A fractional Moser-Trudinger type inequality in one dimension and its critical points

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Abstract

We show a sharp fractional Moser-Trudinger type inequality in dimension 1, i.e. for an interval $I \Subset \mathbb{R}$, $p \in (1, \infty)$ and some $\alpha_p > 0$

$$\sup_{u \in \tilde{H}^{\frac{1}{p},p}(I): \|(-\Delta)^{\frac{1}{2p}} u\|_{L^p(I)} \le 1} \int_I |u|^a e^{\alpha_p |u|^{\frac{p}{p-1}}} dx < \infty \quad \text{if and only if } a = 0.$$

Here $\tilde{H}^{\frac{1}{p},p}(I) = \{u \in L^p(\mathbb{R}) : (-\Delta)^{\frac{1}{2p}} u \in L^p(\mathbb{R}), \mathrm{supp}(u) \subset \bar{I}\}$.

Restricting ourselves to the case $p = 2$ we further consider for $\lambda > 0$ the functional

$$J(u) := \frac{1}{2} \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx - \lambda \int_I \left(e^{\frac{1}{2}u^2} - 1\right) dx, \quad u \in \tilde{H}^{\frac{1}{2},2}(I),$$

and prove that it satisfies the Palais-Smale condition at any lever $c \in (-\infty, \pi)$. We use these results to show that the equation

$$(-\Delta)^{\frac{1}{2}} u = \lambda u e^{\frac{1}{2}u^2} \quad \text{in } I$$

has a positive solution in $\tilde{H}^{\frac{1}{2},2}(I)$ if and only if $\lambda \in (0, \lambda_1(I))$, where $\lambda_1(I)$ is the first eigenvalue of $(-\Delta)^{\frac{1}{2}}$ on I. This extends to the fractional case some previous results proven by Adimurthi for the Laplacian and the p-Laplacian operators.

Finally with a technique of Ruf we show a fractional Moser-Trudinger inequality on $\mathbb{R}$.

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1 Introduction

Let $s \in (0, 1)$. We consider the space of functions $L_s(\mathbb{R})$ defined by

$$L_s(\mathbb{R}) = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}) : \int_{\mathbb{R}} \frac{|u(x)|}{1 + |x|^{1+2s}} dx < \infty \right\}. \tag{1}$$

For a function $u \in L_s(\mathbb{R})$ we define $(-\Delta)^s u$ as a tempered distribution as follows:

$$\langle (-\Delta)^s u, \varphi \rangle := \int_{\mathbb{R}} u (-\Delta)^s \varphi dx, \quad \varphi \in \mathcal{S}, \tag{2}$$

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where $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing smooth functions and for $\varphi \in \mathcal{S}$ we set

$$(-\Delta)^s \varphi := \mathcal{F}^{-1}(|\cdot|^{2s}\hat{\varphi}).$$

Here the Fourier transform is defined by

$$\hat{\varphi}(\xi) \equiv \mathcal{F}\varphi(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \varphi(x)\, dx.$$

Notice that the convergence of the integral in (2) follows from the fact that for $\varphi \in \mathcal{S}$ one has

$$|(-\Delta)^s \varphi(x)| \leq C(1 + |x|^{1+2s})^{-1}.$$

For $s \in (0,1)$ and $p \in [1, \infty]$ we define the Bessel-potential space

$$H^{s,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R} \colon (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R}) \right\}, \tag{3}$$

and its subspace

$$\tilde{H}^{s,p}(I) := \{u \in L^p(\mathbb{R}) : u \equiv 0 \text{ in } \mathbb{R} \setminus I,\ (-\Delta)^{\frac{s}{2}} u \in L^p(\mathbb{R})\}, \tag{4}$$

where $I \Subset \mathbb{R}$ is a bounded interval. Both spaces are endowed with the norm

$$\|u\|^p_{H^{s,p}(\mathbb{R})} := \|u\|^p_{L^p(\mathbb{R})} + \|(-\Delta)^{\frac{s}{2}} u\|^p_{L^p(\mathbb{R})}. \tag{5}$$

Remark 1 *Notice that the standard $H^{s,p}$-norm defined in* (5) *is equivalent to the smaller norm $\|u\|^*_{H^{s,p}(\mathbb{R})} := \|(-\Delta)^{\frac{s}{2}} u\|_{L^p(I)}$ on $\tilde{H}^{s,p}(I)$, see for instance Theorem* 7.1 *in [15].*

1.1 A fractional Moser-Trudinger type inequality

The first result that we shall prove is a fractional Moser-Trudinger type inequality:

Theorem 1.1 *For any $p \in (1, +\infty)$ set $p' = \frac{p}{p-1}$ and*

$$\alpha_p := \frac{1}{2}\left[2\cos\left(\frac{\pi}{2p}\right)\Gamma\left(\frac{1}{p}\right)\right]^{p'}, \quad \Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t}\, dt. \tag{6}$$

Then for any interval $I \Subset \mathbb{R}$ and $\alpha \leq \alpha_p$ we have

$$\sup_{u \in \tilde{H}^{\frac{1}{p},p}(I),\ \|(-\Delta)^{\frac{1}{2p}} u\|_{L^p(I)} \leq 1} \int_I \left(e^{\alpha |u|^{p'}} - 1\right) dx = C_p |I|, \tag{7}$$

and $\alpha = \alpha_p$ is the largest constant for which (7) *holds. In fact for any $a > 0$ we have*

$$\sup_{u \in \tilde{H}^{\frac{1}{p},p}(I),\ \|(-\Delta)^{\frac{1}{2p}} u\|_{L^p(I)} \leq 1} \int_I |u|^a \left(e^{\alpha_p |u|^{p'}} - 1\right) dx = \infty. \tag{8}$$

Remark 2 *Notice that in* (7)*, instead of the standard $H^{\frac{1}{p},p}$-norm defined in* (5)*, we are using the smaller but equivalent norm $\|u\|^*_{H^{\frac{1}{p},p}(\mathbb{R})} := \|(-\Delta)^{\frac{1}{2p}} u\|_{L^p(I)}$ (see Remark 1).*

Theorem 1.1 is a fractional version of the well-known Moser-Trudinger inequality

$$\sup_{u\in W_0^{1,n}(\Omega),\ \|\nabla u\|_{L^n(\Omega)}\le 1}\int_\Omega e^{\alpha|u|^{\frac{n}{n-1}}}dx\le C|\Omega|,\quad \text{for } \alpha\le\alpha_n:=n|S^{n-1}|^{\frac{1}{n-1}}, \tag{9}$$

where $\Omega\subset\mathbb{R}^n$ is a domain of finite measure, see e.g. [25], [31], [32]. Recently A. Iannizzotto and M. Squassina [16, Cor. 2.4] proved a subcritical version of (7) in Theorem 1.1 in the case $p=2$, namely

$$\sup_{u\in\tilde H^{\frac12,2}(I)\,:\,\|(-\Delta)^{\frac14}u\|_{L^2(\mathbb{R})}\le 1}\int_I e^{\alpha u^2}dx\le C_\alpha|I|,\quad \text{for } \alpha<\pi.$$

1.2 Palais-Smale condition and critical points

In the rest of this paper we will focus on the case $p=2$, and denote

$$H:=\tilde H^{\frac12,2}(I),\quad \|u\|_H:=\|(-\Delta)^{\frac14}u\|_{L^2(\mathbb{R})}. \tag{10}$$

By Remark 1 also this norm is equivalent to the full $H^{\frac12,2}$-norm on $\tilde H^{\frac12,2}(I)$. This also follows from the following Poincaré-type inequality (see [28, Lemma 6]):

$$\|u\|^2_{L^2(I)}\le C\|(-\Delta)^{\frac14}u\|^2_{L^2(\mathbb{R})}\quad \text{for } u\in\tilde H^{\frac12,2}(I).$$

We now investigate the existence of critical points of inequality (7) in the case $p=2$. Since we often integrate by parts and $(-\Delta)^s u$ is not in general supported in I even if $u\in C_c^\infty(I)$, it is more natural to consider the slightly weaker inequality

$$\sup_{u\in H,\ \|u\|_H^2\le 2\pi}\int_I\left(e^{\frac12 u^2}-1\right)dx=C|I|, \tag{11}$$

where we use the slightly different norm given in (10). The reason for using the constant $\frac12$ instead of $\alpha_2=\pi$ in the exponential and having $\|u\|_H^2\le 2\pi$ instead of $\|u\|_H^2\le 1$ is mostly cosmetic, and becomes more clear when studying the blow-up behaviour of critical points of (1.1), see e.g. [23] and [20].

We want to investigate the existence of critical points of (11), or more precisely solutions of the non-local equation

$$(-\Delta)^{\frac12}u=\lambda u e^{\frac12 u^2}\quad \text{in } I,\quad u\equiv 0 \text{ in } \mathbb{R}\setminus I, \tag{12}$$

which is the equation satisfied by critical points of the functional $E:M_\Lambda\to\mathbb{R}$, where

$$E(u)=\int_I\left(e^{\frac12 u^2}-1\right)dx,\quad M_\Lambda:=\{u\in H:\|u\|_H^2=\Lambda\},$$

$\Lambda>0$ is given, λ is a Lagrange multiplier and E is well defined on M_Λ thanks to Lemma 2.3 below. Since with this variational interpretation of (12) it is not possible to prescribe λ, we will follow the approach of Adimurthi and see solutions of (12) of critical points of the functional

$$J:H\to\mathbb{R},\quad J(u)=\frac12\|u\|_H^2-\lambda\int_I\left(e^{\frac12 u^2}-1\right)dx. \tag{13}$$

Again J is well-defined on H by Lemma 2.3. Moreover it is differentiable by Lemma 2.5 below, and its derivative is given by

$$\langle J'(u), v\rangle := \frac{d}{dt}J(u+tv)\Big|_{t=0} = (u,v)_H - \lambda\int_I uve^{\frac{1}{2}u^2}dx,$$

for any $u, v \in H$, where

$$(u,v)_H := \int_{\mathbb{R}} (-\Delta)^{\frac{1}{4}}u(-\Delta)^{\frac{1}{4}}v\,dx.$$

In particular we have that if $u \in H$ and $J'(u) = 0$, then u is a weak solution of Problem (12) in the sense that

$$(u,v)_H = \lambda\int_I uve^{\frac{1}{2}u^2}dx, \quad \text{for all } v \in H. \tag{14}$$

That this Hilbert-space definition of (12) is equivalent to the definition in sense of tempered distributions given by (2) is discussed in the introduction of [20].

To find critical points of J we will follow a method of Nehari, as done by Adimurthi [3]. An important point will be to understand whether J satisfies the Palais-Smale condition or not. We will prove the following:

Theorem 1.2 *The functional J satisfies the Palais-Smale condition at any level $c \in (-\infty, \pi)$, i.e. any sequence (u_k) with*

$$J(u_k) \to c \in (-\infty,\pi), \quad \|J'(u_k)\|_{H'} \to 0 \quad \text{as } k\to\infty \tag{15}$$

admits a subsequence strongly converging in H.

Theorem 1.3 *Let $I \subset \mathbb{R}$ be a bounded interval and $\lambda_1(I)$ denote the first eigenvalue of $(-\Delta)^{\frac{1}{2}}$ on $H = \tilde{H}^{\frac{1}{2},2}(I)$. Then for every $\lambda \in (0, \lambda_1(I))$ Problem* (12) *has at least one* positive *solution $u \in H$ in the sense of* (14)*. When $\lambda \geq \lambda_1(I)$ or $\lambda \leq 0$ Problem* (12) *has no non-trivial non-negative solutions.*

To prove Theorem 1.3 one constructs a sequence (u_k) which is almost of Palais-Smale type for J, in the sense that $J(u_k) \to \bar{c}$ for some $\bar{c} \in \mathbb{R}$ and $\langle J'(u_k), u_k\rangle = 0$. Then a modified version of Theorem 1.2 is used, namely Lemma 3.1 below. In order to do so, it is crucial to show that $\bar{c} < \pi$ (Lemma 4.4 below) and this will follow from (8) with $p = a = 2$. Interestingly, in the general case $s > 1$, $n \geq 2$, $p = \frac{n}{s}$, the analog of (8) is known only when s is integer or when $a > p'$ (see [24] and Remark 3 below).

Both Theorems 1.2 and 1.3 were first proven by Adimurthi [3] in dimension $n \geq 2$ with $(-\Delta)^{\frac{1}{2}}$ replaced by the n-Laplacian.

Let us briefly discuss the blow-up behaviour of solutions to (12). Extending previous works in even dimension (see e.g. [4], [12], [23], [27]) the second and third authors and Armin Schikorra [20] studied the blow-up of sequences of solutions to the equation

$$(-\Delta)^{\frac{n}{2}}u = \lambda ue^{\frac{n}{2}u^2} \quad \text{in } \Omega \Subset \mathbb{R}^n$$

with suitable Dirichlet-type boundary conditions when n is odd. The moving plane technique for the fractional Laplacian (see [7]) implies that a non-negative solution to (12) is symmetric and monotone decreasing from the center of I. Then it is not difficult to check that in dimension 1 Theorem 1.5 and Proposition 2.8 of [20] yield:

Theorem 1.4 *Fix $I = (-R, R) \Subset \mathbb{R}$ and let $(u_k) \subset H = \tilde{H}^{\frac{1}{2},2}(I)$ be a sequence of non-negative solutions to*

$$(-\Delta)^{\frac{1}{2}} u_k = \lambda_k u_k e^{\frac{1}{2}u_k^2} \quad \text{in } I, \tag{16}$$

in the sense of (14). *Let $m_k := \sup_I u_k$ and assume that*

$$\Lambda := \limsup_{k\to\infty} \|u_k\|_H^2 < \infty.$$

Then up to extracting a subsequence we have that either

(i) $u_k \to u_\infty$ *in* $C^\ell_{\mathrm{loc}}(I) \cap C^0(\bar{I})$ *for every* $\ell \geq 0$*, where* $u_\infty \in C^\ell_{\mathrm{loc}}(I) \cap C^0(\bar{I}) \cap H$ *solves*

$$(-\Delta)^{\frac{1}{2}} u_\infty = \lambda_\infty u_\infty e^{\frac{1}{2}u_\infty^2} \quad \text{in } I, \tag{17}$$

for some $\lambda_\infty \in (0, \lambda_1(I))$*, or*

(ii) $u_k \to u_\infty$ *weakly in* H *and strongly in* $C^0_{\mathrm{loc}}(\bar{I} \setminus \{0\})$ *where* u_∞ *is a solution to* (17). *Moreover, setting* r_k *such that* $\lambda_k r_k m_k^2 e^{\frac{1}{2}m_k^2}$ *and*

$$\eta_k(x) := m_k(u_k(r_k x) - m_k) + \log 2, \quad \eta_\infty(x) := \log\left(\frac{2}{1+|x|^2}\right), \tag{18}$$

one has $\eta_k \to \eta_\infty$ *in* $C^\ell_{\mathrm{loc}}(\mathbb{R})$ *for every* $\ell \geq 0$ *and* $\Lambda \geq \|u_\infty\|_H^2 + 2\pi$.

The function η_∞ appearing in (18) solves the equation

$$(-\Delta)^{\frac{1}{2}} \eta_\infty = e^{\eta_\infty} \quad \text{in } \mathbb{R},$$

which has been recently interpreted in terms of holomorphic immersions of a disk (or the half-plane) by Francesca Da Lio, Tristan Rivière and the third author [10].

Theorem 1.4 should be compared with the two dimensional case, where the analogous equation $-\Delta u = \lambda u e^{u^2}$ on the unit disk has a more precise blow-up behaviour, see e.g. [5], [4], [12], [21].

1.3 A fractional Moser-Trudinger type inequality on the whole $\mathbb{R}$

When replacing a bounded interval I by $\mathbb{R}$, an estimate of the form (7) cannot hold, for instance because of the scaling of (7), or simply because the quantity $\|(-\Delta)^{\frac{1}{2p}} u\|_{L^p(\mathbb{R})}$ vanishes on constants. This suggests to use the full Sobolev norm including the term $\|u\|_{L^p(I)}$ (see Remark 1). This was done by Bernhard Ruf [30] in the case of $H^{1,2}(\mathbb{R}^2)$. We shall adapt his technique to the case $H^{\frac{1}{2},2}(\mathbb{R})$.

Theorem 1.5 *We have*

$$\sup_{u \in H^{\frac{1}{2},2}(\mathbb{R}),\ \|u\|_{H^{\frac{1}{2},2}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} \left(e^{\pi u^2} - 1\right) dx < \infty, \tag{19}$$

where $\|u\|_{H^{\frac{1}{2},2}(\mathbb{R})}$ *is defined in* (5)*. Moreover, for any* $a > 2$*,*

$$\sup_{u\in H^{\frac{1}{2},2}(\mathbb{R}),\ \|u\|_{H^{\frac{1}{2},2}(\mathbb{R})}\leq 1} \int_{\mathbb{R}} |u|^a \left(e^{\pi u^2}-1\right) dx = \infty. \tag{20}$$

In particular the constant π *in* (19) *is sharp.*

A main ingredient in the proof of (19) is a fractional Pólya-Szegõ inequality which seems to be known only in the L^2 setting, being based mainly on Fourier transform techniques.

Open question 1 *Does an* L^p*-version of Theorem 1.5 hold, i.e. can we replace* $H^{\frac{1}{2},2}$ *with* $H^{\frac{1}{p},p}$ *in* (19)*?*

The reason for taking $a > 2$ in (20) (contrary to (8)) is that the test functions for (20) will be constructed using a cut-off procedure, and due to the non-local nature of the $H^{\frac{1}{2},2}$-norm, giving a precise estimate for the norm of such test functions is difficult.

Open question 2 *In analogy with Theorem 1.1, can one also take* $a \in (0, 2]$ *in* (20)*?*

In the following sections we shall prove Theorems 1.1, 1.2, 1.3 and 1.5. In the appendix we collected some useful results about fractional Sobolev spaces and fractional Laplace operators.

2 Theorem 1.1

2.1 Idea of the proof

The following analog of (7)

$$\sup_{u=c_p I_{\frac{1}{p}}*f\,:\,\mathrm{supp}(f)\subset\bar{I},\ \|f\|_{L^p(I)}\leq 1} \int_I e^{\alpha_p|u|^{p'}} dx = C_p|I|, \quad I_{\frac{1}{p}}(x) := |x|^{\frac{1}{p}-1} \tag{21}$$

is well-known (also in higher dimension), since it follows easily from the Theorem 2 in [2], up to choosing c_p so that

$$c_p(-\Delta)^{\frac{1}{2p}} I_{\frac{1}{p}} = \delta_0, \tag{22}$$

compare to Lemma 2.1 below.

In (21) one requires that the support of $f = (-\Delta)^{\frac{1}{2p}}u$ is bounded; following Adams [2] one would be tempted to write $u = I_{\frac{1}{p}} * (-\Delta)^{\frac{1}{2p}}u$ and apply (21), but the support of $(-\Delta)^{\frac{1}{2p}}u$ is in general not bounded, when u is compactly supported.

In order to circumvent this issue, we rely on a Green representation formula of the form

$$u(x) = \int_I G_{\frac{1}{2p}}(x,y)(-\Delta)^{\frac{1}{2p}}u(y)dy,$$

and show that $|G_{\frac{1}{2p}}(x,y)| \leq I_{\frac{1}{p}}(x-y)$ for $x \neq y$. This might infer from the explicit formular of $G_s(x,y)$, which is known on an interval, see e.g. [6] and [9], but we prefer to follow a more self-contained path, only using the maximum principle.

More delicate is the proof of (8). We will construct functions u supported in $\bar{I}$ with $(-\Delta)^{\frac{1}{2p}}u = f$ for some prescribed function $f \in L^p(I)$ suitably concentrated. Then with a barrier argument we will show that $u \in \tilde{H}^{\frac{1}{p},p}(I)$, i.e. $(-\Delta)^{\frac{1}{2p}}u \in L^p(\mathbb{R})$. This is not obvious because $(-\Delta)^{\frac{1}{2p}}$ is a non-local operator and even if $u \equiv 0$ in I^c, $(-\Delta)^{\frac{1}{2p}}u$ does not vanish outside I, and a priori it could even concentrate on ∂I.

Remark 3 *An alternative approach to* (8) *uses the Riesz potential and a cut-off function ψ, as done in [24] following a suggestion of A. Schikorra. This works in every dimension and for arbitrary powers of $-\Delta$, but is less efficient in the sense that the $\|(-\Delta)^s\psi\|_{L^p}$ is not sufficiently small, and* (8) *(or its higher-order analog) can be proven only for $a > p'$. On the other hand, the approach used here to prove* (8) *for every $a > 0$ does not work for higher-order operators, since for instance if for $\Omega \Subset \mathbb{R}^4$ we take $u \in W^{1,2}_0(\Omega)$ solving $\Delta u = f \in L^2(\Omega)$, then we do not have in general $\Delta u \in W^{2,2}(\mathbb{R}^4)$.*

2.2 Proof of Theorem 1.1

By a simple scaling argument it suffices to prove (7) for a given interval, say $I = (-1, 1)$.

Lemma 2.1 *For $s \in \left(0, \frac{1}{2}\right)$ the fundamental solution of $(-\Delta)^s$ on $\mathbb{R}$ is*

$$F_s(x) = \frac{1}{2\cos(s\pi)\Gamma(2s)|x|^{1-2s}},$$

i.e. $(-\Delta)^s F_s = \delta_0$ in the sense of tempered distributions.

Proof. This follows easily e.g. from Theorem 5.9 in [19]. □

Lemma 2.2 *Fix $s \in \left(0, \frac{1}{2}\right)$. For any $x \in I = (-1, 1)$ let $g_x \in C^\infty(\mathbb{R})$ be any function with $g_x(y) = F_s(x - y)$ for $y \in I^c$. Then there exists $H_s(x, \cdot) \in \tilde{H}^{s,2}(I) + g_x$ unique solution to*

$$\begin{cases} (-\Delta)^s H_s(x, \cdot) = 0 & \text{in } I \\ H_s(x, \cdot) = g_x & \text{in } \mathbb{R} \setminus I \end{cases} \tag{23}$$

and the function

$$G_s(x, y) := F_s(x - y) - H_s(x, y), \qquad (x, y) \in I \times \mathbb{R}$$

is the Green function of $(-\Delta)^s$ on I, i.e. for $x \in I$ it satisfies

$$\begin{cases} (-\Delta)^s G_s(x, \cdot) = \delta_x & \text{in } I \\ G(x, y) = 0 & \text{for } y \in \mathbb{R} \setminus I. \end{cases} \tag{24}$$

Moreover

$$0 < G_s(x, y) \le F_s(x - y) \quad \text{for } y \neq x \in I. \tag{25}$$

Finally, for any function $u \in \tilde{H}^{2s,p}(I)$ ($p \in [1, \infty)$) we have

$$u(x) = \int_I G_s(x, y)(-\Delta)^s u(y)dy, \quad \text{for a.e. } x \in I, \tag{26}$$

where the right-hand side is well defined for a.e. $x \in I$ thanks to (25) *and Fubini's theorem.*

Remark 4 *The first equations in* (23) *above and in* (24) *below are intended in the sense of distribution, compare to* (2).

Proof. The existence and non-negativity of $H_s(x,\cdot)$ for every $x \in I$ follow from Theorem A.2 and Proposition A.3 in the Appendix. The next claim, namely (24), follows at once from Lemma 2.1 and (23).

We show now that $G(x,y) \geq 0$ for every $(x,y) \in I \times I$. We claim that

$$\lim_{y\to\pm 1} H_s(x,y) = H_s(x,\pm 1) = F_s(x \mp 1), \tag{27}$$

hence $G_s(x,y) \to 0$ as $y \to \partial I$, and by Silvestre's maximum principle, Proposition A.6 below, we also have $G_s(x,\cdot) \geq 0$ for every $x \in I$, hence also (25) follows. For the proof of (27) notice that

$$\tilde{H}_s(x,\cdot) := H_s(x,\cdot) - g_x \in \tilde{H}^{s,2}(I)$$

satifies

$$\begin{cases} (-\Delta)^s \tilde{H}_s(x,\cdot) = -(-\Delta)^s g_x & \text{in } I \\ \tilde{H}_s(x,\cdot) = 0 & \text{in } \mathbb{R}\setminus I \end{cases}$$

and $((-\Delta)^s g_x)|_I \in L^\infty(I)$ by Proposition A.7 (we are using that $g_x \in C^\infty(\mathbb{R})$), hence Proposition A.4 gives $\tilde{H}_s(x,y) \to 0$ as $y \to \partial I$, and (27) follows at once.

To prove (26), let us start considering $u \in C_c^\infty(I)$. Then, according to (24), we have

$$u(x) = \langle \delta_x, u\rangle = \langle (-\Delta)^s G_s(x,\cdot), u\rangle = \int_I G_s(x,y)(-\Delta)^s u(y)dy.$$

Given now $u \in \tilde{H}^{2s,p}(I)$, let $(u_k)_{k\in\mathbb{N}} \subset C_c^\infty(I)$ converge to u in $\tilde{H}^{2s,p}(I)$, i.e.

$$u_k \to u, \quad (-\Delta)^s u_k \to (-\Delta)^s u \quad \text{in } L^p(\mathbb{R}), \text{ hence in } L^1(I),$$

see Lemma A.5. Then

$$u \overset{L^1(I)}{\longleftarrow} u_k = \int_I G_s(\cdot,y)(-\Delta)^s u_k(y)dy \overset{L^1(I)}{\longrightarrow} \int_I G_s(\cdot,y)(-\Delta)^s u(y)dy,$$

the convergence on the right following from (25) and Fubini's theorem:

$$\begin{aligned} &\int_I \left| \int_I G_s(x,y)\left[(-\Delta)^s u_k(y) - (-\Delta)^s u(y)\right] dy \right| dx \\ &\leq \int_I \int_I F_s(x-y)\left|(-\Delta)^s u_k(y) - (-\Delta)^s u(y)\right| dxdy \\ &\leq \sup_{y\in I} \|F_s\|_{L^1(I-y)} \|(-\Delta)^s u_k - (-\Delta)^s u\|_{L^1(I)} \to 0 \end{aligned}$$

as $k \to \infty$. Since the convergence in L^1 implies the a.e. convergence (up to a subsequence), (26) follows. ∎

Proof of Theorem 1.1. Set $s = \frac{1}{2p}$. From Lemma 2.2 we get

$$0 \leq (2\alpha_p)^{\frac{p-1}{p}} G_s(x,y) \leq I_{\frac{1}{p}}(x-y) = |x-y|^{\frac{1}{p}-1},$$

where G_s is the Green's function of the interval I defined in Lemma 2.2. Choosing $f := |(-\Delta)^{\frac{1}{2p}} u|\big|_I$ and using (25) and (26), we bound

$$(2\alpha_p)^{\frac{p-1}{p}} |u(x)| \le (2\alpha_p)^{\frac{p-1}{p}} \int_I G_s(x,y) f(y) dy \le I_{\frac{1}{p}} * f(x)$$

and (7) follows at once from (21).

It remains to show (8). The proof is based on the construction of suitable test functions and it is split into steps.

Step 1. Definition of the test functions. We fix $q \ge 1$ and set

$$f(y) = f_q(y) := \frac{1}{2q} |y|^{-\frac{1}{p}} \chi_{[-\frac{1}{2}, -r] \cup [r, \frac{1}{2}]}, \quad r := \frac{e^{-q}}{2}. \tag{28}$$

Notice that

$$\|f\|_{L^p}^p = \frac{2}{(2q)^p} \int_r^{\frac{1}{2}} \frac{dy}{y} = \frac{1}{(2q)^{p-1}}.$$

Now let $u = u_q \in \tilde{H}^{s,2}(I)$ solve

$$\begin{cases} (-\Delta)^s u = f & \text{in } I \\ u \equiv 0 & \text{in } I^c. \end{cases} \tag{29}$$

in the sense of Theorem A.2 in the appendix.

Step 2. Proving that $u \in \tilde{H}^{2s,p}(I)$. According to Proposition A.4 u satisfies

$$|u(x)| \le C \|f\|_{L^\infty} (1 - |x|)^s \quad \text{for } x \in I. \tag{30}$$

We want to prove that $(-\Delta)^s u \in L^p(\mathbb{R})$. Since by Proposition A.7

$$(-\Delta)^s u(x) = C_s \int_I \frac{-u(y)}{|x-y|^{1+2s}} dy, \quad \text{for } |x| > 1$$

and u is bounded, we see immediately that

$$|(-\Delta)^s u(x)| \le \frac{C}{|x|^{1+2s}}, \quad \text{for } |x| \ge 2,$$

hence

$$\|(-\Delta)^s u\|_{L^q(\mathbb{R} \setminus [-2,2])} < \infty \quad \text{for every } q \in [1, \infty). \tag{31}$$

Now we claim that

$$(I) := \|(-\Delta)^s u\|_{L^q([-2,2] \setminus [-1,1])} < \infty, \quad q = \max\{p, 2\}. \tag{32}$$

Again using Proposition A.7, (30) and translating, we have

$$(I) = \left(\int_{[-2,2] \setminus [-1,1]} \left| C \int_{-1}^1 \frac{-u(y) dy}{|y-x|^{1+2s}} \right|^q dx \right)^{\frac{1}{q}} \le C \left(\int_{-1}^0 \left| \int_0^2 \frac{y^s dy}{(y-x)^{1+2s}} \right|^q dx \right)^{\frac{1}{q}},$$

and using the Minkowski inequality

$$\left(\int_{A_1}\left|\int_{A_2}F(x,y)dy\right|^q dx\right)^{\frac{1}{q}} \le \int_{A_2}\left(\int_{A_1}|F(x,y)|^q dx\right)^{\frac{1}{q}} dy,$$

we get

$$(I) \le C\int_0^2 y^s\left(\int_{-1}^0 \frac{dx}{(y-x)^{(1+2s)q}}\right)^{\frac{1}{q}} dy \le C\int_0^2 \frac{dy}{y^{1+s-\frac{1}{q}}} < \infty,$$

since $1+s-\frac{1}{q}<1$. This proves (32).

To conclude that $(-\Delta)^s u \in L^p(\mathbb{R})$ it remains to shows that $(-\Delta)^s u$ does not concentrate on $\partial I = \{-1,1\}$, in the sense that the distribution defined by

$$\begin{aligned}\langle T,\varphi\rangle &:= \int_{\mathbb{R}} u(-\Delta)^s\varphi dx - \int_I f\varphi dx - \int_{I^c} C_s\int_{\mathbb{R}}\frac{-u(y)}{|x-y|^{1+2s}}dy\,\varphi(x)dx\\ &=: \langle T_1,\varphi\rangle - \langle T_2,\varphi\rangle - \langle T_3,\varphi\rangle \quad \text{for } \varphi\in C_c^\infty(\mathbb{R})\end{aligned}$$

vanishes. Notice that $\langle T,\varphi\rangle = 0$ for $\varphi \in C_c^\infty(\mathbb{R}\setminus\partial I)$, since $T_1 = (-\Delta)^s u$, while

$$\langle T_2,\varphi\rangle = \langle(-\Delta)^s u,\varphi\rangle, \quad \langle T_3,\varphi\rangle = 0 \quad \text{for } \varphi\in C_c^\infty(I)$$

by (29), and

$$\langle T_2,\varphi\rangle = 0, \quad \langle T_3,\varphi\rangle = \langle(-\Delta)^s u,\varphi\rangle \quad \text{for } \varphi\in C_c^\infty(I^c)$$

by Proposition A.7, and for $\varphi\in C_c^\infty(\mathbb{R}\setminus\partial I)$ we can split $\varphi = \varphi_1+\varphi_2$ with $\varphi_1\in C_c^\infty(I)$ and $\varphi_2 \in C_c^\infty(I^c)$. In particular $\mathrm{supp}(T)\subset\partial I$.

It is easy to see that T_1 is a distribution of order at most 1, i.e.

$$\left|\int_{\mathbb{R}} u(-\Delta)^s\varphi dx\right| \le C\|\varphi\|_{C^1(\mathbb{R})}, \quad \text{for every } \varphi\in C_c^\infty(\mathbb{R})$$

(use for instance Proposition A.7), and that T_2 and T_3 are distributions of order zero, i.e.

$$|\langle T_i,\varphi\rangle| \le C\|\varphi\|_{L^\infty(\mathbb{R})} \quad \text{for } i=2,3.$$

Since $\mathrm{supp}(T)\subset\partial I$ it follows from Schwartz's theorem (see e.g. [8, Sec. 6.1.5]) that

$$T = \alpha\delta_{-1}+\beta\delta_1+\tilde\alpha D\delta_{-1}+\tilde\beta D\delta_1, \quad \text{for some } \alpha,\beta,\tilde\alpha,\tilde\beta\in\mathbb{R},$$

where $\langle D\delta_{x_0},\varphi\rangle := -\langle\delta_{x_0},\varphi'\rangle = -\varphi'(x_0)$ for $\varphi\in C_c^\infty(\mathbb{R})$.

In order to show that $\tilde\alpha = 0$, take $\varphi\in C_c^\infty(\mathbb{R})$ with

$$\mathrm{supp}(\varphi)\subset(-1,1), \quad \varphi'(0)=1, \quad \varphi(0)=0,$$

and rescale it by setting for $\varphi_\lambda(-1+x) = \lambda\varphi(\lambda^{-1}x)$ for $\lambda>0$. Since T_2 and T_3 have order 0 it follows

$$|\langle T_i,\varphi_\lambda\rangle| \le C\lambda \to 0 \text{ as } \lambda\to 0, \quad \text{for } i=2,3.$$

As for T_1, using Proposition A.7 we get

$$\begin{aligned}\frac{\langle T_1,\varphi_\lambda\rangle}{C_s} &= \int_{(B_{2\lambda}(-1))^c} u(x)\int_{B_\lambda(-1)} \frac{-\varphi_\lambda(y)}{|x-y|^{1+2s}}dydx \\ &\quad + \int_{B_{2\lambda}(-1)} u(x)\int_{(B_{4\lambda}(-1))^c} \frac{\varphi_\lambda(x)}{|x-y|^{1+2s}}dydx \\ &\quad + \int_{B_{2\lambda}(-1)} u(x)\int_{B_{4\lambda}(-1)} \frac{\varphi_\lambda(x)-\varphi_\lambda(y)}{|x-y|^{1+2s}}dydx \\ &=: (I)+(II)+(III).\end{aligned}$$

Since $\|\varphi_\lambda\|_{L^\infty(\mathbb{R})} = C_\varphi\lambda$ and $u \in L^\infty(\mathbb{R})$, one easily bounds $|(I)|+|(II)| \to 0$ as $\lambda \to 0$, and using that $\sup_{\mathbb{R}}|\varphi_\lambda'| = \sup_{\mathbb{R}}|\varphi'|$ we get

$$|(III)| \le \int_{B_{2\lambda}(-1)} |u(x)| \int_{B_{4\lambda}(-1)} \frac{\sup_{\mathbb{R}}|\varphi'|}{|x-y|^{2s}}dydx \le C\lambda^{1-2s}\int_{B_{2\lambda}(-1)}|u(x)|dx \to 0 \quad \text{as } \lambda \to 0.$$

Since for $\lambda \in (0,1)$ we have $\langle T,\varphi\rangle = -\tilde{\alpha}$, by letting $\lambda \to 0$ it follows that $\tilde{\alpha} = 0$. Similarly one can prove that $\tilde{\beta} = 0$.

We now claim that $\alpha, \beta = 0$. Considering

$$\tilde{u}(x) := u(x) - \alpha F_s(x+1) - \beta F_s(x-1),$$

and recalling that $(-\Delta)^s F_s = \delta_0$, one obtains that

$$(-\Delta)^s \tilde{u} = T_1 - \alpha\delta_{-1} - \beta\delta_1 = T_2 + T_3 \in L^2(\mathbb{R}),$$

hence with Proposition A.1

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|\tilde{u}(x)-\tilde{u}(y)|^2}{|x-y|^{1+2s}}dxdy = [\tilde{u}]^2_{W^{2s,2}(\mathbb{R})} = C\|(-\Delta)^s\tilde{u}\|^2_{L^2(\mathbb{R})} < \infty,$$

and this gives a contradiction if $\alpha \neq 0$ or $\beta \neq 0$ since the integral on the left-hand side does not converge in these cases.

Then $T = 0$, i.e. $(-\Delta)^s u =: T_1 = T_2 + T_3$ and from (29), (31) and (32) we conclude that $(-\Delta)^s u \in L^p(\mathbb{R})$, hence $u \in \tilde{H}^{2s,p}(I)$, as wished.

Step 3: Conclusion. Recalling that $(-\Delta)^s u = f$ in I, from (26) we have for $x \in I$

$$\begin{aligned} u(x) &= \int_I G_s(x,y)f(y)dy \\ &= \frac{1}{2q(2\alpha_p)^{\frac{p-1}{p}}}\int_{r<|y|<\frac{1}{2}} \frac{1}{|x-y|^{1-\frac{1}{p}}|y|^{\frac{1}{p}}}dy - \int_{r<|y|<\frac{1}{2}} H_s(x,y)f(y)dy \\ &=: u_1(x) + u_2(x), \end{aligned} \tag{33}$$

where $H_s(x,y)$ is as in Lemma 2.2.

We now want a lower bound for u in the interval $[-r,r]$. We fix $0<x\le r$ and estimate

$$\begin{aligned}u_1(x) &= \frac{1}{2q(2\alpha_p)^{\frac{p-1}{p}}}\left(\int_r^{\frac12}\frac{dy}{(y-x)^{1-\frac1p}y^{\frac1p}}+\int_{-\frac12}^{-r}\frac{dy}{|y-x|^{1-\frac1p}|y|^{\frac1p}}\right)\\
&\ge \frac{1}{2q(2\alpha_p)^{\frac{p-1}{p}}}\left(\int_r^{\frac12}\frac{dy}{y}+\int_r^{\frac12}\frac{dy}{y+x}\right)\\
&= \frac{1}{2q(2\alpha_p)^{\frac{p-1}{p}}}\left(2q+\log\left(\frac{1+2x}{1+\frac{x}{r}}\right)\right)\\
&= \frac{1}{(2\alpha_p)^{\frac{p-1}{p}}}+O(q^{-1}).\end{aligned}$$

Since H_s is bounded on $[-r,r]\times[-\frac12,\frac12]$, we have

$$|u_2(x)|\le C\int_r^{\frac12} f(y)dy\le Cq^{-1}\int_0^{\frac12}|y|^{-\frac1p}dy=O(q^{-1}),\quad x\in[-r,r].$$

Then for $|x|<r$ we have

$$u(x)\ge\frac{1}{(2\alpha_p)^{\frac{p-1}{p}}}+O(q^{-1}),$$

as $q\to\infty$. We now set

$$w_q:=(2q)^{\frac{p-1}{p}}u_q\in\tilde H^{\frac1p,p}(I),$$

so that $\|(-\Delta)^s w_q\|_{L^p(I)}=1$, we compute for $a>0$

$$\int_I |w_q|^a e^{\alpha_p|w_q|^{p'}}dx\ge\int_{-r}^r\left(\frac{q}{\alpha_p}+O(1)\right)^{a/p'}e^{q+O(1)}dx\ge\frac{2rq^{a/p'}e^q}{C}=\frac{q^{a/p'}}{C},$$

and we conclude by letting $q\to\infty$. □

2.3 A few consequences of Theorem 1.1

Lemma 2.3 *Let $u\in H$. Then $u^q e^{pu^2}\in L^1(I)$ for every $p,q>0$.*

Proof. Since $|u|^q\le C(q)e^{|u|^2}$, it is enough to prove the case $q=0$. Given $\varepsilon>0$ (to be fixed later), by Lemma A.5 there exists $v\in C_c^\infty(I)$ such that

$$\|v-u\|_H^2<\varepsilon.$$

Using

$$u^2\le(v-u)^2+v^2+2vu$$

we bound

$$e^{pu^2}\le e^{p(v-u)^2}e^{pv^2}e^{2pvu}, \tag{34}$$

where clearly $e^{pv^2}\in L^\infty(I)$. Using the inequality $|ab|\le\frac12(a^2+b^2)$ we have

$$e^{2puv}\le e^{\frac1\varepsilon p^2\|u\|_H^2 v^2}e^{\varepsilon(\frac{u}{\|u\|_H})^2},$$

and for ε small enough the right-hand side is bounded in $L^2(I)$ thanks to Theorem 1.1. Still by Theorem 1.1 we have $e^{p(u-v)^2} \in L^2(I)$ if $\varepsilon > 0$ is small enough, hence going back to (34) and using that $v \in L^\infty(I)$ is now fixed, we conclude with Hölder's inequality that $e^{pv^2} \in L^1(I)$. $\square$

Lemma 2.4 *For any $q, p \in (1, +\infty)$ the functional*

$$E_{q,p} : H \to \mathbb{R}, \quad E_{q,p}(u) := \int_I |u|^q e^{pu^2} dx$$

is continuous.

Proof. Consider a sequence $u_k \to u$ in H. By Lemma 2.3 (up to changing the exponents) we have that the sequence $f_k := |u_k|^q e^{pu_k^2}$ is bounded in $L^2(I)$. Indeed, it is enough to write $u_k = (u_k - u) + u$ and use the same estimates as in (34) with u instead of v and u_k instead of u. We now claim that $f_k \to f$ in $L^1(I)$. Indeed up to a subsequence $u_k \to u$ a.e., hence $f_k \to f := |u|^q e^{pu^2}$ a.e. Then considering that since f_k is bounded in $L^2(I)$ we have

$$\int_{\{f_k > L\}} f_k \, dx \le \frac{1}{L} \int_{\{f_k > L\}} f_k^2 \, dx \le \frac{C}{L} \to 0 \quad \text{as } L \to +\infty,$$

the claim follows at once from Lemma A.9. $\square$

Lemma 2.5 *The functional $J : H \to \mathbb{R}$ defined in* (13) *is smooth.*

Proof. This follows easily from Lemma 2.4, since the first term on the right-hand side of (13) is simply $\frac{1}{2}\|u\|_H^2$, and the derivatives of the second term are continuous thanks to Lemma 2.4. $\square$

The following lemma is a fractional analog of a well-known result of P-L. Lions [22].

Lemma 2.6 *Consider a sequence $(u_k) \subset H$ with $\|u_k\|_H = 1$ and $u_k \rightharpoonup u$ weakly in H, but not strongly (so that $\|u\|_H < 1$). Then if $u \not\equiv 0$, $e^{\pi u_k^2}$ is bounded in L^p for $1 \le p < \tilde{p} := (1 - \|u\|_H^2)^{-1}$.*

Proof. We split

$$u_k^2 = u^2 - 2u(u - u_k) + (u - u_k)^2.$$

Then $v_k := e^{\pi u_k^2} = v v_{k,1} v_{k,2}$, where $v = e^{\pi |u|^2} \in L^p(I)$ for all $p \ge 1$ by Lemma 2.3, $v_{k,1} = e^{-2\pi u(u-u_k)}$ and $v_{k,2} = e^{\pi(u-u_k)^2}$.

Notice now that from

$$-2p\pi u(u - u_k) \le \pi \left(\frac{p^2}{\varepsilon^2} u^2 + \varepsilon^2 (u - u_k)^2 \right),$$

we get from Lemma 2.3 and Theorem 1.1 that $v_{k,1} \in L^q(I)$ for all $q \ge 1$ if $\varepsilon > 0$ is small enough (depending on q). But again from Theorem 1.1 $v_{2,k}$ is bounded in $L^p(I)$ for all $p < \tilde{p}$ since

$$\|u_k - u\|_H^2 = 1 - 2\langle u_k, u \rangle + \|u\|_H^2 \to 1 - \|u\|_H^2.$$

Therefore by Hölder's inequality we have that v_k is bounded in $L^p(I)$ for all $p < \tilde{p}$. $\square$

3 Proof of Theorem 1.2

For the proof of Theorem 1.2 we will closely follow [3]. Set

$$Q(u) := J(u) - \frac{1}{2}\langle J'(u), u\rangle = \lambda \int_I \left(\left(\frac{u^2}{2} - 1 \right) e^{\frac{1}{2}u^2} + 1 \right) dx. \tag{35}$$

Remark 5 *Notice that the integrand on the right-hand side of* (35) *is strictly convex and has a minimum at* $u = 0$*; in particular*

$$0 = Q(0) < Q(u) \quad \textit{for every } u \in H \setminus \{0\}. \tag{36}$$

Furthermore by Lemma 2.4 the functional Q *is continuous on* H *and by convexity* Q *is also weakly lower semi-continuous.*

Let us also notice that

$$\begin{aligned} \lambda \int_I u^2 e^{\frac{1}{2}u^2}\, dx &= \lambda \int_{\{|u|\le 4\}} u^2 e^{\frac{1}{2}u^2}\, dx + \lambda \int_{\{|u|>4\}} u^2 e^{\frac{1}{2}u^2}\, dx \\ &\le C + \lambda \int_{\{|u>4|\}} u^2 e^{\frac{1}{2}u^2}\, dx \le C + \tilde{C} Q(u) \end{aligned}$$

and hence we have

$$\lambda \int_I u^2 e^{\frac{1}{2}u^2} dx \le C(1 + Q(u)) \quad \text{for every } u \in H. \tag{37}$$

We consider a Palais-Smale sequence $(u_k)_{k\ge 0}$ with $J(u_k) \to c$. From (15) we get

$$\langle J'(u_k), u_k\rangle = o(1)\|u_k\|_H \quad \text{as } k \to \infty,$$

and

$$Q(u_k) = J(u_k) - \frac{1}{2}\langle J'(u_k), u_k\rangle = c + o(1) + o(1)\|u_k\|_H. \tag{38}$$

Then with (37) we have

$$\lambda \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx \le C\left(1 + \|u_k\|_H\right),$$

hence, using that $Q(u_k) \ge 0$

$$\lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1 \right) dx \le C\left(1 + \|u_k\|_H\right),$$

so that

$$J(u_k) \ge \frac{1}{2}\|u_k\|_H^2 - C(1 + \|u_k\|_H).$$

This and the boundedness of $(J(u_k))_{k\ge 0}$ yield that the sequence $(u_k)_{k\ge 0}$ is bounded in H, hence we can extracts a weakly converging subsequence $u_k \rightharpoonup \tilde{u}$ in H. By the compactness of the embedding $H \hookrightarrow L^2$ (see e.g. [11, Theorem 7.1]), up to extracting a further subsequence we can assume that $u_k \to \tilde{u}$ almost everywhere. To complete the proof of the theorem it remains to show that, up to extracting a further subsequence, $u_k \to \tilde{u}$ strongly in H.

By Remark 5 we have

$$0 \le Q(\tilde{u}) \le \liminf_{k\to\infty} Q(u_k) = \liminf_{k\to\infty} \left(J(u_k) - \frac{1}{2}\langle J'(u_k), u_k\rangle \right) = c \tag{39}$$

Thus necessarily $c \ge 0$. In other words the Palais-Smale condition is vacantly true when $c < 0$ because no sequence can satisfy (15).

Let us now consider the case $c = 0$. Clearly (39) implies $Q(u_k) \to Q(\tilde{u}) = 0$. We now claim that

$$u_k^p e^{\frac{1}{2}u_k^2} \to \tilde{u}^p e^{\frac{1}{2}\tilde{u}^2} \quad \text{in } L^1(I) \quad \text{for } 0 \le p < 2. \tag{40}$$

Indeed, up to extracting a further subsequence, from (37) and (39) we get

$$\int_{\{|u_k|>L\}} u_k^p e^{\frac{1}{2}u_k^2} dx \le \frac{1}{L^{2-p}} \int_{\{|u_k|>L\}} u_k^2 e^{\frac{1}{2}u_k^2} dx = O\left(\frac{1}{L^{2-p}}\right),$$

and (40) follows from Lemma A.9 in the appendix. Then, also considering that $Q(\tilde{u}) = 0$, hence $\tilde{u} \equiv 0$, we get

$$\lim_{k\to\infty} \|u_k\|_H^2 = 2\lim_{k\to\infty} \left(J(u_k) + \lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1\right) dx \right) = 2\lambda \int_I \left(e^{\frac{1}{2}\tilde{u}^2} - 1\right) dx = 0, \tag{41}$$

so that $u_k \to 0$ is H and the Palais-Smale condition holds in the case $c = 0$ as well.

The last case is when $c \in (0, \pi)$. We will need the following result which is analogue to Lemma 3.3 in [3].

Lemma 3.1 *Consider a bounded sequence $(u_k) \subset H$ such that u_k converges weakly and almost everywhere to a function $u \in H$. Further assume that:*

1. *there exists $c \in (0, \pi]$ such that $J(u_k) \to c$;*
2. *$\|u\|_H^2 \ge \lambda \int_I u^2 e^{\frac{1}{2}u^2} dx$;*
3. *$\sup_k \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx < \infty$;*
4. *either $u \not\equiv 0$ or $c < \pi$.*

Then

$$\lim_{k\to\infty} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \int_I u^2 e^{\frac{1}{2}u^2} dx.$$

Proof. We assume $u \not\equiv 0$ (if $u \equiv 0$ and $c < \pi$ the existence of $\varepsilon > 0$ in (42) below is obvious). We then have $Q(u) > 0$. On the other hand from assumption 2 we get

$$J(u) = \frac{1}{2}\|u\|_H^2 + Q(u) - \frac{\lambda}{2} \int_I u^2 e^{\frac{1}{2}u^2} dx \ge Q(u) > 0.$$

We also know from the weak convergence of u_k to u in H, the weakly lower semicontinuity of the norm and (40) that

$$J(u) \le \lim_{k\to\infty} J(u_k) = c,$$

where the inequality is strict, unless $u_k \to u$ strongly in $\mathcal{H}$ (in which case the proof is complete). Then one can choose $\varepsilon > 0$ so that

$$\frac{1+2\varepsilon}{\pi} < \frac{1}{c - J(u)}. \tag{42}$$

Notice now that if we set $\beta = \lambda \int_I \left(e^{\frac{1}{2}u^2} - 1 \right) dx$, then

$$\lim_{k\to\infty} \|u_k\|_H^2 = 2c + 2\beta.$$

Then multiplying (42) by $\frac{1}{2}\|u_k\|_H^2$ we have for k large enough

$$\frac{1+\varepsilon}{2\pi}\|u_k\|_H^2 \leq \tilde{p} := \frac{1+2\varepsilon}{2\pi} \lim_{k\to\infty} \|u_k\|_H^2 < \frac{c+\beta}{c-J(u)} = \left(1 - \frac{\|u\|_H^2}{2(c+\beta)}\right)^{-1}.$$

By Lemma 2.6 below applied to $v_k := \frac{u_k}{\|u_k\|_H}$, we get that the sequence $\exp(\tilde{p}\pi v_k^2)$ is bounded in $L^1(I)$, hence $e^{\frac{(1+\varepsilon)}{2}u_k^2}$ is bounded in L^1.
Now we have that

$$\begin{aligned} \int_{\{|u_k|>K\}} u_k^2 e^{\frac{1}{2}u_k^2} dx &= \int_{\{|u_k|>K\}} \left(u_k^2 e^{-\frac{\varepsilon}{2}u_k^2}\right) e^{\frac{1+\varepsilon}{2}u_k^2} dx \\ &\leq o(1) \int_{\{|u_k|>K\}} e^{\frac{1+\varepsilon}{2}u_k^2} dx \end{aligned}$$

with $o(1) \to 0$ as $K \to \infty$, and we conclude with Lemma A.9. □

We now claim

$$\|\tilde{u}\|_H^2 = \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}\tilde{u}^2} dx. \tag{43}$$

First we show that $\tilde{u} \not\equiv 0$. So for the sake of contradiction, we assume that $\tilde{u} \equiv 0$. By Lemma 3.1

$$\lim_{k\to\infty} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = 0.$$

Therefore, also using (40), we obtain $\lim_{k\to\infty} Q(u_k) = 0$. It follows that

$$0 < c = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} \left(Q(u_k) + \frac{1}{2}\langle J'(u_k), u_k \rangle \right) = 0,$$

contradiction, hence $\tilde{u} \not\equiv 0$.

Fix now $\varphi \in C_0^\infty(I) \cap H$. We have $\langle J'(u_k), \varphi \rangle \to 0$ as $k \to \infty$, since (u_k) is a Palais-Smale sequence. But, by weak convergence we have that

$$(u_k, \varphi)_H \to (\tilde{u}, \varphi)_H.$$

Now (40) implies

$$\int_I \varphi u_k e^{\frac{1}{2}u_k^2} dx \to \int_I \varphi \tilde{u} e^{\frac{1}{2}\tilde{u}^2} dx, \quad \text{for every } \varphi \in C_0^\infty(I).$$

Thus we have

$$(\tilde{u},\varphi)_H = \lambda \int_I \varphi \tilde{u} e^{\frac{1}{2}\tilde{u}^2} dx.$$

By density and the fact that $\tilde{u} e^{\frac{1}{2}\tilde{u}^2} \in L^p$ for all $p \geq 1$, we have that

$$(\tilde{u},\tilde{u})_H = \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}\tilde{u}^2} dx,$$

hence (43) is proven. Therefore, we are under the assumptions of Lemma 3.1, which yields

$$\begin{aligned}
\|\tilde{u}\|_H^2 &\leq \liminf_{k\to\infty} \|u_k\|_H^2 \\
&= 2 \liminf_{k\to\infty} \left[J(u_k) + \lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1 \right) dx \right] \\
&= 2 \liminf_{k\to\infty} \left[\frac{\lambda}{2} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx + \frac{1}{2} \langle J'(u_k), u_k \rangle \right] \\
&= \lambda \int_I \tilde{u}^2 e^{\frac{1}{2}\tilde{u}^2} dx \\
&= \|\tilde{u}\|_H^2.
\end{aligned} \tag{44}$$

By Hilbert space theory the convergence of the norms implies that $u_k \to \tilde{u}$ strongly in H, and the Palais-Smale condition is proven.

4 Proof of Theorem 1.3

We start by proving the last claim of Theorem 1.3.

Proposition 4.1 *Let u be a non-negative non-trivial solution to* (12) *for some $\lambda \in \mathbb{R}$. Then $\lambda < \lambda_1(I)$.*

Proof. Let $\varphi_1 \geq 0$ be as in Lemma A.8. Then using φ_1 as a test function in (12) (compare to (14)) yields

$$\lambda_1(I) \int_I u\varphi_1 dx = \lambda \int_I u\varphi_1 e^{\frac{1}{2}|u|^2} dx > \lambda \int_I u\varphi_1 dx.$$

Hence $\lambda < \lambda_1$. Using u as test function in (12) gives at once $\lambda > 0$. □

The rest of the section is devoted to the proof of the existence part of Theorem 1.3.

Define the Nehari manifold

$$N(J) := \left\{ u \in H \setminus \{0\}; \langle J'(u), u \rangle = 0 \right\}.$$

Since, according to (35)-(36), $J(u) = Q(u) > 0$ for $u \in N(J)$, we have

$$a(J) := \inf_{u \in N(J)} J(u) \geq 0.$$

Lemma 4.2 *We have $a(J) > 0$.*

Proof. Assume that $a(J)=0$, then there exists a sequence $(u_k) \subset N(J)$ such that

$$J(u_k) = Q(u_k) \to 0 \quad \text{ as } k \to \infty,$$

From (37) we infer

$$\sup_{k \geq 0} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx < \infty, \tag{45}$$

which, again using the fact that $u_k \in N(J)$, implies that $\|u_k\|_H$ is bounded. Thus, up to extracting a subsequence, we have that u_k weakly converges to a function $u \in H$. From the weak lower semicontinuity of Q we then get

$$0 \leq I(u) \leq \liminf_{k\to\infty} Q(u_k) = 0,$$

thus $I(u) = 0$ and (36) implies $u \equiv 0$. On the other hand, we have from (40) with $\tilde{u}$ replaced by u (which holds with the same proof thanks to (45))

$$\lim_{k\to\infty} \|u_k\|_H^2 = 2 \lim_{k\to\infty} \left\{ J(u_k) + \lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1 \right) dx \right\} = 0, \tag{46}$$

therefore we have strong convergence of u_k to 0.

Now, if we let $v_k = \frac{u_k}{\|u_k\|_H}$ and up to a subsequence we assume $v_k \to v$ weakly in H and almost everywhere, we have

$$1 = \|v_k\|_H^2 = \lim_{k\to\infty} \lambda \int_I e^{\frac{1}{2}u_k^2} v_k^2 dx = \lambda \int_I v^2 dx < \lambda_1 \int_I v^2 dx \leq 1, \tag{47}$$

where in the third equality is justified as follows: From the Sobolev imbedding $v_k \to v$ in all $L^p(I)$ for every $p \in [1,\infty)$, while from (46) and Theorem 1.1 we have $e^{\frac{1}{2}u_k^2} \in L^q(I)$ for any $q \in [1,\infty)$ and $k \geq k_0(q)$, hence from Hölder's inequality we have the desired limit. The last inequality in (47) follows from the Poincaré inequality.

Clearly (47) is a contradiction, hence $a(J) > 0$. □

Lemma 4.3 *For every $u \in H \setminus \{0\}$ there exists a unique $t = t(u) > 0$ such that $tu \in N(J)$. Moreover, if*

$$\|u\|_H^2 \leq \lambda \int_I u^2 e^{\frac{1}{2}u^2} dx, \tag{48}$$

then $t(u) \leq 1$ and $t(u) = 1$ if and only if $u \in N(J)$.

Proof. Fix $u \in H \setminus \{0\}$ and for $t \in (0,\infty)$ define the function

$$f(t) = t^2 \left(\|u\|_H^2 - \lambda \int_I u^2 e^{\frac{1}{2}t^2u^2} dx \right),$$

which can also be written as

$$f(t) = t^2 \left(\|u\|_H^2 - \lambda \int_I u^2 dx \right) - t^2 \lambda \int_I u^2 \left(e^{\frac{1}{2}t^2u^2} - 1 \right) dx.$$

Notice that $tu \in N(J)$ if and only if $f(t) = 0$.

From the inequality

$$u^2 \left(e^{\frac{1}{2}t^2u^2} - 1 \right) \geq t^2 u^4$$

we infer

$$f(t) \leq t^2 \left(\|u\|_H^2 - \lambda \int_I u^2 dx \right) - t^4 \lambda \int_I u^4 dx,$$

hence

$$\lim_{t \to +\infty} f(t) = -\infty.$$

Now notice that the function $t \mapsto \left(e^{\frac{1}{2}t^2u^2} - 1 \right)$ is monotone decreasing on $(0, \infty)$, and by Lemma 2.3 we have $\left(e^{\frac{1}{2}u^2} - 1 \right) \in L^p(I)$ for all $p \in [1, \infty)$, so that

$$u^2 \left(e^{\frac{1}{2}u^2} - 1 \right) \in L^1(I).$$

Then by the dominated convergence theorem we get

$$\lim_{t \to 0} \int_I u^2 \left(e^{\frac{1}{2}t^2u^2} - 1 \right) dx = 0.$$

So one has

$$f(t) = t^2 \left(\|u\|_H^2 - \lambda \int_I u^2 dx \right) + o(t^2) \quad \text{as } t \to 0.$$

Hence, $f(t) > 0$ for t small, since for $\lambda < \lambda_1(I)$

$$\|u\|_H^2 - \lambda \int_I u^2 dx > 0$$

(compare the proof of Lemma A.8). Therefore there exists $t = t(u)$ such that $f(t) = 0$, i.e. $tu \in N(J)$. The uniqueness of such t follows noticing that the function

$$t \mapsto \int_I u^2 e^{\frac{1}{2}t^2u^2} dx$$

is increasing. Keeping this in mind, if we assume (48), then $f(1) \leq 0$, hence $f(t) \leq 0$ for all $t \geq 1$. This implies at once that $t(u) \leq 1$ and $t(u) = 1$ if and only if $u \in N(J)$. □

Lemma 4.4 *We have $a(J) < \pi$.*

Proof. Take $w \in H$ such that $\|w\|_H = 1$ and let $t = t(w)$ be given as in Lemma 4.3 so that $tw \in N(J)$. Then

$$a(J) \leq J(tw) \leq \frac{t^2}{2} \|w\|_H^2 = \frac{t^2}{2}.$$

Now using the monotonicity of $t \mapsto \int_I w^2 e^{\frac{1}{2}t^2w^2} dx$ we have

$$\lambda \int_I w^2 e^{a(J)w^2} dx \leq \lambda \int_I w^2 e^{\frac{1}{2}t^2w^2} dx = \frac{t^2 \|w\|_H^2}{t^2} = 1.$$

Thus

$$\sup_{\|w\|_H = 1} \lambda \int_I w^2 e^{a(J)w^2} dx \leq 1,$$

and Theorem 1.1 implies that $a(J) < \pi$. □

Lemma 4.5 *Let $u \in N(J)$ be such that $J'(u) \neq 0$, then $J(u) > a(J)$.*

Proof. We choose $h \in H$ such that $\langle J'(u), h\rangle = 1$, and for $\alpha \in \mathbb{R}$ we consider the path $\sigma_t(\alpha) = \alpha u - th$, $t \in \mathbb{R}$. Remember that by Lemma 2.5 $J \in C^1(H)$. By the chain rule

$$\frac{d}{dt}J(\sigma_t(\alpha)) = -\langle J'(\sigma_t(\alpha)), h\rangle,$$

therefore, if we let $\alpha \to 1$ and $t \to 0$ we find

$$\frac{d}{dt}J(\sigma_t(\alpha))\Big|_{t=0,\alpha=1} = -\langle J'(u), h\rangle = -1.$$

Hence, there exist, $\delta > 0$ and $\varepsilon > 0$ such that for $\alpha \in [1-\varepsilon, 1+\varepsilon]$ and $t \in (0, \delta]$

$$J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u). \tag{49}$$

Now we consider the function f defined by

$$f_t(\alpha) = \|\sigma_t(\alpha)\|_H^2 - \lambda \int_I \sigma_t(\alpha)^2 e^{\frac{1}{2}\sigma_t(\alpha)^2} dx,$$

which is continuous with respect to t and α by Lemma 2.4. Notice that since $u \in N(J)$ we have

$$f_0(\alpha) = \alpha^2 \int_I u^2 \left(e^{\frac{1}{2}u^2} - e^{\frac{1}{2}\alpha^2 u^2}\right) dx$$

and $f_0(1) = 0$. Since the function $\alpha \mapsto u^2(e^{\frac{1}{2}u^2} - e^{\frac{1}{2}\alpha^2 u^2})$ is decreasing, by continiuity we can find $\varepsilon_1 \in (0, \varepsilon)$ and $\delta_1 \in (0, \delta)$ such that

$$f_t(1-\varepsilon_1) > 0, \quad f_t(1+\varepsilon_1) < 0 \quad \text{for } t \in [0, \delta_1].$$

Then if we fix $t \in (0, \delta_1]$ we can find $\alpha_t \in [1-\varepsilon_1, 1+\varepsilon_1]$ such that $f_t(\alpha_t) = 0$, i.e. $\sigma_t(\alpha_t) \in N(J)$, and from (49) we get

$$a(J) \leq J(\sigma_t(\alpha_t)) < J(\alpha_t u).$$

Since

$$\frac{d}{d\alpha}J(\alpha u) = f_0(\alpha),$$

and $f_0(\alpha) > 0$ for $\alpha < 1$ and $f_0(\alpha) < 0$ for $\alpha > 1$, we get

$$J(\alpha u) \leq J(u) \quad \text{for } \alpha \in \mathbb{R},$$

and we conclude that

$$a(J) \leq J(\sigma_t(\alpha_t)) < J(\alpha_t u) \leq J(u).$$

□

Proof of Theorem 1.3 (completed). To complete the proof it is enough to show the existence of $u_0 \in N(J)$ such that $J(u_0) = a(J)$. We consider then a minimizing sequence $(u_k) \subset N(J)$. We assume that u_k changes sign. Then since $u_k \in N(J)$ we have

$$\||u_k|\|_H^2 < \|u_k\|_H^2 = \lambda \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \lambda \int_I |u_k|^2 e^{\frac{1}{2}|u_k|^2} dx,$$

where we used (62), hence by Lemma 4.3 there exists $t_k = t(|u_k|) < 1$ such that $t_k|u_k| \in N(J)$, whence

$$J(t_k|u_k|) = Q(t_k|u_k|) < Q(|u_k|) = Q(u_k) = J(u_k),$$

where the inequality in the middle depends on the monotonicity of Q. Hence up to replacing u_k with $t_k|u_k|$ we can assume that the minimizing sequence (still denoted by (u_k)) is made of non-negative functions.

Since $J(u_k) = Q(u_k) \le C$ we infer from (37)

$$\int_I u_k^2 e^{\frac{1}{2}u_k^2} dx \le C$$

and for $u_k \in N(J)$ we get

$$\|u_k\|_H \le C.$$

Thus up to a subsequence u_k weakly converges to a function $u_0 \in H$, and up to a subsequence the convergence is also almost everywhere.

We claim that $u_0 \not\equiv 0$. Indeed if $u_0 \equiv 0$, then from (40), we have that $\left(e^{\frac{1}{2}u_k^2} - 1\right) \to 0$ in $L^1(I)$. Thus

$$\lim_{k\to\infty} \|u_k\|_H^2 = 2 \lim_{k\to\infty} \left[J(u_k) + \lambda \int_I \left(e^{\frac{1}{2}u_k^2} - 1\right) dx\right] = 2a(J).$$

Then according to Theorem 1.1, since $a(J) < \pi$ we have that $e^{\frac{1}{2}u_k^2}$ is bounded in L^p for some $p > 1$, hence weakly converging in $L^p(I)$ to $e^{\frac{1}{2}u_0^2}$. From the compactness of the Sobolev embeddings (see [11, Theorem 7.1]), up to a subsequence $u_k^2 \to u_0^2$ strongly in $L^{p'}(I)$, hence

$$\lim_{k\to\infty} \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx = 0,$$

and with Lemma 4.2 and (35) one gets

$$0 < a(J) = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} Q(u_k) = 0,$$

which is a contradiction.

Next we claim that

$$\|u_0\|_H^2 \le \lambda \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx.$$

So we assume by contradiction that this is not the case, i.e.

$$\|u_0\|_H^2 > \lambda \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx.$$

Then from Lemma 3.1, Lemma 4.4 and the weak convergence, we have that

$$\|u_0\|_H^2 \le \liminf_{k\to\infty} \|u_k\|_H^2 = \liminf_{k\to\infty} \lambda \int_I u_k^2 e^{\frac{1}{2}u_k^2} dx = \lambda \int_I u_0^2 e^{\frac{1}{2}u_0^2} dx,$$

again leading to a contradiction.

From Lemma 4.3, we have that there exists $0 < t \le 1$ such that $tu_0 \in N(J)$. Taking Remark 5 into account we get

$$a(J) \le J(tu_0) = Q(tu_0) \le Q(u_0) \le \liminf_{k\to\infty} Q(u_k) = a(J).$$

It follows that $t = 1$, since otherwise the second inequality above would be strict. Then $u_0 \in N(J)$ and $J(u_0) = a(J)$. By Lemma 4.5 we have $J'(u_0) = 0$ □

5 Proof of Theorem 1.5

For $u \in H^{\frac{1}{2},2}(\mathbb{R})$ we set $|u|^* : \mathbb{R} \to \mathbb{R}_+$ to be its non-increasing symmetric rearrangement, whose definition we shall now recall. For a measeurable set $A \subset \mathbb{R}$, we define

$$A^* = \{x \in \mathbb{R} : 2|x| < |A|\}.$$

The set A^* is symmetric (with respect to 0) and $|A^*| = |A|$. For a non-negative measurable function f, such that

$$|\{x \in \mathbb{R} : f(x) > t\}| < \infty \quad \text{for every } t > 0,$$

we define the symmetric non-increasing rearrangement of f by

$$f^*(x) = \int_0^\infty \chi_{\{y\in\mathbb{R}:f(y)>t\}^*}(x)dt.$$

Notice that f^* is even, i.e. $f^*(x) = f^*(-x)$ and non-increasing (on $[0,\infty)$).

We will state here the two properties that we shall use in the proof of Theorem 1.5.

Proposition 5.1 *Given a measurable function $F : \mathbb{R} \to \mathbb{R}$ and a non-negative measurable function $f : \mathbb{R} \to \mathbb{R}$ it holds*

$$\int_{\mathbb{R}} F(f)dx = \int_{\mathbb{R}} F(f^*)dx.$$

The following Pólya-Szegõ type inequality can be found e.g. in [17] (Inequality (3.6)) or [26].

Theorem 5.2 *Let $u \in H^{s,2}(\mathbb{R})$ for $0 < s < 1$. Then*

$$\int_{\mathbb{R}} |(-\Delta)^s |u|^*|^2 dx \le \int_{\mathbb{R}} |(-\Delta)^s u|^2 dx.$$

Now given $u \in H^{\frac{1}{2},2}(\mathbb{R})$, from Proposition 5.1 we get

$$\int_{\mathbb{R}} \left(e^{\pi u^2} - 1\right) dx = \int_{\mathbb{R}} \left(e^{\pi(|u|^*)^2} - 1\right) dx, \quad \||u|^*\|_{L^2} = \|u\|_{L^2},$$

and according to Theorem 5.2

$$\||u|^*\|^2_{H^{\frac{1}{2},2}(\mathbb{R})} = \||u|^*\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}}|u|^*|^2 dx \le \|u\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} |(-\Delta)^{\frac{1}{4}} u|^2 dx = \|u\|^2_{H^{\frac{1}{2},2}(\mathbb{R})}.$$

Therefore in the rest of the proof of (19) we may assume that $u \in H^{\frac{1}{2},2}(\mathbb{R})$ is even, non-increasing on $[0,\infty)$, and $\|u\|_{H^{\frac{1}{2},2}(\mathbb{R})} \le 1$.

We write

$$\int_{\mathbb{R}} \left(e^{\pi u^2} - 1\right) dx = \int_{\mathbb{R}\setminus I} \left(e^{\pi u^2} - 1\right) dx + \int_I \left(e^{\pi u^2} - 1\right) dx =: (I) + (II),$$

where $I = (-1/2, 1/2)$. We start by bounding (I). By monotone convergence

$$(I) = \sum_{k=1}^{\infty} \int_{I^c} \pi^k \frac{u^{2k}}{k!} dx.$$

Since u is even and non-increasing, for $x \neq 0$ we have

$$u^2(x) \leq \frac{1}{2|x|} \int_{-|x|}^{|x|} u^2(y)dy \leq \frac{\|u\|_{L^2}^2}{2|x|}, \tag{50}$$

hence for $k \geq 2$ we bound

$$\int_{I^c} u^{2k} dx \leq 2^{1-k} \|u\|_{L^2(\mathbb{R})}^{2k} \int_{\frac{1}{2}}^{\infty} \frac{1}{x^k} dx = \frac{\|u\|_{L^2(\mathbb{R})}^{2k}}{(k-1)}.$$

It follows that

$$\sum_{k=2}^{\infty} \int_{I^c} \pi^k \frac{u^{2k}}{k!} dx \leq \sum_{k=2}^{\infty} \frac{(\pi \|u\|_{L^2}^2)^k}{k!(k-1)}.$$

Thus, since $\|u\|_{L^2(\mathbb{R})} \leq 1$ we estimate

$$(I) \leq \pi \|u\|_{L^2(\mathbb{R})}^2 \left(1 + \sum_{k=1}^{\infty} \frac{\left(\pi \|u\|_{L^2(\mathbb{R})}^2\right)^k}{(k+1)!k} \right) \leq C.$$

We shall now bound (II). We define the function $v : \mathbb{R} \to \mathbb{R}$ as follows

$$v(x) = \begin{cases} u(x) - u(\frac{1}{2}) & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2}. \end{cases}$$

Then with (50) and the estimate $2a \leq a^2 + 1$, we find

$$\begin{aligned} u^2 &\leq v^2 + 2vu(\tfrac{1}{2}) + u(\tfrac{1}{2})^2 \\ &\leq v^2 + 2v\|u\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}^2 \\ &\leq v^2 + v^2 \|u\|_{L^2(\mathbb{R})}^2 + 1 + \|u\|_{L^2(\mathbb{R})}^2 \\ &\leq v^2 \left(1 + \|u\|_{L^2(\mathbb{R})}^2\right) + 2. \end{aligned} \tag{51}$$

Now, recalling that u is decreasing we have

$$\begin{aligned} \int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2} dy &= \int_I \frac{(u(x) - u(y))^2}{(x-y)^2} dy + \int_{I^c} \frac{(u(x) - u(\frac{1}{2}))^2}{(x-y)^2} dy \\ &\leq \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x-y)^2} dy < \infty \quad \text{for a.e. } x \in I = [-\tfrac{1}{2}, \tfrac{1}{2}], \end{aligned}$$

the last inequality coming from Proposition A.1 and Fubini's theorem. Similarly for a.e. $x \in I^c$

$$\begin{aligned} \int_{\mathbb{R}} \frac{(v(x) - v(y))^2}{(x-y)^2} dy &= \int_I \frac{(u(\frac{1}{2}) - u(y))^2}{(x-y)^2} dy \\ &\leq \int_I \frac{(u(x) - u(y))^2}{(x-y)^2} dy \\ &\leq \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{(x-y)^2} dy. \end{aligned}$$

Integrating with respect to x we obtain

$$
\begin{aligned}
\|(-\Delta)^{\frac{1}{4}}v\|^2_{L^2(\mathbb{R})} &= \frac{1}{C_s^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(v(x)-v(y))^2}{(x-y)^2}dydx \\
&\le \frac{1}{C_s^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(u(x)-u(y))^2}{(x-y)^2}dydx \\
&= \|(-\Delta)^{\frac{1}{4}}u\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where C_s is as in Proposition A.1 below. Thus

$$
\|(-\Delta)^{\frac{1}{4}}v\|^2_{L^2(\mathbb{R})} \le \|(-\Delta)^{\frac{1}{4}}u\|^2_{L^2(\mathbb{R})} \le 1 - \|u\|^2_{L^2(\mathbb{R})}.
$$

Therefore, if we set $w = v\sqrt{1+\|u\|^2_{L^2(\mathbb{R})}}$, we have

$$
\|(-\Delta)^{\frac{1}{4}}w\|^2_{L^2(\mathbb{R})} \le \left(1+\|u\|^2_{L^2(\mathbb{R})}\right)\left(1-\|u\|^2_{L^2(\mathbb{R})}\right) \le 1,
$$

hence, using the Moser-Trudinger inequality on the interval $I = (-1/2, 1/2)$ (Theorem 1.1), one has

$$
\int_I e^{\pi w^2}dx < C,
$$

and using (51)

$$
\int_I e^{\pi u^2}dx \le e^{2\pi}\int_I e^{\pi w^2}dx \le C,
$$

which completes the proof of (19).

It remains to prove (20). Given $q > 2$ consider the function

$$
f = f_q := \frac{1}{2q\sqrt{|x|}}\chi_{\{x\in\mathbb{R}: r<|x|<\delta\}}, \quad \delta := \frac{1}{q}, \quad r := \frac{1}{qe^q}.
$$

Notice that $\|f\|^2_{L^2(\mathbb{R})} = (2q)^{-1}$. Fix a smooth even function $\psi : \mathbb{R} \to [0,1]$ with $\psi \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\mathrm{supp}(\psi) \subset (-1,1)$. For $x \in \mathbb{R}$ we set

$$
u(x) = \psi(x)(F_{\frac{1}{4}} * f)(x),
$$

where $F_{\frac{1}{4}}(x) = (2\pi|x|)^{-\frac{1}{2}}$ is as in Lemma 2.1. Clearly $u \equiv 0$ in $\mathbb{R}\setminus I$, and u is non-negative and even everywhere.

In the rest of the proof $s = \frac{1}{4}$. Notice that $(-\Delta)^s(F_s * f) = f$. This follows easily from Lemma 2.1 and the properties of the Fourier transform, see e.g. [19, Corollary 5.10]. Then we compute

$$
(-\Delta)^s u = f + (-\Delta)^s[(\psi-1)(F_s * f)] =: f + v, \tag{52}
$$

and set $g(x,y) = (\psi-1)(x)F_s(x-y)$. Notice that g is smooth in $\mathbb{R}\times(-\frac{1}{2},\frac{1}{2})$. We write

$$
\begin{aligned}
v(x) &= (-\Delta)^s\int_{\mathbb{R}} g(x,y)f(y)dy \\
&= \int_{\{r<|y|<\delta\}}(-\Delta_x)^s g(x,y)f(y)dy,
\end{aligned}
$$

where we used Proposition A.7 and Fubini's theorem. With Jensen's inequality

$$
\begin{aligned}
\|v\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left| \int_{\{r<|y|<\delta\}} (-\Delta_x)^s g(x,y) f(y) dy \right|^2 dx \\
&\le 2(\delta - r) \int_{\{r<|y|<\delta\}} f(y)^2 \int_{\mathbb{R}} |(-\Delta_x)^s g(x,y)|^2 \, dx dy \\
&\le 2\delta \|f\|^2_{L^2(\mathbb{R})} \sup_{|y|\in[r,\delta]} \int_{\mathbb{R}} |(-\Delta_x)^s g(x,y)|^2 \, dx \\
&\le C(\delta q^{-1}) = O(q^{-2}),
\end{aligned} \tag{53}
$$

where we used that

$$
\sup_{|y|\in[r,\delta]} \int_{\mathbb{R}} |(-\Delta_x)^s g(x,y)|^2 \, dx < \infty.
$$

This in turn can be seen noticing that $(-\Delta_x)^s g(x,y)$ is smooth, hence bounded on $[-R,R]\times[r,\delta]$ for every R, and for $|x|$ large and $r \le |y| \le \delta$, using Proposition A.7

$$
\begin{aligned}
(-\Delta_x)^s g(x,y) &= C_s \int_{\mathbb{R}} \frac{-F_s(x-y) - (\psi(z)-1)F_s(z-y)}{|z-x|^{1+2s}} dz \\
&= C_s \int_{-1}^{1} \frac{-\psi(z)F_s(z-y)}{|z-x|^{1+2s}} dz - (-\Delta)^s F_s(x-y) \\
&= O(|x|^{-1-2s}) \quad \text{uniformly for } |y| \le \frac{1}{2},
\end{aligned}
$$

where we also used that $(-\Delta)^s F_s = 0$ away from the origin, see Lemma 2.1. Actually, with the same estimates we get

$$
\begin{aligned}
\int_{-\delta}^{\delta} |v|^2 dx &\le 2(\delta - r)\|f\|^2_{L^2(\mathbb{R})} \int_{-\delta}^{\delta} \sup_{(x,y)\in[-\delta,\delta]^2} |(-\Delta_x)^s g(x,y)|^2 \, dx \\
&\le C\delta^2 \|f\|^2_{L^2(\mathbb{R})} = O(q^{-3}).
\end{aligned}
$$

Therefore, using Hölder's inequality and that $\mathrm{supp}(f) \subset [-\delta,\delta]$ we get

$$
\|(-\Delta)^s u\|^2_{L^2(\mathbb{R})} = \|f\|^2_{L^2} + \|v\|^2_{L^2} + 2\int_{-\delta}^{\delta} f v dx = \frac{1}{2q} + O(q^{-2}), \quad \text{as } q \to \infty. \tag{54}
$$

We now estimate u. For $0 < x < r$, with the change of variable $\tilde{y} = \sqrt{\frac{y}{x}}$ we have

$$
\begin{aligned}
u(x) &= \frac{1}{2q\sqrt{2\pi}} \int_r^{\delta} \left(\frac{1}{\sqrt{(y-x)y}} + \frac{1}{\sqrt{(y+x)y}} \right) dy \\
&= \frac{1}{q\sqrt{2\pi}} \int_{\sqrt{\frac{r}{x}}}^{\sqrt{\frac{\delta}{x}}} \left(\frac{1}{\sqrt{\tilde{y}^2-1}} + \frac{1}{\sqrt{\tilde{y}^2+1}} \right) d\tilde{y} \\
&= \frac{1}{q\sqrt{2\pi}} \left(\log(\sqrt{\tilde{y}^2-1} + \tilde{y}) \Big|_{\sqrt{\frac{r}{x}}}^{\sqrt{\frac{\delta}{x}}} + \log(\sqrt{\tilde{y}^2+1} + \tilde{y}) \Big|_{\sqrt{\frac{r}{x}}}^{\sqrt{\frac{\delta}{x}}} \right) \\
&= \frac{1}{\sqrt{2\pi}} + O(q^{-1}).
\end{aligned}
$$

Similarly for $r < x < \delta$ we write

$$u(x) \leq \frac{1}{q\sqrt{2\pi}} \left[\int_r^x \frac{dy}{\sqrt{(x-y)y}} + \int_x^\delta \frac{dy}{\sqrt{(x-y)y}} \right]$$
$$= \frac{2}{q\sqrt{2\pi}} \left[\int_{\sqrt{\frac{r}{x}}}^1 \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} + \log(\sqrt{\tilde{y}^2-1}+\tilde{y})\Big|_1^{\sqrt{\frac{\delta}{x}}} \right]$$
$$= \frac{1}{q\sqrt{2\pi}} \left[\log\left(\frac{\delta}{x}\right) + O(1) \right],$$

since $\int_0^1 \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} < \infty$.
When $\delta < x < 1$ similar to the previous computation, and recalling that $0 \leq \psi \leq 1$,

$$u(x) \leq \frac{1}{q\sqrt{2\pi}} \int_r^\delta \frac{dy}{\sqrt{(x-y)y}} = \frac{2}{q\sqrt{2\pi}} \int_{\sqrt{\frac{r}{x}}}^{\sqrt{\frac{\delta}{x}}} \frac{d\tilde{y}}{\sqrt{1-\tilde{y}^2}} = O(q^{-1}).$$

Thus

$$\begin{cases} u(x) = \frac{1}{\sqrt{2\pi}} + O(q^{-1}) & \text{for } 0 < x < r \\ u(x) \leq \frac{2}{q\sqrt{2\pi}} \log\left(\frac{\delta}{x}\right) + O(q^{-1}) & \text{for } r < x < \delta \\ u(x) = O(q^{-1}) & \text{for } \delta < x < 1. \end{cases} \tag{55}$$

Of course the same bounds hold for $x < 0$ since u is even. We now want to estimate $\|u\|^2_{L^2(\mathbb{R})}$. We have

$$\int_0^r u^2 dx = r\left(\frac{1}{2\pi} + O(q^{-1})\right) = O(q^{-2}).$$

For $x \in [r, \delta]$ we have from (55)

$$u(x)^2 \leq \frac{C}{q^2}\left(\log^2\left(\frac{\delta}{x}\right) + \log\left(\frac{\delta}{x}\right) + 1\right) \leq \frac{2C}{q^2}\left(\log^2\left(\frac{\delta}{x}\right) + 1\right).$$

Then, since

$$\int_r^\delta \log^2\left(\frac{\delta}{x}\right) dx = x\left(\log^2\left(\frac{\delta}{x}\right) + 2\log\left(\frac{\delta}{x}\right) + 2\right)\Bigg|_r^\delta \leq 2\delta = O(q^{-1}),$$

we bound

$$\int_r^\delta u^2 dx = O(q^{-3}).$$

Finally, still using (55),

$$\int_\delta^1 u^2 dx = O(q^{-2}).$$

Also considering (54), we conclude

$$\|u\|^2_{L^2(\mathbb{R})} = 2\|u\|^2_{L^2([0,1])} = O(q^{-2}), \quad \|u\|^2_{H^{\frac{1}{2},2}(\mathbb{R})} = \frac{1}{2q} + O(q^{-2}). \tag{56}$$

Setting $w = w_q := u\|u\|^{-1}_{H^{\frac{1}{2},2}(\mathbb{R})}$, and using (55) and (56), we conclude

$$\int_{\mathbb{R}} |w|^{\frac{a}{2}} \left(e^{\pi w^2} - 1\right) dx \geq \int_{-r}^{r} \left(\frac{q + O(1)}{\pi}\right)^{\frac{a}{2}} \left(e^{q+O(1)} - 1\right) dx$$
$$\geq Crq^{\frac{a}{2}} e^{q} = Cq^{\frac{a}{2}-1} \to \infty,$$

as $q \to \infty$ for any $a > 2$. □

A Some useful results

We define

$$W^{s,p}(\mathbb{R}) := \left\{ u \in L^p(\mathbb{R}) : [u]^p_{W^{s,p}(\mathbb{R})} := \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^{1+sp}} dxdy < \infty \right\}. \tag{57}$$

Proposition A.1 *For $s \in (0,1)$ we have, $[u]_{W^{2,s}(\mathbb{R})} < \infty$ if and only if $(-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R})$, and in this case*

$$[u]_{W^{s,2}(\mathbb{R})} = C_s \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R})},$$

where $[u]_{W^{2,s}(\mathbb{R})}$ is as in (57) *and C_s depends only on s. In particular $H^{s,2}(\mathbb{R}) = W^{s,2}(\mathbb{R})$.*

Proof. See e.g. Proposition 4.4 in [11]. □

Define the bilinear form

$$\mathcal{B}_s(u, v) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{1+2s}} dxdy, \quad \text{for } u, v \in H^{s,2}(\mathbb{R}),$$

where the double integral is well defined thanks to Hölder's inequality and Proposition A.1.

The following simple and well-known existence result proves useful. A proof can be found (in a more general setting) in [13].

Theorem A.2 *Given $s \in (0,1)$, $f \in L^2(I)$ and $g : \mathbb{R} \to \mathbb{R}$ such that*

$$\int_I \int_{\mathbb{R}} \frac{(g(x) - g(y))^2}{|x - y|^{1+2s}} dxdy < \infty, \tag{58}$$

there exists a unique function $u \in \tilde{H}^{s,2}(I) + g$ solving the problem

$$\mathcal{B}_s(u, v) = \int_{\mathbb{R}} f v dx \quad \textit{for every } v \in \tilde{H}^{s,2}(I). \tag{59}$$

Moreover such u satisfies $(-\Delta)^s u = \frac{C_s}{2} f$ in I in the sense of distributions, i.e.

$$\int_{\mathbb{R}} u(-\Delta)^s \varphi dx = \frac{C_s}{2} \int_{\mathbb{R}} f \varphi dx \quad \textit{for every } \varphi \in C^\infty_c(I), \tag{60}$$

where C_s is the constant in Proposition A.7.

The following version of the maximum principle is a special case of Theorem 4.1 in [13].

Proposition A.3 *Let $u \in \tilde{H}^{s,2}(I)+g$ solve* (59) *for some $f \in L^2(I)$ with $f \geq 0$ and g satisfying* (58) *and $g \geq 0$ in I^c. Then $u \geq 0$.*

Proof. From Proposition A.1 it easily follows $v := \min\{u, 0\} \in \tilde{H}^{s,2}(I)$. Then according to (59) we have

$$0 \geq \mathcal{B}_s(u,v) = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u^+(x)+v(x)-u^+(y)-v(y))(v(x)-v(y))}{|x-y|^{1+2s}}dxdy$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(v(x)-v(y))^2}{|x-y|^{1+2s}}dxdy,$$

where we used that $u^+v = 0$. It follows at once that $v \equiv 0$, hence $u \geq 0$. □

Proposition A.4 *Let $u \in \tilde{H}^{s,2}(I)$ be as in Theorem A.2 (with $g = 0$), where we further assume $f \in L^\infty(I)$. Then*

$$|u(x)| \leq C\|f\|_{L^\infty(I)}(\mathrm{dist}(x, \partial I))^s$$

for every $x \in I$. In particular u is bounded in I and continuous at ∂I.

Proof. This proof is inspired from [27], where a much stronger result is proven, i.e. $u/(\mathrm{dist}(\cdot, \partial I))^s \in C^\alpha(\bar{I})$ for some $\alpha > 0$.

To prove the proposition we assume as usual that $I = (-1, 1)$ and recall that

$$w(x) := \begin{cases} (1-|x|^2)^s & \text{for } x \in (-1,1) \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

belongs to $\tilde{H}^{s,2}(I)$ and solves $(-\Delta)^s w = \gamma_s$ for a positive constant γ_s, in the sense of Theorem A.2, i.e. (59) holds with $u = w$ and $f \equiv \gamma_s$ (see e.g. [14]). Then

$$-\frac{(-\Delta)^s w}{\gamma_s} \leq \frac{(-\Delta)^s u}{\|f\|_{L^\infty(I)}} \leq \frac{(-\Delta)^s w}{\gamma_s}$$

and Proposition A.3 gives at once

$$-\frac{\|f\|_{L^\infty(I)}}{\gamma_s} w \leq u \leq \frac{\|f\|_{L^\infty(I)}}{\gamma_s} w \quad \text{in } I.$$

We conclude noticing that $0 \leq w(x) \leq 2^s(\mathrm{dist}(x, \partial I))^s$. □

The following density result is known for an arbitrary domain in $\mathbb{R}^n$. On the other hand, its proof is quite complex in such a generality, hence we provide a short elementary proof which fits the case of an interval.

Lemma A.5 *For $s \in (0,1)$ and $p \in [1,\infty)$ the sets $C_c^\infty(I)$ ($I \Subset \mathbb{R}$ is a bounded interval) is dense in $\tilde{H}^{s,p}(I)$.*

Proof. Without loss of generality we consider $I = (-1, 1)$. Given $u \in \tilde{H}^{s,p}(I)$ and $\lambda > 1$, set $u_\lambda(x) := u(\lambda x)$. We claim that $u_\lambda \to u$ in $\tilde{H}^{s,p}(I)$ as $\lambda \to 1$. Indeed

$$\|u_\lambda - u\|^p_{H^{s,p}(\mathbb{R})} = \|u - u_\lambda\|^p_{L^p(\mathbb{R})} + \|\lambda^s f_\lambda - f\|^p_{L^p(\mathbb{R})},$$

where $f = (-\Delta)^{\frac{s}{2}} u$ and $f_\lambda(x) := f(\lambda x)$. Since $f \in L^p(\mathbb{R})$ it follows that $\|\lambda^s f_\lambda - f\|_{L^p(\mathbb{R})} \to 0$ as $\lambda \to 1$, since this is obviously true for $f \in C^0(\mathbb{R})$ with compact support, and for a general $f \in L^p(\mathbb{R})$ it can be proven by approximation in the following standard way. Given $\varepsilon > 0$ choose $f_\varepsilon \in C^0(\mathbb{R})$ with compact support and $\|f_\varepsilon - f\|_{L^p(\mathbb{R})} \le \varepsilon$. Then by the Minkowski inequality

$$\begin{aligned}\|\lambda^s f_\lambda - f\|_{L^p(\mathbb{R})} &\le \|\lambda^s f_\lambda - \lambda^s f_{\varepsilon,\lambda}\|_{L^p(\mathbb{R})} + \|\lambda^s f_{\varepsilon,\lambda} - f_\varepsilon\|_{L^p(\mathbb{R})} + \|f_\varepsilon - f\|_{L^p(\mathbb{R})}\\ &\le \varepsilon \lambda^{s-\frac{1}{p}} + \|\lambda^s f_{\varepsilon,\lambda} - f_\varepsilon\|_{L^p(\mathbb{R})} + \varepsilon,\end{aligned}$$

and it suffices to let $\lambda \to 1$ and $\varepsilon \to 0$. Similarly $\|u - u_\lambda\|^p_{L^p(\mathbb{R})} \to 0$ as $\lambda \to 1$.

Now given $\delta > 0$ fix $\lambda > 1$ such that $\|u_\lambda - u\|_{H^{s,p}(\mathbb{R})} < \delta$ and let ρ be a mollifying kernel, i.e. a smooth non-negative function supported in I with $\int_I \rho dx = 1$. Also set $\rho_\varepsilon(x) := \varepsilon^{-1}\rho(\varepsilon^{-1}x)$. Then noticing that u_λ is supported in $[-\lambda^{-1}, \lambda^{-1}] \Subset I$, for $\varepsilon > 0$ sufficiently small we have that $\rho_\varepsilon * u_\lambda \in C^\infty_c(I)$. To conclude the proof notice that

$$\rho_\varepsilon * u_\lambda \to u_\lambda \text{ in } \tilde{H}^{s,p}(I) \text{ as } \varepsilon \to 0,$$

since

$$(-\Delta)^{\frac{s}{2}}(\rho_\varepsilon * u_\lambda) = \rho_\varepsilon * (-\Delta)^{\frac{s}{2}} u_\lambda \to (-\Delta)^{\frac{s}{2}} u_\lambda \quad \text{in } L^p(\mathbb{R}) \text{ as } \varepsilon \to 0,$$

and use the Minkowski inequality to conclude that $\rho_\varepsilon * u_\lambda \to u$ in $\tilde{H}^{s,p}(I)$ as $\varepsilon \to 0$ and $\lambda \downarrow 1$.
□

Proposition A.6 *Let $I \Subset \mathbb{R}$ be a bounded interval and $s \in (0,1)$. Let $u \in L_s(\mathbb{R})$ satisfy $(-\Delta)^s u \ge 0$ in I (i.e. $\langle u, (-\Delta)^s \varphi\rangle \ge 0$ for every $\varphi \in C^\infty_c(I)$ with $\varphi \ge 0$), $u \ge 0$ in I^c and*

$$\liminf_{x\to\partial I} u(x) \ge 0. \tag{61}$$

Then $u \ge 0$ in I. More precisely, either $u > 0$ in I, or $u \equiv 0$ in $\mathbb{R}$.

Proof. This is a special case of Proposition 2.17 in [29]. □

Remark 6 *The statement of Proposition 2.17 in [29] is slightly different, since it assumes u to be lower-semicontinuous in $\bar{I}$. On the other hand, lower semicontinuity inside I already follows from [29, Prop. 2.15]. What really matters is condition* (61)*. That an assumption of this kind (possibly weaker) is needed follows for instance from the example of Lemma 3.2.4 in [1].*

The following way of computing the fractional Laplacian of a sufficiently regular function is often used.

Proposition A.7 *For an interval $I \subset \mathbb{R}$, let $s \in (0, \frac{1}{2})$ and $u \in L_s(\mathbb{R}) \cap C^{0,\alpha}(I)$ for some $\alpha \in (2s, 1]$, or $s \in [\frac{1}{2}, 1)$ and $u \in L_s(\mathbb{R}) \cap C^{1,\alpha}(I)$ for some $\alpha \in (2s-1, 1]$. Then $((-\Delta)^s u)|_I \in C^0(I)$ and*

$$(-\Delta)^s u(x) = C_s P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy := C_s \lim_{\varepsilon\to 0} \int_{\mathbb{R}\setminus[x-\varepsilon, x+\varepsilon]} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy$$

for every $x \in I$. This means that

$$\langle (-\Delta)^s u, \varphi\rangle = C_s \int_{\mathbb{R}} \varphi(x)\, P.V. \int_{\mathbb{R}} \frac{u(x)-u(y)}{|x-y|^{1+2s}} dy\, dx, \quad \textit{for every } \varphi \in C^\infty_c(I).$$

Proof. See e.g. [29, Prop. 2.4] □

Lemma A.8 *Let $\varphi_1 \in H = \tilde{H}^{\frac{1}{2},2}(I)$ be an eigenfunction corresponding to the first eigenvalue $\lambda_1(I)$ of $(-\Delta)^{\frac{1}{2}}$ on I. Then φ_1 does not change sign and the corresping eigenspace has dimension 1.*

Proof. Recall that the first eigenvalue $\lambda_1(I)$ can be characterised by minimizing the following functional

$$F(u) = \frac{\|u\|_H^2}{\int_I u^2 dx},$$

that is,

$$\lambda_1(I) = \min_{u \in H\setminus\{0\}} F(u).$$

On the other hand using Proposition A.1 we get that for any $u \in H$

$$\|u\|_H^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2} dxdy \geq \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(|u(x)|-|u(y)|)^2}{(x-y)^2} dxdy = \||u|\|_H^2, \tag{62}$$

hence, $F(|u|) \leq F(u)$, and $F(u) = F(|u|)$ if and only if u is non-negative or non-positive. Therefore if $F(\varphi_1) = \lambda_1$, then φ_1 does not change sign. Any other eigenfunction corresponding to λ_1 must also have fixed sign, hence it cannot be orthogonal to φ_1, therefore it is a multiple of φ_1. □

Lemma A.9 *Consider a sequence of non-negative functions $(f_k) \subset L^1(I)$ with $f_k \to f$ a.e. and with*

$$\int_{\{f_k > L\}} f_k dx = o(1), \tag{63}$$

with $o(1) \to 0$ as $L \to \infty$ uniformly with respect to k. Then $f_k \to f$ in $L^1(I)$.

Proof. From the dominated convergence theorem

$$\min\{f_k, L\} \to \min\{f, L\} \quad \text{in } L^1(I),$$

and the convergence of f_k to f in L^1 follows at once from (63) and the triangle inequality. □

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