{\partial} + \boldsymbol{l}_x} \mathcal{A}_X^{0,1} \otimes A),$$

where in local holomorphic coordinates $z_1, \ldots, z_n$

$$x = \sum_{i,j} x_{ij} d\overline{z}_i \frac{\partial}{\partial z_j}, \qquad \boldsymbol{l}_x(f) = \sum_{i,j} x_{ij} \frac{\partial f}{\partial z_j} d\overline{z}_i.$$

Equivalently we can interpret every element of $A_X^{0,1}(\Theta_X)$ as a morphism of vector bundles $T_X^{0,1} \to T_X^{1,0}$ and then also as a variation of the almost complex structure of $X$. The Maurer-Cartan equation becomes exactly the integrability condition of the Newlander-Nirenberg theorem (see e.g. [1], [4]). If we are interested only to infinitesimal deformations, the proof of the isomorphism $\mathrm{Def}_{KS_X} \to \mathrm{Def}_X$ can be done without using almost complex structures and therefore without Newlander-Nirenberg theorem: for full details see either [7] or [3].

**Definition 2.3.** A compact complex manifold $X$ is said to have *unobstructed deformations* if the functor $\mathrm{Def}_X$ is smooth. This is equivalent to the fact that the Kuranishi family of $X$ is based on a smooth germ.

As an application of the above results we sketch a proof (due to Deligne, Goldman and Millson) of the following theorem.

**Theorem 2.4** (Bogomolov-Tian-Todorov)**.** *Let $X$ be a compact Kaehler manifold with trivial canonical bundle. Then $X$ has unobstructed deformations.*

*Proof.* It is sufficient to prove that Kodaira-Spencer DGLA $KS_X$ is quasiisomorphic to an abelian DGLA. Let $n$ be the dimension of $X$ and let $\omega \in \Gamma(X, \Omega^n_X)$ be a nowhere vanishing holomorphic $n$-form; the isomorphism $\lrcorner\omega\colon \Theta_X \to \Omega^{n-1}_X$ extends to an isomorphism of complexes

$$i\colon (A^{0,*}_X(\Theta_X), \overline{\partial}) \to (A^{n-1,*}_X, \overline{\partial})$$

and then induces a structure of DGLA on $A^{n-1,*}_X = \bigoplus_p A^{n-1,p}$ isomorphic to $KS_X$. A straightforward local computation (see [14] for a proof) shows that, if $\alpha, \beta \in A^{n-1,*}_X$ are $\partial$-closed, then their bracket $[\alpha, \beta]$ is $\partial$-exact. In particular

$$Q^* = \ker\partial \cap A^{n-1,*}_X$$

is a DGL subalgebra of $A^{n-1,*}_X$. Consider the complex $(R^*, \overline{\partial})$, where

$$R^p = \frac{\ker\partial \cap A^{n-1,p}_X}{\partial A^{n-2,p}_X}$$

endowed with the trivial bracket: the projection $Q^* \to R^*$ is a morphism of DGLA. It is therefore sufficient to prove that the DGLA morphisms

$$A^{n-1,*}_X \longleftarrow Q^* \longrightarrow R^*$$

are quasiisomorphisms. According to the $\partial\overline{\partial}$-lemma, $\overline{\partial}(\ker\partial) \subset Image(\partial)$ and then for every $p$ the three cohomology groups

$$H^p(R^*) = \frac{\ker\partial \cap A^{n-1,p}_X}{\partial A^{n-2,p}_X}, \qquad H^p(A^{n-1,*}_X) = \frac{\ker\overline{\partial} \cap A^{n-1,p}_X}{\overline{\partial} A^{n-1,p-1}_X},$$

$$H^p(Q^*) = \frac{\ker\partial \cap \ker\overline{\partial} \cap A^{n-1,p}_X}{\overline{\partial}(\ker\partial \cap A^{n-1,p-1}_X)}$$

are isomorphic. □

*Remark* 2.5. For smooth projective manifolds over an algebraically closed field of characteristic 0 the Kodaira-Spencer DGLA is conveniently replaced with al $L_\infty$ structure on the Cech resolution of the tangent sheaf on an affine cover. This $L_\infty$-algebra governs infinitesimal deformations [3] and the Bogomolov-Tian-Todorov theorem can be proved in a completely algebraic way [10].

## Lecture 3. Deformations of holomorphic maps (after Donatella Iacono)

The basic theorems of Kodaira and Spencer [12], [11] about deformations of complex manifolds have been extended to deformations of holomorphic maps by Horikawa in the papers [5], [6]. In this section we describe the construction, made by Donatella Iacono in her thesis [7], of the differential graded Lie algebra governing infinitesimal deformations of a holomorphic map of complex manifolds.

**Definition 3.1.** Let $f : X \to Y$ be a holomorphic map and $A \in \mathbf{Art}$. An *infinitesimal deformation of $f$ over $Spec(A)$* is a commutative diagram of complex spaces

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & S, & \end{array}$$

where $S = Spec(A)$, $(X_A, \pi, S)$ and $(Y_A, \mu, S)$ are infinitesimal deformations of $X$ and $Y$, respectively, $\mathcal{F}$ is a holomorphic map that restricted to the fibers over the closed point of $S$ coincides with $f$.

**Definition 3.2.** Let

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & S & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X'_A & \xrightarrow{\mathcal{F}'} & Y'_A \\ & {\scriptstyle\pi'}\searrow \quad \swarrow{\scriptstyle\mu'} & \\ & S & \end{array}$$

be two infinitesimal deformations of $f$. They are *equivalent* if there exist biholomorphic maps $\phi : X_A \to X'_A$ and $\psi : Y_A \to Y'_A$ (that are equivalence of infinitesimal deformations of $X$ and $Y$, respectively) such that the following diagram is commutative:

$$\begin{array}{ccc} X_A & \xrightarrow{\mathcal{F}} & Y_A \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X'_A & \xrightarrow{\mathcal{F}'} & Y'_A. \end{array}$$

**Definition 3.3.** The *functor of infinitesimal deformations* of a holomorphic map $f : X \to Y$ is

$$\mathrm{Def}(f) : \mathbf{Art} \to \mathbf{Set},$$

$$A \longmapsto \mathrm{Def}(f)(A) = \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{infinitesimal deformations of} \\ f \text{ over } Spec(A) \end{array} \right\}.$$

We want to find a differential graded Lie algebra $H$ such that $\mathrm{Def}_H \simeq \mathrm{Def}(f)$. To do this, it is convenient to define first the deformation functor associated with a pair of morphisms of differential graded Lie algebras.

Given morphisms of differential graded Lie algebras $h : L \to M$ and $g : N \to M$:

$$\begin{array}{ccc} & & L \\ & & \downarrow{\scriptstyle h} \\ N & \xrightarrow{g} & M, \end{array}$$

we define the functor

$$\mathrm{Def}_{(h,g)} : \mathbf{Art} \to \mathbf{Set},$$

$$\mathrm{Def}_{(h,g)}(A) = \{(x, y, e^p) \in (L^1 \otimes m_A) \times (N^1 \otimes m_A) \times \exp(M^0 \otimes m_A) |$$
$$dx + \frac{1}{2}[x, x] = 0,\ dy + \frac{1}{2}[y, y] = 0,\ g(y) = e^p * h(x)\}/ \approx,$$

where the equivalence relation $\approx$ is defined by:

$$(x_1, y_1, e^{p_1}) \approx (x_2, y_2, e^{p_2})$$

if and only if there exist $a \in (L \otimes A)^0$, $b \in (N \otimes A)^0$ and $c \in (M \otimes A)^{-1}$ such that

$$x_2 = e^a * x_1, \qquad y_2 = e^b * y_1$$

and

$$e^{p_2} = e^{g(b)} e^T e^{p_1} e^{-h(a)}, \quad \text{where} \quad T = dc + [g(y_1), c].$$

Notice that if $N = M = 0$, then $\mathrm{Def}_{(h,g)}$ reduces to $\mathrm{Def}_L$.

In the above set-up, define the differential graded Lie algebra

$$M[t,dt] = M \otimes \mathbb{C}[t,dt].$$

For every $s \in \mathbb{C}$, the evaluation morphism of dg-algebras

$$\mathbb{C}[t,dt] \xrightarrow{e_s} \mathbb{C}, \qquad e_s(t) = s,\ e_s(dt) = 0,$$

induces a quasiisomorphism of DGLA's

$$M[t,dt] \xrightarrow{e_s} M.$$

Denote by

$$H = \{(l,n,m(t,dt)) \in L \times N \times M[t,dt] \mid h(l) = e_1(m(t,dt)),\ g(n) = e_0(m(t,dt))\}.$$

It is clear that $H$ is a differential graded Lie algebra.

**Theorem 3.4** (Iacono)**.** *In the notation above, there exists an isomorphism of functors*

$$\operatorname{Def}_H \simeq \operatorname{Def}_{(h,g)}.$$

As a second step we look for two morphisms of DGLA $h,g$ such that $\operatorname{Def}_{(h,g)}$ is isomorphic to the deformation functor of a holomorphic map. Consider the DGLA $A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y)$ and the morphism

$$g = (p^*,q^*) : A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y) \to A^{0,*}_{X\times Y}(\Theta_{X\times Y}),$$

where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the projections.

The solutions $n = (n_1,n_2)$ of the Maurer-Cartan equation in $N = A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y)$ correspond to infinitesimal deformations of both $X$ (induced by $n_1$) and $Y$ (induced by $n_2$). Moreover the image $g(n)$ satisfies the Maurer-Cartan equation in $M = A^{0,*}_{X\times Y}(\Theta_{X\times Y})$ and so it is associated with an infinitesimal deformation of $X \times Y$, that is exactly the one obtained as product of the deformations of $X$ (induced by $n_1$) and of $Y$ (induced by $n_2$). Define the DGLA $L = A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma))$ by the following exact sequence

$$0 \to A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma)) \to A^{0,*}_{X\times Y}(\Theta_{X\times Y}) \to A^{0,*}_{\Gamma}(N_{\Gamma|X\times Y}) \to 0,$$

where $N_{\Gamma|X\times Y}$ is the normal bundle of the graph $\Gamma \subset X \times Y$ of the map $f$. Then we are in the following situation:

$$\begin{array}{ccc} & & A^{0,*}_{X\times Y}(\Theta_{X\times Y}(-log\,\Gamma)) \\ & & \Big\downarrow h \\ A^{0,*}_X(\Theta_X) \times A^{0,*}_Y(\Theta_Y) & \xrightarrow{g=(p^*,q^*)} & A^{0,*}_{X\times Y}(\Theta_{X\times Y}). \end{array}$$

**Theorem 3.5** (Iacono)**.** *In the notation above, there exists an isomorphism of functors*

$$\operatorname{Def}(f) \simeq \operatorname{Def}_{(h,g)}.$$

*Proof.* See [7, 9]. □

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