An application of Q-curvature to an embedding of critical type

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Let $\Omega \subset \mathbb{R}^{2m}$ be open, bounded and with smooth boundary, and let a sequence $\lambda_k \to 0^+$ be given. Consider a sequence $(u_k)_{k\in\mathbb{N}}$ of positive smooth solutions to

$$\begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2} & \text{in } \Omega \\ u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Solutions to (1) arise from the Adams-Moser-Trudinger inequality [1, 10, 13]:

$$\sup_{u\in H_0^m(\Omega),\, \|u\|^2_{H_0^m}\le \Lambda_1} \fint_\Omega e^{mu^2} dx = c_0(m) < +\infty, \tag{2}$$

where $c_0(m)$ is a dimensional constant, and $H_0^m(\Omega)$ is the Beppo-Levi space defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_0^m} := \|\Delta^{\frac{m}{2}} u\|_{L^2} = \left(\int_\Omega |\Delta^{\frac{m}{2}} u|^2 dx\right)^{\frac{1}{2}}, \tag{3}$$

and we set $\Delta^{\frac{m}{2}} u := \nabla\Delta^{\frac{m-1}{2}} u$ for m odd. In fact critical points of (2) under the constraint $\|u\|^2 = \Lambda_1$ solve (1). Then we have the following concentration-compactness result:

Theorem 1 *Let (u_k) be a sequence of solutions to (1) such that*

$$\int_\Omega u_k(-\Delta)^m u_k dx = \lambda_k \int_\Omega u_k^2 e^{mu_k^2} dx \to \Lambda < \infty, \quad \text{as } k\to\infty. \tag{4}$$

Then either

(i) $\Lambda = 0$ and $u_k \to 0$ in $C^{2m-1,\alpha}(\Omega)$, or

(ii) There exists $I \in \mathbb{N}\backslash\{0\}$ such that $\Lambda \ge I\Lambda_1$, where $\Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m})$, and there is a finite set $S = \{x^{(1)}, \ldots, x^{(I)}\}$ such that

$$u_k \to 0 \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash S),$$

and

$$\lambda_k u_k^2 e^{mu_k^2} \rightharpoonup \sum_{i=1}^I \alpha_i \delta_{x^{(i)}}, \quad \alpha_i \ge \Lambda_1.$$

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Theorem 1 has been proven by Adimurthi and M. Struwe [3] and Adimurthi and O. Druet [2] in the case $m = 1$, and by F. Robert and M. Struwe [11] for $m = 2$. Recently O. Druet [6] for the case $m = 1$, and M. Struwe [12] for $m = 2$ improved the previous results by showing that in case (ii) of Theorem 1 we have $\Lambda = L\Lambda_1$ for some positive $L \in \mathbb{N}$. Whether the same holds true for $m > 2$ is still an open question.

Part (ii) of the theorem shows an interesting threshold phenomenon: below the critical energy level Λ_1 we always have compactness. Moreover Λ_1 is the total Q-curvature of the sphere (see [8] for a short discussion of Q-curvature). We shall briefly explain how this remarkable connection with Riemannian geometry arises. It is easy to see that if we are not in case (i) of the theorem, then $\sup_\Omega u_k \to \infty$ as $k \to \infty$. Then we blow up, i.e. we define the scaled functions

$$\eta_k(x) := u_k(x_k)(u_k(x_k + r_k x) - u_k(x_k)) + \log 2,$$

where x_k is such that $u_k(x_k) = \max_\Omega u_k$ and $r_k \to 0$ is a scaling factor suitably chosen. Then it turns out that

$$\eta_k(x) \to \eta_0(x) \quad \text{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}), \quad (k \to \infty), \tag{5}$$

where η_0 is a solution of the Liouville-type equation

$$(-\Delta)^m \eta = (2m-1)! e^{2m\eta} \quad \text{on } \mathbb{R}^{2m}, \quad \int_{\mathbb{R}^{2m}} e^{2m\eta} dx < \infty. \tag{6}$$

We recall (see e.g. [8]) that if η solves $(-\Delta)^m \eta = V e^{2m\eta}$ on $\mathbb{R}^{2m}$, then the conformal metric $g_\eta := e^{2\eta}|dx|^2$ has Q-curvature V, where $|dx|^2$ denotes the Euclidean metric. Now the problem is to understand what is the solution η_0 or (equivalently) what is the conformal metric g_{η_0}. A bunch of solution to (6) is given by the so-called *standard solutions*

$$\eta_{\lambda,x_0}(x) = \log \frac{2\lambda}{1 + \lambda^2|x - x_0|^2}, \quad \lambda > 0, x_0 \in \mathbb{R}^{2m}.$$

These are "spherical" solutions, as the metric $e^{2\eta_{\lambda,x_0}}|dx|^2$ can be obtained by pulling-back the metric of the round sphere S^{2m} onto $\mathbb{R}^{2m}$ via the stereographic projection and a Möbius diffeomorphism.

While Chen and Li [5] proved that in the case $m = 1$ the only solutions to (6) are the standard solutions, Chang and Chen [4] show that for any $m > 1$ (6) possesses many other solutions. Therefore the problem of understanding η_0 starts to appear quite subtle. In fact we claim the following

Proposition 2 *For any $m > 1$ the function η_0 given by* (5) *is a standard solution to* (6).

Proposition 2 yields

$$\begin{aligned} \lim_{k\to\infty} \int_\Omega \lambda_k u_k^2 e^{m u_k^2} dx &\geq (2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx \\ &= (2m-1)! \int_{\mathbb{R}^{2m}} Q_{S^{2m}} \mathrm{dvol}_{g_{S^{2m}}} = \Lambda_1, \end{aligned}$$

This is the basic reason why $\alpha_i \geq \Lambda_1$ in case (ii) of Theorem 1.

The proof of Proposition 2 is based on a classification result for the solution to (6) due to the author [8] and on sharp gradient estimates for u_k. Let us start with the latter.

Proposition 3 *For any $R > 0$, $1 \leq \ell \leq 2m-1$ there exists k_0 such that*

$$u_k(x_k) \int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \leq C(Rr_k)^{2m-\ell}, \quad \text{for all } k \geq k_0. \tag{7}$$

Equivalently

$$\int_{B_R(0)} |\nabla^\ell \eta_k| dx \leq CR^{2m-\ell}. \tag{8}$$

The key idea in proving (7) is that

$$\|\Delta^m(u_k^2)\|_{L^1(\Omega)} \leq C, \tag{9}$$

so that one can write u_k^2 in terms of the Green function for Δ^m on Ω (it is convenient to work with u_k^2 because (7) is quadratic in u_k). Estimate (9) is an easy consequence of

Proposition 4 *For every $1 \leq \ell \leq 2m-1$, $\nabla^\ell u_k$ belongs to the Lorentz space $L^{(2m/\ell,2)}(\Omega)$ and*

$$\|\nabla^\ell u_k\|_{(2m/\ell,2)} \leq C.$$

This follows by interpolation theory once we observe that (4) implies that $\Delta^m u_k$ is bounded in the Zygmund space $L(\log L)^{\frac{1}{2}}$. It is instructive to remark that if we decided to be a bit sloppy and consider that (4) gives bounds for $\Delta^m u_k$ in $L^1(\Omega)$, then we would obtain the bounds $\|\nabla^\ell u_k\|_{(2m/\ell,\infty)} \leq C$ ($L^{(p,\infty)}$ being the Marcinkievicz space, or weak-L^p). On the other hand those bounds are too weak to prove (9), hence Proposition 3.

We now turn to the classification result for solutions to (6).

Theorem 5 ([8]) *Let η be a solution to (6) and set*

$$v(x) := \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y|}{|x-y|} e^{2mu(y)}\right) dt,$$

where γ_m is such that $(-\Delta)^m \frac{1}{\gamma_m} \log \frac{1}{|x|} = \delta_0$. Then $\eta = v + p$, where p is a polynomial of degree at most $2m-2$ and

$$\lim_{|x|\to\infty} \Delta^j v(x) = 0, \quad 1 \leq j \leq m-1.$$

Moreover the following are equivalent:

(i) η is a standard solution,

(ii) p is constant.

Moreover if η is not a standard solution there exist $1 \leq j \leq m-1$ and a constant $a \neq 0$ such that

$$\lim_{|x|\to\infty} \Delta^j \eta(x) = a. \tag{10}$$

Taking the limit in (8) and applying Theorem 5 implies that η_0 is a standard solution.

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