Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature, prescribed volume and asymptotic behavior

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Abstract

We study the solutions $u \in C^\infty(\mathbb{R}^{2m})$ of the problem

$$(-\Delta)^m u = \bar{Q} e^{2mu}, \text{ where } \bar{Q} = \pm(2m-1)!, \quad V := \int_{\mathbb{R}^{2m}} e^{2mu} dx < \infty, \tag{1}$$

particularly when $m > 1$. Problem (1) corresponds to finding conformal metrics $g_u := e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ with constant Q-curvature $\bar{Q}$ and finite volume V. Extending previous works of Chang-Chen, and Wei-Ye, we show that both the value V and the asymptotic behavior of $u(x)$ as $|x| \to \infty$ can be simultaneously prescribed, under certain restrictions. When $\bar{Q} = (2m-1)!$ we need to assume $V < \mathrm{vol}(S^{2m})$, but surprisingly for $\bar{Q} = -(2m-1)!$ the volume V can be chosen arbitrarily.

1 Introduction

We consider the equation

$$(-\Delta)^m u = (2m-1)! e^{2mu} \text{ in } \mathbb{R}^{2m}, \tag{2}$$

where $u \in C^\infty(\mathbb{R}^{2m})$ and satisfies

$$V := \int_{\mathbb{R}^{2m}} e^{2mu} dx < \infty. \tag{3}$$

Equation (2) has been widely studied because of its geometric meaning. Indeed if u solves (2), then the conformal metric $g_u := e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ (here $|dx|^2$ denotes the Euclidean metric on $\mathbb{R}^{2m}$) has constant Q-curvature equal to $(2m-1)!$. For a brief discussion of the geometric meaning of (2) and a survey of related previous works we refer to the introduction of [12] and the references therein. Here we only mention some relevant facts, necessary to contextualize the results of our present work.

First of all the assumption that $u \in C^\infty(\mathbb{R}^{2m})$ is not restrictive, since any weak solution $u \in L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$ of (2) with right-hand side in $L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$ is smooth, see e.g. [12, Corollary 8].

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Also the particular choice of the constant $(2m-1)!$ in (2) is not restrictive, since it can be changed by considering $u+C$ for $C\in\mathbb{R}$.

Next we recall that Problem (2)-(3) possesses the following explicit radially symmetric solutions

$$u(x)=\log\left(\frac{2\lambda}{1+\lambda^2|x-x_0|^2}\right),\quad \lambda>0,\ x_0\in\mathbb{R}^{2m},$$

which are called spherical solutions, since they are obtained (up to a Möbius transformation) by pulling back the round metric of S^{2m} onto $\mathbb{R}^{2m}$ via the stereographic projection.

While in dimension 2, i.e. for $m=1$, such spherical solutions exhaust the set of solutions to (2)-(3), as proven by W. Chen and C. Li [4], in the case $m\geq 2$ A. Chang and W. Chen [2] showed that non-spherical solutions do exist. In fact they proved that for any $m\geq 2$ and every $V\in(0,\mathrm{vol}(S^{2m}))$ there exists a (non-spherical) solution to (2)-(3). This suggests to investigate the properties of such solutions. Building upon the previous work of A. Chang and P. Yang [2], C-S. Lin for $m=2$ and L. Martinazzi for $m>2$ proved:

Theorem A ([9], [12]) *If u solves* (2)-(3)*, then u has the asymptotic behavior*

$$u(x)=-\alpha\log(|x|)-P(x)+C+o(1),\quad o(1)\to 0 \text{ as } |x|\to\infty, \tag{4}$$

where $\alpha=\frac{2V}{\mathrm{vol}(S^{2m})}$ and P is a polynomial of degree at most $2m-2$ bounded from below. Moreover P is constant if and only if u is spherical. When $m=2$ one has $V\in(0,\mathrm{vol}(S^4)]$ and $V=\mathrm{vol}(S^4)$ if and only if u is spherical.

J. Wei and D. Ye complemented the result of C-S. Lin by showing, among other things:

Theorem B ([17]) *For any $V\in(0,\mathrm{vol}(S^4))$ and $P(x)=\sum_{j=1}^4 a_jx_j^2$ with $a_j>0$, Problem (2)-*(3) *has a solution with asymptotic expansion* (4) *for some $C\in\mathbb{R}$.*

The first result which we prove here is an extension of the result of J. Wei and D. Ye to the case $m>2$. We will prove the existence of solutions to (2)-(3) having the asymptotic behavior (4) where P will be any given polynomial of degree at most $2m-2$ satisfying

$$\lim_{|x|\to\infty} x\cdot\nabla P(x)=\infty, \tag{5}$$

while $\alpha>0$ is determined by $V\in(0,\mathrm{vol}(S^{2m}))$. More precisely, define

$$\mathcal{P}_m:=\left\{P \text{ polynomial in } \mathbb{R}^{2m}:\deg P\leq 2m-2,\ (5)\text{ holds}\right\}.$$

It is worth noticing that (5) is equivalent to the apparently stronger condition

$$\liminf_{|x|\to\infty}\frac{P(x)}{|x|^a}>0 \quad\text{and}\quad \liminf_{|x|\to\infty}\frac{x\cdot\nabla P(x)}{|x|^a}>0,\quad \text{for some } a>0. \tag{6}$$

Indeed (5) implies the second inequality of (6) by a subtle result of E. Gorin (see [6, Theorem 3.1]), and the second inequality in (6) implies the first one, since one can write

$$P(x)=\int_0^{|x|}\frac{d}{dr}P\left(r\frac{x}{|x|}\right)dr+P(0).$$

A simple example of polynomial belonging to $\mathcal{P}_m$ is

$$P(x) = \sum_{j=1}^{2m} a_j x_j^{2i_j} + p(x),$$

where $a_j > 0$, $i_j \in \{1, 2, \dots, m-1\}$ for $1 \le j \le 2m$, and p is a polynomial of degree at most $2\min\{i_j\} - 1$, but in general $\mathcal{P}_m$ contains polynomials whose higher degree monomials do not split in such a simple way.

Theorem 1.1 *For any integer $m \ge 2$, given $P \in \mathcal{P}_m$ and $V \in (0, \mathrm{vol}(S^{2m}))$, there exists a solution of* (2)-(3) *having the asymptotic behavior* (4) *with $\alpha = \frac{2V}{\mathrm{vol}(S^{2m})}$.*

The restriction $V < \mathrm{vol}(S^{2m})$ in Theorem 1.1 is necessary when $m = 2$ because of the result of C-S. Lin (Theorem A), but appears to be only a technical issue when $m \ge 3$. In fact for $m = 3$ L. Martinazzi recently proved that there are solutions to (2)-(3) with V arbitrarily large, see [14]. The crucial step in which we need V to be smaller than $\mathrm{vol}(S^{2m})$ is Theorem 4.2 below, a compactness result which follows form the blow-up analysis of sequences of prescribed Q-curvature in open domains of $\mathbb{R}^{2m}$ (Theorem 4.1 below) proven by L. Martinazzi, and inspired by previous works of H. Brézis and F. Merle [1] and F. Robert [16]. This compactness is used to prove the a priori bounds necessary to run the fixed point argument of [17], which we closely follow. For $m > 2$ it remains open whether one can prescribe $P \in \mathcal{P}_m$ and $V \ge \mathrm{vol}(S^{2m})$ in Theorem 1.1.

From the work of Brézis-Merle we also borrow a simple but fundamental critical estimate, whose generalization is Lemma A.2 below, which is used in Lemma 3.6 below.

As we shall now show, things go differently when the prescribed Q-curvature is negative. Consider the equation

$$(-\Delta)^m u = -(2m-1)! e^{2mu} \quad \text{in } \mathbb{R}^{2m}, \tag{7}$$

whose solutions give rise to metrics $g_u = e^{2u}|dx|^2$ of Q-curvature $-(2m-1)!$ in $\mathbb{R}^{2m}$. One can easily verify that under the assumption (3) Equation (7) has no solutions when $m = 1$, see e.g. [11, Proposition 6]. On the other hand, when $m \ge 2$ we have:

Theorem C ([11]) *For every $m \ge 2$ there is some $V > 0$ such that Problem* (7)-(3) *has a radially symmetric solution. Every solution to* (7)-(3) *(a priori not necessarily radially symmetric) has the asymptotic behavior given by* (4) *where $\alpha = -\frac{2V}{\mathrm{vol}(S^{2m})}$ and P is a* non-constant *polynomial of degree at most $2m-2$ bounded from below.*

Notice that, contrary to Chang-Chen's result [2], the existence part of Theorem C does not allow to prescribe V. Moreover its proof is based on an ODE argument which only produces radially symmetric solutions. It is then natural to address the following question: For which values of V and which polynomials P does Problem (7)-(3) have a solution with asymptotic behavior (4) (with $\alpha = -\frac{2V}{\mathrm{vol}(S^{2m})}$)? In analogy with Theorem 1.1 we will show:

Theorem 1.2 *For any integer $m \ge 2$, given $P \in \mathcal{P}_m$ and $V > 0$, there exists a solution of* (7)-(3) *having the asymptotic behavior* (4) *for $\alpha = -\frac{2V}{\mathrm{vol}(S^{2m})}$.*

The remarkable fact which allows for large values of V in Theorem 1.2 (but not in Theorem 1.1) is that, as shown in [13], when the Q-curvature is negative, compactness is obtained even for large volumes, compare Theorems 4.1 and 4.2 below. This in turn depends on Theorem C above, and in particular on the fact that the polynomial in the expansion (4) of a solution to (7)-(3) is necessarily non-constant.

About the assumption that $P \in \mathcal{P}_m$ in Theorems 1.1 and 1.2, we do not claim nor believe that it is optimal, but it is technically convenient in the crucial Lemma 3.5 below, where it is needed in (22). Since a solution to (2)-(3) or (7)-(3) must satisfy (4) for $\alpha = \pm\frac{2V}{\mathrm{vol}(S^{2m})}$, a necessary condition on P and V is

$$\int_{\mathbb{R}^{2m}\setminus B_1} e^{-2m(P(x)+\alpha\log|x|)}dx < \infty, \tag{8}$$

but it is unknown whether this condition is also sufficient to guarantee the existence of a solution to (2)-(3) or (7)-(3) with asymptotic expansion (4), at least in the negative case, or for $V < \mathrm{vol}(S^{2m})$ in the positive case.

Also replacing (5) with the weaker assumption

$$\lim_{|x|\to\infty} P(x) = \infty \tag{9}$$

(which implies the first inequality in (6), hence (8)) creates problems, since (9) does not imply (5) when $\deg P \geq 4$, see e.g. Proposition A.4 in the appendix, and as already noticed (5) is crucial in Lemma 3.5 below.

Finally, we remark that new difficulties arise when recasting the above problems in odd dimension. For instance in dimension 3 T. Jin, A. Maalaoui, J. Xiong and the second author studied in [8] the non-local problem

$$(-\Delta)^{\frac{3}{2}}u = 2e^{3u} \quad \text{in } \mathbb{R}^3, \quad V := \int_{\mathbb{R}^3} e^{3u}dx < \infty,$$

proving the existence of some non-spherical solutions with asymptotic behavior as in (4). Whether also in this case one can show an analog to Theorems 1.1 and 1.2 above is an open question.

Notation In the following C will denote a generic positive constant, whose dependence will be specified when necessary, and whose value can change from line to line. We will also write

$$B_r(x) := \{y \in \mathbb{R}^{2m} : |y-x| < r\}, \quad B_r := B_r(0).$$

2 Strategy of the proof of Theorems 1.1 and 1.2

Fix $u_0 \in C^\infty(\mathbb{R}^{2m})$ such that $u_0(x) = \log|x|$ for $|x| \geq 1$. Integration by parts yields

$$\int_{\mathbb{R}^{2m}} (-\Delta)^m u_0 dx = -\gamma_m,$$

where γ_m is defined by

$$(-\Delta)^m \log\frac{1}{|x|} = \gamma_m\delta_0 \text{ in } \mathbb{R}^{2m}, \text{ i.e. } \gamma_m = \frac{(2m-1)!}{2}\mathrm{vol}(S^{2m}). \tag{10}$$

Let V, $\alpha = \pm\frac{2V}{\mathrm{vol}(S^{2m})}$ and $P \in \mathcal{P}_m$ be given as in Theorem 1.1 or 1.2. We would like to find a solution to (2) or (7) of the form

$$u = -\alpha u_0 - P + v + C, \tag{11}$$

for a suitable choice of $C \in \mathbb{R}$ and of a smooth function $v(x) = o(1)$ as $|x| \to \infty$. Define

$$K = \frac{\alpha\gamma_m}{V} e^{-2mP-2m\alpha u_0} = \mathrm{sign}(\alpha)(2m-1)! e^{-2mP-2m\alpha u_0}, \tag{12}$$

and notice that (5) implies

$$|K(x)| \leq C_1 e^{-C_2|x|^a} \tag{13}$$

for some $C_1, C_2 > 0$.

Now if we assume (3), then the constant C in (11) is determined by the function v. Indeed (3) implies

$$V = \int_{\mathbb{R}^{2m}} e^{2mu} dx = \frac{e^{2mC}}{(2m-1)!} \int_{\mathbb{R}^{2m}} |K| e^{2mv} dx,$$

hence

$$C = c_v := -\frac{1}{2m} \log\left(\frac{1}{(2m-1)!V} \int_{\mathbb{R}^{2m}} |K| e^{2mv} dx\right) = -\frac{1}{2m} \log\left(\frac{1}{\alpha\gamma_m} \int_{\mathbb{R}^{2m}} K e^{2mv} dx\right). \tag{14}$$

An easy computation shows that u given by (11) satisfies

$$(-\Delta)^m u = \mathrm{sign}(\alpha)(2m-1)! e^{2mu}$$

and (3) if and only if $C = c_v$ and

$$(-\Delta)^m v = K e^{2m(v+c_v)} + \alpha(-\Delta)^m u_0. \tag{15}$$

Then we will use a fixed point method in the spirit of [17] to find a solution v to (15) in the Banach space

$$C_0(\mathbb{R}^{2m}) := \left\{ f \in C^0(\mathbb{R}^{2m}) : \lim_{|x|\to\infty} f(x) = 0 \right\}, \quad \|f\|_{C_0} := \sup_{\mathbb{R}^{2m}} |f|,$$

and of course v will also be smooth by elliptic estimates. In order to run the fixed-point argument we introduce the following weighted Sobolev spaces.

Definition 2.1 *For $k \in \mathbb{N}$, $\delta \in \mathbb{R}$ and $p \geq 1$ we set $M^p_{k,\delta}(\mathbb{R}^{2m})$ to be the completion of $C^\infty_c(\mathbb{R}^{2m})$ in the norm*

$$\|f\|_{M^p_{k,\delta}} := \sum_{|\beta|\leq k} \|(1+|x|^2)^{\frac{(\delta+|\beta|)}{2}} D^\beta f\|_{L^p(\mathbb{R}^{2m})}.$$

We also set $L^p_\delta(\mathbb{R}^{2m}) := M^p_{0,\delta}(\mathbb{R}^{2m})$. Finally we set

$$\Gamma^p_\delta(\mathbb{R}^{2m}) := \left\{ f \in L^p_{2m+\delta}(\mathbb{R}^{2m}) : \int_{\mathbb{R}^{2m}} f dx = 0 \right\},$$

whenever $\delta p > -2m$, so that $L^p_{2m+\delta}(\mathbb{R}^{2m}) \subset L^1(\mathbb{R}^{2m})$ and the above integral is well defined.

Lemma 2.1 *Fix $p \geq 1$ and $\delta > -\frac{2m}{p}$. For $v \in C_0(\mathbb{R}^{2m})$ and c_v as in (14) we have*

$$S(v) := Ke^{2m(v+c_v)} + \alpha(-\Delta)^m u_0 \in \Gamma^p_\delta(\mathbb{R}^{2m}),$$

and the map $S : C_0(\mathbb{R}^{2m}) \to \Gamma^p_\delta(\mathbb{R}^{2m})$ is continuous.

Proof. This follows easily from (13) and dominated convergence. □

Lemma 2.2 (Theorem 5 in [10]) *For $1 < p < \infty$ and $\delta \in \left(-\frac{2m}{p}, -\frac{2m}{p} + 1\right)$, the operator $(-\Delta)^m$ is an isomorphism from $M^p_{2m,\delta}(\mathbb{R}^{2m})$ to $\Gamma^p_\delta(\mathbb{R}^{2m})$.*

The following Lemma will be proven in Section A.2 below.

Lemma 2.3 *For $\delta > -\frac{2m}{p}$, $p \geq 1$, the embedding $E : M^p_{2m,\delta}(\mathbb{R}^{2m}) \hookrightarrow C_0(\mathbb{R}^{2m})$ is compact.*

Fix $p \in (1, \infty)$ and $\delta \in \left(-\frac{2m}{p}, -\frac{2m}{p} + 1\right)$. Then by Lemma 2.1, Lemma 2.2 and Lemma 2.3, one can define a compact map

$$T := E \circ ((-\Delta)^m)^{-1} \circ S : C_0(\mathbb{R}^{2m}) \to C_0(\mathbb{R}^{2m}) \tag{16}$$

given by $Tv = \bar{v}$ where $\bar{v}$ is the only solution to

$$(-\Delta)^m \bar{v} = Ke^{2m(v+c_v)} + \alpha(-\Delta)^m u_0,$$

and compactness follows from the continuity of S and $((-\Delta)^m)^{-1}$ and the compactness of E.

If v is a fixed point of T, then it solves (15) and $u = v + c_v - P - \alpha u_0$ is a solution of (2) or (7) (depending on the sign of K in (12)) and (3), with asymptotic expansion (4). Then in order to prove Theorems 1.1 and 1.2 it remains to prove that T has a fixed point, and we shall do that using the following fixed-point theorem.

Lemma 2.4 (Theorem 11.3 in [5]) *Let T be a compact mapping of a Banach space X into itself, and suppose that there exists a constant M such that*

$$\|x\|_X < M$$

for all $x \in X$ and $t \in (0, 1]$ satisfying $tTx = x$. Then T has a fixed point.

In order to apply Lemma 2.4 to the operator T defined in (16) we will prove in Section 3 the following a priori bound, which completes the proof of Theorems 1.1 and 1.2.

Proposition 2.5 *For any $0 < t \leq 1$ and $v \in C_0(\mathbb{R}^{2m})$ such $tTv = v$ we have*

$$\|v\|_{C_0(\mathbb{R}^{2m})} \leq M, \tag{17}$$

with M independent of v and t.

3 A priori estimates and proof of Proposition 2.5

Throughout this section let $t \in (0,1]$ and $v \in C_0(\mathbb{R}^{2m})$ be fixed and satisfy $tTv = v$, that is

$$(-\Delta)^m v = t(Ke^{2m(v+c_v)} + \alpha(-\Delta)^m u_0),$$

where c_v is as in (14). Also define

$$\bar{w} := v + c_v + \frac{\log t}{2m}. \tag{18}$$

Lemma 3.1 *We have*

$$v(x) = -\frac{t}{\gamma_m}\int_{\mathbb{R}^{2m}} \log(|x-y|)K(y)e^{2m(v(y)+c_v)}dy + t\alpha u_0(x). \tag{19}$$

Proof. Let $\tilde{v}(x)$ be defined as the right-hand side of (19). Then for $|x| \geq 1$, using (14) we write

$$\tilde{v}(x) = \frac{t}{\gamma_m}\int_{\mathbb{R}^{2m}} K(y)e^{2m(v(y)+c_v)}(\log|x| - \log|x-y|)dy$$

We first show that

$$\lim_{|x|\to\infty} \tilde{v}(x) = 0. \tag{20}$$

Let $R > 1$ be fixed. Then for $|x| > 2R$, we split

$$\tilde{v}(x) = \sum_{i=1}^{5} I_i, \quad I_i := \frac{t}{\gamma_m}\int_{A_i} K(y)e^{2m(v(y)+c_v)}\log\left(\frac{|x|}{|x-y|}\right)dy,$$

where

$$\begin{aligned}
A_1 &:= B_R(0)\\
A_2 &:= B_1(x)\\
A_3 &:= B_{|x|/2}(x) \setminus B_1(x)\\
A_4 &:= (B_{2|x|}(x) \setminus B_{|x|/2}(x)) \setminus B_R(0)\\
A_5 &:= \mathbb{R}^{2m} \setminus B_{2|x|}(x),
\end{aligned}$$

and we will show that $I_i \to 0$ as $|x| \to \infty$ for $1 \leq i \leq 5$.

For $i = 1$, since $\lim_{|x|\to\infty} \log\left(\frac{|x|}{|x-y|}\right) = 0$ uniformly with respect to $y \in B_R(0)$, from the dominated convergence theorem we get

$$|I_1| \leq C\int_{B_R(0)} |K(y)| \left|\log\left(\frac{|x|}{|x-y|}\right)\right| dy \to 0 \quad \text{as } |x| \to \infty.$$

From (13) we also have

$$\begin{aligned}
|I_2| &\leq C\int_{B_1(x)} |K(y)| \left(\log|x| + |\log|x-y||\right) dy\\
&\leq C\|K\|_{L^\infty(B_1(x))}\left(\log|x| + \|\log|\cdot|\|_{L^1(B_1(0))}\right)\\
&\to 0, \quad \text{as } |x| \to \infty.
\end{aligned}$$

Since (13) yields $K\log(|\cdot|) \in L^1(\mathbb{R}^{2m})$, we infer with the dominated convergence theorem

$$\begin{aligned}
|I_3| &\leq C\int_{\{1\leq|x-y|<|x|/2\}} |K(y)|\left(\log|x| + \log(|x|/2)\right) dy \\
&\leq C\int_{\{1\leq|x-y|<|x|/2\}} |K(y)|\left(\log|2y| + \log(|y|)\right) dy \\
&\to 0, \quad \text{as } |x|\to\infty.
\end{aligned}$$

Using that $\frac{1}{2} < \frac{|x|}{|x-y|} < 2$ on A_4 and that $K \in L^1(\mathbb{R}^{2m})$ we find that for every $\varepsilon > 0$ it is possible to choose R so large that

$$|I_4| \leq C\int_{A_4} |K(y)|\left|\log\left(\frac{|x|}{|x-y|}\right)\right| dy \leq C\int_{A_4}|K|dy \leq C\int_{\mathbb{R}^{2m}\setminus B_R(0)}|K|dy \leq \varepsilon.$$

Finally, again using that $K\log(|\cdot|) \in L^1(\mathbb{R}^{2m})$ with the dominated convergence theorem we get

$$\begin{aligned}
|I_5| &\leq C\int_{\{|x-y|>2|x|\}} |K(y)|(\log|x| + \log|x-y|)dy \\
&\leq C\int_{\{|x-y|>2|x|\}} |K(y)|(\log|y| + \log|2y|)dy \\
&\to 0, \quad \text{as } |x|\to\infty.
\end{aligned}$$

Since ε can be chosen arbitrarily small, (20) is proven. Since $v \in C_0(\mathbb{R}^{2m})$, and $\Delta^m\tilde{v} = \Delta^m v$, the difference $w := v - \tilde{v}$ satisfies

$$\Delta^m w = 0 \quad \text{in } \mathbb{R}^{2m}, \qquad \lim_{|x|\to\infty} w(x) = 0.$$

Then by the Liouville theorem for polyharmonic functions (see e.g. Theorem 5 in [12]) w is a polynomial, and since it vanishes at infinity, it must be identically zero, i.e. $v \equiv \tilde{v}$. $\square$

By Lemma 2.2 and (13), we have

$$\begin{aligned}
\frac{1}{C}\|v\|_{M^p_{2m,\delta}} &\leq \|(-\Delta)^m v\|_{L^p_{2m+\delta}} \\
&= \|Ke^{2m\bar{w}} + t\alpha(-\Delta)^m u_0\|_{L^p_{2m+\delta}} \\
&\leq \|K\|_{L^p_{2m+\delta}}\|e^{2m\bar{w}}\|_{L^\infty} + \alpha\|(-\Delta)^m u_0\|_{L^p_{2m+\delta}} \\
&\leq C\|e^{2m\bar{w}}\|_{L^\infty} + C,
\end{aligned}$$

with C independent of t and v, and together with Lemma 3.3 and Lemma 3.6 below we obtain

$$\|v\|_{M^p_{2m,\delta}} \leq C,$$

where C is independent of v and t. Now Proposition 2.5 follows at once from the continuity of the embedding $M^p_{2m,\delta}(\mathbb{R}^{2m}) \hookrightarrow C_0(\mathbb{R}^{2m})$ (see Lemma 2.3).

Remark. An alternative way of getting uniform bounds on $\|v\|_{C_0}$ is to get uniform upper bounds of $\bar{w}$ and use them in (19).

Using Lemma 3.1 one can prove the following decay estimate for the derivatives of v at infinity.

Lemma 3.2 *For $1 \le \ell \le 2m-1$ we have*

$$\lim_{|x|\to\infty} |x|^\ell \nabla^\ell v(x) = \lim_{|x|\to\infty} |x|^\ell \nabla^\ell \bar{w}(x) = 0.$$

Proof. Notice that $\nabla v = \nabla \bar{w}$, so it is enough to work with v.

Using (19) for $|x| > 1$ one can compute

$$\nabla^\ell v(x) = \frac{1}{\gamma_m} \int_{\mathbb{R}^{2m}} K(y) e^{2m\bar{w}(y)} \Big(\nabla^\ell \log(|x|) - \nabla^\ell \log(|x-y|) \Big)\, dy.$$

Fix $\varepsilon > 0$ and $R_1 > 1$ such that

$$\int_{\mathbb{R}^{2m}\setminus B_{R_1}} |K| e^{2m\bar{w}} dy < \varepsilon.$$

For $|x| > 2R_1$, we split $\mathbb{R}^{2m}$ in to three disjoint domains:

$$A_1 := B_{R_1}(0), \quad A_2 := B_{|x|/2}(x), \quad A_3 := \mathbb{R}^{2m} \setminus (A_1 \cup A_2).$$

Then

$$|x|^\ell \nabla^\ell v(x) = \frac{1}{\gamma_m} \sum_{i=1}^3 I_i, \quad I_i := |x|^\ell \int_{A_i} K(y) e^{2m\bar{w}(y)} \Big(\nabla^\ell \log(|x|) - \nabla^\ell \log(|x-y|) \Big)\, dy.$$

Since R_1 is fixed, for $|x|$ large enough we have by the mean-value theorem

$$\Big| \nabla^\ell \log(|x|) - \nabla^\ell \log(|x-y|) \Big| \le |y| \sup_{B_{|y|}(x)} \Big| \nabla^{\ell+1} \log(|z|) \Big| \le \frac{C}{|x|^{\ell+1}} \quad \text{for } y \in A_1,$$

hence with (14) we get

$$|I_1| \le \frac{C}{|x|} \int_{A_1} |K| e^{2m\bar{w}} dy \le \frac{C}{|x|} |\alpha| \gamma_m \to 0, \quad \text{as } |x| \to \infty.$$

Since K goes to zero rapidly at infinity, $\bar{w}$ is bounded, and $|x-y| \le |x|/2$ on A_2, we have

$$\begin{aligned} |I_2| &\le C \|K\|_{L^\infty(A_2)} \|e^{2m\bar{w}}\|_{L^\infty} |x|^\ell \int_{A_2} \left(\frac{1}{|x|^\ell} + \frac{1}{|x-y|^\ell} \right) dy \\ &\le C \|K\|_{L^\infty(A_2)} \|e^{2m\bar{w}}\|_{L^\infty} |x|^{2m} \\ &\to 0, \qquad \text{as } |x| \to \infty. \end{aligned}$$

On A_3 we have $|x-y| \ge |x|/2$, which implies $\frac{|x|^\ell}{|x-y|^\ell} \le 2^\ell$. Hence

$$|I_3| \le C(1+2^\ell) \int_{A_3} |K| e^{2m\bar{w}} dy < C\varepsilon.$$

Since ε is arbitrarily small, the proof is complete. $\square$

Lemma 3.3 *The function $\bar{w}$ given by (18) is locally uniformly upper bounded, i.e. for every $R > 0$ there exists $C = C(R)$ such that $\bar{w} \leq C$ in B_R.*

Proof. Since u_0 is a fixed function and locally bounded, it is enough to prove that $w := \bar{w} - t\alpha u_0$ is locally uniformly upper bounded. Now

$$(-\Delta)^m w = tKe^{2m(v+c_v)} = Qe^{2mw},$$

where $Q = Ke^{2mt\alpha u_0}$.
We bound

$$\int_{B_R} e^{2mw}dx = t\int_{B_R} e^{2m(v+c_v)-2mt\alpha u_0}dx \leq C(R)\int_{B_R} |K|e^{2m(v+c_v)}dx \leq C(R)|\alpha|\gamma_m,$$

where we used (14) and that $|K|$ is positive and continuous.

In addition in the case when $Q > 0$ we have

$$\int_{B_R} Qe^{2mw}dx \leq \int_{B_R} Ke^{2m(v+c_v)}dx < \alpha\gamma_m < (2m-1)!|S^{2m}|.$$

Moreover Lemma 3.1 gives

$$\Delta w(x) = -\frac{t}{\gamma_m}\int_{\mathbb{R}^{2m}} \frac{2m-2}{|x-y|^2} K(y)e^{2m(v(y)+c_v)}dy$$

and with Fubini's theorem we get

$$\begin{aligned}
\int_{B_R} |\Delta w(x)|dx &= \frac{t}{\gamma_m}(2m-2)\int_{\mathbb{R}^{2m}} |K(y)|e^{2m(v(y)+c_v)}\left(\int_{B_R}\frac{dx}{|x-y|^2}\right)dy \\
&\leq C\int_{\mathbb{R}^{2m}} |K(y)|e^{2m(v(y)+c_v)}\left(\int_{B_R(y)}\frac{dx}{|x-y|^2}\right)dy \\
&\leq CR^{2m-2}.
\end{aligned}$$

Therefore Theorem 4.2 implies that there exists $C = C(R) > 0$ (independent of w) such that

$$\sup_{B_{R/2}} w \leq C.$$

∎

A consequence of the local uniform upper bounds of $\bar{w}$ is the following local uniform bound for the derivatives of v:

Lemma 3.4 *For every $R > 0$ there exists a conastant $C = C(R) > 0$ independent of v and t such that for $1 \leq \ell \leq 2m-1$ we have*

$$\sup_{B_R} |\nabla^\ell v| \leq C.$$

Proof. Let $x \in B_R$. Then from (19) and Lemma 3.3, we have

$$\begin{aligned}|\nabla^\ell (v - t\alpha u_0)| &\le C \int_{\mathbb{R}^{2m}} |K(y)| e^{2m\bar{w}(y)} \frac{1}{|x-y|^\ell} dy \\ &\le C \|K\|_{L^\infty} \|e^{2m\bar{w}}\|_{L^\infty(B_{2R})} \int_{B_{2R}} \frac{1}{|x-y|^\ell} dy + \frac{C}{R^\ell} \int_{\mathbb{R}^{2m}\setminus B_{2R}} |K| e^{2m\bar{w}} dy \\ &\le C(R),\end{aligned}$$

where the last integral is bounded using (14). Since u_0 is smooth, α is fixed and $t \in (0,1]$, then the lemma follows. $\square$

Now to prove uniform upper bounds for $\bar{w}$ outside a fixed compact set, first we will need the following result, which relies on a Pohozaev-type identity.

Lemma 3.5 *For given $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon) > 0$ only depending on K (and not on v or t) such that*

$$\int_{\mathbb{R}^{2m}\setminus B_{R_0}} |K| e^{2m\bar{w}} dx < \varepsilon.$$

Proof. Taking $R \to \infty$ in Lemma A.1 and noticing that the first term on the right-hand side of (28) vanishes thanks to (13) and last two terms vanish thanks to Lemma 3.2, we find

$$\int_{\mathbb{R}^{2m}} (x\cdot \nabla K) e^{2m\bar{w}} dx + 2m \int_{\mathbb{R}^{2m}} K e^{2m\bar{w}} dx - 2mt\alpha \int_{B_1} (x\cdot \nabla v)(-\Delta)^m u_0 dx = 0. \tag{21}$$

Thanks to (6) we can find $C_1 > 0$ and $R_1 \ge 1$ such that

$$x\cdot \nabla |K(x)| = -2m\,(x\cdot \nabla P(x) + \alpha)\,|K(x)| \le -\frac{1}{C_1}|x|^a |K(x)| \quad \text{for } |x| \ge R_1. \tag{22}$$

Then for some $R \ge R_1$ to be fixed later we bound

$$\begin{aligned}\frac{1}{C_1} R^a \int_{\mathbb{R}^{2m}\setminus B_R} |K| e^{2m\bar{w}} dx &\le \frac{1}{C_1} \int_{\mathbb{R}^{2m}\setminus B_R} |x|^a |K(x)| e^{2m\bar{w}} dx \\ &\le - \int_{\mathbb{R}^{2m}\setminus B_R} x\cdot \nabla |K(x)| e^{2m\bar{w}} dx \\ &= 2m \int_{\mathbb{R}^{2m}} |K| e^{2m\bar{w}} dx + \int_{B_R} (x\cdot \nabla |K(x)|) e^{2m\bar{w}} dx \\ &\quad - 2mt|\alpha| \int_{B_1} (x\cdot \nabla v(x))(-\Delta)^m u_0 dx \\ &=: (I) + (II) + (III),\end{aligned} \tag{23}$$

where in the equality on the third line we used (21). Now using (14) and (18), we compute $(I) = 2mt|\alpha|\gamma_m$, and using Lemma 3.4 we bound

$$\begin{aligned}(I) + (II) + (III) &\le C_1 + \int_{B_R} (x\cdot \nabla |K(x)|) e^{2m\bar{w}} dx \\ &\le C_1 + \int_\Omega (x\cdot \nabla |K(x)|) e^{2m\bar{w}} dx\end{aligned}$$

where

$$\Omega := \left\{x \in \mathbb{R}^{2m} : x \cdot \nabla P(x) + \alpha < 0\right\}.$$

From (22) we infer that $\Omega \subset B_{R_1}$. Then with Lemma 3.3 we find

$$(I) + (II) + (III) \leq C_1 + \sup_{x \in B_{R_1}} (|x \cdot \nabla K(x)|) \int_{B_{R_1}} e^{2m\bar{w}} dx \leq C_2 = C_2(R_1),$$

where C_2 does not depend on t or v. To complete the proof it suffices to take $R_0 = R$ so large that

$$\frac{R^a}{C_1} \geq \frac{C_2}{\varepsilon}.$$

□

To prove uniform upper bound of $\bar{w}$ on the complement of a compact set, we use the Kelvin transform. For $R > 1$ define

$$\xi_R(x) := \bar{w}\left(\frac{Rx}{|x|^2}\right), \quad 0 < |x| \leq 1. \tag{24}$$

Lemma 3.6 *There exists $\varepsilon > 0$ sufficiently small such that if $R_0 = R_0(\varepsilon) > 1$ is as in Lemma 3.5, then $\xi(x) := \xi_{R_0}(x)$ is uniformly upper bounded on B_1, i.e. $\bar{w}$ is uniformly upper bounded in $\mathbb{R}^{2m} \setminus B_{R_0}$.*

Proof. Using (31) for $n = 2m$ and $k = m$ and recalling that

$$(-\Delta)^m \bar{w} = K e^{2m\bar{w}} \quad \text{in } \mathbb{R}^{2m} \setminus B_1,$$

we have

$$\begin{aligned}
(-\Delta)^m \xi(x) &= \frac{R_0^{2m}}{|x|^{4m}} ((-\Delta)^m \bar{w}) \left(\frac{R_0 x}{|x|^2}\right) \\
&= \left(\frac{R_0}{|x|^2}\right)^{2m} K\left(\frac{R_0 x}{|x|^2}\right) e^{2m\xi(x)} \\
&=: f(x).
\end{aligned}$$

Then with the change of variable $y = \frac{R_0 x}{|x|^2}$ and Lemma 3.5 we obtain for $R_0 = R_0(\varepsilon)$ large enough (and $\varepsilon > 0$ to be fixed later)

$$\int_{B_1} f(x) dx < \varepsilon.$$

We write $\bar{\xi} := \xi_1 + \xi_2$, where

$$\begin{cases} (-\Delta)^m \xi_1 = f & \text{in } B_1 \\ (-\Delta)^k \xi_1 = 0 & \text{on } \partial B_1 \text{ for } k = 0, 1, 2, .., m-1 \end{cases}$$

and

$$\begin{cases} (-\Delta)^m \xi_2 = 0 & \text{in } B_1 \\ (-\Delta)^k \xi_2 = (-\Delta)^k \xi & \text{on } \partial B_1 \text{ for } k = 1, 2, .., m-1 \\ \xi_2 = \xi^+ := \max\{\xi, 0\} & \text{on } \partial B_1. \end{cases}$$

Iteratively using the *maximum principle* it is easy to see that

$$\xi \le \bar{\xi} \text{ in } B_1. \tag{25}$$

Now fix $\varepsilon > 0$ small enough (and consequently $R_0 = R_0(\varepsilon) > 0$ large enough) so that by Lemma A.2 below, there exists $p > 1$ such that $e^{2m\xi_1}$ is bounded in $L^p(B_1)$. As usual this bound, as well as ε, R_0 are independent of t and v.

Since $|\Delta^k \xi_2|$ is uniformly bounded on ∂B_1 for $k = 0, 1, 2, ..., m-1$ by Lemma 3.4 and $\bar{w}^+$ is uniformly bounded on ∂B_{R_0} by Lemma 3.3, so that ξ^+ is uniformly bounded on ∂B_1, by the maximum principle we get uniform bounds of ξ_2 in B_1. Hence, noticing that

$$\frac{R_0^{2m}}{|x|^{4m}} K\left(\frac{R_0 x}{|x|^2}\right) \le C \quad \text{for } x \in B_1$$

by (13), and using (25), we can bound

$$\begin{aligned}
\|f\|_{L^p(B_1)} &\le C\|e^{2m\xi}\|_{L^p(B_1)} \\
&\le C\|e^{2m\bar{\xi}}\|_{L^p(B_1)} \\
&\le C\|e^{2m\xi_1}\|_{L^p(B_1)}\|e^{2m\xi_2}\|_{L^\infty(B_1)} \\
&\le C.
\end{aligned}$$

Consequently by elliptic estimates and Sobolev embedding there exists a conastant $C > 0$ (independent of v and t) such that

$$\|\xi_1\|_{L^\infty(B_1)} \le C'\|\xi_1\|_{W^{2m,p}(B_1)} \le C,$$

and therefore

$$\xi \le \bar{\xi} \le |\xi_1| + |\xi_2| \le C \quad \text{in } B_1,$$

with C not depending on v and t. □

4 Local uniform upper bounds for the equation $(-\Delta)^m u = Ke^{2mu}$

Here we state a slightly simplified version of Theorem 1 from [13] which we will use to prove the uniform upper bound of Theorem 4.2 below. This theorem was originally proved by F. Robert [16] in dimension 4 and under the assumption $V_k > 0$, and is a delicate counterpart to the blow-up analysis initiated by H. Brézis and F. Merle [1] in dimension 2. The crucial fact which we shall use is that in order to lose compactness V_0 must be positive somewhere *and* $\|V_k e^{2mu_k}\|_{L^1}$ must approach or go above $\Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m})$.

Theorem 4.1 ([13]) *Let $\Omega \subseteq \mathbb{R}^{2m}$ be a connected set. Let $(u_k) \subset C^{2m}_{\mathrm{loc}}(\Omega)$ be such that*

$$(-\Delta)^m u_k = V_k e^{2mu_k} \quad \text{in} \quad \Omega$$

where $V_k \to V_0$ in $C^0_{\mathrm{loc}}(\Omega)$ and, for some $C_1, C_2 > 0$,

$$\int_\Omega e^{2mu_k} dx \le C_1, \quad \int_\Omega |\Delta u_k| dx \le C_2.$$

Then one of the following is true:

(i) up to a subsequence $u_k \to u_0$ *in* $C^{2m-1}_{\mathrm{loc}}(\Omega)$ *for some* $u_0 \in C^{2m}(\Omega)$, *or*

(ii) there is a finite (possibly empty) set $S = \{x^{(1)}, ..., x^{(I)}\} \subset \Omega$ *such that* $V_0(x^{(i)}) > 0$ *for* $1 \le i \le I$, *and up to a subsequence* $u_k \to -\infty$ *locally uniformly in* $\Omega \setminus S$, *and*

$$V_k e^{2mu_k} dx \rightharpoonup \sum_{i=1}^{I} \alpha_i \delta_{x^{(i)}}$$

in the sense of measures in Ω, *where*

$$\alpha_i = L_i \Lambda_1 \text{ for some } L_i \in \mathbb{N} \setminus \{0\}, \quad \Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m}).$$

In particular, in case (ii) for any open set $\Omega_0 \Subset \Omega$ *with* $S \subset \Omega_0$ *we have*

$$\int_{\Omega_0} V_k e^{2mu_k} \to L\Lambda_1 \text{ for some } L \in \mathbb{N}, \text{ and } L = 0 \Leftrightarrow S = \emptyset. \tag{26}$$

Theorem 4.2 *Let* $u \in C^{2m}(B_R)$ *solve*

$$(-\Delta)^m u = K e^{2mu} \quad \text{in } B_R$$

for a function $K \in C^0(B_R)$ *and assume that for given* $C_1, C_2 > 0$ *one has*

(a) $\int_{B_R} e^{2mu} dx \le C_1$,

(b) $\int_{B_R} |\Delta u| dx \le C_2$,

(c_1*) either* $\int_{B_R} K e^{2mu} dx \le \Lambda$ *for some* $\Lambda < (2m-1)!|S^{2m}|$, *or*

(c_2*)* $K \le 0$ *in* B_R.

Then

$$\sup_{B_{R/2}} u \le C$$

where C *only depends on* R, C_1, C_2, Λ *(in case (*c_1*) holds and not (*c_2*)) and* K.

Proof. Assume that there is a sequence of functions $u_n \in C^{2m}(B_R)$ and a sequence of points $x_n \in B_{R/2}$ such that u_n satisfies the conditions (a), (b), and (c_1) or (c_2), and assume that

$$\lim_{n\to\infty} u_n(x_n) = \infty. \tag{27}$$

Then we can apply Theorem 4.1 with $V_k = K$ for every k, and because of (27), we clearly are in case (ii) of the theorem. Assume that $S \neq \emptyset$. Then $K > 0$ on S, hence condition (c_2) does not hold. On the other hand condition (c_1) contradicts (26). Then $S = \emptyset$, hence $u_k \to -\infty$ uniformly in $B_{R/2}$, contradicting (27). ∎

A Appendix

A.1 Some useful lemmas

Lemma A.1 (Pohozaev-type identity) *Consider $K \in C^1(\overline{B_R})$ for some $R > 1$, and let $u_0 \in C^{2m}(\mathbb{R}^{2m})$ be such that* $\mathrm{supp}(\Delta^m u_0) \subseteq \overline{B_1}$. *Let $\bar{w} \in C^{2m}(\overline{B_R})$ be a solution of*

$$(-\Delta)^m \bar{w} = K e^{2m\bar{w}} + t\alpha(-\Delta)^m u_0.$$

Then we have

$$\begin{aligned}\int_{B_R} (x\cdot \nabla K) e^{2m\bar{w}} dx + 2m \int_{B_R} K e^{2m\bar{w}} dx - 2mt\alpha \int_{B_1} (x\cdot \nabla \bar{w})(-\Delta)^m u_0 dx \\ = R \int_{\partial B_R} K e^{2m\bar{w}} d\sigma - mR \int_{\partial B_R} |\Delta^{\frac{m}{2}} \bar{w}|^2 d\sigma - 2m \int_{\partial B_R} f d\sigma,\end{aligned} \tag{28}$$

where,

$$f(x) := \sum_{j=0}^{m-1} (-1)^{m+j} \frac{x}{R} \cdot \left(\Delta^{j/2} (x\cdot \nabla \bar{w}) \Delta^{(2m-1-j)/2} \bar{w} \right) \quad \text{on } \partial B_R,$$

and for k odd $\Delta^{k/2} := \nabla \Delta^{(k-1)/2}$.

Proof. Integrating by parts we find

$$\begin{aligned}2m \int_{B_R} (1 + x\cdot \nabla \bar{w}) K e^{2m\bar{w}} dx &= \int_{B_R} K \,\mathrm{div}(x e^{2m\bar{w}}) dx \\ &= -\int_{B_R} (x\cdot \nabla K) e^{2m\bar{w}} dx + R \int_{\partial B_R} K e^{2m\bar{w}} d\sigma.\end{aligned}$$

Now

$$\int_{B_R} (x\cdot \nabla \bar{w}) K e^{2m\bar{w}} dx = \int_{B_R} (x\cdot \nabla \bar{w})(-\Delta)^m \bar{w} dx - t\alpha \int_{B_1} (x\cdot \nabla \bar{w})(-\Delta)^m u_0 dx, \tag{29}$$

and integrating by parts m times the first term on the right-hand side of (29) we find

$$\int_{B_R} (x\cdot \nabla \bar{w})(-\Delta)^m \bar{w} dx = \int_{B_R} \Delta^{\frac{m}{2}} (x\cdot \nabla \bar{w}) \Delta^{\frac{m}{2}} \bar{w} dx + \int_{\partial B_R} f d\sigma =: I \tag{30}$$

Using

$$\Delta^{\frac{m}{2}} (x\cdot \nabla \bar{w}) \Delta^{\frac{m}{2}} \bar{w} = \frac{1}{2} \mathrm{div}(x |\Delta^{\frac{m}{2}} \bar{w}|^2)$$

(see e.g. [15, Lemma 14] for the simple proof) and using the divergence theorem we obtain

$$I = \frac{1}{2} \int_{\partial B_R} R |\Delta^{\frac{m}{2}} \bar{w}|^2 d\sigma + \int_{\partial B_R} f d\sigma,$$

and putting together the above equations we conclude. □

The proof of the following lemma can be found in [12] (Theorem 7). It extends to arbitrary dimension Theorem 1 of [1].

Lemma A.2 *Let $f \in L^1(B_R)$ and let v solve*

$$\begin{cases} (-\Delta)^m v = f & \text{in } B_R \subset \mathbb{R}^{2m}, \\ \Delta^k v = 0 & \text{on } \partial B_R \text{ for } k = 0, 1, \ldots, m-1. \end{cases}$$

Then, for any $p \in \left(0, \frac{\gamma_m}{\|f\|_{L^1(B_R)}}\right)$, we have $e^{2mp|v|} \in L^1(B_R)$ and

$$\int_{B_R} e^{2mp|v|} dx \leq C(p) R^{2m},$$

where γ_m is definde by (10).

Lemma A.3 *Given $u \in C^\infty(\mathbb{R}^n)$, define $\tilde{u}(x) := u\left(\frac{x}{|x|^2}\right)$ for $x \in \mathbb{R}^n \setminus \{0\}$. Then for any $k \in \mathbb{N}$ we have*

$$\Delta^k \left(\frac{1}{|x|^{n-2k}} \tilde{u}(x) \right) = \frac{1}{|x|^{n+2k}} (\Delta^k u) \left(\frac{x}{|x|^2} \right), \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{31}$$

Proof. We shall prove the lemma by induction on $k \in \mathbb{N}$. Notice that for $k = 0$ (31) is trivial.

For a smooth function f and $g(x) := |x|^2$, we have the formula

$$\Delta^{k+1}(fg) = g\Delta^{k+1} f + 2(k+1)(n+2k)\Delta^k f + 4(k+1)x \cdot \nabla(\Delta^k f),$$

which can be easily proven by induction on $k \in \mathbb{N}$. Choosing

$$f(x) = \frac{\tilde{u}(x)}{|x|^{n-2k}}$$

and assuming that (31) is true for a given $k \in \mathbb{N}$, we compute

$$\begin{aligned} \Delta^{k+1} \left(\frac{\tilde{u}(x)}{|x|^{n-2(k+1)}} \right) &= \Delta^{k+1}(fg) \\ &= g\Delta(\Delta^k f) + 2(k+1)(n+2k)\Delta^k f + 4(k+1)x \cdot \nabla(\Delta^k f) \\ &= |x|^2 \Delta \left(\frac{1}{|x|^{n+2k}} (\Delta^k u) \left(\frac{x}{|x|^2} \right) \right) + 2(k+1)(n+2k) \frac{1}{|x|^{n+2k}} (\Delta^k u) \left(\frac{x}{|x|^2} \right) \\ &\quad + 4(k+1)x \cdot \nabla \left(\frac{1}{|x|^{n+2k}} (\Delta^k u) \left(\frac{x}{|x|^2} \right) \right) \\ &= \frac{1}{|x|^{n+2(k+1)}} (\Delta^{k+1} u) \left(\frac{x}{|x|^2} \right), \end{aligned}$$

hence completing the induction. □

A.2 Proof of Lemma 2.3

For any $R \geq 1$ set

$$A_R := \{x \in \mathbb{R}^{2m} : R < |x| < 2R\}, \quad A := A_1 = \{x \in \mathbb{R}^{2m} : 1 < |x| < 2\}.$$

Given $f \in W^{2m,p}(A_R)$, define

$$\tilde{f}(x) := f(Rx), \quad \text{for } x \in A.$$

For $|\beta| \leq 2m$, we have

$$\begin{aligned}\int_A |D^\beta \tilde{f}(x)|^p dx &= R^{p|\beta|} \int_A |(D^\beta f)(Rx)|^p dx \\ &= R^{p|\beta|-2m} \int_{A_R} |D^\beta f(x)|^p dx.\end{aligned}$$

From the embedding $W^{2m,p}(A) \hookrightarrow C^0(A)$ there exists a constant $S > 0$, such that

$$\|u\|_{C^0(A)} \leq S\|u\|_{W^{2m,p}(A)}, \quad \text{for all } u \in W^{2m,p}(A).$$

Hence

$$\begin{aligned}\|f\|_{C^0(A_R)} &= \|\tilde{f}\|_{C^0(A)} \\ &\leq S\|\tilde{f}\|_{W^{2m,p}(A)} \\ &= S \sum_{|\beta|\leq 2m} \|D^\beta \tilde{f}\|_{L^p(A)} \\ &= S \sum_{|\beta|\leq 2m} R^{|\beta|-2m/p} \|D^\beta f\|_{L^p(A_R)} \\ &\leq CS \sum_{|\beta|\leq 2m} R^{-2m/p-\delta} \|(1+|x|^2)^{\frac{\delta+|\beta|}{2}} D^\beta f\|_{L^p(A_R)} \\ &\leq CSR^{-\gamma}\|f\|_{M^p_{2m,\delta}}, \quad \gamma = 2m/p + \delta > 0.\end{aligned} \tag{32}$$

Since $R \geq 1$ is arbitrary (32) and on B_2 we have

$$\|f\|_{C^0(B_2)} \leq S'\|f\|_{W^{2m,p}(B_2)} \leq CS'\|f\|_{M^p_{2m,\delta}}, \tag{33}$$

we conclude that $M^p_{2m,\delta}(\mathbb{R}^{2m}) \subset C_0(\mathbb{R}^{2m})$, and actually

$$\sup_{n\in\mathbb{N}} \|f_n\|_{M^p_{2m,\delta}} < \infty \quad \Rightarrow \quad \lim_{R\to\infty} \sup_{n\in\mathbb{N}} \|f_n\|_{C^0(A_R)} = 0. \tag{34}$$

By (32) and (33), on any compact set $\Omega \Subset \mathbb{R}^{2m}$ the sequence $\|f_n\|_{W^{2m,p}(\Omega)}$ is bounded and from the compact embedding $W^{2m,p}(\Omega) \hookrightarrow C^0(\Omega)$, we can extract a subsequence converging in $C^0(\Omega)$. Then up to choosing $\Omega = B_n$ and extracting a diagonal subsequence we have $f_n \to f$ locally uniformly for a continuous function f, and actually $f \in C_0(\mathbb{R}^{2m})$ and the convergence is globally uniform thanks to (34). □

A.3 Condition (9) does not imply (5)

Proposition A.4 *For $n \geq 2$ there exists a polynomial P of degree 4 in $\mathbb{R}^n$ satisfying (9) but not (5).*

Proof. In $\mathbb{R}^2$ consider $P(x) = P(x_1, x_2) = x_1^2 + x_2^4 - \beta x_1 x_2^2$, with $\beta < 2$. Then

$$P(x) \geq x_1^2 + x_2^4 - \beta\left(\frac{x_1^2}{2} + \frac{x_2^4}{2}\right) = \left(1 - \frac{\beta}{2}\right)(x_1^2 + x_2^4),$$

so that P satisfies (9). Moreover

$$x \cdot \nabla P(x) = 2x_1^2 + 4x_2^4 - 3\beta x_1 x_2^2.$$

Choosing $x = (ax_2^2, x_2)$ we obtain

$$(ax_2^2, x_2) \cdot \nabla P(ax_2^2, x_2) = x_2^4(2a^2 - 3\beta a + 4).$$

Then, since for $|\beta| > \sqrt{2}\frac{4}{3}$ the polynomial $2a^2 - 3\beta a + 4$ has positive discriminant, fixing $\beta \in (-\infty, -\sqrt{2}\frac{4}{3}) \cup \left(\sqrt{2}\frac{4}{3}, 2\right)$ and a such that $2a^2 - 3\beta a + 4 < 0$ we see that

$$\liminf_{|x|\to\infty} x \cdot \nabla P(x) \leq \lim_{|x_2|\to\infty} (ax_2^2, x_2) \cdot \nabla P(ax_2^2, x_2) = -\infty.$$

This proves the proposition for $n = 2$. For $n > 2$ it suffices to consider

$$\tilde{P}(x_1, x_2, \ldots, x_n) = P(x_1, x_2) + \sum_{j=3}^{n} x_j^2,$$

where P is as before. □

References

[1] Brézis, Haim; Merle, Frank: *Uniform estimates and blow-up behaviour for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations **16** (1991), 1223-1253.

[2] Chang, Sun-Yung Alice; Chen, Wenxiong: *A note on a class of higher order conformally covariant equations*, Discrete Contin. Dynam. Systems **63** (2001), 275-281.

[3] Chang, Sun-Yung Alice; Yang, Paul: *On uniqueness of solutions of n-th order differential equations in conformal geometry*, Math. Res. Lett. **4** (1997), 91-102.

[4] Chen, Wenxiong; Li, Congming: *Classification of solutions of some nonlinear elliptic equations*, Duke Math. J. **63** (3) (1991), 615-622.

[5] Gilbarg, David; Trudinger, Neil: *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp. ISBN: 3-540-41160-7

[6] Gorin, Evgeniĭ A.: *Asymptotic properties of polynomials and algebraic functions of several variables*, Russ. Math. Surv. **16**(1) (1961), 93-119.

[7] Graham, C. Robin; Jenne, Ralph; Mason, Lionel; Sparling, George: *Conformally invariant powers of the Laplacian, I: existence*, J. London Math. Soc. **46** no.2 (1992), 557-565.

[8] Jin, Tianling; Maalaoui, Ali; Martinazzi, Luca; Xiong, Jingag: *Existence and asymptotics for solutions of a non-local Q-curvature equation in dimension three*, to appear in Calc. Var. Partial Differential Equations (2013).

[9] Lin, Chang-Shou: *A classification of solutions of a conformally invariant fourth order equation in* $\mathbb{R}^n$, Comment. Math. Helv. **73** (1998), no. 2, 206-231.

[10] McOwen, Robert C.: *The behavior of the Laplacian on weighted Sobolev spaces.* Comm. Pure Appl. Math. **32** (1979), no. 6, 783-795.

[11] Martinazzi, Luca: *Conformal metrics on* $\mathbb{R}^{2m}$ *with constant Q-curvature*, Rend. Lincei. Mat. Appl. **19** (2008), 279-292.

[12] Martinazzi, Luca: *Classification of solutions to the higher order Liouville's equation on* $\mathbb{R}^{2m}$. Math. Z. **263** (2009), no. 2, 307-329

[13] Martinazzi, Luca: *Quantization for the prescribed Q-curvature equation on open domains.* Commun. Contemp. Math. **13** (2011), no. 3, 533-551.

[14] Martinazzi, Luca: *Conformal metrics on* $\mathbb{R}^{2m}$ *with constant Q-curvature and large volume,* Ann. Inst. Henri Poincaré (C) **30** (2013), 969-982.

[15] Martinazzi, Luca; Petrache, Mircea: *Asymptotics and quantization for a mean-field equation of higher order.* Comm. Partial Differential Equations **35** (2010), no. 3, 443-464.

[16] Robert, Frédéric: *Quantization effects for a fourth order equation of exponential growth in dimension four*, Proc. Roy. Soc. Edinburgh Sec. A **137** (2007), 531-553.

[17] Wei, Juncheng; Ye, Dong: *Nonradial solutions for a conformally invariant fourth order equation in* $\mathbb{R}^4$. Calc. Var. Partial Differential Equations **32** (2008), no. 3, 373-386.