The Moser-Trudinger inequality and its extremals on a disk via energy estimates

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Abstract

We study the Dirichlet energy of non-negative radially symmetric critical points u_μ of the Moser-Trudinger inequality on the unit disc in $\mathbb{R}^2$, and prove that it expands as

$$4\pi + \frac{4\pi}{\mu^4} + o(\mu^{-4}) \le \int_{B_1} |\nabla u_\mu|^2 dx \le 4\pi + \frac{6\pi}{\mu^4} + o(\mu^{-4}), \quad \text{as } \mu \to \infty,$$

where $\mu = u_\mu(0)$ is the maximum of u_μ. As a consequence, we obtain a new proof of the Moser-Trudinger inequality, of the Carleson-Chang result about the existence of extremals, and of the Struwe and Lamm-Robert-Struwe multiplicity result in the supercritical regime (only in the case of the unit disk).

Our results are stable under sufficiently weak perturbations of the Moser-Trudinger functional. We explicitly identify the critical level of perturbation for which, although the perturbed Moser-Trudinger inequality still holds, the energy of its critical points converges to 4π *from below*. We expect, in some of these cases, that the existence of extremals does not hold, nor the existence of critical points in the supercritical regime.

1 Introduction

Consider the Moser-Trudinger inequality in dimension two (see [16, 17, 22]):

Theorem A (Moser [16]) *For $\Omega \subset \mathbb{R}^2$ with finite measure $|\Omega|$ we have*

$$\sup_{u \in H_0^1(\Omega) : \|\nabla u\|_{L^2}^2 \le 4\pi} \int_\Omega e^{u^2} dx \le C|\Omega|. \tag{1}$$

Moreover the constant 4π is sharp.

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As noticed by Moser, the subcritical inequality

$$\sup_{u\in H_0^1(\Omega):\|\nabla u\|_{L^2}^2\le\alpha}\int_\Omega e^{u^2}dx\le\frac{|\Omega|}{1-\frac{\alpha}{4\pi}},\qquad (I_\alpha)$$

is easy to obtain for $\alpha<4\pi$. Indeed, by symmetrization and scaling one reduces to the case of the unit disk $\Omega=B_1$ and $u=u(r)$ radially symmetric. Then, by the fundamental theorem of calculus and Hölder's inequality, one bounds

$$|u(r)|^2\le\left(\int_r^1|u'(\rho)|d\rho\right)^2\le\int_r^1 2\pi\rho|u'(\rho)|^2d\rho\int_r^1\frac{d\rho}{2\pi\rho}\le\frac{\|\nabla u\|_{L^2}^2}{2\pi}\log\frac{1}{r},\qquad (2)$$

hence if $\|\nabla u\|_{L^2}^2\le\alpha<4\pi$,

$$\int_{B_1}e^{u^2}dx\le\int_0^1 2\pi re^{\frac{\alpha}{2\pi}\log\frac{1}{r}}dr=2\pi\int_0^1 r^{1-\frac{\alpha}{2\pi}}dr=\frac{\pi}{1-\frac{\alpha}{4\pi}}.$$

The difficult part of Theorem A is to prove that (1) also holds with the critical constant 4π. To do that Moser considers a special class of functions, which are now known as Moser-functions or broken-line functions, and notices that for such functions (1) holds (and it fails if we replace 4π by a larger constant). Further he shows that any function for which (2) is close to an identity at one point must be close to a Moser function in a suitable sense.

The existence of maximizers (usually called *extremals*) for the Moser-Trudinger inequality has been pioneered for $\Omega=B_1$ by L. Carleson and A. Chang [4]:

Theorem B (Carleson-Chang [4]) *When $\Omega=B_1$ is the unit disk, the inequality* (1) *admits an extremal.*

The original proof of Theorem B is based on estimating

$$F(u):=\int_{B_1}e^{u^2}dx$$

on a sequence u_k maximizing the supremum in (1), and showing, in a very clever way, that $\limsup_{k\to\infty}F(u_k)\le\pi(1+e)$ if the sequence blows-up. On the other hand, this cannot be the case, since the authors exhibit a function u^* such that $F(u^*)>\pi(1+e)$. Then the sequence (u_k) is precompact and converges to a maximizer. This method has been extended to several more general cases, starting from the works of Struwe [20], Flucher [7] and Li [11].

In this paper we shall give an alternative approach to Theorems A and B, based on estimating the Dirichlet energy of the extremals of subcritical inequalities. Indeed it is easy to prove that the subcritical inequality (I_α) has a maximizer u_α for every $\alpha < 4\pi$, see Proposition 6 below. Such extremal satisfies

$$-\Delta u_\alpha = \lambda_\alpha u_\alpha e^{u_\alpha^2}, \tag{3}$$

for a positive Lagrange multiplier λ_α. The crucial question is whether u_α converges as $\alpha \uparrow 4\pi$. The answer is affirmative and follows easily from the energy estimate of the next theorem, which is the core of our argument.

Theorem 1 *Let $(u_k) \subset H_0^1(B_1)$ be any sequence (possibly unbounded) of radially symmetric and positive solutions*[1] *to*

$$-\Delta u_k = \lambda_k u_k e^{u_k^2}, \tag{4}$$

for some $\lambda_k > 0$. Assume

$$\mu_k := u_k(0) = \max_{B_1} u_k \to \infty, \quad \text{as } k \to \infty. \tag{5}$$

Then

$$4\pi + \frac{4\pi}{\mu_k^4} + o(\mu_k^{-4}) \le \|\nabla u_k\|_{L^2}^2 \le 4\pi + \frac{6\pi}{\mu_k^4} + o(\mu_k^{-4}). \tag{6}$$

To prove Theorem 1 we build up on a technique introduced in [13] and perform a Taylor expansion of the solutions u_k near the origin, which needs to be precise enough to obtain (6), see Section 3.

Consider now a mildly perturbed, though completely equivalent version of Theorem A, namely for $\alpha \in (0, 4\pi]$ replace (1) and (I_α) with

$$\sup_{u \in H_0^1(\Omega): \|\nabla u\|_{L^2}^2 \le \alpha} \int_\Omega (1 + g(u)) e^{u^2} dx \le C_{g,\alpha}, \tag{I_α^g}$$

where

$$g \in C^1(\mathbb{R}), \quad \inf_{\mathbb{R}} g > -1, \quad g(t) = g(-t) \quad \text{and} \quad \lim_{|t| \to \infty} g(t) = 0. \tag{7}$$

We want to investigate whether an analog of Theorem 1 holds for positive critical points of (I_α^g), and consequently whether $(I_{4\pi}^g)$ admits an extremal. As we shall now see, this is the case if g decays well enough at infinity. More precisely, observe that the critical points of (I_α^g) satisfy

$$-\Delta u = \lambda \left(1 + g(u) + \frac{g'(u)}{2u}\right) u e^{u^2} = \lambda(1 + h(u)) u e^{u^2}, \tag{8}$$

for some $\lambda \in \mathbb{R}$, where we set

$$h(t) := g(t) + \frac{g'(t)}{2t}, \quad t \in \mathbb{R} \setminus \{0\}. \tag{9}$$

[1] Actually the radial symmetry follows from positivity and the moving plane technique.

We further assume

$$\inf_{(0,\infty)} h > -1, \quad \sup_{(0,\infty)} h < \infty, \quad \lim_{t\to\infty} t^2 h(t) = 0 \tag{10}$$

and

$$\lim_{t\to\infty} \sup_{|s|\le 1} t^4 \left| h\left(t + \frac{s(8\log t + 1)}{t}\right) - h(t) \right| = 0. \tag{11}$$

A typical function g that we have in mind is $g(t) = |t|^{-p}$ near infinity for some $p > 2$. More generally one can take a function $\chi \in C^\infty([0,\infty))$ with $\chi \equiv 0$ on $[0,1]$, $\chi \equiv 1$ on $[2,\infty)$, and consider for $R > 0$ sufficiently large

$$g(t) = a\chi(R^{-1}|t|)\log^q(|t|)|t|^{-p}, \quad a, q \in \mathbb{R},\, p > 2, \tag{12}$$

or even the oscillating function

$$g(t) = a\chi(R^{-1}|t|)\cos(\log|t|)|t|^{-p}, \quad a \in \mathbb{R},\, p > 2. \tag{13}$$

Then we have the following generalized versions of Theorems 1 and B.

Theorem 2 *Let $(u_k) \subset H_0^1(B_1)$ be a sequence of radially symmetric and positive solutions to*

$$-\Delta u_k = \lambda_k(1 + h(u_k))u_k e^{u_k^2}, \tag{14}$$

with $\lambda_k > 0$ and $h : (0,\infty) \to \mathbb{R}$ satisfying (10)-(11). *Assume that* (5) *holds. Then*

$$4\pi + \frac{4\pi}{\mu_k^4} + o(\mu_k^{-4}) \le \|\nabla u_k\|_{L^2}^2 \le 4\pi + \frac{4\pi + 2\pi(1 + \sup h)}{\mu_k^4} + o(\mu_k^{-4}). \tag{15}$$

Corollary 3 *If g satisfies* (7) *and h as in* (9) *satisfies* (10) *and* (11), *then $(I_{4\pi}^g)$ with $\Omega = B_1$ admits an extremal.*

It is natural to ask how sharp conditions (10) and (11) are. The following example shows that the quadratic decay is indeed critical.

Theorem 4 *Let $h : \mathbb{R} \to [-1/2, 1/2]$ satisfy $h(t) = -at^{-2}$ for $t \ge R$ for some $a > 0$ and $R > 0$ fixed, and let $(u_k) \subset H_0^1(B_1)$ be a sequence of radially symmetric positive solutions to* (14) *satisfying* (5). *Then*

$$4\pi + \frac{4\pi - 4\pi a}{\mu_k^4} + o(\mu_k^{-4}) \le \|\nabla u_k\|_{L^2}^2 \le 4\pi + \frac{4\pi + 2\pi(1 + \sup h) - 4\pi a}{\mu_k^4} + o(\mu_k^{-4}).$$

In particular for $a > \frac{3}{2} + \frac{\sup h}{2}$ we can find a value $\bar{\mu}$ such that for any positive solution u to (8) *with $u(0) \ge \bar{\mu}$ we have $\|\nabla u\|_{L^2}^2 < 4\pi$.*

Open problem 1 *Can one find a function h as in Theorem 4 and satisfying* (9) *for some g as in* (7) *such that $\|\nabla u\|_{L^2}^2 < 4\pi$ for* every *positive u solving* (8)*?*

The function h given in Theorem 4 (and a corresponding function g can be easily constructed) covers the case when $u(0)$ is sufficiently large but one should also rule out the possibility that some "small" solutions have energy at least 4π. If the above question has a positive answer, for such functions g and h one would have that $(I^g_{4\pi})$ admits no extremal. The non-existence of extremals for a very mildly perturbed Moser-Trudinger inequality originally motivated our interest in Theorems 2 and 4. In [18] Pruss showed the existence of a function g as in (7) such that the inequality $(I^g_{4\pi})$ does not have extremals. However his construction of g is quite implicit and we do not know its asymptotic behaviour at infinity. More generally the following appears to be open:

Open problem 2 *For which functions g as in* (7) *does the perturbed Moser-Trudinger inequality $(I^g_{4\pi})$ have an extremal?*

Finally we remark that the following result is an immediate consequence of Theorems 1 and 2:

Theorem 5 *Set*

$$E(u) := \int_{B_1} (1 + g(u))e^{u^2} dx, \quad M_\Lambda := \left\{u \in H^1_0(B_1) : \|\nabla u\|^2_{L^2} = \Lambda\right\},$$

where g is as in Theorem 2. Then there exists $\Lambda^* > 4\pi$ such that for every $\Lambda \in (4\pi, \Lambda^*)$ the functional $E|_{M_\Lambda}$ has at least two critical points.

Theorem 5 for a general smoothly bounded domain $\Omega \subset \mathbb{R}^2$ and with $g \equiv 0$ was proven in [10, 20] using variational methods, geometric flows, a sharp quantization estimate, and a monotonicity technique. See also [5], where the existence of one critical point with energy Λ for every Λ slightly larger that 4π was proven using a fixed point method via Lyapunov-Schmidt reduction.

The paper is organized as follows. In Section 2 we will show how the energy estimates of Theorems 1 and 2 imply Theorems A, B and Corollary 3, while the proofs of Theorems 1, 2, and 4 are contained in Sections 3, 4 and 5 respectively. Finally, in the last section, we collect some open problems. While attempting to avoid repetitions, we had to allow some redundancy to keep the paper reader-friendly. The proof of Theorem 1 is the most detailed, and some parts of it will be reused when proving Theorems 2 and 4.

Notations For a non-vanishing function $f : (0, \infty) \to \mathbb{R}$ we use the Peano notation $o(f(t))$ and $O(f(t))$ to denote functions such that $o(f(t))/f(t) \to 0$ and $|O(f(t))/f(t)| \leq C$ as $|t| \to \infty$.

Since all function we use are radially symmetric, we will use the notation $u(x) = u(r)$ with $x \in \mathbb{R}^2$, $r = |x|$, and also write $\Delta u(r) = r^{-1}(ru'(r))'$.

2 Proof Theorems A and B using Theorem 1

In this section we prove Theorems A and B starting from the subcritical inequality (I_α) and the energy estimate in Theorem 1. In fact we will be more general and work directly with (I_α^g), showing that Corollary 3 follows from Theorem 2.

Proposition 6 *Assume that g and h satisfy (7), (9) and the first condition in (10). Then for any $\alpha < 4\pi$ the inequality (I_α^g) has an extremal $u_\alpha > 0$ satisfying (8) for some $\lambda \in \left(0, \frac{\lambda_1(\Omega)}{1+\inf h}\right)$. Here $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω with Dirichlet boundary condition. If $\Omega = B_1$, then u_α can be taken radially symmetric and decreasing.*

Proof. Let $(u_k) \subset H_0^1(\Omega)$ with $\|\nabla u_k\|_{L^2}^2 \le \alpha$ be a maximizing sequence for (I_α^g). By the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ we have that, up to a subsequence, $u_k \to u_\alpha$ weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and almost everywhere.

Fix now $\alpha' \in (\alpha, 4\pi)$. Since $\tilde{u}_k := \sqrt{\frac{\alpha'}{\alpha}} u_k$ satisfies $\|\nabla \tilde{u}_k\|_{L^2}^2 \le \alpha'$, using $(I_{\alpha'})$, we have for any $L > 0$

$$\int_{\{x\in\Omega : u_k \ge L\}} (1+g(u_k)) e^{u_k^2} dx \le e^{-\left(\frac{\alpha'}{\alpha}-1\right)L^2} (1+\sup_{\mathbb{R}} g) \int_\Omega e^{\tilde{u}_k^2} dx = O\left(e^{-\left(\frac{\alpha'}{\alpha}-1\right)L^2}\right) = o(1)$$

as $L \to \infty$, uniformly in k. Then, by Lemma 7 we infer that

$$\begin{aligned}\int_\Omega (1+g(u_\alpha)) e^{u_\alpha^2} dx &= \lim_{k\to\infty} \int_\Omega (1+g(u_k)) e^{u_k^2} dx \\ &= \sup_{u \in H_0^1(\Omega),\, \|\nabla u\|_{L^2}^2 \le \alpha} \int_\Omega (1+g(u)) e^{u^2} dx.\end{aligned}$$

Since

$$\|\nabla u_\alpha\|_{L^2}^2 \le \limsup_{k\to\infty} \|\nabla u_k\|_{L^2}^2 \le \alpha,$$

we have that indeed u_α is an extremal for (I_α^g). Since (9)-(10) imply that $(1+g(t))e^{t^2}$ is increasing for $t \ge 0$, we have that $\|\nabla u_\alpha\|_{L^2}^2 = \alpha$. In particular u_α solves the Euler-Lagrange equation (8) for some $\lambda \in \mathbb{R}$. Multiplying (8) by u_α and integrating we obtain

$$\int_\Omega |\nabla u_\alpha|^2 dx = \lambda \int_\Omega (1+h(u_\alpha)) u_\alpha^2 e^{u_\alpha^2} dx > \lambda (1+\inf h) \int_\Omega u_\alpha^2 dx,$$

and using the variational characterization of $\lambda_1(\Omega)$ we infer $\lambda \in \left(0, \frac{\lambda_1(\Omega)}{1+\inf h}\right)$.

That u_α has a sign follows by considering $|u_\alpha|$, which is also an extremal, also satisfying (8) hence by the maximum principle it never vanishes. In particular also u_α never vanishes, and by continuity it has a sign.

Finally, if $\Omega = B_1$, the claim about the symmetry of u_α follows at once by choosing u_k radially symmetric and decreasing, which is possible by symmetrization. □

Proof of Theorems A and B assuming Theorem 1, and of Corollary 3 using Theorem 2. Set $\alpha_k = 4\pi - \frac{1}{k}$ and let $u_k = u_{\alpha_k} > 0$ be the radially symmetric extremal to $(I^g_{\alpha_k})$ with $\Omega = B_1$ given by Proposition 6. According to (15) we have

$$\limsup_{k\to\infty} u_k(0) = \limsup_{k\to\infty} \max_{B_1} u_k < \infty,$$

otherwise for some k large enough we would have

$$4\pi + \frac{4\pi}{u_k^4(0)} + o(u_k^{-4}(0)) \leq \|\nabla u_k\|^2_{L^2} = 4\pi - \frac{1}{k},$$

which is a contradiction. Then $u_k(0) = \max_{B_1} |u_k| \leq C$ and by elliptic estimates we have $u_k \to u_\infty$ in $C^1(\bar{\Omega})$. It is now easy to see that u_∞ is an extremal for $(I^g_{4\pi})$. Indeed

$$\int_{B_1} (1 + g(u_k))e^{u_k^2}dx \uparrow \int_{B_1} (1 + g(u_\infty))e^{u_\infty^2}dx, \quad \text{as } k \to \infty,$$

and if there was a function $v \in H^1_0(B_1)$ with $\|\nabla v\|^2_{L^2} \leq 4\pi$ and

$$\int_{B_1} (1 + g(v))e^{v^2}dx > \int_{B_1} (1 + g(u_\infty))e^{u_\infty^2}dx,$$

we could find (for instance by monotone convergence) k large such that $\tilde{u}_k := \sqrt{\frac{\alpha_k}{4\pi}}v$ satisfies

$$\int_{B_1} (1 + g(\tilde{u}_k))e^{\tilde{u}_k^2}dx > \int_{B_1} (1 + g(u_\infty))e^{u_\infty^2}dx \geq \int_{B_1} (1 + g(u_k))e^{u_k^2},$$

which would contradict the maximality of u_k, since $\|\nabla \tilde{u}_k\|^2_{L^2} \leq \alpha_k$. Then u_∞ is an extremal $(I^g_{4\pi})$. This also implies $(I^g_{4\pi})$ (hence (1)) for $\Omega = B_1$, and by symmetrization and scaling, also for any domain Ω with finite measure. This completes the proof. □

Lemma 7 *Let $|\Omega| < \infty$, and consider a sequence of non-negative functions $(f_k) \subset L^1(\Omega)$ with $f_k \to f$ a.e. and with*

$$\int_{\{f_k > L\}} f_k dx = o(1), \tag{16}$$

with $o(1) \to 0$ as $L \to \infty$ uniformly with respect to k. Then $f_k \to f$ in $L^1(B_1)$.

Proof. By Fatou's lemma (16) implies $f \in L^1(\Omega)$. From the dominated convergence theorem

$$\min\{f_k, L\} \to \min\{f, L\} \quad \text{in } L^1(\Omega),$$

and the convergence of f_k to f in L^1 follows at once from (16) and the triangle inequality. □

3 Proof of Theorem 1

Let u_k and $\mu_k = u_k(0) \to \infty$ be as in Theorem 1. In order to estimate $\|\nabla u_k\|_{L^2}$, after a well-known scaling (see (18) below) we reduce to study a function η_k which solves a perturbed version of the Liouville equation, namely (19). We will make a Taylor expansion of the right-hand side of (19) up to order μ_k^{-6} (Lemma 10) and expand $\eta_k = \eta_0 + \frac{w_0}{\mu_k^2} + \frac{z_0}{\mu_k^4} + \frac{\phi_k}{\mu_k^6}$. Inspired from [13] (where the Taylor expansion was made only up to order μ_k^{-4}), we will prove uniform bounds on the error term ϕ_k up to sufficiently large scales. This can be achieved by ODE theory and a fixed point argument, see Lemma 11. Together with the asymptotic behaviour of w_0, which is explicit thanks to Lemma 9, this implies

$$\|\nabla u_k\|_{L^2}^2 = 4\pi + O(\mu_k^{-4}),$$

but with no information about the sign of the error $O(\mu_k^{-4})$. In order to obtain the more precise estimate (6) we shall need the asymptotic behaviour of the function z_0, which is not given by an explicit formula. For this we will use the somewhat surprising Lemma 16 (also see Corollary 18).

3.1 Taylor expansions and behaviour at large scales

We will start with the following standard blow-up procedure. Set $r_k > 0$ such that

$$r_k^2 \lambda_k \mu_k^2 e^{\mu_k^2} = 4 \tag{17}$$

and rescale u_k to a new function η_k defined on $B_{r_k^{-1}}$ as

$$\eta_k(x) := \mu_k(u_k(r_k x) - \mu_k). \tag{18}$$

Notice that

$$\begin{cases} -\Delta \eta_k = 4\left(1 + \frac{\eta_k}{\mu_k^2}\right) e^{2\eta_k + \frac{\eta_k^2}{\mu_k^2}} & \text{in } [0, r_k^{-1}] \\ \eta_k(0) = \eta_k'(0) = 0 \end{cases} \tag{19}$$

and, as $\mu_k \to \infty$, the nonlinearity on the right-hand side approaches $4e^{2\eta_k}$. More precisely one has:

Lemma 8 ([6, 13]) *Let r_k, η_k be as in* (5), (17) *and* (18), *with η_k solving* (19). *Then as $k \to \infty$ we have $r_k \to 0$,*

$$\eta_k(x) \to \eta_0(x) := -\log(1 + |x|^2) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^2), \tag{20}$$

and η_0 solves

$$-\Delta \eta_0 = 4e^{2\eta_0} \quad \text{in } \mathbb{R}^2. \tag{21}$$

Moreover

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_{Rr_k}} \lambda_k u_k^2 e^{u_k^2} dx = \int_{\mathbb{R}^2} 4e^{2\eta_0} dx = 4\pi. \tag{22}$$

One easily sees that (22) implies

$$\liminf_{k\to\infty} \|\nabla u_k\|_{L^2}^2 \geq 4\pi. \tag{23}$$

In order to improve (23) to

$$\lim_{k\to\infty} \|\nabla u_k\|_{L^2}^2 = 4\pi, \tag{24}$$

in [13] Malchiodi and the second author investigated the blow-up behaviour of the sequence u_k up to a higher order of precision.

Lemma 9 ([13]) *Set $w_k := \mu_k^2(\eta_k - \eta_0)$. Then we have $w_k \to w_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where*

$$w_0(r) := \eta_0(r) + \frac{2r^2}{1+r^2} - \frac{1}{2}\eta_0^2(r) + \frac{1-r^2}{1+r^2}\int_1^{1+r^2} \frac{\log t}{1-t}dt \tag{25}$$

is the unique solution to the ODE

$$\begin{cases} -\Delta w_0 = 4e^{2\eta_0}(\eta_0 + \eta_0^2 + 2w_0) \text{ in } \mathbb{R}^2 \\ w_0(0) = w_0'(0) = 0. \end{cases} \tag{26}$$

Moreover $w_0(r) = \eta_0(r) + O(1)$ as $r \to \infty$ and in fact

$$\int_{\mathbb{R}^2} \Delta w_0 dx = -4\pi. \tag{27}$$

One can further prove that

$$w_0'(r) = -\frac{2}{r} + O\left(\frac{\log^2 r}{r^3}\right) \tag{28}$$

as $r \to \infty$, which will be important in our analysis. This follows from the explicit expression (25) but can also be deduced from the structure of equation (26), see Corollary 17.

To prove Theorem 1 we need to further expand the right-hand side of (19), namely we write

$$\eta_k = \eta_0 + \frac{w_0}{\mu_k^2} + \frac{z_k}{\mu_k^4},$$

for an unknown (locally bounded) error z_k, and formally compute

$$\begin{aligned} -\Delta\eta_k &= 4\left(1 + \frac{\eta_k}{\mu_k^2}\right)e^{2\eta_k + \frac{\eta_k^2}{\mu_k^2}} \\ &= 4e^{2\eta_0}\left[1 + \frac{\eta_0 + \eta_0^2 + 2w_0}{\mu_k^2} + \frac{w_0 + 2w_0^2 + 4\eta_0 w_0 + 2w_0\eta_0^2 + \eta_0^3 + \frac{1}{2}\eta_0^4 + 2z_k}{\mu_k^4}\right] \\ &\quad + O(\mu_k^{-6}). \end{aligned}$$

This suggests to define z_0 as the only radial solution to the Cauchy problem

$$\begin{cases} -\Delta z_0 = 4e^{2\eta_0}(w_0 + 2w_0^2 + 4\eta_0 w_0 + 2w_0\eta_0^2 + \eta_0^3 + \frac{1}{2}\eta_0^4 + 2z_0) \text{ in } \mathbb{R}^2 \\ z_0(0) = z_0'(0) = 0. \end{cases} \tag{29}$$

Even though we do not have an explicit formula for z_0, we will show

$$z_0(r) = \beta \log(r) + O(1), \quad \text{as } r \to \infty, \tag{30}$$

for some constant β. In fact we will prove

$$\beta = \frac{1}{2\pi}\int_{\mathbb{R}^2} \Delta z_0 dx = -6 - \frac{\pi^2}{3}, \tag{31}$$

which will be crucial in the proof of Proposition 12. To simplify our exposition of the proof, we postpone the analysis of the asymptotic behaviour of z_0 to the end of the section, see Lemmas 15, 16 and Corollary 18.

The problem now is to use η_0, w_0 and z_0 to approximate η_k in a good sense (up to error $O(\mu_k^{-6}\log^6 r)$) and for sufficiently large radii. For this we will use a method inspired from [13, Lemma 5].

Lemma 10 *Let $s_k \le e^{\mu_k}$ and $\phi : [0, s_k] \to \mathbb{R}$ be given so that $\phi = o(\mu_k^6)$ uniformly on $[0, s_k]$. Set*

$$\eta := \eta_0 + \frac{w_0}{\mu_k^2} + \frac{z_0}{\mu_k^4} + \frac{\phi}{\mu_k^6}$$

and

$$\Phi_k(r, \phi) := \mu_k^6 \left[4\left(1 + \frac{\eta}{\mu_k^2}\right) e^{2\eta + \frac{\eta^2}{\mu_k^2}} + \Delta\eta_0 + \frac{\Delta w_0}{\mu_k^2} + \frac{\Delta z_0}{\mu_k^4}\right]. \tag{32}$$

Then

$$\Phi_k(r, \phi) = 4e^{2\eta_0}\left(2\phi + o(1)\phi + O(\mu_k^{-2}\xi^2)\phi + O(\xi^6)\right), \quad \textit{uniformly for } r \in [0, s_k],$$

where

$$\xi(r) := 1 + \log(1 + r). \tag{33}$$

Proof. By Lemma 8, Lemma 9 and (30) we have $|\eta_0| + |w_0| + |z_0| = O(\xi)$. Moreover the assumptions on s_k imply $\mu_k^{-1}\xi = O(1)$ uniformly on $[0, s_k]$. This will be used several times throughout the proof. In order to expand the exponential term in $\Phi_k(r, \phi)$ we write

$$\begin{aligned} \psi &:= 2\eta + \frac{\eta^2}{\mu_k^2} - 2\eta_0 \\ &= \frac{2w_0 + \eta_0^2}{\mu_k^2} + \frac{2z_0 + 2\eta_0 w_0}{\mu_k^4} + \frac{2\phi}{\mu_k^6} + o(1)\frac{\phi}{\mu_k^6} + O(\mu_k^{-6}\xi^2). \end{aligned} \tag{34}$$

Similarly

$$\psi^2 = \frac{4w_0^2 + 4w_0\eta_0^2 + \eta_0^4}{\mu_k^4} + o(1)\frac{\phi}{\mu_k^6} + O(\mu_k^{-2}\xi^2)\frac{\phi}{\mu_k^6} + O(\mu_k^{-6}\xi^4),$$
$$\psi^3 = O(\mu_k^{-6}\xi^6) + o(1)\frac{\phi}{\mu_k^6} + O(\mu_k^{-4}\xi^4)\frac{\phi}{\mu_k^6}.$$

From (34) we easily get that ψ_k is uniformly bounded for $r \in [0, s_k]$ and we can write

$$\begin{aligned} e^\psi - 1 - \psi - \frac{\psi^2}{2} &= O(\max\{1, e^\psi\})\ \psi^3 \\ &= O(\mu_k^{-6}\xi^6) + o(1)\frac{\phi}{\mu_k^6} + O(\mu_k^{-4}\xi^4)\frac{\phi}{\mu_k^6}. \end{aligned}$$

Therefore

$$\begin{aligned} e^\psi = 1 &+ \frac{2w_0 + \eta_0^2}{\mu_k^2} + \frac{2z_0 + 2w_0^2 + 2\eta_0 w_0 + 2w_0\eta_0^2 + \frac{1}{2}\eta_0^4}{\mu_k^4} \\ &+ \frac{2\phi}{\mu_k^6} + o(1)\frac{2\phi}{\mu_k^6} + O(\mu_k^{-2}\xi^2)\frac{\phi}{\mu_k^6} + O(\mu_k^{-6}\xi^6). \end{aligned}$$

To obtain the Taylor expansion of $\Phi_k(r, \phi)$, we also need to multiply this term by

$$1 + \frac{\eta}{\mu_k^2} = 1 + \frac{\eta_0}{\mu_k^2} + \frac{w_0}{\mu_k^4} + O(\mu_k^{-6}\xi) + o(1)\frac{\phi}{\mu_k^6},$$

and finally, using (21), (26) and (29) we obtain

$$\Phi_k(r, \phi) = 4e^{2\eta_0}\left(2\phi + o(1)\phi + O(\mu_k^{-2}\xi^2)\phi + O(\xi^6)\right),$$

as was to be shown. □

Proposition 11 *There exist $M > 0$ and $T > 0$ such that*

$$\eta_k = \eta_0 + \frac{w_0}{\mu_k^2} + \frac{z_0}{\mu_k^4} + \frac{\phi_k}{\mu_k^6}$$

with

$$|\phi_k(r)| \le M\xi(r), \quad \textit{for } r \in [0, e^{\mu_k}], \quad |\phi_k'(r)| \le \frac{M}{r}, \quad \textit{for } r \in [T, e^{\mu_k}] \tag{35}$$

for k large (depending on M and T), where ξ is as in (33).

Proof. This follows from a fixed-point argument and the uniqueness of solutions of ODEs.

From Lemma 8 we have that for every interval $[0, T]$, $\phi_k = o(\mu_k^6)$ uniformly in $[0, T]$, hence by Lemma 10

$$\begin{cases} -\Delta\phi_k = \Phi_k(r, \phi_k) = 4e^{2\eta_0}(2\phi_k + o(1)\phi_k + O(1)) \\ \phi_k(0) = \phi_k'(0) = 0 \end{cases} \tag{36}$$

with $o(1) \to 0$ and $|O(1)| \le C$ uniformly in $[0,T]$, and from ODE theory it follows that ϕ_k is uniformly bounded in $[0,T]$. In particular there exists a constant $C(T)$ such that

$$|\phi_k(r)| \le C(T), \quad |\phi_k'(r)| \le C(T), \quad \text{for } r \in [0,T], \tag{37}$$

uniformly in k.

Define the norm

$$\|f\| = \sup_{r\in(T,e^{\mu_k}]} \left| \frac{f(r)}{\log r - \log T} \right| .$$

For a large constant $M > 0$ to be fixed later, we will work with the following set of functions

$$\mathcal{B}_M = \left\{ \phi \in C^0([T, e^{\mu_k}]) : \|\phi - \phi_k(T)\| \le M \right\}.$$

Notice that for $\phi \in \mathcal{B}_M$ we have

$$|\phi(r)| \le |\phi_k(T)| + |\phi(r) - \phi_k(T)| \le C(T) + M(\log r - \log T) \tag{38}$$

for any $r \in [T, e^{\mu_k}]$. In particular

$$\frac{|\phi|}{\mu_k^6} \le \frac{C(T) + M\mu_k}{\mu_k^6} = o(1)$$

uniformly on $[0, e^{\mu_k}]$ for k large enough. Then by Lemma 10 we have

$$\Phi_k(r,\phi) = 4e^{2\eta_0}(2\phi + O(\xi^6)) \tag{39}$$

where $|O(\xi^6)| \le C\xi^6$ uniformly for $k \ge k_0(T,M)$ sufficiently large.

Let now $F_k : \mathcal{B}_M \to C^0([T, e^{\mu_k}])$ (for a fixed k) associate to a function ϕ the solution $\bar\phi$ of

$$\begin{cases} -\frac{1}{r}(r\bar\phi'(r))' = \Phi_k(r,\phi(r)) \quad \text{for } T \le r \le e^{\mu_k} \\ \bar\phi(T) = \phi_k(T) \\ \bar\phi'(T) = \phi_k'(T). \end{cases} \tag{40}$$

We will show that F_k sends $\mathcal{B}_M$ into itself for suitable choices of M and T, and is compact. Indeed for $\phi \in \mathcal{B}_M$ one can integrate (40) and use (37)-(39) to get

$$\begin{aligned} |t\bar\phi'(t)| &= \left| T\phi_k'(T) - \int_T^t r\Phi_k(r,\phi(r))dr \right| \\ &\le TC(T) + \int_T^t \frac{8r|\phi(r)|}{(1+r^2)^2}dr + \int_T^t \frac{4Cr\xi^6(r)}{(1+r^2)^2}dr \\ &\le (T + o_T(1))C(T) + Mo_T(1) + o_T(1), \end{aligned} \tag{41}$$

where

$$|o_T(1)| \le \int_T^\infty \frac{8r(1+\log r)}{(1+r^2)^2}dr + \int_T^\infty \frac{4Cr\xi^6(r)}{(1+r^2)^2}dr \to 0, \quad \text{as } T \to \infty.$$

Integrating again we infer

$$\begin{aligned}|\bar{\phi}(r)-\phi_k(T)| &\leq \int_T^r |\bar{\phi}'(t)|dt\\ &\leq \int_T^r \frac{(T+o_T(1))C(T)+(M+1)o_T(1)}{t}dt\\ &= ((T+o_T(1))C(T)+(M+1)o_T(1))(\log r-\log T).\end{aligned}$$

First choosing T so large that $|o_T(1)|\leq \frac{1}{2}$, and then M such that

$$\left(T+\frac{1}{2}\right)C(T)+\frac{1}{2}\leq \frac{M}{2}, \tag{42}$$

we conclude that

$$|\bar{\phi}(r)-\phi_k(T)|\leq M(\log r-\log T), \tag{43}$$

hence $\bar{\phi}\in\mathcal{B}_M$. Then F_k sends $\mathcal{B}_M$ into itself. Moreover it is compact by the theorem of Ascoli-Arzelà since for a sequence $(\psi_n)\subset\mathcal{B}_M$ and $\bar{\psi}_n:=F_k(\psi_n)$ we have

$$(\bar{\psi}_n(r)(\log r-\log T))'=\bar{\psi}_n'(\log r-\log T)+\frac{\bar{\psi}_n(r)}{r}$$

which is uniformly bounded on $[T,e^{\mu_k}]$ by (41)-(43), so that up to a subsequence

$$\bar{\psi}_n(r)(\log r-\log T)\to\bar{\psi}_\infty(r)(\log r-\log T),\quad \text{uniformly},$$

i.e. $\|\bar{\psi}_n-\psi_\infty\|\to 0$ for some $\psi_\infty\in\mathcal{B}_M$. Therefore, by the fixed-point theorem of Caccioppoli-Schauder (see e.g. [8, Corollary 11.2]) F_k has a fixed point $\phi\in\mathcal{B}_M$, which solves (36). Then, by uniqueness for the Cauchy problem, we have $\phi=\phi_k$ in $[T,e^{\mu_k}]$, whence the bounds

$$|\phi_k(r)|\leq C(T)+M(\log r-\log T),\quad \text{for } T\leq r\leq e^{\mu_k}, \tag{44}$$

which is another way of writing the first inequality in (35) (a priori the identity $\phi=\phi_k$ holds as long as ϕ_k is defined, i.e. up to r_k^{-1}; on the other hand, the reader can easily verify that $\eta_k>-\mu_k^2$ as long (44) holds, so that in particular $r_k^{-1}>e^{\mu_k}$). The second inequality in (35) follows from (41) and (42). □

3.2 Proof of Theorem 1 completed

We are now in a position to use the Taylor expansion computed in the previous section to estimate the Dirichlet energy of u_k.

Proposition 12 *Given a sequence (s_k) with $s_k\in[\mu_k^p,e^{\mu_k}]$ for some $p>2$, we have*

$$\int_{B_{r_k s_k}}\lambda_k u_k^2 e^{u_k^2}dx=4\pi+\frac{4\pi}{\mu_k^4}+o(\mu_k^{-4}). \tag{45}$$

Proof. We start writing

$$\begin{aligned}(I) &:= \int_{B_{r_k s_k}} \lambda_k u_k^2 e^{u_k^2} dx = 4\int_{B_{s_k}} \left(1+\frac{\eta_k}{\mu_k^2}\right)^2 e^{2\eta_k + \frac{\eta_k^2}{\mu_k^2}} dx \\ &= \int_{B_{s_k}} \left(1+\frac{\eta_k}{\mu_k^2}\right)\left(-\Delta\eta_0 - \frac{\Delta w_0}{\mu_k^2} - \frac{\Delta z_0}{\mu_k^4} + \frac{\Phi_k(r,\phi_k)}{\mu_k^6}\right) dx,\end{aligned}$$

where Φ_k is as in (32). Using Lemma 10 and Proposition 11 we have on $[0, s_k]$

$$1+\frac{\eta_k}{\mu_k^2} = 1 + \frac{\eta_0}{\mu_k^2} + \frac{w_0}{\mu_k^4} + O(\mu_k^{-5}),$$

and

$$\Phi_k(r,\phi_k) = O(e^{2\eta_0}\xi^6),$$

where ξ is as in (33). In particular

$$\int_{B_{s_k}} |\Phi_k(r,\phi_k)| dx \leq C\int_{\mathbb{R}^2} \frac{\xi^6(x)}{(1+|x|^2)^2} dx \leq C.$$

Similarly

$$\max\{|\Delta\eta_0|, |\Delta w_0|, |\Delta z_0|\} = O(e^{2\eta_0}\xi^4),$$

so that

$$\int_{B_{s_k}} \xi \max\{|\Delta\eta_0|, |\Delta w_0|, |\Delta z_0|\} dx \leq C\int_{\mathbb{R}^2} \frac{\xi^5(x)}{(1+|x|^2)^2} dx \leq C.$$

Summing up one gets

$$\begin{aligned}(I) &= \int_{B_{s_k}} \left(-\Delta\eta_0 - \frac{\eta_0\Delta\eta_0 + \Delta w_0}{\mu_k^2} - \frac{w_0\Delta\eta_0 + \eta_0\Delta w_0 + \Delta z_0}{\mu_k^4}\right) dx + O(\mu_k^{-5}) \\ &=: (I_0) + \frac{(I_2)}{\mu_k^2} + \frac{(I_4)}{\mu_k^4} + O(\mu_k^{-5}).\end{aligned}$$

Now we compute

$$(I_0) = \int_{B_{s_k}} 4e^{2\eta_0} dx = 4\pi\left(1 - \frac{1}{1+s_k^2}\right) = 4\pi + o(\mu_k^{-4}).$$

Using the divergence theorem, and (28) we get

$$\begin{aligned}(I_2) &= \int_{B_{s_k}} 4e^{2\eta_0}\eta_0 dx - 2\pi s_k w_0'(s_k) \\ &= 4\pi\left(\frac{\log(1+s_k^2)}{1+s_k^2} + \frac{1}{1+s_k^2} - 1\right) + 4\pi + O(s_k^{-2}\log^2 s_k) \\ &= o(\mu_k^{-4}).\end{aligned}$$

From (31) we get

$$-\int_{B_{s_k}} \Delta z_0 dx = 2\pi\left(6+\frac{\pi^2}{3}\right)+o(1),$$

while a direct computation shows that

$$-\int_{B_{s_k}} (w_0\Delta\eta_0+\eta_0\Delta w_0)\,dx = 4\int_{B_{s_k}} e^{2\eta_0}(w_0+\eta_0^2+\eta_0^3+2w_0\eta_0)dx = -8\pi-\frac{2}{3}\pi^3+o(1),$$

hence $(I_4) = 4\pi + o(1)$, and we conclude by summing up. □

Remark. The freedom in the choice of the sequence $s_k \in [\mu_k^p, e^{\mu_k}]$ in Proposition 12 implies that

$$\int_{B_{r_k e^{\mu_k}}\setminus B_{r_k\mu_k^p}} \lambda_k u_k^2 e^{u_k^2} dx = o(\mu_k^{-4})$$

for any $p > 2$.

Open problem 3 *Is there any geometric meaning to the term* $\frac{4\pi}{\mu_k^4}$ *in* (45)*, in particular to its positivity?*

From Lemma 8 we know that the first 4π appearing on the right-hand side of (45) can be seen as the area of S^2, since $-\Delta\eta_0$ is the conformal factor of the pull-back of the metric of S^2 onto $\mathbb{R}^2$ via stereographic projection. The second 4π appearing in (45) depends on the asymptotic behavior of z_0, but we do not have a geometric interpretation.

While Proposition 12 gives a lower bound on $\|\nabla u_k\|_{L^2}$, we will now prove an upper bound. First of all we shall observe $\eta_k(r) \le \eta_0(r)$ for sufficiently large r, which was proved in [13]. The next lemma gives a more general statement which will turn out to be useful also in the next sections.

Lemma 13 *Let* $\bar\eta_k : [0, r_k^{-1}] \to \mathbb{R}$ *be a sequence of* C^2 *functions satisfying* $\Delta\bar\eta_k \le 0$*. Assume further that* $\bar\eta_k$ *has an expansion of the form*

$$\bar\eta_k = \eta_0 + \frac{w}{\mu_k^2} + \psi_k \qquad \text{in } [0,\mu_k^2], \tag{46}$$

with $w : [0,+\infty) \to \mathbb{R}$*,* $\psi_k : [0, r_k^{-1}) \to \mathbb{R}$ *satisfying*

$$w(\mu_k^2) \le -1, \tag{47}$$

$$\int_{\mathbb{R}^2} \Delta w\, dx < 0, \tag{48}$$

and

$$\sup_{[0,\mu_k^2]} |\psi_k| + \int_{B_{\mu_k^2}} |\Delta\psi_k| dx = o(\mu_k^{-2}). \tag{49}$$

Then $\bar\eta_k \le \eta_0$ *in* $[\mu_k^2, r_k^{-1}]$*, for* k *sufficiently large.*

Proof. By (46), (48) and (49) we compute

$$\int_{B_{\mu_k^2}} \Delta\bar\eta_k dx = \int_{B_{\mu_k^2}} \Delta\eta_0 dx + \frac{1}{\mu_k^2}\int_{B_{\mu_k^2}} \Delta w dx + o(\mu_k^{-2})$$
$$= -4\pi + \frac{1}{\mu_k^2}\int_{\mathbb{R}^2} \Delta w dx + o(\mu_k^{-2}) < -4\pi.$$

Since $\Delta\bar\eta_k \le 0$, for $r \in [\mu_k^2, r_k^{-1}]$ we get

$$\int_{B_r} \Delta\bar\eta_k dx \le \int_{B_{\mu_k^2}} \Delta\bar\eta_k dx < -4\pi < \int_{B_r} \Delta\eta_0 dx$$

and by the divergence theorem we deduce $\bar\eta_k'(r) \le \bar\eta_0'(r)$. Finally (46), (47) and (49) guarantee that $\bar\eta_k(\mu_k^2) \le \eta_0(\mu_k^2)$ for large k, and the conclusion follows from the fundamental theorem of calculus. □

Clearly, by (27) and Proposition 11, Lemma 13 applies to η_k.

Proposition 14 *For some $p > 2$ let $s_k \in [\mu_k^p, e^{\mu_k}]$. Then we have*

$$\int_{B_1 \setminus B_{s_k r_k}} \lambda_k u_k^2 e^{u_k^2} dx \le \frac{2\pi}{\mu_k^4} + o(\mu_k^{-4}), \quad \text{as } k \to \infty.$$

Proof. With the usual scaling, we have to prove that

$$(I) := 4\int_{B_{\frac{1}{r_k}} \setminus B_{s_k}} \left(1 + \frac{\eta_k}{\mu_k^2}\right)^2 e^{2\eta_k + \frac{\eta_k^2}{\mu_k^2}} dx \le \frac{2\pi}{\mu_k^4} + o(\mu_k^{-4}).$$

By Lemma 13 for $r \in [s_k, r_k^{-1}]$ and for k large enough we have $\eta_k \le \eta_0$. Let us set $t_k := \sqrt{e^{\mu_k^2} - 1}$ and $\tilde t_k := \sqrt{\mu_k^{2p} - 1}$. We claim that, for k large enough, $\frac{1}{r_k} \le t_k$. Otherwise, as soon as $t_k \ge e^{\mu_k}$, we would have

$$u_k(r_k t_k) = \mu_k + \frac{\eta_k(t_k)}{\mu_k} \le \mu_k + \frac{\eta_0(t_k)}{\mu_k} = 0,$$

which contradicts the positivity of u_k in B_1. Hence

$$(I) \le \int_{B_{t_k} \setminus B_{\tilde t_k}} \left(1 + \frac{\eta_0}{\mu_k^2}\right)^2 e^{2\eta_0 + \frac{\eta_0^2}{\mu_k^2}} dx = 2\pi \int_{\tilde t_k}^{t_k} r\left(1 + \frac{\eta_0}{\mu_k^2}\right)^2 e^{2\eta_0 + \frac{\eta_0^2}{\mu_k^2}} dr =: (II).$$

With the changes of variable $s = -\eta_0(r) = \log(1 + r^2)$ and $\tau = \frac{s}{\mu_k} - \frac{\mu_k}{2}$, we get

$$\begin{aligned}(II) &= \pi e^{-\frac{\mu_k^2}{4}} \int_{2p\log\mu_k}^{\mu_k^2} \left(1 - \frac{s}{\mu_k^2}\right)^2 e^{(\frac{s}{\mu_k} - \frac{\mu_k}{2})^2} ds \\ &= \pi e^{-\frac{\mu_k^2}{4}} \int_{\frac{2p\log\mu_k}{\mu_k} - \frac{\mu_k}{2}}^{\frac{\mu_k}{2}} \left(\frac{\mu_k}{4} - \tau + \frac{\tau^2}{\mu_k}\right) e^{\tau^2} d\tau.\end{aligned} \tag{50}$$

Since $p > 2$ we have

$$-e^{-\frac{\mu_k^2}{4}} \int_{\frac{2p\log\mu_k}{\mu_k}-\frac{\mu_k}{2}}^{\frac{\mu_k}{2}} \tau e^{\tau^2} d\tau = -\frac{1}{2} + o(\mu_k^{-4}).$$

Moreover it is simple to verify (using e.g. de l'Hôpital rule) that

$$e^{-\frac{\mu_k^2}{4}} \int_{\frac{2p\log\mu_k}{\mu_k}-\frac{\mu_k}{2}}^{\frac{\mu_k}{2}} \frac{\mu_k}{4} e^{\tau^2} d\tau = \frac{1}{4} + \frac{1}{2\mu_k^2} + \frac{3}{\mu_k^4} + o(\mu_k^{-4}),$$
$$e^{-\frac{\mu_k^2}{4}} \int_{\frac{2p\log\mu_k}{\mu_k}-\frac{\mu_k}{2}}^{\frac{\mu_k}{2}} \frac{\tau^2 e^{\tau^2}}{\mu_k} d\tau = \frac{1}{4} - \frac{1}{2\mu_k^2} - \frac{1}{\mu_k^4} + o(\mu_k^{-4}),$$

and, summing up, we conclude

$$(I) \leq (II) = 2\pi\mu_k^{-4} + o(\mu_k^{-4}).$$

$\square$

Proof of Theorem 1 (completed). Integrating by parts and using (4) we can write

$$\|\nabla u_k\|_{L^2}^2 = -\int_{B_1} u_k \Delta u_k dx = \int_{B_1} \lambda_k u_k^2 e^{u_k^2} dx.$$

Then Theorem 1 follows at once from Propositions 12 and 14. $\square$

3.3 Some ODE theory and a crucial formula

We conclude this section with some general lemmas analyzing the asymptotic behaviour of w_0 and z_0. In particular we will prove (28), (30), (31).

Lemma 15 *Let $f \in C^0(\mathbb{R}^2)$ be radially symmetric and satisfy $f(r) = O(\log^q r)$ as $r \to \infty$ for some $q \geq 0$. If $w \in C^2(\mathbb{R}^2)$ is a radially symmetric solution of*

$$-\Delta w = 4e^{2\eta_0}(f + 2w), \tag{51}$$

where η_0 is as in (21), then $\Delta w \in L^1(\mathbb{R}^2)$ and we have

$$\begin{aligned} w(r) &= \beta \log r + O(1) \\ w'(r) &= \frac{\beta}{r} + O\left(\frac{\log^{\bar{q}} r}{r^3}\right), \end{aligned} \tag{52}$$

as $r \to \infty$, where $\bar{q} = \max\{1, q\}$ and

$$\beta := \frac{1}{2\pi} \int_{\mathbb{R}^2} \Delta w dx.$$

Proof. We start by proving

$$|w(r)| \le C \log r, \tag{53}$$

for some $C > 0$ and r sufficiently large. We consider the functions $\varphi(r) = rw'(r)$ and $y(r) = (w(r), \varphi(r))$. Then we can rewrite (51) as

$$y'(r) = F(r, y(r))$$

with

$$F(r, w, \varphi) = \left(\frac{\varphi}{r}, -4re^{2\eta_0(r)}(f(r) + 2w)\right).$$

If we choose R_0 sufficiently large, so that

$$4r^2 e^{2\eta_0(r)} \max\{|f(r)|, 2\} \le \frac{1}{\sqrt{2}}, \quad \text{for } r \ge R_0,$$

then

$$|F(r, y)| \le \frac{1}{r}(1 + |y|) \qquad \forall\, r \ge R_0.$$

In particular we have

$$|y(r)| \le |y(R_0)| + \int_{R_0}^{r} |F(s, y(s))| ds \le |y(R_0)| + \log r - \log R_0 + \int_{R_0}^{r} \frac{|y(s)|}{s} ds.$$

By Grönwall's lemma this yields

$$|y(r)| \le (|y(R_0)| + \log r - \log R_0)\frac{r}{R_0} \le C(R_0) r \log r.$$

In particular,

$$|\varphi'(r)| \le re^{2\eta_0(r)}(|f(r)| + 2|w(r)|) \le C(q, R_0)\frac{\log r}{r^2} \in L^1((R_0, +\infty))$$

so that $\varphi(r) = rw'(r)$ is bounded and $|w(r)| \le |w(R_0)| + C \log r$ for for $r \ge R_0$.

Now we prove (52). By the divergence theorem we have

$$\begin{aligned} 2\pi r w'(r) &= \int_{B_r} \Delta w dx \\ &= 2\pi\beta - \int_{\mathbb{R}^2 \setminus B_r} \Delta w dx \\ &= 2\pi\beta + O(r^{-2} \log^{\bar{q}} r), \end{aligned}$$

where we used that, thanks to (53), $-\Delta w = O(r^{-4} \log^{\bar{q}} r)$. This gives the second identity in (52). The first one follows with the fundamental theorem of calculus. □

Lemma 16 *Let f, w and β be as in Lemma 15. Then*

$$\beta = -\frac{2}{\pi}\int_{\mathbb{R}^2} \frac{|x|^2-1}{(1+|x|^2)^3} f(x)dx.$$

Proof. Let us define

$$\psi(x) := \frac{|x|^2-1}{1+|x|^2},$$

which solves

$$-\Delta\psi = 8e^{2\eta_0}\psi \qquad \text{in } \mathbb{R}^2.$$

Then for $r > 0$

$$\begin{aligned} 4\int_{B_r} \frac{|x|^2-1}{(1+|x|^2)^3} f(x)dx &= 4\int_{B_r} \psi e^{2\eta_0} f dx \\ &= 4\int_{B_r} \psi e^{2\eta_0}(f+2w)dx - 8\int_{B_r} \psi e^{2\eta_0} w dx \\ &= -\int_{B_r} \psi\Delta w dx + \int_{B_r} w\Delta\psi dx =: (I). \end{aligned}$$

By the divergence theorem and (52) we compute

$$\begin{aligned} (I) &= 2\pi r[\psi'(r)w(r) - \psi(r)w'(r)] \\ &= 2\pi r[O(r^{-3}\log r) - (1+O(r^{-2}))r^{-1}(\beta + o(1))] \\ &= -2\pi\beta + o(1), \end{aligned}$$

with $o(1) \to 0$ as $r \to \infty$. Letting $r \to \infty$ we conclude. □

We can now apply Lemma 15 and Lemma 16 to the solutions w_0 and z_0 of (26) and (29).

Corollary 17 *Let w_0 be the solution to* (26)*. Then w_0' has asymptotic behaviour* (28)*.*

Proof. The ODE in (26) corresponds to (51) with

$$f = \eta_0 + \eta_0^2 = O(\log^2|x|).$$

Hence (28) follows from Lemma 15 and (27). □

Corollary 18 *Let z_0 be the solution to* (29)*. Then z_0 has asymptotic behaviour* (30)-(31)*.*

Proof. The ODE in (29) corresponds to (51) with

$$f = w_0 + 2w_0^2 + 4\eta_0 w_0 + 2\eta_0^2 w_0 + \eta_0^3 + \frac{1}{2}\eta_0^4 = O(\log^4|x|).$$

A straightforward computation shows that

$$\int_{\mathbb{R}^2} \frac{|x|^2-1}{(1+|x|^2)^3}\eta_0^3(x)dx = -\frac{21}{4}\pi$$

and

$$\int_{\mathbb{R}^2} \frac{|x|^2-1}{(1+|x|^2)^3}\eta_0^4(x)dx = \frac{45}{2}\pi.$$

Using the explicit expression (25) of w_0 and integrating by parts we find

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{|x|^2-1}{(1+|x|^2)^3}w_0(x)dx &= \frac{\pi^3}{18} - \frac{7}{12}\pi,\\
\int_{\mathbb{R}^2} \frac{|x|^2-1}{(1+|x|^2)^3}w_0(x)\eta_0(x)dx &= \left(\frac{125}{72} - \frac{2}{3}Z(3)\right)\pi - \frac{2}{27}\pi^3,\\
\int_{\mathbb{R}^2} \frac{|x|^2-1}{(1+|x|^2)^3}w_0(x)\eta_0^2(x)dx &= \left(\frac{16}{9}Z(3) - \frac{409}{54}\right)\pi + \frac{35}{162}\pi^3 + \frac{\pi^5}{45},
\end{aligned}$$

where Z denotes the Euler-Riemann zeta function. Finally, integrating by parts twice, we find

$$\int_{\mathbb{R}^2} w_0^2\frac{|x|^2-1}{(1+|x|^2)}dx = \left(\frac{625}{216} - \frac{4}{9}Z(3)\right)\pi - \frac{1}{81}\pi^3 - \frac{\pi^5}{45}.$$

Therefore, by Lemma 16, (52) holds with

$$\beta = -\frac{2}{\pi}\int_{\mathbb{R}^2} \frac{|x|^2-1}{(1+|x|^2)^3}f(x)dx = -6 - \frac{\pi^2}{3}.$$

□

4 Proof of Theorem 2

Let u_k be as in the statement of the theorem, and set r_k and η_k as before in (17)-(18).

$$\begin{aligned}
-\Delta\eta_k &= \lambda_k r_k^2\mu_k e^{\mu_k^2}(1+h(u_k))u_k e^{2\eta_k+\frac{\eta_k^2}{\mu_k^2}}\\
&= 4e^{2\eta_0}\left(1+h\left(\mu_k+\frac{\eta_k}{\mu_k}\right)\right)\left(1+\frac{\eta_k}{\mu_k^2}\right)e^{2(\eta_k-\eta_0)+\frac{\eta_k^2}{\mu_k^2}}.
\end{aligned} \tag{54}$$

A very mild perturbation in the proof of Lemma 8 gives:

Lemma 19 *The conclusion of Lemma 8 still holds if we replace the ODE in* (19) *by* (54), *for some function h with $h(t) \to 0$ as $t \to \infty$.*

Set now

$$\delta_k := \max\left\{\sup_{s\in[-1,1]}\left|h\left(\mu_k + \frac{s(8\log\mu_k+1)}{\mu_k}\right) - h(\mu_k)\right|, \frac{1}{\mu_k^6}, \frac{h(\mu_k)}{\mu_k^2}\right\}.$$

Assuming (10)-(11) we have $\delta_k = o(\mu_k^{-4})$. We also introduce the function

$$\zeta_0(r) = -1 + \frac{1}{1+r^2},$$

solution to

$$-\Delta\zeta_0 = 4e^{2\eta_0}(1+2\zeta_0). \tag{55}$$

Lemma 20 *Let $s_k \le \mu_k^4$ and $\phi : [0, s_k] \to \mathbb{R}$ be given so that $\phi = o(\delta_k^{-1})$ uniformly on $[0, s_k]$. Set*

$$\eta := \eta_0 + \frac{w_0}{\mu_k^2} + \frac{z_0}{\mu_k^4} + h(\mu_k)\zeta_0 + \delta_k\phi$$

and

$$\Phi_k^h(r,\phi) := \frac{4\left(1+h\left(\mu_k+\frac{\eta}{\mu_k}\right)\right)\left(1+\frac{\eta}{\mu_k^2}\right)e^{2\eta+\frac{\eta^2}{\mu_k^2}} + \Delta\eta_0 + \frac{\Delta w_0}{\mu_k^2} + \frac{\Delta z_0}{\mu_k^4} + h(\mu_k)\Delta\zeta_0}{\delta_k}. \tag{56}$$

Then

$$\Phi_k^h(r,\phi) = 4e^{2\eta_0}\left(2\phi + o(1)\phi + O(\xi^6)\right), \quad \text{in } [0, s_k], \tag{57}$$

where ξ is as in (33).

Proof. The proof is similar to the one of Lemma 10. Using the logarithmic growth of η_0, w_0, z_0, the bound on s_k, and the definition of δ_k, we expand

$$\begin{aligned}
\psi &:= 2\eta + \frac{\eta^2}{\mu_k^2} - 2\eta_0 \\
&= \frac{2w_0+\eta_0^2}{\mu_k^2} + \frac{2z_0+2\eta_0 w_0}{\mu_k^4} + 2h(\mu_k)\zeta_0 + 2\delta_k\phi + o(1)\delta_k\phi + O(\delta_k\xi^2), \\
\psi^2 &= \frac{4w_0^2 + 4w_0\eta_0^2 + \eta_0^4}{\mu_k^4} + o(1)\delta_k\phi + O(\delta_k\xi^4), \\
\psi^3 &= O(\delta_k\xi^6) + o(1)\delta_k\phi.
\end{aligned}$$

Then ψ is uniformly bounded for $r \in [0, s_k]$ and we can write

$$e^\psi - 1 - \psi - \frac{\psi^2}{2} = o(1)\delta_k\phi + O(\delta_k\xi^6).$$

Therefore

$$e^{\psi} = 1 + \frac{2w_0 + \eta_0^2}{\mu_k^2} + \frac{2z_0 + 2w_0^2 + 2\eta_0 w_0 + 2w_0\eta_0^2 + \frac{1}{2}\eta_0^4}{\mu_k^4} + 2h(\mu_k)\zeta_0 + 2\delta_k\phi \\ + o(1)\delta_k\phi + O(\delta_k\xi^6). \tag{58}$$

Furthermore

$$1 + \frac{\eta}{\mu_k^2} = 1 + \frac{\eta_0}{\mu_k^2} + \frac{w_0}{\mu_k^4} + o(1)\delta_k\phi + O(\delta_k\xi), \tag{59}$$

and, since $|\eta(r)| \leq 8\log\mu_k + 1$ for $r \in [0, \mu_k^4]$ and k large, the definition of δ_k gives

$$1 + h\left(\mu_k + \frac{\eta}{\mu_k}\right) = 1 + h(\mu_k) + O(\delta_k). \tag{60}$$

Finally, multiplying (58) by (59)-(60) and using (21), (26), (29) and (55), we obtain (57). □

Remark. Our choice of the bound $s_k \leq \mu_k^4$ is strictly connected to the regularity assumptions on h. If one replaces (11) with the simpler (but stronger) assumption

$$\lim_{t\to\infty} \sup_{|s|\leq L} t^4|h(t+s) - h(t)| = 0 \qquad \forall\, L > 0, \tag{61}$$

then it is possible to obtain

$$\Phi_k^h(r,\phi) = 4e^{2\eta_0}\left(2\phi + o(1)\phi + O(\mu_k^{-2}\xi^2)\phi + O(\xi^6)\right) \quad \text{in } [0, e^{\mu_k}],$$

precisely as in Lemma 10. However, considering as a model problem $h(t) = t^{-p}$ for large t, (61) is satisfied only for $p > 3$, while the condition (11) allows to consider any $p > 2$. Alternatively, the scale of the Taylor expansions can be improved by considering further terms in the expansion (60), see Section 5.

Proposition 21 *There exist $M > 0$ and $T > 0$ such that*

$$\eta_k = \eta_0 + \frac{w_0}{\mu_k^2} + \frac{z_0}{\mu_k^4} + h(\mu_k)\zeta_0 + \delta_k\phi_k$$

with $|\phi_k| \leq M\xi$ on $[0, \mu_k^4]$ and $|\phi_k'(r)| \leq \frac{M}{r}$ on $[T, \mu_k^4]$, where ξ is as in (33).

Proof. Nothing changes from the proof of Proposition 11, since the structure and bounds of the equation

$$-\Delta\phi_k = \Phi_k^h(r, \phi_k)$$

satisfied by ϕ_k, as given by Lemma 20, are the same. □

Proposition 22 *Given a sequence (s_k) with $s_k \in [\mu_k^p, \mu_k^4]$ for some $p \in (2,4]$, we have*

$$\int_{B_{r_k s_k}} \lambda_k(1+h(u_k))u_k^2 e^{u_k^2}dx = 4\pi + \frac{4\pi}{\mu_k^4} + o(\mu_k^{-4}).$$

Proof. We start writing

$$\begin{aligned}
(I) &:= \int_{B_{r_k s_k}} \lambda_k(1+h(u_k))u_k^2 e^{u_k^2}dx \\
&= 4\int_{B_{s_k}} \left(1+\frac{\eta_k}{\mu_k}\right)^2 \left(1+h\left(\mu_k+\frac{\eta_k}{\mu_k}\right)\right) e^{2\eta_k+\frac{\eta_k^2}{\mu_k^2}}dx \\
&= \int_{B_{s_k}} \left(1+\frac{\eta_k}{\mu_k^2}\right)\left(-\Delta\eta_0 - \frac{\Delta w_0}{\mu_k^2} - \frac{\Delta z_0}{\mu_k^4} - h(\mu_k)\Delta\zeta_0 + \delta_k\Phi_k^h(r,\phi_k)\right)dx.
\end{aligned}$$

Using Lemma 20 and Proposition 21 we have on $[0, s_k]$

$$1+\frac{\eta_k}{\mu_k^2} = 1 + \frac{\eta_0}{\mu_k^2} + \frac{w_0}{\mu_k^4} + O(\delta_k),$$

and

$$\Phi_k^h(r,\phi_k) = O(e^{2\eta_0}\xi^6).$$

Arguing as in Proposition 12 we get

$$\begin{aligned}
(I) &= \int_{B_{s_k}} \left(-\Delta\eta_0 - \frac{\eta_0\Delta\eta_0 + \Delta w_0}{\mu_k^2} - \frac{w_0\Delta\eta_0 + \eta_0\Delta w_0 + \Delta z_0}{\mu_k^4} - h(\mu_k)\Delta\zeta_0\right)dx + O(\delta_k) \\
&=: (I_0) + \frac{(I_2)}{\mu_k^2} + \frac{(I_4)}{\mu_k^4} - h(\mu_k)\int_{B_{s_k}} \Delta\zeta_0 dx + O(\delta_k).
\end{aligned}$$

As before we have

$$\begin{aligned}
(I_0) &= 4\pi + o(\mu_k^{-4}) \\
(I_2) &= O(s_k^{-2}\log^2 s_k) = o(\mu_k^{-2}) \\
(I_4) &= 4\pi + o(1).
\end{aligned}$$

Finally,

$$h(\mu_k)\int_{B_{s_k}} \Delta\zeta_0 dx = 2\pi h(\mu_k) r\zeta_0'(s_k) = h(\mu_k)O(s_k^{-3}) = o(\mu_k^{-4}),$$

and we conclude. □

Proof of Theorem 2 (completed). Again integrating by parts we infer for some $p \in (2,4]$

$$\begin{aligned}
\|\nabla u_k\|_{L^2}^2 &= -\int_{B_1} u_k\Delta u_k dx \\
&= \int_{B_{\mu_k^p r_k}} \lambda_k(1+h(u_k))u_k^2 e^{u_k^2}dx + \int_{B_1\setminus B_{\mu_k^p r_k}} \lambda_k(1+h(u_k))u_k^2 e^{u_k^2}dx \\
&=: (I) + (II).
\end{aligned}$$

The term (I) is bounded from above and below by Proposition 22. For the term (II) we use that $\eta_k(r) \leq \eta_0(r)$ for $r \geq \mu_k^p$ and k large enough, which follows from Lemma 13. Then the proof of Proposition 14 can still be applied and we infer

$$(II) \leq \frac{2\pi(1+\sup h)}{\mu_k^4} + o(\mu_k^{-4}).$$

Summing up (I) and (II) we conclude. □

5 Proof of Theorem 4

Since the perturbation $h(t)$ is now of order t^{-2}, its presence will change the Taylor expansion of the right-hand side of (54) already at order μ_k^{-2}. As a consequence we will see that the function $\mu_k^2(\eta_k - \eta_0)$ will converge to a new function w_a, solution to

$$\begin{cases} -\Delta w_a = 4e^{2\eta_0}(\eta_0 + \eta_0^2 - a + 2w_a) \text{ in } \mathbb{R}^2 \\ w_a(0) = w_a'(0) = 0. \end{cases} \tag{62}$$

Since

$$-\Delta(w_a - w_0) = 4e^{2\eta_0}(-a + 2(w_a - w_0)),$$

we have $w_a - w_0 = -a\zeta_0$.

Also the function z_0 will be replaced by z_a which satisfies

$$\begin{cases} -\Delta z_a = 4e^{2\eta_0}(a(\eta_0 - \eta_0^2 - 2w_a) + w_a + 2w_a^2 + 4\eta_0 w_a + 2\eta_0^2 w_a + \eta_0^3 + \frac{1}{2}\eta_0^4 + 2z_a) \text{ in } \mathbb{R}^2 \\ z_a(0) = z_a'(0) = 0, \end{cases}$$

and differs from z_0 by the solution to

$$\begin{cases} -\Delta(z_a - z_0) = 4e^{2\eta_0}[2a^2(\zeta_0 + \zeta_0^2) + a(\eta_0 - \eta_0^2 - 2w_0 + \zeta_0(-2\eta_0^2 - 4\eta_0 - 4w_0 - 1) \\ \qquad\qquad + 2(z_a - z_0)] \text{ in } \mathbb{R}^2 \\ z_a(0) - z_0(0) = z_a'(0) - z_0'(0) = 0. \end{cases}$$

Then with Lemma 16 we have

$$z_a(r) - z_0(r) = \beta \log r + O(1), \quad \frac{1}{2\pi}\int_{\mathbb{R}^2} \Delta(z_a - z_0)dx = \beta, \tag{63}$$

with $\beta = \beta_1 + \beta_2$, where for $\psi_0(x) := \frac{|x|^2-1}{(1+|x|^2)^3}$,

$$\begin{aligned} \beta_1 &= -\frac{2a}{\pi}\int_{\mathbb{R}^2} (\eta_0 - \eta_0^2 - 2w_0 + \zeta_0(-2\eta_0^2 - 4\eta_0 - 4w_0 - 1))\psi_0 dx \\ \beta_2 &= -\frac{2a^2}{\pi}\int_{\mathbb{R}^2} 2(\zeta_0 + \zeta_0^2)\psi_0 dx. \end{aligned}$$

One can compute

$$\frac{2}{\pi}\int_{\mathbb{R}^2}\zeta_0^2\psi_0 dx = \frac{1}{3}, \qquad \frac{2}{\pi}\int_{\mathbb{R}^2}(-2w_0)\psi_0 dx = \frac{7}{3}-\frac{2\pi^2}{9},$$
$$\frac{2}{\pi}\int_{\mathbb{R}^2}\eta_0\psi_0 dx = -1, \qquad \frac{2}{\pi}\int_{\mathbb{R}^2}(-\eta_0^2)\psi_0 dx = -3,$$
$$\frac{2}{\pi}\int_{\mathbb{R}^2}(-\zeta_0)\psi_0 dx = \frac{1}{3}, \qquad \frac{2}{\pi}\int_{\mathbb{R}^2}(-4w_0\zeta_0)\psi_0 dx = -\frac{67}{27}+\frac{2\pi^2}{9},$$
$$\frac{2}{\pi}\int_{\mathbb{R}^2}(-4\eta_0\zeta_0)\psi_0 dx = -\frac{34}{9}, \qquad \frac{2}{\pi}\int_{\mathbb{R}^2}(-2\eta_0^2\zeta_0)\psi_0 dx = \frac{151}{27},$$

hence

$$\beta_2 = 0, \quad \beta = \beta_1 = 2a. \tag{64}$$

Similar to Lemma 10 we get

Lemma 23 *Let $s_k \le e^{\mu_k}$ and $\phi : [0, s_k] \to \mathbb{R}$ be given so that $\phi = o(\mu_k^6)$ uniformly on $[0, s_k]$. Set*

$$\eta := \eta_0 + \frac{w_a}{\mu_k^2} + \frac{z_a}{\mu_k^4} + \frac{\phi}{\mu_k^6}$$

and (using that $h(t) = -at^{-2}$ for t large)

$$\Phi_k^a(r,\phi) := \mu_k^6\left[4\left(1-\frac{a}{\mu_k^2}\left(1+\frac{\eta}{\mu_k^2}\right)^{-2}\right)\left(1+\frac{\eta}{\mu_k^2}\right)e^{2\eta+\frac{\eta^2}{\mu_k^2}} + \Delta\eta_0 + \frac{\Delta w_a}{\mu_k^2} + \frac{\Delta z_a}{\mu_k^4}\right]. \tag{65}$$

Then as $k \to \infty$

$$\Phi_k^a(r,\phi) = 4e^{2\eta_0}\left(2\phi + o(1)\phi + O(\mu_k^{-2}\xi^2)\phi + O(\xi^6)\right), \quad r \in [0, s_k],$$

where ξ is as in (33).

Proof. The proof is identical to the one of Lemma 10, just replacing w_0 and z_0 with w_a and z_a respectively, and noticing that after the Taylor expansion of the exponential in (65) we have to consider

$$h\left(\mu_k + \frac{\eta}{\mu_k}\right) = -\frac{a}{\mu_k^2}\left(1+\frac{\eta}{\mu_k^2}\right)^{-2} = -\frac{a}{\mu_k^2} + \frac{2a\eta_0}{\mu_k^4} + O(\mu_k^{-6}\xi^2),$$

as $k \to \infty$. □

With the same proof of Proposition 11 (using Lemma 23 instead of Lemma 10) we get:

Proposition 24 *There exist $M > 0$ and $T > 0$ such that*

$$\eta_k = \eta_0 + \frac{w_a}{\mu_k^2} + \frac{z_a}{\mu_k^4} + \frac{\phi_k}{\mu_k^6}$$

with ϕ_k satisfying (35) *for k large.*

Proposition 25 *Given a sequence (s_k) with $s_k \in [\mu_k^p, e^{\mu_k}]$ for some $p > 2$, and $h(t) = -at^{-2}$ for t large, we have*

$$\int_{B_{r_k s_k}} \lambda_k(1 + h(u_k))u_k^2 e^{u_k^2} dx = 4\pi + \frac{4\pi - 4\pi a}{\mu_k^4} + o(\mu_k^{-4}).$$

Proof. Write as in the proof of Theorem 1

$$\begin{aligned}(I) &:= \int_{B_{r_k s_k}} \lambda_k(1 + h(u_k))u_k^2 e^{u_k^2} dx \\ &= \int_{B_{s_k}} \left(1 + \frac{\eta_k}{\mu_k^2}\right)\left(-\Delta\eta_0 - \frac{\Delta w_a}{\mu_k^2} - \frac{\Delta z_a}{\mu_k^4} + \frac{\Phi_k^a(r, \phi_k)}{\mu_k^6}\right) dx,\end{aligned}$$

where Φ_k^a is as in (65). Proceeding as in the Proof of Theorem 1 and using Lemma 23 and Proposition 24 we have

$$\begin{aligned}(I) &= \int_{B_{s_k}} \left(-\Delta\eta_0 - \frac{\eta_0\Delta\eta_0 + \Delta w_a}{\mu_k^2} - \frac{w_a\Delta\eta_0 + \eta_0\Delta w_a + \Delta z_a}{\mu_k^4}\right) dx + O(\mu_k^{-5}) \\ &=: (I_0) + \frac{(I_2^a)}{\mu_k^2} + \frac{(I_4^a)}{\mu_k^4} + O(\mu_k^{-5}).\end{aligned}$$

As before we have $(I_0) = 4\pi + o(\mu_k^{-4})$, while for (I_2^a) replacing w_0 with w_a leads us to the extra term

$$-a\int_{B_{s_k}} \Delta\zeta_0 dx = -2\pi s_k a\zeta_0'(s_k) = O(s_k^{-2}) = o(\mu_k^{-2}).$$

Therefore we have again $(I_2^a) = (I_2) + o(\mu_k^{-2}) = o(\mu_k^{-2})$.

As for the remaining term we have

$$\begin{aligned}(I_4^a) &= (I_4) + a\int_{B_{s_k}} (\zeta_0\Delta\eta_0 + \eta_0\Delta\zeta_0)dx - \int_{B_{s_k}} \Delta(z_a - z_0) \\ &= 4\pi + ao(1) - 2\pi\beta + o(1) \\ &= 4\pi - 4\pi a + o(1),\end{aligned}$$

where we used (63) and (64). Summing up we conclude. ☐

Proof of Theorem 4 (completed). As in the proof of Theorems 1 and 2, it suffices to add the estimate of Proposition 25 to the estimate of Proposition 14. For the latter we use Lemma 13. ☐

6 A few more open problems

Open problem 4 *Can one extend Theorem 1 (and its perturbed versions) to critical points of*

$$\sup_{u\in H_0^{1,n}(B_1):\|\nabla u\|^n_{L^n}\le n^{n-1}\omega_{n-1}}\int_\Omega e^{u^{\frac{n}{n-1}}}dx\le C_n$$

in dimension $n>2$? (Here ω_{n-1} is the volume of S^{n-1}.)

In this direction, there are some results of Adimurthi [2] and Adimurthi-Yang [3] on the solutions to

$$-\Delta_n u=\lambda u|u|^{n-2}e^{u^{n'}}\quad \text{in } \Omega\Subset\mathbb{R}^n, \tag{66}$$

from which one is led to conjecture that

$$\int_{B_1}|\nabla u_k|^n dx=n^{n-1}\omega_{n-1}+o(1),\quad \text{as } k\to\infty,$$

where (u_k) is a blowing-up sequence of positive radial solution to (66), and it would be interesting to understand the sign of the error term $o(1)$.

Open problem 5 *Can one extend Theorem 1 to the higher-order problem*

$$(-\Delta)^m u=\lambda u e^{mu^2},\quad u\in H_0^m(B_1),\quad B_1\subset\mathbb{R}^{2m}, \tag{67}$$

particularly to study the existence of extremals of the Adams inequality (see [1]) on a ball?

In this direction, the works [14, 15, 19, 21] suggest that for a blowing up sequence of solutions to (67)

$$\int_{B_1}|\nabla^m u_k|^2dx=\Lambda_1+o(1),\quad \text{as } k\to\infty,$$

where $\Lambda_1=(m-1)!\omega_{2m}$ is the total Q-curvature of S^{2m}.

Similarly one could consider the case $m=\frac{n}{2}$ with n odd, which has the additional difficult of $(-\Delta)^{\frac{n}{2}}$ being non-local. In this direction see [9] and [12].

Finally we do not know what happens when we drop the assumptions (10)-(11).

Open problem 6 *Is it possible to find functions g and h as in* (7)-(9) *such that the energy expansion of the solutions to* (8) *is of the form*

$$\|\nabla u_\mu\|^2_{L^2}=4\pi+\frac{A}{\mu_k^p}+o(\mu_k^{-p}),\quad \mu_k=u_k(0)\to\infty$$

for some $A\neq 0$, $p<4$? Can one even take $p\le 2$?

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