# Blow-up behaviour of a fractional Adams-Moser-Trudinger type inequality in odd dimension

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**Abstract**

Given a smoothly bounded domain $\Omega \Subset \mathbb{R}^n$ with $n \geq 1$ odd, we study the blow-up of bounded sequences $(u_k) \subset H^{\frac{n}{2}}_{00}(\Omega)$ of solutions to the non-local equation

$$(-\Delta)^{\frac{n}{2}} u_k = \lambda_k u_k e^{\frac{n}{2} u_k^2} \quad \text{in } \Omega,$$

where $\lambda_k \to \lambda_\infty \in [0, \infty)$, and $H^{\frac{n}{2}}_{00}(\Omega)$ denotes the Lions-Magenes spaces of functions $u \in L^2(\mathbb{R}^n)$ which are supported in $\Omega$ and with $(-\Delta)^{\frac{n}{4}} u \in L^2(\mathbb{R}^n)$. Extending previous works of Druet, Robert-Struwe and the second author, we show that if the sequence $(u_k)$ is not bounded in $L^\infty(\Omega)$, a suitably rescaled subsequence $\eta_k$ converges to the function $\eta_0(x) = \log\left(\frac{2}{1+|x|^2}\right)$, which solves the prescribed non-local $Q$-curvature equation

$$(-\Delta)^{\frac{n}{2}} \eta = (n-1)! e^{n\eta} \quad \text{in } \mathbb{R}^n$$

recently studied by Da Lio-Martinazzi-Rivière when $n = 1$, Jin-Maalaoui-Martinazzi-Xiong when $n = 3$, and Hyder when $n \geq 5$ is odd. We infer that blow-up can occur only if $\Lambda := \limsup_{k\to\infty} \|(-\Delta)^{\frac{n}{4}} u_k\|^2_{L^2} \geq \Lambda_1 := (n-1)!|S^n|$.

## 1 Introduction

In this paper we study some compactness properties of the embedding of $H^{\frac{n}{2}}_{00}(\Omega)$ into Orlicz spaces, where $\Omega$ is a smoothly bounded domain in $\mathbb{R}^n$. In order to introduce the relevant function spaces we start by recalling various definitions of fractional Laplacians.

Let $\mathcal{S}(\mathbb{R}^n)$ denote Schwarz space of smooth and rapidly decreasing functions on $\mathbb{R}^n$. For a function $u \in \mathcal{S}(\mathbb{R}^n)$ and for $s \in (0, \infty)$, we define

$$(-\Delta)^{\frac{s}{2}} u := (|\cdot|^{2s} u^\wedge)^\vee.$$

Here the Fourier transform is defined via

$$u^\wedge(\xi) \equiv \mathcal{F}u(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, u(x) dx.$$

and $u^\vee$ is its inverse.

For $s \in (0, 2)$ one can also prove (see e.g. [12]) that for a certain constant $c_{n,s} \in \mathbb{R}$

$$(-\Delta)^{\frac{s}{2}} u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x+h) - u(x)}{|h|^{n+s}}\, dh.$$

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In order to define the operator $(-\Delta)^s$ on a space larger than the Schwarz space, set for $s > 0$

$$L_s(\mathbb{R}^n) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1+|x|^{n+s}} dx < \infty \right\}. \tag{1}$$

Then for $u \in L_s(\mathbb{R}^n)$ we can define $(-\Delta)^s u$ as a tempered distribution as follows:

$$\langle (-\Delta)^{\frac{s}{2}} u, \varphi \rangle := \int_{\mathbb{R}^n} u (-\Delta)^{\frac{s}{2}} \varphi dx, \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

This is due to the fact that for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ one has $(1+|x|^{n+s})|(-\Delta)^{\frac{s}{2}}\varphi(x)| \le C$ for a constant depending on $\varphi$ but not on $x$, see [18, Proposition 2.2] and [32].

We can now define the space

$$H^s(\mathbb{R}^n) := \{ u \in L^2(\mathbb{R}^n) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n) \},$$

endowed with the norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \|u\|^2_{L^2(\mathbb{R}^n)} + \|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^n)},$$

where with the expression $(-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^n)$ we mean that the tempered distribution $(-\Delta)^{\frac{s}{2}} u$ can be represented by a square-summable function.

Given a bounded set $\Omega \Subset \mathbb{R}^n$ we also define its subspace

$$H^s_{00}(\Omega) := \{ u \in H^s(\mathbb{R}^n) : u \equiv 0 \text{ on } \Omega^c \}.$$

In particular we will consider the space $X(\Omega) := H^{\frac{n}{2}}_{00}(\Omega)$ for $n$ is odd, endowed with the norm

$$\|u\|^2_X := \|(-\Delta)^{\frac{n}{4}} u\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^n |\hat{u}(\xi)|^2 d\xi.$$

The norms $\|\cdot\|_X$ and $\|\cdot\|_{H^{\frac{n}{2}}(\mathbb{R}^n)}$ are equivalent on $H^{\frac{n}{2}}_{00}(\Omega)$ by a Poincaré-type inequality. The space $H^s_{00}(\Omega)$ is also known as Lions-Magenes space, and is sometimes denoted by $\tilde{H}^s(\Omega)$, or even $L^{s,2}_0(\Omega)$.

We recall the following fractional version of the Adams-Moser-Trudinger inequality, see [23, Theorem 1]:

**Theorem 1.1.** *For any integer $n > 0$ there exists a constant $C_n > 0$ such that for every open set $\Omega \subset \mathbb{R}^n$ with finite volume $|\Omega|$ one has*

$$\sup_{u \in X(\Omega),\, \|u\|^2_X \le \Lambda_1} \int_\Omega e^{\frac{n}{2}u^2} dx \le C_n |\Omega|, \tag{2}$$

*where $\Lambda_1 := (n-1)!|S^n|$.*

When $n = 2$ the above theorem is a special case of the Moser-Trudinger inequality [35], and when $n > 2$ is even it is a special case of Adams' inequality [1].

In this paper we want to study the blow-up behavior of extremals of (2), i.e. weak solutions $u \in X(\Omega)$ of the Euler-Lagrange equation

$$(-\Delta)^{\frac{n}{2}} u = \lambda u e^{\frac{n}{2}u^2}, \quad \text{for some } \lambda \in \mathbb{R}, \tag{3}$$

which can be intended in the following sense:

**Definition 1.2.** *Given $f \in L^2(\Omega)$, a function with $u \in X(\Omega) + \mathbb{R}$ (i.e. $u + c \in X(\Omega)$ for some $c \in \mathbb{R}$) is a weak solution to*

$$(-\Delta)^{\frac{n}{2}} u = f \quad \text{in } \Omega \tag{4}$$

*if*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{4}} u \, (-\Delta)^{\frac{n}{4}} \varphi dx = \int_{\mathbb{R}^n} f \varphi dx, \quad \forall \varphi \in X(\Omega). \tag{5}$$

**Remark 1.3.** *It follows from Theorem 1.1 that for $u \in X(\Omega)$ one has $e^{u^2} \in L^p(\Omega)$ for every $p \in [1, \infty)$ (see also [27, Theorem 9.1]). In particular the right-hand side of (3) belongs to $L^p(\Omega)$ for $p \in [1, \infty)$.*

The Lagrange multiplier $\lambda$ in (3) can be computed by testing the equation with $\varphi = u$ (in the spirit of (5)). This leads to

$$\|(-\Delta)^{\frac{n}{4}} u\|^2_{L^2(\mathbb{R}^n)} = \lambda \int_\Omega u^2 e^{\frac{n}{2}u^2} dx, \tag{6}$$

whence $\lambda > 0$, unless $u \equiv 0$.

We are interested in the study of the blowing-up behavior of a sequence of continuous solution to the following problem :

$$\begin{cases} (-\Delta)^{\frac{n}{2}} u_k = \lambda_k u_k e^{\frac{n}{2}u_k^2} & \text{in } \Omega \\ u_k \in X(\Omega) \end{cases} \tag{7}$$

where $\lambda_k \geq 0$.

**Remark 1.4.** *It follows from Remark 1.3, from the estimates in [15] and bootstrapping, that every solution $u$ to (3) belongs to $C^{\frac{n-1}{2},\alpha}(\bar{\Omega}) \cap C^\infty(\Omega)$ for some $\alpha \in (0,1)$, and in fact the function $d^{-\frac{n}{2}} u : \Omega \to \mathbb{R}$, where $d$ is the distance function from $\partial\Omega$, can be extended to a function in $C^\infty(\bar{\Omega})$. In particular, $\sup_\Omega u_k \in \mathbb{R}$.*

The main result of this paper can be stated as follows :

**Theorem 1.5.** *Consider a* bounded *sequence $(u_k)_{k \in \mathbb{N}} \subset X(\Omega)$ of solutions to (7). Set $m_k := \sup_\Omega |u_k|$ and*

$$\Lambda := \limsup_{k \to \infty} \|u_k\|^2_X < \infty.$$

*Up to possibly replacing $u_k$ with $-u_k$ we can assume that $m_k = \sup_\Omega u_k$ for every $k$. Assume also that $0 < \lambda_k \leq \bar{\lambda}$ for some $\bar{\lambda} < \infty$ and that $\lambda_k \to \lambda_\infty$ as $k \to \infty$. Then up to extracting a subsequence one of the following holds:*

*(i)* $\lim_{k\to\infty} m_k < \infty$ *and $u_k$ converges to $u_\infty$ in $C^\ell_{\mathrm{loc}}(\Omega) \cap C^{\frac{n-1}{2}}(\bar{\Omega})$ for any $\ell \in \mathbb{N}$ and $u_\infty \in C^\ell_{\mathrm{loc}}(\Omega) \cap C^{\frac{n-1}{2}}(\bar{\Omega}) \cap X(\Omega)$ solves*

$$(-\Delta)^{\frac{n}{2}} u_\infty = \lambda_\infty u_\infty e^{\frac{n}{2}u_\infty^2} \quad \text{in } \Omega.$$

*(ii)* $\lim_{k\to\infty} m_k = \infty$, $\Lambda \geq \Lambda_1$, *with $\Lambda_1$ as in Theorem 1.1, and setting $r_k$ such that*

$$\lambda_k m_k^2 e^{\frac{n}{2}m_k^2} r_k^n = 2^n (n-1)!, \tag{8}$$

*and*

$$\eta_k(x) := m_k(u_k(x_k + r_k x) - m_k), \quad \eta_0(x) := \log\left(\frac{2}{1+|x|^2}\right), \tag{9}$$

*one has $\eta_k + \log 2 \to \eta_0$ in $C^\ell_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\ell \geq 0$.*

Since Theorem 1.5 was proven in [2], [14], [28] and [22] when $n$ is even, we shall only consider the remaining case $n$ odd. In some proofs we will focus on the case $n \geq 3$, but simple modifications make every argument work for the case $n = 1$. In fact the case $n = 1$ is a slightly simpler, since comparison principles and in particular the Harnack inequality are available.

The general strategy of the proof is similar to the one in the even-dimensional case, but some new difficulties arise due to the nonlocal nature of the operator $(-\Delta)^{\frac{n}{2}}$, as we shall now describe.

One would like to shows that in case of blow-up (Case (ii) in Theorem 1.5) the functions $\eta_k$ converge to a function $\eta_0 \in L_n(\mathbb{R}^n)$ solving

$$(-\Delta)^{\frac{n}{2}}\eta_0 = (n-1)! e^{n\eta_0} \quad \text{in } \mathbb{R}^n, \quad V := \int_{\mathbb{R}^n} e^{n\eta_0} dx < \infty, \tag{10}$$

and then prove that, among all solutions to (10), $\eta_0$ has the special form given by (9).

The first problem is that the local convergence of $\eta_k$ to a function $\eta_0$ rests on local gradient bounds for $\eta_k$ not depending on $k$ (when $n = 1, 2$ such bound are not necessary, thanks to the Harnack inequality). This is the content of Propositions 2.4 and 2.5, one of the crucial parts of the paper. In particular we will show that for $s < n$,

$$\int_{B_\rho(x_0)} |u_k (-\Delta)^{\frac{s}{2}} u_k|\, dx \leq C\rho^{n-s}, \quad \text{for } B_{10\rho}(x_0) \subset \Omega. \tag{11}$$

In the previous work [22] an analogous estimate was obtained by noticing that $(-\Delta)^{\frac{n}{2}}(u_k^2)$ is uniformly bounded in $L^1(\Omega)$ when $n$ is even. Unfortunately this was based on an explicit expansion of $(-\Delta)^{\frac{n}{2}}(u_k^2)$ as sum of partial derivatives of $u_k$, which is of course not possible when $n$ is odd. Here instead we reduce (11) to a the bound

$$\|u_k(-\Delta)^{\frac{s}{2}} u_k\|_{L^{(\frac{n}{s},\infty)}(B_\rho(x_0))} \leq C,$$

which will be proven writing $u_k(-\Delta)^{\frac{s}{2}} u_k$ in terms of the Riesz potential. The formal heuristic argument goes as follows. Write formally

$$\begin{aligned} u_k(-\Delta)^{\frac{s}{2}} u_k &= u_k I_{n-s}(-\Delta)^{\frac{n}{2}} u_k \\ &=: (I_{\frac{n}{2}}(-\Delta)^{\frac{n}{4}} u_k) I_{n-s}(\theta(-\Delta)^{\frac{n}{2}} u_k) + I_{n-s}(u_k(-\Delta)^{\frac{n}{2}} u_k) + E \\ &=: A + B + E, \end{aligned} \tag{12}$$

where $\theta \in C_0^\infty(B_{2\rho}(x_0))$ is a cut-off functions, $I_t$ denotes the Riesz potential, and $E$ is an error term, which can be bounded using a commutator-type estimate. Then one has to bound the term $A$ in $L^{(\frac{n}{s},\infty)}(B_\rho(x_0))$ using that $(-\Delta)^{\frac{n}{4}} u_k$ is bounded in $L^2(\mathbb{R}^n)$, while $(-\Delta)^{\frac{n}{2}} u_k$ is bounded in $L \log^{\frac{1}{2}} L(\Omega)$. These are borderline estimates, for instance because $I_{\frac{n}{2}}$ fails to send $L^2$ into $L^\infty$. Using elementary tricks we are able to circumvent this problem, obtaining Propositions 3.1 and 3.2. In order to bound $B$ one uses the PDE, and in particular that $u_k(-\Delta)^{\frac{n}{2}} u_k$ is bounded in $L^1(\Omega)$. Finally, to move from the formal argument to a rigorous one, and in particular to replace the first identity in (12) with a correct identity, we have to approximate $u_k$ with functions in $C_c^\infty(\mathbb{R}^n)$. The necessary technical results are contained in Section 3 and the appendix.

The second problem, still related to the non-local nature of $(-\Delta)^{\frac{n}{2}}$, is that uniform estimates on the derivatives of the blown-up functions $\eta_k$ do indeed guarantee that $\eta_k \to \eta_0$ in $C^\ell_{\text{loc}}(\mathbb{R}^n)$ (up to the additive constant $\log 2$ which we shall now ignore) for $\ell \leq n-1$, but why should the convergence

$$(-\Delta)^{\frac{n}{2}}\eta_k \to (-\Delta)^{\frac{n}{2}}\eta_0 \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \text{ as } k \to \infty \tag{13}$$

hold? Indeed (13) means that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}\eta_k\,(-\Delta)^{\frac{n}{2}}\varphi dx \to \int_{\mathbb{R}^n}\eta_0\,(-\Delta)^{\frac{n}{2}}\varphi dx \quad \text{for every } \varphi\in\mathcal{S}(\mathbb{R}^n), \tag{14}$$

and since even for $\varphi\in C_c^\infty(\mathbb{R}^n)$ we have that $(-\Delta)^{\frac{n}{2}}\varphi$ is not compactly supported, the local convergence of $\eta_k$ is not sufficient to guarantee (14). A priori it could happen that while $\eta_k\to\eta_0$ very strongly in a compact set, "at infinity" $\eta_k$ has a wild behaviour. To rule this out we shall prove uniform bounds of $\eta_k$ in $L_s(\mathbb{R}^n)$ for any $s>0$, which is the content of Proposition 2.7. Here we critically use that $\eta_k$ is uniformly upper bounded by construction, and the local bounds on the derivatives of $\eta_k$.

At this point it will be easy to conclude that $\eta_k$ locally converges to a function $\eta_0\in L_n(\mathbb{R}^n)$ solving (10). Now we are faced with the problem of determining $\eta_0$. Indeed, similarly to what was shown in [5], also in odd dimension 3 or higher, Problem (10) has many solutions, as shown in [19] (when $n=3$) and [17] (for any $n\geq 3$ odd). Here we are able to use the following recent result of Ali Hyder, together with the previous bounds to show that among all solutions of (10) actually $\eta_0$ is a special one, precisely the one given in (9).

**Theorem 1.6** (A. Hyder [18]). *Let $\eta_0\in L_n(\mathbb{R}^n)$ solve (10). Then $\eta_0$ can be decomposed as $\eta_0=v+P$, where $P$ is a polynomial of degree at most $n-1$, and $v(x)=-\alpha\log(|x|)+o(\log|x|)$ as $|x|\to\infty$. Moreover $P$ is constant if and only if*

$$\eta_0(x)=\log\frac{2\lambda}{1+\lambda^2|x-x_0|^2}, \quad \textit{for some } \lambda>0, \quad x_0\in\mathbb{R}^n. \tag{15}$$

Indeed, if $\eta_0$ is not of the form (15), then $\eta_0$ at infinity behaves like a logarithm plus a polynomial, only the former belonging to $L_s(\mathbb{R}^n)$ for $s$ small. This is in contradiction to the fact that $\eta_0\in L_s(\mathbb{R}^n)$ for every $s>0$. This argument is different from the one used in the even dimensional case, first introduced in [28] and then also applied in [22] and other works.

In the case $n=1$ Theorem 1.6 is not necessary because Da Lio-Martinazzi-Rivière [10] proved that every function $\eta_0\in L_1(\mathbb{R})$ solving (10) for $n=1$ has necessarily the form (15).

It has to be mentioned that in even dimension the analog of Theorem 1.5 was complemented in [14], [20], [24] and [34] by a quantization result, saying that in case of blow-up

$$\Lambda=\int_\Omega|\nabla^{\frac{n}{2}}u_\infty|^2dx+L\Lambda_1 \quad \text{for an integer } L>0.$$

In other words the energy loss in the weak limit is an integer multiple of the fixed quantity $\Lambda_1$. Although it is natural to expect this to hold true also in our non-local case, we remark that in the local case the proofs make abundant use of ODE techniques, which are not available when dealing with fractional Laplacians. On the other hand in the case of half-harmonic maps, precise energy quantization was obtained in [7].

## Notation

The space $C^\alpha(\Omega)\equiv C^{\alpha_0,\tilde{\alpha}}(\Omega)$, for $\alpha=\alpha_0+\tilde{\alpha}$ with $\tilde{\alpha}\in(0,1]$, $\alpha_0\in\mathbb{N}_0$, is the space of $\alpha_0$-times differentiable functions with $\alpha_0$th derivative Hölder continous of order $\tilde{\alpha}$. We define the semi-norm

$$[f]_{C^\alpha(\Omega)}=\sup_{x\neq y\in\Omega}\frac{|\nabla^{\alpha_0}f(x)-\nabla^{\alpha_0}f(y)|}{|x-y|^{\tilde{\alpha}}},$$

and the norm

$$\|f\|_{C^\alpha(\Omega)}:=\sum_{k=0}^{\alpha_0}\|\nabla^k f\|_{L^\infty(\Omega)}+[f]_{C^\alpha(\Omega)}.$$

## 2 Proof of Theorem 1.5

**Proposition 2.1.** *If* $\sup_k m_k \le C$ *then up to a subsequence* $u_k \to u_\infty$ *in* $C^\ell_{\text{loc}}(\Omega) \cap C^{\frac{n-1}{2}}(\bar{\Omega})$ *for every* $\ell > 0$*, where* $u_\infty$ *solves* (7)*.*

*Proof.* This follows from Lemma A.6, from the estimates in [15] (compare also to [28]), and the theorem of Arzelà-Ascoli. □

We shall now assume that, up to a subsequence, $m_k \to \infty$ as $k \to \infty$ and we consider $x_k \in \Omega$ so that

$$m_k \equiv \sup_\Omega u_k = u_k(x_k) \to \infty \quad \text{as } k \to \infty. \tag{16}$$

### 2.1 Rescaling and Convergence

**Lemma 2.2.** *Let* $r_k$ *and* $x_k$ *be defined by* (8) *and* (16) *respectively. Then we have*

$$\lim_{k\to\infty} \frac{\operatorname{dist}(x_k, \partial\Omega)}{r_k} = +\infty$$

*Proof.* For the sake of contradiction, we assume that

$$\lim_{k\to\infty} \frac{\operatorname{dist}(x_k, \partial\Omega)}{r_k} < \infty.$$

Let us assume that

$$0 < \lim_{k\to\infty} \frac{\operatorname{dist}(x_k, \partial\Omega)}{r_k} < \infty. \tag{17}$$

If the above limit vanishes then the argument is similar. We set $\Omega_k = \{r_k^{-1}(x - x_k); x \in \Omega\}$. Then

$$v_k(x) := \frac{u_k(r_k x + x_k)}{m_k}$$

satisfies

$$\begin{cases} (-\Delta)^{\frac{n}{2}} v_k = \frac{2^n (n-1)!}{m_k^2} v_k e^{\frac{n}{2} m_k^2 (v_k^2 - 1)} & \text{in } \Omega_k \\ v_k \in X(\Omega_k). \end{cases} \tag{18}$$

Notice that,

$$\|(-\Delta)^{\frac{n}{4}} v_k\|_{L^2(\mathbb{R}^n)} = (m_k)^{-1} \|(-\Delta)^{\frac{n}{4}} u_k\|_{L^2(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0.$$

Then by the Sobolev embedding, Proposition A.3 using also (44), the boundedness of the Riesz transform, and that $(-\Delta)^{\frac{1}{2}} = I_{\frac{n}{2}-1}(-\Delta)^{\frac{n}{4}}$,

$$\begin{aligned} \|\nabla v_k\|_{L^n(\mathbb{R}^n)} &= c\|\mathcal{R}(-\Delta)^{\frac{1}{2}} v_k\|_{L^n(\mathbb{R}^n)} \prec \|(-\Delta)^{\frac{1}{2}} v_k\|_{L^n(\mathbb{R}^n)} \\ &\prec \|I_{\frac{n}{2}-1}(-\Delta)^{\frac{n}{4}} v_k\|_{L^n(\mathbb{R}^n)} \le C\|(-\Delta)^{\frac{n}{4}} v_k\|_{L^2(\mathbb{R}^n)} \xrightarrow{k\to\infty} 0. \end{aligned} \tag{19}$$

On the other hand, by (17) there exists some $R > 0$ so that $B_{4R}(0) \subset \Omega_k$ for all $k \in \mathbb{N}$. Then

$$\|(-\Delta)^{\frac{n}{2}} v_k\|_{L^\infty(B_{3R}(0))} \xrightarrow{k\to\infty} 0.$$

This implies that for any $\alpha \in (0, n)$, Lemma A.6,

$$[v_k]_{C^\alpha(B_{2R}(0))} \le C.$$

So recalling that $|v_k| \leq 1$, by Arzelà-Ascoli (up to a subsequence) we have that $v_k \to v$ in $C^{n-1}(B_R)$ for some $v$. Since at the same time $\nabla v_k \to 0$ in $L^n(\mathbb{R}^n)$ and $v_k(0) = 1$, we know that $v \equiv 1$ in $B_R$.

On the other hand, take $R_1 > R$ so that $B_{\frac{R_1}{2}}(0) \cap \partial\Omega_k \neq 0$ for all but possibly finitely many $k \in \mathbb{N}$. Using (19), and noticing that $v_k \equiv 0$ on a fixed part of positive measure of $B_{R_1}$, we know that $v_k \to 0$ in $L^n(B_{R_1}(0))$, hence $v \equiv 0$ in $B_R$. This contradicts $v \equiv 1$. □

**Lemma 2.3.** *Let $m_k$ be as in* (16). *Then we have*

$$u_k(x_k + r_k x) - m_k \to 0 \text{ in } C^{n-1}_{\text{loc}}(\mathbb{R}^n) \text{ as } k \to \infty. \tag{20}$$

*Proof.* Let $\tilde{u}_k := u_k(x_k + r_k x)$. Then $\tilde{u}_k \in C^0_c(\mathbb{R}^n) \cap X(\Omega_k)$ and

$$\sup_{x \in \mathbb{R}^n} |\tilde{u}_k(x)| = \tilde{u}_k(0) = m_k \in [0, \infty).$$

As above by Sobolev embedding, $\tilde{u}_k \in W^{1,n}_0(\Omega_k)$

$$\begin{aligned}\limsup_{k\to\infty} \|\nabla \tilde{u}_k\|_{L^n(\mathbb{R}^n)} &\leq C \limsup_{k\to\infty} \|(-\Delta)^{\frac{n}{4}} \tilde{u}_k\|_{L^2(\mathbb{R}^n)} \\ &= C \limsup_{k\to\infty} \|(-\Delta)^{\frac{n}{4}} u_k\|_{L^2(\mathbb{R}^n)} \\ &\leq C(\Lambda).\end{aligned}$$

and from (42), (43) and (44) below for $k$ large enough we get

$$\begin{aligned}\|(-\Delta)^{\frac{1}{2}} \tilde{u}_k\|_{L^n(\mathbb{R}^n)} &= \sum_{i=1}^n \|\mathcal{R}_i\mathcal{R}_i(-\Delta)^{\frac{1}{2}} \tilde{u}_k\|_{L^n(\mathbb{R}^n)} \\ &\leq C \sum_{i=1}^n \|\mathcal{R}_i(-\Delta)^{\frac{1}{2}} \tilde{u}_k\|_{L^n(\mathbb{R}^n)} \\ &\leq C\|\nabla \tilde{u}_k\|_{L^n(\mathbb{R}^n)} \\ &\leq C(\Lambda).\end{aligned} \tag{21}$$

Notice that

$$|(-\Delta)^{\frac{n}{2}} \tilde{u}_k| \leq \frac{C}{m_k} \quad \text{in } \Omega_k.$$

Finally, by Lemma 2.2 for any $\varphi \in C^\infty_c(\mathbb{R}^3)$, for all sufficiently large $k$ depending on the size of the support of $\varphi$,

$$\left| \int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{4}} \tilde{u}_k \, (-\Delta)^{\frac{n}{4}} \varphi \, dx \right| \leq C \, \frac{1}{m_k} \int_{\mathbb{R}^3} |\varphi| \, dx. \tag{22}$$

Let $g_k := (-\Delta)^{\frac{1}{2}} \tilde{u}_k$, bounded in $L^n(\mathbb{R}^n)$, according to (21). There is a weakly convergent subsequence $g_k \rightharpoonup g$ in $L^n(\mathbb{R}^n)$. Moreover, we have for any $\varphi \in C^\infty_c(\mathbb{R}^n)$, by (22)

$$\int_{\mathbb{R}^n} g \, (-\Delta)^{\frac{n-1}{2}} \varphi \, dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} g_k \, (-\Delta)^{\frac{n-1}{2}} \varphi \, dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{4}} \tilde{u}_k \, (-\Delta)^{\frac{n}{4}} \varphi \, dx \xrightarrow{k\to\infty} 0.$$

Consequently, $g \in C^\infty(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$, and pointwise $(-\Delta)^{\frac{n-1}{2}} g \equiv 0$. This implies that $g \equiv 0$. Indeed by elliptic estimates (see e.g. [21, Proposition 4]) and Hölder's inequality it follows that

$$\|g\|_{L^\infty(B_1)} \leq C\|g\|_{L^1(B_2)} \leq \tilde{C}\|g\|_{L^n(B_2)},$$

which scaled gives

$$\|g\|_{L^\infty(B_R)} \leq \tilde{C} R^{-1} \|g\|_{L^n(B_{2R})} \to 0 \quad \text{as } R \to \infty.$$

So we have obtained, that $(-\Delta)^{\frac{1}{2}} \tilde{u}_k \rightharpoonup 0$ in $L^n(\mathbb{R}^n)$. Then, using (44) and (45) we also have

$$\nabla \tilde{u}_k = \mathcal{R}(-\Delta)^{\frac{1}{2}} \tilde{u}_k \rightharpoonup 0 \quad \text{in } L^n(\mathbb{R}^n).$$

Since $\tilde{u}_k$ is uniformly bounded in $H^{\frac{n}{2}}(\mathbb{R}^n)$, since $n \geq 3$, we also have strong convergence in $W^{1,2}_{\text{loc}}(\mathbb{R}^n)$. In particular up to choosing a subsequence, for any $R > 1$,

$$\nabla \tilde{u}_k \to 0 \quad \text{in } L^2(B_R). \tag{23}$$

On the other hand, observe the following: For any $R > 1$, for all large $k \in \mathbb{N}$, we have $B_{2R} \subset \Omega_k$. From (22), Lemma A.6 we obtain

$$\|\nabla \tilde{u}_k\|_{C^{n-2,\alpha}(B_R)} \leq C$$

for a uniform constant $C$ and $\alpha \in (0,1)$.

Since $\tilde{u}_k(0) = m_k$, we have

$$\|\tilde{u}_k - m_k\|_{L^\infty(B_R)} \leq \|\nabla \tilde{u}_k\|_{L^\infty(B_R)} \leq C,$$

and consequently we have shown that

$$\|\tilde{u}_k - m_k\|_{C^{n-1,\alpha}(B_R)} \leq C$$

Now Arzelà-Ascoli gives (up to a further subsequence) $C^{n-1}(B_R)$-convergence of $\tilde{u}_k - m_k$, and using (23) we have that $\tilde{u}_k - m_k \to 0$ in $C^{n-1}(B_R)$. Since $R$ is arbitrary the proof is complete. □

## 2.2 Gradient-type estimates

Note that from (6)

$$\limsup_{k\to\infty} \|u_k(-\Delta)^{\frac{n}{2}} u_k\|_{L^1(\Omega)} \leq \Lambda.$$

Moreover, as in [22, Proof of Lemma 5], we know that for the Orlicz space $L\log^{\frac{1}{2}} L(\Omega)$,

$$\limsup_{k\to\infty} \|(-\Delta)^{\frac{n}{2}} u_k\|_{L\log^{\frac{1}{2}} L(\Omega)} \leq C(\Lambda, \Omega).$$

We will now need the following crucial estimate applied to $u = u_k$ and $\rho = Rr_k$ for a given $R > 0$ and $k$ so large that $B_{10\rho}(x_k) \subset \Omega$ (compare to Lemma 2.2).

**Proposition 2.4.** *Let $\Omega$ be a smoothly bounded domain, and consider $u \in X(\Omega)$ such that $(-\Delta)^{\frac{n}{2}} u = f$ weakly in $\Omega$ for some $f \in L\log^{\frac{1}{2}} L(\Omega) \cap L^\infty(\Omega)$. Assume moreover that*

$$\|u\, f\|_{L^1(\Omega)} + \|f\|_{L\log^{\frac{1}{2}} L(\Omega)} + \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)} \leq C_1. \tag{24}$$

*Then for a constant depending $C_2$ depending only on $C_1$ and $s \in (0,n)$ we have*

$$\sup_{B_{4\rho}(x_0)\subset\Omega} \rho^{s-n} \int_{B_\rho(x_0)} |u(-\Delta)^{\frac{s}{2}} u|\, dx \leq C_2$$

*Proof.* We will use the Lorentz spaces $L^{(p,q)}$, for which we refer the reader to the appendix. Using the Hölder-type inequality (see [26])

$$\|gh\|_{L^1(\Omega)} \leq \|g\|_{L^{(\frac{n}{n-s},1)}(\Omega)} \|h\|_{L^{(\frac{n}{s},\infty)}(\Omega)},$$

we get (for $B_\rho = B_\rho(x_0)$, to simplify the notation)

$$\begin{aligned}\rho^{s-n} \int_{B_\rho} |u(-\Delta)^{\frac{s}{2}} u| dx &\leq \rho^{s-n} \|\chi_{B_\rho}\|_{L^{(\frac{n}{n-s},1)}(B_\rho)} \|u(-\Delta)^{\frac{s}{2}} u\|_{L^{(\frac{n}{s},\infty)}(B_\rho)} \\ &= C\|u(-\Delta)^{\frac{s}{2}} u\|_{L^{(\frac{n}{s},\infty)}(B_\rho)},\end{aligned}$$

so that it remains to show the bound

$$\sup_{B_{4\rho}\subset\Omega} \|u(-\Delta)^{\frac{s}{2}} u\|_{L^{(\frac{n}{s},\infty)}(B_\rho)} \leq C_2.$$

For $\varepsilon > 0$ we denote with $u^\varepsilon \in C_c^\infty(\mathbb{R}^n)$ the usual mollification.

Consider now a cut-off function $\theta_{B_1} \in C^\infty(B_2)$, $\theta_{B_1} \equiv 1$ in $B_1$ and $0 \leq \theta_{B_1} \leq 1$ everywhere. Set $\theta_{B_{2\rho}} := \theta_{B_1}(\cdot/2\rho) \in C_c^\infty(B_{4\rho})$. Then since $u^\varepsilon \in C_c^\infty(\mathbb{R}^n)$ we have for $s \in (0, n)$ pointwise in $B_\rho$:

$$\begin{aligned}|u(-\Delta)^{\frac{s}{2}} u^\varepsilon| =&|uI_{n-s}(-\Delta)^{\frac{n}{2}} u^\varepsilon| \\ \leq&|uI_{n-s}(\theta_{B_{2\rho}}(-\Delta)^{\frac{n}{2}} u^\varepsilon)| + |uI_{n-s}((1-\theta_{B_{2\rho}})(-\Delta)^{\frac{n}{2}} u^\varepsilon)| \\ \leq&|uI_{n-s}(\theta_{B_{2\rho}}(-\Delta)^{\frac{n}{2}} u^\varepsilon)| \\ &+ |uI_{n-s}((1-\theta_{B_{2\rho}})(-\Delta)^{\frac{n}{2}} u^\varepsilon) - I_{n-s}(u(1-\theta_{B_{2\rho}})(-\Delta)^{\frac{n}{2}} u^\varepsilon)| \\ &+ |I_{n-s}(u(-\Delta)^{\frac{n}{2}} u^\varepsilon)| \\ =&:I + II + III.\end{aligned}$$

By Proposition 3.1 and Proposition 3.2, using that $u = I_{\frac{n}{2}}(-\Delta)^{\frac{n}{4}} u$ we infer

$$\|I\|_{L^{(\frac{n}{s},\infty)}(\mathbb{R}^n)} \prec \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L\log^{\frac{1}{2}} L(B_{4\rho})}. \tag{25}$$

From the disjoint-support commutator estimate, see Proposition 3.4, we have

$$\|II\|_{L^{(\frac{n}{s},\infty)}(B_\rho)} \prec \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)}^2. \tag{26}$$

Since the support of $u$ is contained in $\Omega$, by the Sobolev inequality

$$\|III\|_{L^{(\frac{n}{s},\infty)}(\mathbb{R}^n)} \prec \|u(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^1(\Omega)} \tag{27}$$

Combining the estimates (27), (25), (26) we arrive at

$$\begin{aligned}\|u(-\Delta)^{\frac{s}{2}} u^\varepsilon\|_{L^{(\frac{n}{2},\infty)}(B_\rho)} \prec& \|u(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^1(\Omega)} + \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)} \ \|(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L\log^{\frac{1}{2}} L(B_{4\rho})} \\ &+ \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)}^2 \\ \overset{(24)}{\leq}& \|u(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^1(\Omega)} + C_1 \ \|(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L\log^{\frac{1}{2}} L(B_{4\rho})} + (C_1)^2.\end{aligned}$$

It remains to take $\varepsilon \to 0$, but some care is needed, since $(-\Delta)^{\frac{n}{2}} u$ is in general not a function, but a distribution.

Firstly, since $B_{4\rho} \subset \Omega$, for $\varepsilon < \rho$ we have that

$$(-\Delta)^{\frac{n}{2}} u^\varepsilon = ((-\Delta)^{\frac{n}{2}} u)^\varepsilon \quad \text{in } B_{4\rho}.$$

In particular, for $\varepsilon < \rho$

$$\|(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L \log^{\frac{1}{2}} L(B_{4\rho})} \prec \|(-\Delta)^{\frac{n}{2}} u\|_{L \log^{\frac{1}{2}} L(B_{4\rho})} \le \|(-\Delta)^{\frac{n}{2}} u\|_{L \log^{\frac{1}{2}} L(\Omega)} \le C_1.$$

For the remaining term $\|u(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^1(\Omega)}$, we need to argue as follows. Firstly, since $u^\varepsilon$ is the usual mollification, we have

$$\|(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^2(\mathbb{R}^n)} \le \varepsilon^{-\frac{n}{2}} \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)} \tag{28}$$

Moreover, since $\Omega$ is smooth and bounded and $u \in X(\Omega)$, the results by [15], see also [29, Theorem 1.2], using that

$$\|(-\Delta)^{\frac{n}{2}} u\|_{L^\infty(\Omega)} =: C_3 < \infty,$$

then if we set $\Omega_{-\varepsilon} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$

$$\|u\|_{L^\infty(\Omega \backslash \Omega_{-\varepsilon})} \prec \varepsilon^{\frac{n}{2}} C_3.$$

In particular with (28) we have

$$\begin{aligned}\|u(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^1(\Omega)} &\le \|u(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^1(\Omega_{-\varepsilon})} + |\Omega \backslash \Omega_{-\varepsilon}|^{\frac{1}{2}} \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)} \\ &= \|u(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^1(\Omega_{-\varepsilon})} + o(1) \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Now note again that

$$(-\Delta)^{\frac{n}{2}} u^\varepsilon = ((-\Delta)^{\frac{n}{2}} u)^\varepsilon \quad \text{pointwise in } \Omega_{-\varepsilon}.$$

Consequently,

$$\|u(-\Delta)^{\frac{n}{2}} u^\varepsilon\|_{L^1(\Omega)} \le \|u((-\Delta)^{\frac{n}{2}} u)^\varepsilon\|_{L^1(\Omega_{-\varepsilon})} + o(1) \xrightarrow{\varepsilon \to 0} \|u\ (-\Delta)^{\frac{n}{2}} u\|_{L^1(\Omega)}.$$

This concludes the proof of Lemma 2.4. □

## 2.3 Convergence of $\eta_k$

Let $\eta_k$ be as in (9). From Proposition 2.4 we now infer:

**Proposition 2.5.** *For every $s \in (0, n)$ there exists $C > 0$ such that for every $R > 0$ and $k$ large enough (depending on $R$ and $s$) we have*

$$\int_{B_R} |(-\Delta)^{\frac{s}{2}} \eta_k| dx \le C R^{n-s}. \tag{29}$$

*Proof.* According to Lemma 2.3 we have

$$m_k \le 2u_k \quad \text{on } B_{Rr_k}(x_k) \text{ for } k \text{ large enough},$$

hence with Proposition 2.4 applied with $u = u_k$ and $\rho = Rr_k$ we obtain (note that $(-\Delta)^{\frac{s}{2}}(m_k^2) = 0$)

$$\begin{aligned}\int_{B_R} |(-\Delta)^{\frac{s}{2}} \eta_k| dx &= \frac{m_k}{r_k^{n-s}} \int_{B_{Rr_k}(x_k)} |(-\Delta)^{\frac{s}{2}} u_k| dx \\ &\le \frac{2}{r_k^{n-s}} \int_{B_{Rr_k}(x_k)} |u_k(-\Delta)^{\frac{s}{2}} u_k| dx \\ &\le C R^{n-s},\end{aligned}$$

as claimed. □

**Proposition 2.6.** *For every $B_R \subset \mathbb{R}^n$ and any $\alpha \in [0,1)$ there exists a constant $C_{R,\alpha}$ so that*

$$\|\eta_k\|_{C^{n-1+\alpha}(B_R)} \le C_{R,\alpha}.$$

*for $k$ large enough.*

*Proof.* We have that $|(-\Delta)^{\frac{n}{2}}\eta_k| \le C(R)$ in $B_R$, in the sense that

$$\left|\int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{4}}\eta_k (-\Delta)^{\frac{n}{4}}\varphi dx\right| \le C\|\varphi\|_{L^1(B_R)}, \quad \text{for } \varphi \in C_c^\infty(B_R).$$

This can be rewritten as

$$\left|\int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}}\eta_k (-\Delta)^{\frac{n-1}{2}}\varphi dx\right| \le C\|\varphi\|_{L^1(B_R)}, \quad \text{for } \varphi \in C_c^\infty(B_R), \tag{30}$$

which means that the function $\psi_k := (-\Delta)^{\frac{1}{2}}\eta_k$ satisfies

$$|(-\Delta)^{\frac{n-1}{2}}\psi_k| \le C_R \quad \text{in } B_R$$

in the sense of distributions (notice that $(-\Delta)^{\frac{n-1}{2}}$ is an integer power of $-\Delta$ since $n$ is odd). This, together with the estimate

$$\|\psi_k\|_{L^1(B_R)} \le CR^{n-1}$$

given by Proposition 2.5, and standard elliptic estimates (see e.g. Proposition 4 and Lemma 20 in [21]) implies that

$$\|\psi_k\|_{C^{n-2,\alpha}(B_{R/2})} \le C_{R,\alpha} \quad \text{for } 0 \le \alpha < 1,$$

as claimed. Together with Harnack's inequality (see [16]) we get

$$\|\eta_k\|_{C^{n-1,\alpha}(B_{R/4})} \le C_{R,\alpha} \quad \text{for } 0 \le \alpha < 1,$$

and replacing $R$ with $4R$ we conclude. □

**Proposition 2.7.** *The sequence $(\eta_k)$ is uniformly bounded in $L_s(\mathbb{R}^n)$ for any $s > 0$.*

*Proof.* Since by Proposition 2.6 the sequence $(\eta_k)$ is bounded in $L^\infty(B_1)$, it is easy to see that boundedness of $(\eta_k)$ in $L_s(\mathbb{R}^n)$ for some $s > 0$ implies boundedness in $L_{s'}(\mathbb{R}^n)$ for every $s' > s$. Therefore without loss of generality we can assume that $s < 1$. We then have

$$(-\Delta)^{\frac{s}{2}}\eta_k(x) = C_{n,s}\int_{\mathbb{R}^n} \frac{\eta_k(y) + \eta_k(x)}{|x-y|^{n+s}}\, dy. \tag{31}$$

Consequently, for an arbitrary $\varphi \in C_c^\infty(B_1)$ using (29)

$$\begin{aligned}
C\|\varphi\|_{L^\infty(B_1)} &\overset{(29)}{\ge} \left|\int_{B_1} (-\Delta)^{\frac{s}{2}}\eta_k\, \varphi\, dx\right| \\
&\overset{(31)}{\ge} \int_{B_1}\int_{B_2} \frac{\eta_k(x) - \eta_k(y)}{|x-y|^{n+s}} dy\, \varphi(x)\, dx \\
&\quad + \int_{B_1}\int_{B_2^c} \frac{1}{|x-y|^{n+s}} dy\, \eta_k(x)\, \varphi(x)\, dx \\
&\quad + \int_{B_1}\int_{B_2^c} \frac{-\eta_k(y)}{|x-y|^{n+s}} dy\, \varphi(x)\, dx \\
&=: I + II + III.
\end{aligned}$$

Since by Proposition 2.6,

$$|\eta_k(x) - \eta_k(y)| \prec |x-y| \quad \forall x, y \in B_2,$$

we have that

$$|I| \prec \int_{B_1} |\varphi(x)| \int_{|x-y|\leq 3} |x-y|^{-n+1-s}\, dy\, dx \prec \|\varphi\|_1.$$

Since we also have $|\eta_k(x)| \leq C$ for all $x \in B_2$,

$$|II| \prec \int_{B_1} |\varphi(x)| \int_{|x-y|>1} |x-y|^{-n-s}\, dy\, dx \prec \|\varphi\|_1.$$

Finally, since $-\eta_k(y) = |\eta_k(y)|$, we arrive at

$$\int_{B_1} \int_{B_2^c} \frac{|\eta_k(y)|}{|x-y|^{n+s}} dy\ \varphi(x)\, dx \leq C,$$

for a constant depending on $\varphi$ and $s$, but independent of $k$. Taking $\varphi(x)$ to be non-negative and so that $\varphi \equiv 1$ on $B_{1/2}$, we arrive at

$$\int_{|y|>2} \frac{|\eta_k(y)|}{1+|y|^{n+s}} dy \leq C.$$

Since again by Proposition 2.6 for a $C$ uniform in $k$,

$$\int_{|y|<2} \frac{|\eta_k(y)|}{1+|y|^{n+s}} dy \leq C.$$

we have shown that

$$\sup_{k\in\mathbb{N}} \|\eta_k\|_{L_s(\mathbb{R}^n)} \leq C.$$

□

**Proposition 2.8.** *Up to a subsequence, $\eta_k + \log 2 \to \eta_0 = \log(\frac{2}{1+|\cdot|^2})$ in $C^\ell_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\ell \geq 0$, and*

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_{Rr_k}(x_k)} \lambda_k u_k^2 e^{\frac{n}{2}u_k^2} dx = (n-1)! \int_{\mathbb{R}^n} e^{n\eta_0} dx = \Lambda_1. \tag{32}$$

*Proof.* Let $\eta_0$ be the pointwise limit of $\eta_k + \log 2$, which exists up to a subsequence, by Proposition 2.6 and Arzelà-Ascoli's theorem. In fact the limit is in $C^\ell_{\mathrm{loc}}(\mathbb{R}^n)$ for every $\ell \geq 0$ since with Proposition 2.7 one can bootstrap regularity for the operator $(-\Delta)^{\frac{n}{2}}$, see e.g. [19, Corollary 24]. It follows from Proposition 2.7 that

$$\eta_0 \in L_s(\mathbb{R}^n) \quad \text{for every } s > 0. \tag{33}$$

We then have

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} e^{n(\eta_k+\log 2)} \varphi dx = \int_{\mathbb{R}^n} e^{n\eta_0} \varphi dx \quad \text{for every } \varphi \in C^\infty_c(\mathbb{R}^n).$$

We will show that moreover

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \eta_k\ (-\Delta)^{\frac{n}{2}} \varphi dx = \int_{\mathbb{R}^n} \eta_0 (-\Delta)^{\frac{n}{2}} \varphi dx \quad \text{for every } \varphi \in C^\infty_c(\mathbb{R}^n). \tag{34}$$

Then $\eta_0$ satisfies $(-\Delta)^{\frac{n}{2}}\eta_0 = (n-1)!e^{n\eta_0}$ as a distribution, and in fact also as tempered distribution. Then from to Theorem 1.6 we infer that $\eta_0 = v + P$ where $|v| \le C(1+\log(1+|\cdot|)) \in L_s(\mathbb{R}^n)$ for every $s > 0$, and $P$ is polynomial bounded from above. It is easy to see that if $P$ is not constant, then $P \notin L_s(\mathbb{R}^n)$ for any $s > 0$, which contadicts (33). Therefore $P$ is constant and $\eta_0$ is as in (15). It remains to determine $\lambda$ and $x_0$ in (15), but this is easy since $\eta_k(0) = 0 = \max_{\mathbb{R}^n} \eta_k$, so that $\eta_0(0) = \log 2 = \max_{\mathbb{R}^n} \eta_0$, i.e. $x_0 = 0$, $\lambda = 1$ and $\eta_0(x) = \log \frac{2}{1+|x|^2}$.

In order to obtain (34), Assume that for some $R > 0$, $\operatorname{supp} \varphi \subset B_R$ and let $\theta > 1$. Then,

$$\begin{aligned}\int \eta_0 (-\Delta)^{\frac{n}{2}}\varphi \, dx - \int \eta_k (-\Delta)^{\frac{n}{2}}\varphi \, dx &= \int_{B_{\theta R}} (\eta_0 (-\Delta)^{\frac{n}{2}}\varphi - \eta_k \, (-\Delta)^{\frac{n}{2}}\varphi) \, dx \\ &\quad + \int_{\mathbb{R}^n \setminus B_{\theta R}} (\eta_0 (-\Delta)^{\frac{n}{2}}\varphi - \eta_k \, (-\Delta)^{\frac{n}{2}}\varphi) \, dx \\ &=: I + II.\end{aligned}$$

Notice that by the disjoint support $\varphi$, see Lemma 3.5,

$$\begin{aligned}|II| &\prec \|\varphi\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_{\theta R}} \frac{|\eta_0(x)| + |\eta_k(x)|}{|x|^{2n}} \, dx \\ &\prec \|\varphi\|_{L^1(\mathbb{R}^n)} \, \theta^{s-n} \int_{\mathbb{R}^n \setminus B_{\theta R}} \frac{|\eta_0(x)| + |\eta_k(x)|}{1+|x|^{n+s}} \, dx\end{aligned}$$

and the uniform bound of $\eta_0$ and $\eta_k$ in $L_s(\mathbb{R}^n)$ implies that

$$|II| \prec \theta^{s-n},$$

for a constant independent of $k$. On the other hand, $\eta_0 - \eta_k \to 0$ uniformly in $B_{\theta R}$, which implies that $\lim_{k\to\infty} I = 0$. Consequently,

$$\lim_{k\to\infty} \left| \int \eta_0 (-\Delta)^{\frac{n}{2}}\varphi \, dx - \int \eta_k \, (-\Delta)^{\frac{n}{2}}\varphi \, dx \right| \prec \theta^{s-n},$$

for any $\theta > 1$, and letting $\theta \to \infty$ we conclude the proof of (34).

Finally, using Lemma 2.3 and the definition of $r_k$, we obtain

$$\begin{aligned}\int_{B_{Rr_k}(x_k)} \lambda_k u_k^2 e^{\frac{n}{2}u_k^2} dx &= \int_{B_R} r_k^n \lambda_k u_k^2(r_k \cdot) e^{\frac{n}{2}m_k^2} e^{\frac{n}{2}(u_k(r_k\cdot)-m_k)^2} e^{n\eta_k} dx \\ &= \int_{B_R} r_k^n \lambda_k m_k^2 (1+o(1)) e^{\frac{n}{2}m_k^2} e^{o(1)} e^{n\eta_k} dx \\ &= 2^n (n-1)! \int_{B_R} (1+o(1)) e^{n\eta_k} dx \\ &= (n-1)! \int_{B_R} e^{n\eta_0} dx + o(1)\end{aligned}$$

with $o(1) \to 0$ as $k \to \infty$. Now letting also $R \to \infty$ and noticing that

$$\int_{\mathbb{R}^n} e^{n\eta_0} dx = |S^n|,$$

we infer (32). $\square$

## 3 Borderline and commutator estimates

We have the following two borderline estimates:

**Proposition 3.1.** *Let $g \in L^2(\mathbb{R}^n)$, $f \in L\log^{1/2} L(\mathbb{R}^n)$, $s \in (0,n)$. Then*

$$\|I_{n-s}(f\, I_{\frac{n}{2}} g)\|_{(\frac{n}{s},\infty)} \prec \|g\|_{L^2} \|f\|_{L\log^{1/2} L}$$

*Proof.* By Fubini's theorem and Proposition A.4,

$$\begin{aligned}
\|f I_{\frac{n}{2}} g\|_{L^1} &\prec \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(z)|}{|z-y|^{n/2}} dz\ |f(y)| dy \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{|z-y|^{n/2}} dy\ |g(z)| dz \\
&= \||g|\ I_{\frac{n}{2}} |f|\|_{L^1} \\
&\leq \|g\|_{L^2} \|I_{\frac{n}{2}} |f|\|_{L^2} \\
&\prec \|g\|_{L^2} \|f\|_{L(\log^{1/2} L)},
\end{aligned}$$

so that by Proposition A.3,

$$\|I_{n-s}(f\ I_{\frac{n}{2}}(g))\|_{(\frac{n}{s},\infty)} \prec \|f\ I_{\frac{n}{2}} g\|_{L^1} \prec \|g\|_{L^2} \|f\|_{L(\log^{1/2} L)}.$$

□

**Proposition 3.2.** *Let $g \in L^2(\mathbb{R}^n)$, $f \in L\log^{1/2} L(\mathbb{R}^n)$, $s \in (0,n)$. Then*

$$\|I_{\frac{n}{2}} g\, I_{n-s} f\|_{(\frac{n}{s},\infty)} \prec \|g\|_{L^2} \|f\|_{L\log^{1/2} L}$$

*Proof.* We write

$$I_{\frac{n}{2}} g\, I_{n-s} f(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k(x,y,z) g(z) f(y) dz dy, \quad k(x,y,z) := \frac{1}{|x-z|^{\frac{n}{2}} |x-y|^s}.$$

For $\varepsilon \in (0,s)$ we can now bound (cf. [31])

$$k(x,y,z) \leq |x-z|^{-\frac{n}{2}-\varepsilon} |x-y|^{\varepsilon - s} + C|y-z|^{-\frac{n}{2}} |x-y|^{-s} =: II + III.$$

Indeed if $|y-z| \geq 2|x-z|$, then $|x-z| \leq |x-y|$ (by the triangular inequality), hence $I \leq II$. If $|y-z| \leq 2|x-z|$, then $I \leq C\, III$. Therefore we have

$$|I_{\frac{n}{2}} g\, I_{n-s} f| \leq I_{\frac{n}{2}-\varepsilon}(|g|)\, I_{n-s+\varepsilon}(|f|) + C I_{n-s}(|f|\, I_{\frac{n}{2}}(|g|)).$$

The first term on the right-hand side can be bounded as

$$\|I_{\frac{n}{2}-\varepsilon} g\, I_{n+\varepsilon-s} f\|_{(\frac{n}{s},\infty)} \prec \|I_{\frac{n}{2}-\varepsilon}(|g|)\|_{L^{\frac{n}{\varepsilon}}}\ \|I_{n+\varepsilon-s}(|f|)\|_{(\frac{n}{s-\varepsilon},\infty)} \prec \|g\|_{L^2}\ \|f\|_{L^1},$$

while the second term can be bounded by Proposition 3.1. □

### 3.1 Disjoint-support estimates

When $\operatorname{supp}\varphi \subset K$ for a compact set $K$ then in general we have no information on the support of $(-\Delta)^{\frac{s}{2}}\varphi$, since $(-\Delta)^{\frac{s}{2}}$ is a non-local operator. In particular $(-\Delta)^{\frac{s}{2}}\varphi(x) \neq 0$ also for $x$ far away from $K$. However, there is a decay of $|(-\Delta)^{\frac{s}{2}}\varphi(x)|$ as $\operatorname{dist}(x,K) \to \infty$. We shall call this pseudo-local behavior of $(-\Delta)^{\frac{s}{2}}$. It has been used allover the literature, for statements in the following form see [4].

**Definition 3.3** (Cut-off functions)**.** *With $\theta_{B_1}$ we will denote a fixed smooth function $\theta_{B_1} \in C_c^\infty(B_2)$ with $\theta_{B_1} \equiv 1$ in $B_1$ and $0 \leq \theta_{B_1} \leq 1$ everywhere. Define*

$$\theta_{B_\rho}(x) := \theta_{B_1}(x/\rho) \in C_c^\infty(B_{2\rho}), \quad \theta_{A_\rho} := \theta_{B_\rho} - \theta_{B_{\frac{\rho}{2}}} \in C_c^\infty(B_{2\rho} \setminus B_{\frac{\rho}{2}}).$$

In the proof of Proposition 2.4 we used the following "disjoint-support" commutator estimate. Compared to the usual commutator estimates [6, 30] the estimates here are simpler, due to the disjoint support. Note that going through the proof, one may obtain a BMO-estimate, which is false for the commutator without disjoint support, see [13].

**Proposition 3.4.** *Define the commutator $[u, I_t](v) := uI_tv - I_t(uv)$. Then for any $u, v \in C_c^\infty(\mathbb{R}^n)$,*

$$\|[u, I_{n-s}]((1-\theta_{B_{2\rho}})(-\Delta)^{\frac{n}{4}}f)\|_{L^{(\frac{n}{s},\infty)}(B_\rho)} \prec \|(-\Delta)^{\frac{n}{4}}u\|_{L^2(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.$$

*Proof.* By scaling we can assume that $\rho = 1$. With $\theta_\ell := \theta_{A_{2^\ell}} \in C_c^\infty(B_{2^{\ell+1}} \backslash B_{2^{\ell-1}})$ as in Definition 3.3 we have pointwise in $\mathbb{R}^n$

$$1 - \theta_{B_2} = \sum_{l=2}^{\infty} \theta_\ell.$$

Moreover for $t \in (0,\infty)$ and $p \in [1,\infty]$ we have

$$\|(-\Delta)^{\frac{t}{2}}\theta_\ell\|_{L^p(\mathbb{R}^n)} \leq C_{t,p}\ 2^{\ell(\frac{n}{p}-t)}.$$

Since $u, v \in C_c^\infty(\mathbb{R}^n)$, we have then

$$|[u, I_{n-s}]((1-\theta_{B_2})(-\Delta)^{\frac{n}{4}}f)| \leq \sum_{\ell=2}^{\infty} |uI_{n-s}(\theta_\ell(-\Delta)^{\frac{n}{4}}f) - I_{n-s}(u\theta_\ell(-\Delta)^{\frac{n}{4}}f)|. \tag{35}$$

Now set

$$u_\ell := \theta_{B_{2^{\ell+2}}}(u - (u)_{B_{2^{\ell+2}}}).$$

Since $\theta_{B_{2^{\ell+2}}} \equiv 1$ in $B_{2^{\ell+2}} \supset B_1 \cup \operatorname{supp}\theta_\ell$, and the constant $(u)_{B_{2^{\ell+2}}}$ commutes with $I_{n-s}$, multiplying each term on the right-hand side of (35) by $\theta_{B_{2^{\ell+2}}}$ and summing and subtracting the term

$$\theta_{B_{2^{\ell+1}}}(u)_{B_{2^{\ell+1}}} I_{n-s}(\theta_\ell(-\Delta)^{\frac{n}{4}}f),$$

we find

$$\begin{aligned}|[u, I_{n-s}]((1-\theta_{B_2})(-\Delta)^{\frac{n}{4}}f)| &\leq \sum_{\ell=2}^{\infty} |u_\ell I_{n-s}(\theta_\ell(-\Delta)^{\frac{n}{4}}f)| + \sum_{\ell=2}^{\infty} |I_{n-s}(u_\ell\theta_\ell(-\Delta)^{\frac{n}{4}}f)| \\ &=: \sum_{\ell=2}^{\infty} ((I)_\ell + (II)_\ell) \quad \text{in } B_1.\end{aligned}$$

Now, by Hölder inequality and Lemma 3.7

$$\|u_\ell I_{n-s}(\theta_\ell(-\Delta)^{\frac{n}{4}} f)\|_{L^{(\frac{n}{s},\infty)}(B_1)} \prec \|u_\ell\|_{L^{\frac{n}{s}}(B_1)} \|I_{n-s}(\theta_\ell(-\Delta)^{\frac{n}{4}} f)\|_{L^\infty(B_1)}$$
$$\prec \|u_\ell\|_{L^{\frac{n}{s}}(B_1)}\ 2^{-\ell s} \|f\|_{L^2(\mathbb{R}^n)}$$

Note that for any $p < \infty$,

$$\|u_\ell\|_{p,\mathbb{R}^n} \prec C_p 2^{\ell \frac{n}{p}} [u]_{BMO} \prec 2^{\ell \frac{n}{p}} \|(-\Delta)^{\frac{n}{4}} u\|_{2,\mathbb{R}^n}.$$

Taking $p > \frac{n}{s}$ and $\delta = \frac{n}{p}$

$$\|u_\ell\|_{\frac{n}{s},B_1} \prec \|u_\ell\|_{p,B_1} \prec 2^{\ell\delta}\ \|(-\Delta)^{\frac{n}{4}} u\|_{2,\mathbb{R}^n}.$$

Together, we arrive at

$$\|u_\ell I_{n-s}(\theta_\ell(-\Delta)^{\frac{n}{4}} f)\|_{(\frac{n}{s},\infty),B_1} \prec 2^{\ell(\delta-s)}\ \|(-\Delta)^{\frac{n}{4}} u\|_{2,\mathbb{R}^n}\ \|f\|_{2,\mathbb{R}^n},$$

and for $\delta < s$, this ensures

$$\sum_{\ell=2}^{\infty} (I)_\ell \prec \|(-\Delta)^{\frac{n}{4}} u\|_{2,\mathbb{R}^n}\ \|f\|_{2,\mathbb{R}^n}. \tag{36}$$

It remains to treat $(II)_\ell$, and we do that with Lemma 3.8:

$$\|I_{n-s}(u_\ell\ \theta_\ell(-\Delta)^{\frac{n}{4}} f)\|_{(\frac{n}{s},\infty),B_1} \prec \|I_{n-s}(u_\ell\ \theta_\ell(-\Delta)^{\frac{n}{4}} f)\|_{\frac{n}{s},B_1}$$
$$\prec \max_{t\in[0,\frac{n}{2}]} \|(-\Delta)^{\frac{t}{2}} u_\ell\|_2\ (2^\ell)^{t-\frac{n}{2}-s}\ \|f\|_{2,\mathbb{R}^n}$$

Now for any $t \in [0, \frac{n}{2}]$, by the construction of $u_\ell$ and Poincaré and Sobolev-embeddings,

$$\|(-\Delta)^{\frac{t}{2}} u_\ell\|_2 \prec (2^\ell)^{\frac{n}{2}-t} \|(-\Delta)^{\frac{n}{4}} u\|_{L^2(\mathbb{R}^n)}.$$

This leads to

$$\|I_{n-s}(u_\ell \theta_\ell(-\Delta)^{\frac{n}{4}} f)\|_{(\frac{n}{2},\infty),B_1} \prec 2^{-s\ell}\ \|(-\Delta)^{\frac{n}{4}} u\|_{2,\mathbb{R}^n}\ \|f\|_{2,\mathbb{R}^n}.$$

Again, this ensures

$$\sum_{\ell=2}^{\infty} (II)_\ell \prec \|(-\Delta)^{\frac{n}{4}} u\|_{2,\mathbb{R}^n}\ \|f\|_{2,\mathbb{R}^n}. \tag{37}$$

□

**Lemma 3.5.** *Let $\varphi \in C_c^\infty(K)$ for some compact set $K$ and let $\Omega \subset \mathbb{R}^n$ be an open set containing $K$ with $\operatorname{dist}(\partial\Omega, K) \geq d$ for some $d > 0$. Then for any $p \in [1,\infty]$ and $s \in (0,\infty)$ we have*

$$\|(-\Delta)^{\frac{s}{2}} \varphi\|_{L^p(\mathbb{R}^n\backslash\Omega)} \leq C_{n,s,p} d^{n-(n+s)p} \|\varphi\|_{L^1(K)},$$

*and for any $s \in (0,n)$ and $p > \frac{n}{n-s}$ we have*

$$\|I_s\varphi\|_{L^p(\mathbb{R}^n\backslash\Omega)} \leq C_{n,s,p} d^{-(\frac{n}{p'}-s)}\ \|\varphi\|_{L^1(K)}.$$

*Proof.* Since convolution and multiplication are transformed into each other under Fourier transform and $(|\cdot|^s)^\wedge = c|\cdot|^{-s-n}$, for $x$ away from the support of $\varphi$ we have

$$(-\Delta)^{\frac{s}{2}}\varphi(x) = c_{n,s}|\cdot|^{-n-s} * \varphi(x).$$

In particular

$$|\chi_{\mathbb{R}^n\setminus\Omega}(-\Delta)^{\frac{s}{2}}\varphi| \leq \left(|\cdot|^{-n-s}\chi_{|\cdot|\geq\frac{d}{2}}\right) * \varphi.$$

Now the first claim follows by Young's inequality:

$$\|\chi_{\mathbb{R}^n\setminus\Omega}(-\Delta)^{\frac{s}{2}}\varphi\|_{L^p(\mathbb{R}^n)} \prec \||\cdot|^{-n-s}\chi_{|\cdot|\geq\frac{d}{2}}\|_{L^p(\mathbb{R}^n)}\ \|\varphi\|_{L^1(\mathbb{R}^n)}.$$

The proof of the second claim is very similar. □

**Lemma 3.6.** *Consider two functions $\theta_1, \theta_2 \in C_c^\infty(\mathbb{R}^n)$. Suppose that $\theta_1$ and $\theta_2$ have disjoint support, i.e. for some $d > 0$,*

$$\operatorname{dist}(\operatorname{supp}\theta_1, \operatorname{supp}\theta_2) \geq d. \tag{38}$$

*For $s \in (0, n)$ let the operator $T$ be defined via*

$$Tf := \theta_1\, I_s(\theta_2 f), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

*Then for any $t > 0$ the operator $T(-\Delta)^{\frac{t}{2}}$, originally defined on $\mathcal{S}(\mathbb{R}^n)$, extends to a linear bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ whenever $1 + \frac{1}{q} - \frac{1}{p} \in [0, 1]$, with the estimate*

$$\|T(-\Delta)^{\frac{t}{2}}f\|_{L^q(\mathbb{R}^n)} \leq C_{\theta_1,\theta_2,p,q,t}\|f\|_{L^p(\mathbb{R}^n)}$$

*Proof.* First set

$$k(x, y) := |x - y|^{s-n}\,\theta_1(x)\,\theta_2(y).$$

Notice that based on $\theta_1$ and $\theta_2$ and the disjoint support of the two functions (38) we can find $\theta_3 \in C_c^\infty(\mathbb{R}^n)$, $\theta_3 \equiv 0$ in the ball $B_{d/2}$ so that

$$k(x, y) = \theta_3(x - y)|x - y|^{s-n}\ \theta_1(x)\ \theta_2(y)$$

Note that

$$\theta_3(\cdot)|\cdot|^{s-n} \in C^\infty(\mathbb{R}^n)$$

In particular, $k(\cdot, y) \in C_c^\infty(\mathbb{R}^n)$ for any $y \in \mathbb{R}^n$ and $k(x, \cdot) \in C_c^\infty(\mathbb{R}^n)$ for any $x \in \mathbb{R}^n$. Morever, $(-\Delta)_x^{\frac{t}{2}}k(x, \cdot) \in C_c^\infty(\mathbb{R}^n)$ for any $x \in \mathbb{R}^n$. Then for $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} T(-\Delta)^{\frac{t}{2}}f(x) &= \int_{\mathbb{R}^n} k(x, y)(-\Delta)_y^{\frac{t}{2}}f(y)\,dy \\ &= \int_{\mathbb{R}^n} (-\Delta)_y^{\frac{t}{2}}k(x, y)\,f(y)\,dy, \end{aligned}$$

where we integrated by parts.

Setting

$$\tilde{k}(x, y) := (-\Delta)_y^{\frac{t}{2}}k(x, y),$$

and using that $(-\Delta)^{\frac{t}{2}}\varphi(y)$ decays like $|y|^{-n-t}$ at infinity if $\varphi$ is compactly supported, we bound

$$\sup_{x\in\mathbb{R}^n}\|\tilde{k}(x, \cdot)\|_{L^r(\mathbb{R}^n)} < \infty, \quad \sup_{y\in\mathbb{R}^n}\|\tilde{k}(\cdot, y)\|_{L^r(\mathbb{R}^n)} < \infty$$

for every $r \in [1,\infty]$. Then, by a straightforward adaption of Young's convolution inequality, if $1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}$ we get

$$\begin{aligned}\|T(-\Delta)^{\frac{t}{2}}f\|_{L^q(\mathbb{R}^n)} &\leq \left(\sup_{x\in\mathbb{R}^n}\|\tilde{k}(x,\cdot)\|_{L^r(\mathbb{R}^n)}+\sup_{y\in\mathbb{R}^n}\|\tilde{k}(\cdot,y)\|_{L^r(\mathbb{R}^n)}\right)\|f\|_{L^p(\mathbb{R}^n)}\\ &= C_{\theta_1,\theta_2,t,p,q}\,\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

□

In some special cases we need to compute the constant in the Lemma above.

**Lemma 3.7.** *For any $p \in [1,\infty]$, $q \in [1,\infty)$, any $\rho > 0$, $k \geq 2$, $s \in (0,n)$, we have the following estimate for any $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\|I_s(\theta_{A_{2^k\rho}}(-\Delta)^{\frac{t}{2}}f)\|_{L^p(B_\rho)} \prec (2^k\rho)^{s-t-\frac{n}{q}}\ \rho^{\frac{n}{p}}\ \|f\|_{L^q(\mathbb{R}^n)}. \tag{39}$$

*Similarly, for any $g \in C_c^\infty(B_\rho)$,*

$$\|(-\Delta)^{\frac{t}{2}}(\theta_{A_{2^k\rho}}I_sg)\|_{L^{q'}(\mathbb{R}^n)} \prec (2^k\rho)^{s-t-\frac{n}{q}}\ \rho^{\frac{n}{p}}\|g\|_{L^{p'}(B_\rho)}.$$

*Proof.* The second estimate follows from the first one by duality. Indeed

$$\begin{aligned}\|(-\Delta)^{\frac{t}{2}}(\theta_{A_{2^k\rho}}I_sg)\|_{L^{q'}(\mathbb{R}^n)} &\prec \sup_{f\in\mathcal{S}(\mathbb{R}^n),\|f\|_{L^q(\mathbb{R}^n)}\leq 1}\int_{\mathbb{R}^n} f\,(-\Delta)^{\frac{t}{2}}(\theta_{A_{2^k\rho}}I_sg)\,dx\\ &= \sup_{f\in\mathcal{S}(\mathbb{R}^n),\|f\|_{L^q(\mathbb{R}^n)}\leq 1}\int_{\mathbb{R}^n} I_s(\theta_{A_{2^k\rho}}(-\Delta)^{\frac{t}{2}}f)\,g\,dx\\ &\prec (2^k\rho)^{s-t-\frac{n}{q}}\ \rho^{\frac{n}{p}}\ \|g\|_{L^{p'}(\mathbb{R}^n)},\end{aligned}$$

where we used integration by parts twice, cf. (46), and (39).

The estimate (39) for $p \in [1,\infty)$ follows via Hölder's inequality from the case $p=\infty$ which we shall now prove. Up to scaling we can take $\rho = 1$ and then (39) reduces to

$$\|I_s(\theta_{A_{2^k}}(-\Delta)^{\frac{t}{2}}f)\|_{L^\infty(B_1)} \prec (2^k)^{s-t-\frac{n}{q}}\|f\|_{L^q(\mathbb{R}^n)}. \tag{40}$$

For $k=2$ (40) follows from Lemma 3.6:

$$\|I_s(\theta_{A_4}(-\Delta)^{\frac{t}{2}}f)\|_{L^\infty(B_1)} \leq C_1\|f\|_{L^q(\mathbb{R}^n)},$$

with $C_1$ depending on $s,t,n,q$ and the chosen cut-off function $\theta_{B_1}$ (fixed in Definition 3.3). The case $k>2$ follows from the case $k=2$ by scaling:

$$\begin{aligned}\|I_s(\theta_{A_{2^{k+2}}}(-\Delta)^{\frac{t}{2}}f)\|_{L^\infty(B_1)} &\leq \|I_s(\theta_{A_{2^{k+2}}}(-\Delta)^{\frac{t}{2}}f)\|_{L^\infty(B_{2^k})}\\ &= (2^k)^{s-t}\|I_s(\theta_{A_4}(-\Delta)^{\frac{t}{2}}f(2^k\cdot))\|_{L^\infty(B_1)}\\ &\leq C_1(2^k)^{s-t}\|f(2^k\cdot)\|_{L^q(\mathbb{R}^n)}\\ &= C_1(2^k)^{s-t-\frac{n}{q}}\|f\|_{L^q(\mathbb{R}^n)}.\end{aligned}$$

□

Considering above $\theta_{A_{2^k\rho}}g$ instead of $\theta_{A_{2^k\rho}}$ we also have the following:

**Lemma 3.8.** *For any $\rho > 0$, $p \in (1,\infty)$*

$$\|I_s(\theta_{A_{2^k\rho}} g(-\Delta)^{\frac{n}{4}} f)\|_{L^p(B_\rho)} \prec \max_{t\in[0,\frac{n}{2}]} \|(-\Delta)^{\frac{t}{2}} g\|_{L^2(\mathbb{R}^n)} (2^k\rho)^{t-\frac{n}{2}+s-n} \rho^{\frac{n}{p}}\ \|f\|_{L^2(\mathbb{R}^n)},$$

*for any $f, g \in \mathcal{S}(\mathbb{R}^n)$.*

*Proof.* By duality, the claim follows if we show for any $\varphi \in C_c^\infty(B_\rho)$

$$\|(-\Delta)^{\frac{n}{4}} (\theta_{A_{2^k\rho}} g(I_s\varphi))\|_{L^2(\mathbb{R}^n)} \prec \max_{t\in[0,\frac{n}{2}]} \|(-\Delta)^{\frac{t}{2}} g\|_{L^2(\mathbb{R}^n)} (2^k\rho)^{t-\frac{n}{2}+s-m} \rho^{\frac{n}{p}}\ \|\varphi\|_{L^{p'}(\mathbb{R}^n)}. \tag{41}$$

By the definition of the three-term-commutator $H_{\frac{n}{2}}$, Hölder inequality for a small $t > 0$, and the related estimates, see Theorem 3.9,

$$\begin{aligned}
\|(-\Delta)^{\frac{n}{4}} (\theta_{A_{2^k\rho}} g(I_s\varphi))\|_{L^2(\mathbb{R}^n)} \prec & \|(-\Delta)^{\frac{n}{4}} g\|_{L^2(\mathbb{R}^n)}\ \|\theta_{A_{2^k\rho}} I_s\varphi\|_{L^\infty(\mathbb{R}^n)} \\
& + \|g\|_{L^{\frac{2n}{n-2t}}(\mathbb{R}^n)}\ \|(-\Delta)^{\frac{n}{4}} (\theta_{A_{2^k\rho}} I_s\varphi)\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \\
& + \|H_{\frac{n}{2}}(g, \theta_{A_{2^k\rho}}(I_s\varphi))\|_{L^2(\mathbb{R}^n)} \\
\prec & \|(-\Delta)^{\frac{n}{4}} g\|_{L^2(\mathbb{R}^n)}\ \|\theta_{A_{2^k\rho}} I_s\varphi\|_{L^\infty(\mathbb{R}^n)} \\
& + \|g\|_{L^{\frac{2n}{n-2t}}(\mathbb{R}^n)}\ \|(-\Delta)^{\frac{n}{4}} (\theta_{A_{2^k\rho}} I_s\varphi)\|_{L^{\frac{n}{t}}(\mathbb{R}^n)} \\
& + \|(-\Delta)^{\frac{n}{4}} g\|_{L^2(\mathbb{R}^n)}\ \|(-\Delta)^{\frac{n}{4}} (\theta_{A_{2^k\rho}} I_s\varphi)\|_{L^2(\mathbb{R}^n)}
\end{aligned}$$

By the Sobolev inequality,

$$\|g\|_{L^{\frac{2n}{n-2t}}(\mathbb{R}^n)} \prec \|(-\Delta)^{\frac{t}{2}} g\|_{L^2(\mathbb{R}^n)},$$

and from Lemma 3.5 with $\operatorname{supp}\varphi \subset B_\rho$

$$\|\theta_{A_{2^k\rho}} I_s\varphi\|_{L^\infty(\mathbb{R}^n)} \prec (2^k\rho)^{s-n} \|\varphi\|_{L^1(\mathbb{R}^n)} \prec (2^k\rho)^{s-n} \rho^{\frac{n}{p'}}\ \|\varphi\|_{L^{p'}(\mathbb{R}^n)}.$$

The remaining terms can be estimated with Lemma 3.7, and (41) follows. □

Let for $t > 0$ the three term commutator given as

$$H_t(a,b) := (-\Delta)^{\frac{t}{2}}(ab) - b(-\Delta)^{\frac{t}{2}} a - a(-\Delta)^{\frac{t}{2}} b.$$

A version similar to $H$ was first was introduced in [9]. For subsequent similar results and extended arguments see also [8, 31],[4, Lemma A.5],[11].

**Theorem 3.9.** *Given $p \in (1,\infty)$, $t \geq 0$, $p_1, p_2 \in (1, \frac{n}{t}]$ satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{t}{n},$$

*it holds*

$$\|H_t(a,b)\|_{L^p(\mathbb{R}^n)} \prec \|(-\Delta)^{\frac{t}{2}} a\|_{L^{p_1}(\mathbb{R}^n)}\ \|(-\Delta)^{\frac{t}{2}} b\|_{L^{p_2}(\mathbb{R}^n)}, \quad \textit{for } a, b \in \mathcal{S}(\mathbb{R}^n).$$

# A Appendix

## A.1 The Riesz transform and Riesz potential

We define the Riesz potential of $u$ for $s \in (0,n)$ and $u \in \mathcal{S}(\mathbb{R}^n)$

$$I_s u := |\cdot|^{s-n} * u,$$

By the density of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, the Riesz potential $I_s$ can be extended to an operator mapping $L^p(\mathbb{R}^n)$ into $L^{\frac{np}{n-s}}(\mathbb{R}^n)$ whenever $p, \frac{np}{n-s} \in (1,\infty)$. Up to a constant, the Riesz potential $I_s$ is the inverse of the fractional laplacian $(-\Delta)^{\frac{s}{2}}$, in the sense that for a constant $c_{n,s} \in \mathbb{R}$

$$(-\Delta)^{\frac{s}{2}} I_s f = I_s (-\Delta)^{\frac{s}{2}} f = c_{n,s} f \quad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

The Riesz transform $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n)$ is defined as

$$\mathcal{R}u(x) := \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1}} u(y)\, dy, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

and by density can be extended to a continuous operator from $L^p(\mathbb{R}^n)$ into itself:

$$\|\mathcal{R}u\|_{L^p(\mathbb{R}^n)} \leq c_{p,n} \|u\|_{L^p(\mathbb{R}^n)} \quad \text{for } u \in L^p(\mathbb{R}^n). \tag{42}$$

One crucial properties of the Riesz transform is the that

$$\sum_{i=1}^{n} \mathcal{R}_i \mathcal{R}_i = c_n\, Id, \tag{43}$$

and

$$\mathcal{R}(-\Delta)^{\frac{1}{2}} f = c_n \nabla f, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{44}$$

We also recall the following property:

**Lemma A.1** ("Integration by parts"). *For any $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$, $p \in [1,\infty]$ so that $\mathcal{R}f \in L^p(\mathbb{R}^n)$, $\mathcal{R}g \in L^{p'}(\mathbb{R}^n)$ it holds*

$$\int_{\mathbb{R}^n} \mathcal{R}f\, g\, dx = -\int_{\mathbb{R}^n} f\, \mathcal{R}g\, dx. \tag{45}$$

*For any $f \in L^p(\mathbb{R}^n)$, $g \in L^{p'}(\mathbb{R}^n)$, $p \in (1,\infty)$ so that $(-\Delta)^{\frac{s}{2}} f \in L^p(\mathbb{R}^n)$, $(-\Delta)^{\frac{s}{2}} g \in L^{p'}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} f\, g\, dx = \int_{\mathbb{R}^n} f\, (-\Delta)^{\frac{s}{2}} g\, dx. \tag{46}$$

Note that together (45) and (46) imply the usual integration by parts formula

$$\int_{\mathbb{R}^n} \nabla f\, g\, dx = c \int_{\mathbb{R}^n} \mathcal{R}(-\Delta)^{\frac{1}{2}} f\, g\, dx = -c \int_{\mathbb{R}^n} f\, \mathcal{R}(-\Delta)^{\frac{1}{2}} g\, dx = -\int_{\mathbb{R}^n} f\, \nabla g\, dx.$$

## A.2 Lorentz spaces and Sobolev inequality

**Definition A.2.** *For $1 < p < \infty$ and $1 \leq q \leq \infty$, we define the Lorentz space $L^{(p,q)}(\mathbb{R}^n)$ as the space of measurable functions $f$ for which*

$$\|f\|_{L^{(p,q)}} := p^{1/q} \|\lambda |\{|f| > \lambda\}|^{1/p}\|_{L^q(\frac{d\lambda}{\lambda})} < \infty$$

It is important to notice that $L^{(p,p)} = L^p$ and $L^{(p,q)} \subset L^{(p,r)}$ if $r \geq q$.

**Proposition A.3** (Sobolev inequality)**.** *Let $1 < p < \frac{n}{\alpha}$ and $1 \leq r \leq \infty$. If $f \in L^{(p,q)}(\mathbb{R}^n)$, then $I_\alpha f \in L^{(q,r)}(\mathbb{R}^n)$ for $q = \frac{np}{n-\alpha p}$. Moreover, there exists $C >$ such that*

$$\|I_\alpha f\|_{L^{(p,r)}(\mathbb{R}^n)} \leq C \|f\|_{L^{(q,r)}(\mathbb{R}^n)}.$$

*For $p = 1$, $I_\alpha$ maps $L^1(\mathbb{R}^n)$ into $L^{(q,\infty)}(\mathbb{R}^n)$ for $q = \frac{n}{n-\alpha}$. For $p = \frac{n}{\alpha}$, $I_\alpha$ is bounded from $L^{(p,1)}$ into $L^\infty(\mathbb{R}^n)$.*

From [3, Corollary 6.16] we have

**Proposition A.4.** *$I_\alpha$ is a bounded linear operator from $L \log^r L(\mathbb{R}^n)$ to $L^{(\frac{n}{n-\alpha}, \frac{1}{r})}(\mathbb{R}^n)$ whenever $r \leq 1$, $\alpha \in (0, n)$.*

## A.3 Interior estimates

The following are a few estimates which could be seen as $L^p$-theory for the fractional Laplacian. Since we only need interior estimates, the proofs are long, but elementary – just relying on the definitions of Riesz potential, Riesz transform and fractional Laplacian.

**Lemma A.5.** *Let $\Omega \subset \mathbb{R}^n$ be open. Then for any $h \in H^{\frac{n}{2}}(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{4}} h \, (-\Delta)^{\frac{n}{4}} \varphi \, dx = 0 \quad \forall \varphi \in C_c^\infty(\Omega), \tag{47}$$

*we have $h \in C^\infty_{\mathrm{loc}}(\Omega)$, and for any compact set $K \Subset \Omega$ and any $\ell \in \mathbb{N}_0$, $\alpha \in (0, 1]$ we have*

$$[\nabla^\ell h]_{C^{0,\alpha}(K)} \leq C_{\ell,\alpha,K,\Omega} \, \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)}.$$

*Proof.* The smoothness $h$, i.e. $h \in C^\infty_{\mathrm{loc}}(\Omega)$, follows via an approximation argument from the a priori estimates below. Notice that (47) can be rewritten as

$$\int_{\mathbb{R}^n} \nabla h \cdot \nabla (-\Delta)^{\frac{n-2}{2}} \varphi \, dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{48}$$

Fix now $K \subset\subset K_1 \subset\subset K_2 \subset\subset \Omega$. For arbitary $\psi \in C_c^\infty(K_1)$ we have for $k \in \mathbb{N}_0$,

$$\Delta^k \psi = (-\Delta)^{\frac{n-2}{2}} I_{n-2} \Delta^k \psi.$$

Thus, taking a cutoff-function $\eta_{K_2} \in C_c^\infty(\Omega)$, $\eta_{K_2} \equiv 1$ on $K_2$,

$$\Delta^k \psi = (-\Delta)^{\frac{n-2}{2}} (\eta_{K_1} I_{n-2} \Delta^k \psi) + (-\Delta)^{\frac{n-2}{2}} ((1 - \eta_{K_1}) I_{n-2} \Delta^k \psi).$$

Thus for any $\psi \in C_c^\infty(K_1)$, using (48) with $\varphi := \eta_{K_2} I_{n-2} \Delta^k \psi$, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \nabla h \cdot \nabla \Delta^k \psi \, dx = & \int_{\mathbb{R}^n} \nabla h \cdot \nabla (-\Delta)^{\frac{n-2}{2}} ((1 - \eta_{K_2}) I_{n-2} \Delta^k \psi) \, dx \\
\leq & \|\nabla h\|_{L^n(\mathbb{R}^n)} \|\nabla (-\Delta)^{\frac{n-2}{2}} ((1 - \eta_{K_2}) I_{n-2} \Delta^k \psi)\|_{L^{n'}(\mathbb{R}^n)} \\
\leq & \|\nabla h\|_{L^n(\mathbb{R}^n)} \|(-\Delta)^{\frac{n-1}{2}} ((1 - \eta_{K_2}) I_{n-2} \Delta^k \psi)\|_{L^{n'}(\mathbb{R}^n)} \\
\leq & C_{K_1,K_2} \, \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)} \, \|\psi\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

The second-to-last step follows again from $\nabla = \mathcal{R}(-\Delta)^{\frac{1}{2}}$ and because the Riesz transform $\mathcal{R}$ is bounded on $L^{n'}$. In the last step we used that the support of $1-\eta_{K_2}$ and $\psi$ are disjoint to apply Lemma 3.6, and Sobolev inequality. Classical regularity theory of elliptic PDE ensures that $h$ belongs to any Sobolev space $W^{\ell,p}_{\mathrm{loc}}(K_1)$ for any $\ell \in \mathbb{N}$, $p \in (1,\infty)$ together with the estimates

$$\|h\|_{W^{\ell,p}(K)} \leq C_{\ell,p,K,\Omega} \left(\|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)} + \|h\|_{L^2(\mathbb{R}^n)}\right) \prec \|h\|_{H^{\frac{n}{2}}(\mathbb{R}^n)},$$

and

$$\|\nabla h\|_{W^{\ell+1,p}(K)} \leq C_{\ell,p,K,\Omega} \, \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)}$$

The latter implies via the Morrey-Sobolev imbedding that for any $\alpha \in (0,1)$, $l \in \mathbb{N}_0$

$$[\nabla^\ell h]_{C^{0,\alpha}(K)} \leq C_{\ell,\alpha,K,\Omega} \, \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)}.$$

□

**Lemma A.6.** *Let $\Omega$ be an open set of $\mathbb{R}^n$. Then for any $h \in H^{\frac{n}{2}}(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{n}{4}} h \, (-\Delta)^{\frac{n}{4}} \varphi \, dx = \int_{\mathbb{R}^n} f \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega),$$

*and for any $\ell \in \{0,1,\ldots,n-1\}$, $\alpha \in (0,1)$ we have on any compact $K \subset\subset \Omega$,*

$$[\nabla^\ell h]_{C^{0,\alpha}(K)} \leq C_{\ell,\alpha,\Omega,K} \, \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)} + C_{\ell,\Omega,K} \, \|f\|_{L^\infty(\Omega)}.$$

*Proof.* The proof is very similar to the one of Lemma A.5. Fix again $K \subset\subset K_1 \subset\subset K_2 \subset\subset \Omega$. We use that the following equation (note that $n-1$ is even and thus $(-\Delta)^{\frac{n-1}{2}}$ is the classical $(n-1)$-Laplacian),

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{1}{2}} h \, \Delta^{\frac{n-1}{2}} \varphi \, dx = \int_{\mathbb{R}^n} f \varphi \, dx \quad \forall \varphi \in C_c^\infty(\Omega).$$

Elliptic theory implies $(-\Delta)^{\frac{1}{2}} h \in W^{n-1,p}_{loc}(\Omega)$ for any $p \in (1,\infty)$, with the estimate

$$\|(-\Delta)^{\frac{1}{2}} h\|_{W^{n-1,p}(K_2)} \prec \|f\|_{L^\infty(\Omega)} + \|(-\Delta)^{\frac{1}{2}} h\|_{L^n(\mathbb{R}^n)} \prec \|f\|_{L^\infty(\Omega)} + \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)}. \tag{49}$$

Here again we used that $(-\Delta)^{\frac{1}{2}} h \in L^n(\mathbb{R}^n)$ by Sobolev embedding. With this in mind, we can write $\nabla h$ in terms of the Riesz transform $\mathcal{R}$ and $(-\Delta)^{\frac{1}{2}}$,

$$\nabla h = \mathcal{R}(\eta_{K_1}(-\Delta)^{\frac{1}{2}} h) + \mathcal{R}((1-\eta_{K_1})(-\Delta)^{\frac{1}{2}} h), \tag{50}$$

where we have a cutoff function $\eta_{K_1} \in C_c^\infty(K_2)$ and $\eta_{K_1} \equiv 1$ in $K_1$. The first term on the right-hand side belongs to $W^{n-1,p}(\mathbb{R}^n)$ by (49) and the boundedness of the Riesz transform, and we have

$$\|\mathcal{R}(\eta_{K_1}(-\Delta)^{\frac{1}{2}} h)\|_{W^{n-1,p}(\mathbb{R}^n)} \prec \|f\|_\infty + \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)}.$$

The second term on the right-hand side of (50) is smooth in $K$, by the disjoint support of $\chi_K$ and $(1-\eta_{K_1})$. Indeed, by Lemma 3.5 for any $\ell \geq 0$,

$$\|\nabla^\ell \mathcal{R}((1-\eta_{K_1})(-\Delta)^{\frac{1}{2}} h)\|_{L^\infty(K)} \prec C_{K,K_1} \, \|(-\Delta)^{\frac{1}{2}} h\|_{L^n(\mathbb{R}^n)} \prec C_{K,K_1} \, \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)}.$$

Together, we have shown that for any $0 \leq \ell \leq n-1$, $p \in (1,\infty)$,

$$\|\nabla h\|_{W^{\ell,p}(K)} \prec \|f\|_{L^\infty(\Omega)} + \|(-\Delta)^{\frac{n}{4}} h\|_{L^2(\mathbb{R}^n)}.$$

Now the Sobolev-Morrey embedding gives the claim. □

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