A threshold phenomenon for embeddings of H_0^m into Orlicz spaces

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Abstract

Given an open bounded domain $\Omega \subset \mathbb{R}^{2m}$ with smooth boundary, we consider a sequence $(u_k)_{k\in\mathbb{N}}$ of positive smooth solutions to

$$\begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{m u_k^2} & \text{in } \Omega \\ u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda_k \to 0^+$. Assuming that the sequence is bounded in $H_0^m(\Omega)$, we study its blow-up behavior. We show that if the sequence is not precompact, then

$$\liminf_{k\to\infty} \|u_k\|^2_{H_0^m} := \liminf_{k\to\infty} \int_\Omega u_k (-\Delta)^m u_k dx \geq \Lambda_1,$$

where $\Lambda_1 = (2m-1)!\mathrm{vol}(S^{2m})$ is the total Q-curvature of S^{2m}.

1 Introduction and statement of the main result

Let $\Omega \subset \mathbb{R}^{2m}$ be open, bounded and with smooth boundary, and let a sequence $\lambda_k \to 0^+$ be given. Consider a sequence $(u_k)_{k\in\mathbb{N}}$ of smooth solutions to

$$\begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{m u_k^2} & \text{in } \Omega \\ u_k > 0 & \text{in } \Omega \\ u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega. \end{cases} \quad (1)$$

Assume also that

$$\int_\Omega u_k (-\Delta)^m u_k dx = \lambda_k \int_\Omega u_k^2 e^{m u_k^2} dx \to \Lambda \geq 0 \quad \text{as } k \to \infty. \quad (2)$$

In this paper we shall prove

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Theorem 1 *Let (u_k) be a sequence of solutions to (1), (2). Then either*

(i) $\Lambda = 0$ *and* $u_k \to 0$ *in* $C^{2m-1,\alpha}(\Omega)$,[1] *or*

(ii) We have $\sup_\Omega u_k \to \infty$ *as* $k \to \infty$. *Moreover there exists* $I \in \mathbb{N}\backslash\{0\}$ *such that* $\Lambda \geq I\Lambda_1$, *where* $\Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m})$, *and up to a subsequence there are* I *converging sequences of points* $x_{i,k} \to x^{(i)}$ *and of positive numbers* $r_{i,k} \to 0$, *the latter defined by*

$$\lambda_k r_{i,k}^{2m} u_k^2(x_{i,k}) e^{m u_k^2(x_{i,k})} = 2^{2m}(2m-1)!, \tag{3}$$

such that the following is true:

1. *For every* $1 \leq i \leq I$ *we have* $\lim_{k\to\infty} \frac{\mathrm{dist}(x_{i,k},\partial\Omega)}{r_{i,k}} = +\infty$.
2. *If we define*

   $$\eta_{i,k}(x) := u_k(x_{i,k})(u_k(x_{i,k} + r_{i,k}x) - u_k(x_{i,k})) + \log 2$$

   for $1 \leq i \leq I$, *then*

   $$\eta_{i,k}(x) \to \eta_0(x) = \log\frac{2}{1+|x|^2} \quad \text{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}) \quad (k \to \infty). \tag{4}$$

3. *For every* $1 \leq i \neq j \leq I$ *we have* $\lim_{k\to\infty} \frac{|x_{i,k}-x_{j,k}|}{r_{i,k}} = \infty$.
4. *Set* $R_k(x) := \inf_{1\leq i \leq I} |x - x_{i,k}|$. *Then*

   $$\lambda_k R_k^{2m}(x) u_k^2(x) e^{m u_k^2(x)} \leq C, \tag{5}$$

   where C *does not depend on* x *or* k.

Finally $u_k \rightharpoonup 0$ *in* $H^m(\Omega)$ *and* $u_k \to 0$ *in* $C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash\{x^{(1)},\ldots,x^{(I)}\})$.

Solutions to (1) arise from the Adams-Moser-Trudinger inequality [Ada] (see also [Mos], [Tru] and [BW]):

$$\sup_{u\in H_0^m(\Omega),\, \|u\|^2_{H_0^m}\leq\Lambda_1} \fint_\Omega e^{mu^2}dx = c_0(m) < +\infty, \tag{6}$$

where $c_0(m)$ is a dimensional constant, and $H_0^m(\Omega)$ is the Beppo-Levi defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm[2]

$$\|u\|_{H_0^m} := \|\Delta^{\frac{m}{2}} u\|_{L^2} = \left(\int_\Omega |\Delta^{\frac{m}{2}} u|^2 dx\right)^{\frac{1}{2}}, \tag{7}$$

and we used the following notation:

$$\Delta^{\frac{m}{2}} u := \begin{cases} \Delta^n u \in \mathbb{R} & \text{if } m = 2n \text{ is even,} \\ \nabla\Delta^n u \in \mathbb{R}^{2m} & \text{if } m = 2n+1 \text{ is odd.} \end{cases} \tag{8}$$

[1] Here and in the following $\alpha \in [0,1)$ is an arbitrary Hölder exponent.

[2] The norm in (7) is equivalent to the usual Sobolev norm $\|u\|_{H^m} := \left(\sum_{\ell=0}^m \|\nabla^\ell u\|_{L^2}\right)^{\frac{1}{2}}$, thanks to elliptic estimates.

In fact (1) is the Euler-Lagrange equation of the functional

$$F(u) := \frac{1}{2}\int_\Omega |\Delta^{\frac{m}{2}}u|^2 dx - \frac{\lambda}{2m}\int_\Omega e^{mu^2}dx$$

(where $\lambda = \lambda_k$ plays the role of a Lagrange multiplier), which is well defined and smooth thanks to (6), but does not satisfy the Palais-Smale condition. For a more detailed discussion, in the context of Orlicz spaces, we refer to [Str3].

The function η_0 which appears in (4) is a solution of the higher-order Liouville's equation

$$(-\Delta)^m \eta_0 = (2m-1)! e^{2m\eta_0}, \quad \text{on } \mathbb{R}^{2m}. \tag{9}$$

We recall (see e.g. [Mar1]) that if u solves $(-\Delta)^m u = Ve^{2mu}$ on $\mathbb{R}^{2m}$, then the conformal metric $g_u := e^{2u}g_{\mathbb{R}^{2m}}$ has Q-curvature V, where $g_{\mathbb{R}^{2m}}$ denotes the Euclidean metric. This shows a surprising relation between Equation (1) and the problem of prescribing the Q-curvature. In fact η_0 has also a remarkable geometric interpretation: If $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection, then

$$e^{2\eta_0} g_{\mathbb{R}^{2m}} = (\pi^{-1})^* g_{S^{2m}}, \tag{10}$$

where $g_{S^{2m}}$ is the round metric on S^{2m}. Then (10) implies

$$(2m-1)!\int_{\mathbb{R}^{2m}} e^{2m\eta_0}dx = \int_{S^{2m}} Q_{S^{2m}} \mathrm{dvol}_{g_{S^{2m}}} = (2m-1)!|S^{2m}| = \Lambda_1. \tag{11}$$

This is the reason why $\Lambda \geq I\Lambda_1$ in case (ii) of Theorem 1 above, compare Proposition 7.

Theorem 1 has been proven by Adimurthi and M. Struwe [AS] and Adimurthi and O. Druet [AD] in the case $m = 1$, and by F. Robert and M. Struwe [RS] for $m = 2$. The extraction of a blow-up profile from a concentrating sequence of solutions to a nonlinear PDE was pioneered by J. Sack and K. Uhlenbeck [SU] and Wente [Wen]. Their ideas were later expanded in various ways by M. Struwe [Str1], [Str2], H. Brezis and J. M. Coron [BC1], [BC2] who, in particular, first wrote down separation conditions like conditions 1 and 3 in part (ii) of Theorem 1 (see also the works of T. H. Parker [Par], E. Hebey and F. Robert [HR] and many others). For further motivations and references we refer to M. Struwe [Str5]. Here, instead, we want to point out the main ingredients of our approach. Crucial to the proof of Theorem 1 are the gradient estimates in Lemma 6 and the blow-up procedure of Proposition 7. For the latter, we rely on a concentration-compactness result from [Mar2] and a classification result from [Mar1], which imply, together with the gradient estimates, that at the finitely many concentration points $\{x^{(1)}, \ldots, x^{(I)}\}$, the profile of u_k is η_0, hence an energy not less that Λ_1 accumulates, namely

$$\lim_{R\to 0}\limsup_{k\to\infty}\int_{B_R(x^{(i)})} \lambda_k u_k^2 e^{mu_k^2}dx \geq \Lambda_1, \quad \text{for every } 1 \leq i \leq I.$$

As for the gradient estimates, if one uses (1) and (2) to infer $\|\Delta^m u_k\|_{L^1(\Omega)} \leq C$, then elliptic regularity gives $\|\nabla^\ell u_k\|_{L^p(\Omega)} \leq C(p)$ for every $p \in [1, 2m/\ell)$. These bounds, though, turn out to be too weak for Lemma 6 (see also the remark after Lemma 5). One has, instead, to fully exploit the integrability of $\Delta^m u_k$ given by

(2), namely $\|\Delta^m u_k\|_{L(\log L)^{1/2}(\Omega)} \leq C$, where $L(\log L)^{1/2} \subsetneq L^1$ is the Zygmund space. Then an interpolation result from [BS] gives uniform estimates for $\nabla^\ell u_k$ in the Lorentz space $L^{(2m/\ell,2)}(\Omega)$, $1 \leq \ell \leq 2m-1$, which are sharp for our purposes (see Lemma 5).

We remark that when $m = 1$, things simplify dramatically, as we can simply integrate by parts (2) and get

$$\|\nabla u_k\|_{L^{(2,2)}(\Omega)} = \|\nabla u_k\|_{L^2(\Omega)} \leq C.$$

In the case $m = 2$, F. Robert and M. Struwe [RS] proved a slightly weaker form of our Lemma 6 by using subtle estimates in the BMO space, whose generalization to arbitrary dimensions appears quite challenging. Our approach, on the other hand, is simpler and more transparent.

Recently O. Druet [Dru] for the case $m = 1$, and M. Struwe [Str4] for $m = 2$ improved the previous results by showing that in case (ii) of Theorem 1 we have $\Lambda = L\Lambda_1$ for some positive $L \in \mathbb{N}$. Whether the same holds true for $m > 2$ is still an open question. In is also unknown whether $L = I$ in case $m = 1, 2$.

In the following, the letter C denotes a generic positive constant, which may change from line to line and even within the same line.

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2 Proof of Theorem 1

Assume first that $\sup_\Omega u_k \leq C$. Then $\Delta^m u_k \to 0$ uniformly, since $\lambda_k \to 0$. By elliptic estimates we infer $u_k \to 0$ in $W^{2m,p}(\Omega)$ for every $1 \leq p < \infty$, hence $u_k \to 0$ in $C^{2m-1,\alpha}(\Omega)$, $\Lambda = 0$ and we are in case (i) of Theorem 1.

From now on, following the approach of [RS], we assume that, up to a subsequence, $\sup_\Omega u_k \to \infty$ and show that we are in case (ii) of the theorem. In Section 2.1 we analyze the asymptotic profile at blow-up points. In Section 2.2 we sketch the inductive procedure which completes the proof.

2.1 Analysis of the first blow-up

Let $x_k = x_{1,k} \in \Omega$ be a point such that $u_k(x_k) = \max_\Omega u_k$, and let $r_k = r_{1,k}$ be as in (3). Integrating by parts in (2), we find $\|\Delta^{\frac{m}{2}} u_k\|_{L^2(\Omega)} \leq C$ which, together with the boundary condition and elliptic estimates (see e.g. [ADN]), gives

$$\|u_k\|_{H^m(\Omega)} \leq C. \tag{12}$$

Lemma 2 *We have*

$$\lim_{k\to\infty} \frac{\operatorname{dist}(x_k, \partial\Omega)}{r_k} = +\infty.$$

Proof. Set

$$\overline{u}_k(x) := \frac{u_k(r_k x + x_k)}{u_k(x_k)} \quad \text{for } x \in \Omega_k := \{r_k^{-1}(x - x_k) : x \in \Omega\}.$$

Then $\overline{u}_k$ satisfies

$$\begin{cases} (-\Delta)^m \overline{u}_k = \dfrac{2^{2m}(2m-1)!}{u_k^2(x_k)} \overline{u}_k e^{m u_k^2(x_k)(\overline{u}_k^2 - 1)} & \text{in } \Omega_k \\ \overline{u}_k > 0 & \text{in } \Omega_k \\ \overline{u}_k = \partial_\nu \overline{u}_k = \ldots = \partial_\nu^{m-1} \overline{u}_k = 0 & \text{on } \partial\Omega_k. \end{cases}$$

Assume for the sake of contradiction that up to a subsequence we have

$$\lim_{k \to \infty} \frac{\operatorname{dist}(x_k, \partial\Omega)}{r_k} = R_0 < +\infty.$$

Then, passing to a further subsequence, $\Omega_k \to \mathcal{P}$, where $\mathcal{P}$ is a half-space. Since

$$\|\Delta^m \overline{u}_k\|_{L^\infty(\Omega_k)} \le \frac{C}{u_k^2(x_k)} \to 0 \quad \text{as } k \to \infty,$$

we see that, up to a subsequence, $\overline{u}_k \to \overline{u}$ in $C^{2m-1,\alpha}_{\text{loc}}(\overline{\mathcal{P}})$, where

$$\overline{u}(0) = \overline{u}_k(0) = 1$$

and

$$\begin{cases} (-\Delta)^m \overline{u} = 0 & \text{in } \mathcal{P} \\ \overline{u} = \partial_\nu \overline{u} = \ldots = \partial_\nu^{m-1} \overline{u} = 0 & \text{on } \partial\mathcal{P}. \end{cases}$$

By (12) and the Sobolev imbedding $H^{m-1}(\Omega) \hookrightarrow L^{2m}(\Omega)$, we find

$$\int_{\Omega_k} |\nabla \overline{u}_k|^{2m} dx = \frac{1}{u_k(x_k)^{2m}} \int_\Omega |\nabla u_k|^{2m} dx \le \frac{C}{u_k(x_k)^{2m}} \to 0, \quad \text{as } k \to \infty.$$

Then $\nabla \overline{u} \equiv 0$, hence $\overline{u} \equiv const = 0$ thanks to the boundary condition. That contradicts $\overline{u}(0) = 1$. □

Lemma 3 *We have*

$$u_k(x_k + r_k x) - u_k(x_k) \to 0 \quad \text{in } C^{2m-1}_{\text{loc}}(\mathbb{R}^{2m}) \text{ as } k \to \infty. \tag{13}$$

Proof. Set

$$v_k(x) := u_k(x_k + r_k x) - u_k(x_k), \quad x \in \Omega_k$$

Then v_k solves

$$(-\Delta)^m v_k = 2^{2m}(2m-1)! \frac{\overline{u}_k(x)}{u_k(x_k)} e^{m u_k^2(x_k)(\overline{u}_k^2 - 1)} \le 2^{2m} \frac{(2m-1)!}{u_k(x_k)} \to 0. \tag{14}$$

Assume that $m > 1$. By (12) and the Sobolev embedding $H^{m-2}(\Omega) \hookrightarrow L^m(\Omega)$, we get

$$\|\nabla^2 v_k\|_{L^m(\Omega_k)} = \|\nabla^2 u_k\|_{L^m(\Omega)} \le C. \tag{15}$$

Fix now $R > 0$ and write $v_k = h_k + w_k$ on $B_R = B_R(0)$, where $\Delta^m h_k = 0$ and w_k satisfies the Navier-boundary condition on B_R. Then, (14) gives

$$w_k \to 0 \quad \text{in } C^{2m-1,\alpha}(B_R). \tag{16}$$

This, together with (15) implies

$$\|\Delta h_k\|_{L^m(B_R)} \le C. \tag{17}$$

Then, since $\Delta^{m-1}(\Delta h_k) = 0$, we get from Proposition 12

$$\|\Delta h_k\|_{C^\ell(B_{R/2})} \le C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{18}$$

By Pizzetti's formula (45),

$$\fint_{B_R} h_k dx = h_k(0) + \sum_{i=1}^{m-1} c_i R^{2i} \Delta^i h_k(0),$$

and (18), together with $|h_k(0)| = |w_k(0)| \le C$ and $h_k \le -w_k \le C$, we find

$$\fint_{B_R} |h_k| dx \le C.$$

Again by Proposition 12 it follows that

$$\|h_k\|_{C^\ell(B_{R/2})} \le C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{19}$$

By Ascoli-Arzelà's theorem, (16) and (19), we have that up to a subsequence

$$v_k \to v \quad \text{in } C^{2m-1,\alpha}(B_{R/2}),$$

where $\Delta^m v \equiv 0$ thanks to (14). We can now apply the above procedure with a sequence of radii $R_k \to \infty$, extract a diagonal subsequence $(v_{k'})$, and find a function $v \in C^\infty(\mathbb{R}^{2m})$ such that

$$v \le 0, \quad \Delta^m v \equiv 0, \quad v_{k'} \to v \quad \text{in } C^{2m-1,\alpha}_{\text{loc}}(\mathbb{R}^{2m}). \tag{20}$$

By Fatou's Lemma

$$\|\nabla^2 v\|_{L^m(\mathbb{R}^{2m})} \le \liminf_{k\to\infty} \|\nabla^2 v_{k'}\|_{L^m(\Omega_k)} \le C. \tag{21}$$

By Theorem 13 and (20), v is a polynomial of degree at most $2m-2$. Then (20) and (21) imply that v is constant, hence $v \equiv v(0) = 0$. Therefore the limit does not depend on the chosen subsequence $(v_{k'})$, and the full sequence (v_k) converges to 0 in $C^{2m-1}_{\text{loc}}(\mathbb{R}^{2m})$, as claimed.

When $m = 1$, Pizzetti's formula and (14) imply at once that, for every $R > 0$, $\|v_k\|_{L^1(B_R)} \to 0$, hence $v_k \to 0$ in $W^{2,p}(B_{R/2})$ as $k \to \infty$, $1 \le p < \infty$. □

Now set

$$\eta_k(x) := u_k(x_k)[u_k(r_k x + x_k) - u_k(x_k)] + \log 2 \le \log 2. \tag{22}$$

An immediate consequence of Lemma 3 is the following

Corollary 4 *The function* η_k *satisfies*

$$(-\Delta)^m \eta_k = V_k e^{2m a_k \eta_k}, \tag{23}$$

where

$$V_k(x) = 2^{m(1-\overline{u}_k)}(2m-1)!\overline{u}_k(x) \to (2m-1)!, \quad a_k = \frac{1}{2}(\overline{u}_k + 1) \to 1$$

in $C^0_{\text{loc}}(\mathbb{R}^{2m})$.

Lemma 5 *For every $1 \leq \ell \leq 2m-1$, $\nabla^\ell u_k$ belongs to the Lorentz space $L^{(2m/\ell,2)}(\Omega)$ and*

$$\|\nabla^\ell u_k\|_{(2m/\ell,2)} \leq C. \tag{24}$$

Proof. We first show that $f_k := (-\Delta)^m u_k$ is bounded in $L(\log L)^{\frac{1}{2}}(\Omega)$, where

$$L(\log L)^\alpha(\Omega) := \left\{ f \in L^1(\Omega) : \|f\|_{L(\log L)^\alpha} := \int_\Omega |f| \log^\alpha(2+|f|)dx < \infty \right\}.$$

Indeed, set $\log^+ t := \max\{0, \log t\}$ for $t > 0$. Then, using the simple inequalities

$$\log(2+t) \leq 2 + \log^+ t, \quad \log^+(ts) \leq \log^+ t + \log^+ s, \quad t, s > 0,$$

one gets

$$\log(2 + \lambda_k u_k e^{mu_k^2}) \leq 2 + \log^+ \lambda_k + \log^+ u_k + mu_k^2 \leq C(1+u_k)^2.$$

Then, since $f_k \geq 0$, we have

$$\begin{aligned} \|f_k\|_{L(\log L)^{\frac{1}{2}}} &\leq \int_\Omega f_k \log^{\frac{1}{2}}(2+f_k)dx \\ &\leq C \int_{\{x\in\Omega: u_k(x)\geq 1\}} \lambda_k u_k^2 e^{mu_k} dx + C|\Omega| \leq C \end{aligned}$$

by (2), as claimed. Now (24) follows from Theorem 10. □

Remark. The inequality (24) is intermediate between the L^1 and the $L \log L$ estimates. Indeed, the bound of $f_k := (-\Delta)^m u_k$ in L^1 implies $\|\nabla^\ell u_k\|_{L^p} \leq C$ for every $1 \leq \ell \leq 2m-1$, $1 \leq p < \frac{2m}{\ell}$, and actually $\|\nabla^\ell u_k\|_{(2m/\ell,\infty)} \leq C$ (compare [Hél, Thm. 3.3.6]), but that is not enough for our purposes (Lemma 6 below). On the other hand, was f_k bounded in $L(\log L)$, we would have $\|\nabla^\ell u_k\|_{(2m/\ell,1)} \leq C$, which implies $\|u_k\|_{L^\infty} \leq C$ (compare [Hél, Thm. 3.3.8]). But we know that this is not the case in general.

Actually, the cases $1 \leq \ell \leq m$ in (24) follow already from (12) and the improved Sobolev embeddings, see [O'N]. What really matters here are the cases $m < \ell < 2m$. In fact, when $m = 1$ Lemma 5 reduces to (12).

The following lemma replaces and sharpens Proposition 2.3 in [RS].

Lemma 6 *For any $R > 0$, $1 \leq \ell \leq 2m-1$ there exists $k_0 = k_0(R)$ such that*

$$u_k(x_k) \int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \leq C(Rr_k)^{2m-\ell}, \quad \textit{for all } k \geq k_0.$$

Proof. We first claim that

$$\|\Delta^m(u_k^2)\|_{L^1(\Omega)} \leq C. \tag{25}$$

To see that, observe that

$$|\Delta^m(u_k^2)| \leq 2u_k(-\Delta)^m u_k + C \sum_{\ell=1}^{2m-1} |\nabla^\ell u_k||\nabla^{2m-\ell} u_k|. \tag{26}$$

The term $2u_k(-\Delta)^m u_k$ is bounded in L^1 thanks to (2). The other terms on the right-hand side of (26) are bounded in L^1 thanks to Lemma 5 and the Hölder-type inequality of O'Neil [O'N].[3] Hence (25) is proven.

Now set $f_k := (-\Delta)^m(u_k^2)$, and for any $x \in \Omega$, let G_x be the Green's function for $(-\Delta)^m$ on Ω with Dirichlet boundary condition. Then

$$u_k^2(x) = \int_\Omega G_x(y) f_k(y) dy.$$

Thanks to [DAS, Thm. 12], $|\nabla^\ell G_x(y)| \le C|x-y|^{-\ell}$, hence

$$|\nabla^\ell(u_k^2)(x)| \le \int_\Omega |\nabla^\ell_x G_x(y)||f_k(y)|dy \le C \int_\Omega \frac{|f_k(y)|}{|x-y|^\ell} dy.$$

Let μ_k denote the probability measure $\frac{|f_k(y)|}{\|f_k\|_{L^1(\Omega)}}dy$. By Fubini's theorem

$$\begin{aligned}
\int_{B_{Rr_k}(x_k)} |\nabla^\ell(u_k^2)(x)|dx &\le C\|f_k\|_{L^1(\Omega)} \int_{B_{Rr_k}(x_k)} \int_\Omega \frac{1}{|x-y|^\ell} d\mu_k(y) dx \\
&\le C \int_\Omega \int_{B_{Rr_k}(x_k)} \frac{1}{|x-y|^\ell} dx d\mu_k(y) \\
&\le C \sup_{y\in\Omega} \int_{B_{Rr_k}(x_k)} \frac{1}{|x-y|^\ell} dx \le C(Rr_k)^{2m-\ell}.
\end{aligned}$$

To conclude the proof, observe that Lemma 3 implies that on $B_{Rr_k}(x_k)$, for $1 \le \ell \le 2m-1$, we have $r_k^\ell \nabla^\ell u_k \to 0$ uniformly, hence

$$\begin{aligned}
u_k(x_k)|\nabla^\ell u_k| &\le Cu_k|\nabla^\ell u_k| \le C\bigg(|\nabla^\ell(u_k^2)| + \sum_{j=1}^{\ell-1} |\nabla^j u_k||\nabla^{\ell-j}u_k|\bigg) \\
&\le C|\nabla^\ell(u_k^2)| + o(r_k^{-\ell}), \quad \text{as } k \to \infty.
\end{aligned}$$

Integrating over $B_{Rr_k}(x_k)$ and using the above estimates we conclude. □

Proposition 7 *Let η_k be as in (22). Then, up to selecting a subsequence, $\eta_k(x) \to \eta_0(x) = \log\frac{2}{1+|x|^2}$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, and*

$$\lim_{R\to\infty}\lim_{k\to\infty} \int_{B_{Rr_k}(x_k)} \lambda_k u_k^2 e^{mu_k^2} dx = \lim_{R\to\infty}(2m-1)! \int_{B_R(0)} e^{2m\eta_0} dx = \Lambda_1. \quad (27)$$

Proof. Fix $R > 0$, and notice that, thanks to Lemma 3 and (23),

$$\begin{aligned}
\int_{B_R(0)} V_k e^{2ma_k\eta_k} dx &= \int_{B_{Rr_k}(x_k)} u_k(x_k) u_k \lambda_k e^{mu_k^2} dx \qquad (28)\\
&\le (1+o(1)) \int_{B_{Rr_k}(x_k)} u_k^2 \lambda_k e^{mu_k^2} dx \le \Lambda + o(1),
\end{aligned}$$

where V_k and a_k are as in Corollary 4, and $o(1) \to 0$ as $k \to \infty$.

[3] If $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, and $f \in L^{(p,q)}$, $g \in L^{(p',q')}$, then $\|fg\|_{L^1} \le \|f\|_{(p,q)}\|g\|_{(p',q')}$.

Step 1. We claim that $\eta_k \to \overline{\eta}$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where $\overline{\eta}$ satisfies

$$(-\Delta)^m \overline{\eta} = (2m-1)! e^{2m\overline{\eta}}. \tag{29}$$

Then, letting $R \to \infty$ in (28), from Corollary 4 and Fatou's lemma we infer $e^{2m\overline{\eta}} \in L^1(\mathbb{R}^{2m})$.

Let us prove the claim. Consider first the case $m > 1$. From Corollary 4, Theorem 1 in [Mar2], and (28), together with $\eta_k \leq \log 2$ (which implies that $S_1 = \emptyset$ in Theorem 1 of [Mar2]), we infer that up to subsequences either

(i) $\eta_k \to \overline{\eta}$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$ for some function $\overline{\eta} \in C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, or

(ii) $\eta_k \to -\infty$ locally uniformly in $\mathbb{R}^{2m}$, or

(iii) there exists a closed set $S_0 \neq \emptyset$ of Hausdorff dimension at most $2m-1$ and numbers $\beta_k \to +\infty$ such that

$$\frac{\eta_k}{\beta_k} \to \varphi \text{ in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash S_0),$$

where

$$\Delta^m \varphi \equiv 0, \quad \varphi \leq 0, \quad \varphi \not\equiv 0 \quad \text{on } \mathbb{R}^{2m}, \quad \varphi \equiv 0 \text{ on } S_0. \tag{30}$$

Since $\eta_k(0) = \log 2$, (ii) can be ruled out. Assume now that (iii) occurs. From Liouville's theorem and (30) we get $\Delta\varphi \not\equiv 0$, hence for some $R > 0$ we have $\int_{B_R} |\Delta\varphi| dx > 0$ and

$$\lim_{k\to\infty} \int_{B_R} |\Delta\eta_k| dx = \lim_{k\to\infty} \beta_k \int_{B_R} |\Delta\varphi| dx = +\infty. \tag{31}$$

On the other hand, we infer from Lemma 6

$$\int_{B_R} |\nabla^\ell \eta_k| dx = u_k(x_k) r_k^{\ell-2m} \int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \leq C R^{2m-\ell}, \tag{32}$$

contradicting (31) when $\ell = 2$ and therefore proving our claim.

When $m = 1$, Theorem 3 in [BM] implies that only Case (i) or Case (ii) above can occur. Again Case (ii) can be ruled out, since $\eta_k(0) = \log 2$, and we are done.

Step 2. We now prove that $\overline{\eta}$ is a standard solution of (29), i.e. there are $\lambda > 0$ and $x_0 \in \mathbb{R}^{2m}$ such that

$$\overline{\eta}(x) = \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}. \tag{33}$$

For $m = 1$ this follows at once from [CL]. For $m > 1$, if $\overline{\eta}$ didn't have the form (33), according to [Mar1, Thm. 2] (see also [Lin] for the case $m = 2$), there would exist $j \in \mathbb{N}$ with $1 \leq j \leq m-1$, and $a < 0$ such that

$$\lim_{|x|\to\infty} (-\Delta)^j \overline{\eta}(x) = a.$$

This would imply

$$\lim_{k\to\infty}\int_{B_R(0)}|\Delta^j\eta_k|dx=|a|\cdot\mathrm{vol}(B_1(0))R^{2m}+o(R^{2m})\quad\text{as } R\to\infty,$$

contradicting (32) for $\ell=2j$. Hence (33) is established. Since $\eta_k\leq\eta_k(0)=\log 2$, it follows immediately that $x_0=0$, $\lambda=1$, i.e. $\overline{\eta}=\eta_0$, and (27) follows from (11), (28) and Fatou's lemma. □

2.2 Exhaustion of the blow-up points and proof of Theorem 1

For $\ell\in\mathbb{N}$ we say that (H_ℓ) holds if there are ℓ sequences of converging points $x_{i,k}\to x^{(i)}$, $1\leq i\leq\ell$ such that

$$\sup_{x\in\Omega}\lambda_k R_{\ell,k}^{2m}(x)u_k^2(x)e^{mu_k^2(x)}\leq C,\tag{34}$$

where

$$R_{\ell,k}(x):=\inf_{1\leq i\leq\ell}|x-x_{i,k}|.$$

We say that (E_ℓ) holds if there are ℓ sequences of converging points $x_{i,k}\to x^{(i)}$ such that, if we define $r_{i,k}$ as in (3), the following hold true:

(E_ℓ^1) For all $1\leq i\neq j\leq\ell$

$$\lim_{k\to\infty}\frac{\mathrm{dist}(x_{i,k},\partial\Omega)}{r_{i,k}}=\infty,\qquad\lim_{k\to\infty}\frac{|x_{i,k}-x_{j,k}|}{r_{i,k}}=\infty.$$

(E_ℓ^2) For $1\leq i\leq\ell$ (4) holds true.

(E_ℓ^3) $\lim_{R\to\infty}\lim_{k\to\infty}\int_{\cup_{i=1}^{\ell}B_{Rr_{i,k}}(x_{i,k})}\lambda_k u_k^2 e^{mu_k^2}dx=\ell\Lambda_1$.

To prove Theorem 1 we show inductively that (H_I) and (E_I) hold for some positive $I\in\mathbb{N}$ (with the same sequences $x_{i,k}\to x^{(i)}$, $1\leq i\leq I$), following the approach of [AD] and [RS]. First observe that (E_1) holds thanks to Lemma 2 and Proposition 7. Assume now that for some $\ell\geq 1$ (E_ℓ) holds and (H_ℓ) does not. Choose $x_{\ell+1,k}\in\Omega$ such that

$$\lambda_k R_{\ell,k}^{2m}(x_{\ell+1,k})u_k^2(x_{\ell+1,k})e^{mu_k^2(x_{\ell+1,k})}=\lambda_k\max_{\Omega}R_{\ell,k}^{2m}u_k^2e^{mu_k^2}\to\infty\quad\text{as } k\to\infty\tag{35}$$

and define $r_{\ell+1,k}$ as in (3). It easily follows from (35) that

$$\lim_{k\to\infty}\frac{|x_{\ell+1,k}-x_{i,k}|}{r_{\ell+1,k}}=\infty,\quad 1\leq i\leq\ell.\tag{36}$$

Moreover, thanks to (E_ℓ^2) and (35), we also have

$$\lim_{k\to\infty}\frac{|x_{\ell+1,k}-x_{i,k}|}{r_{i,k}}=\infty\quad\text{for } 1\leq i\leq\ell.$$

We now need to replace Lemma 2 and Lemma 3 with the lemma below.

Lemma 8 *Under the above assumptions and notation, we have*

$$\lim_{k\to\infty} \frac{\mathrm{dist}(x_{\ell+1,k}, \partial\Omega)}{r_{\ell+1,k}} = \infty \tag{37}$$

and

$$u_k(x_{\ell+1,k} + r_{\ell+1,k}x) - u_k(x_{\ell+1,k}) \to 0 \quad \textit{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}), \quad \textit{as } k \to \infty. \tag{38}$$

Proof. To simplify the notation, let us write $y_k := x_{\ell+1,k}$ and $\rho_k := r_{\ell+1,k}$. Evaluating the right-hand side of (35) at the point $y_k + \rho_k x$ we get

$$\begin{aligned}&\Big(\inf_{1\le i\le \ell} |y_k - x_{i,k} + \rho_k x|^{2m}\Big) u_k^2(y_k + \rho_k x) e^{m u_k^2(y_k+\rho_k x)}\\ &\le \Big(\inf_{1\le i\le \ell} |y_k - x_{i,k}|^{2m}\Big) u_k^2(y_k) e^{m u_k^2(y_k)},\end{aligned}$$

Hence, setting $\overline{u}_{\ell+1,k}(x) := \frac{u_k(y_k+\rho_k x)}{u_k(y_k)}$, we have that

$$\overline{u}^2_{\ell+1,k}(x) e^{m u_k^2(y_k)(\overline{u}^2_{\ell+1,k}(x)-1)} \le \frac{\inf_{1\le i\le\ell} |y_k - x_{i,k}|^{2m}}{\inf_{1\le i\le \ell} |y_k - x_{i,k} + \rho_k x|^{2m}} = 1 + o(1), \tag{39}$$

where $o(1) \to 0$ as $k \to \infty$ locally uniformly in x, as (36) immediately implies. Then (37) follows as in the proof of Lemma 2, since (39) implies

$$(-\Delta)^m \overline{u}_{\ell+1,k} = \frac{2^{2m}(2m-1)!}{u_k^2(y_k)} \overline{u}_{\ell+1,k} e^{m u_k^2(y_k)(\overline{u}^2_{\ell+1,k}-1)} = o(1), \tag{40}$$

where $o(1) \to 0$ as $k \to \infty$ uniformly locally in $\mathbb{R}^{2m}$.

Define now $v_k(x) := u_k(x_{\ell+1,k} + r_{\ell+1,k}x) - u_k(x_{\ell+1,k})$, and observe that

$$u_k(y_k + \rho_k x) \to \infty \quad \text{locally uniformly in } \mathbb{R}^{2m},$$

thanks to (35) and (36). This and (40) imply that we can replace (14) in the proof of Lemma 3 with

$$(-\Delta)^m v_k = 2^{2m}(2m-1)! \frac{\overline{u}_k^2}{u_k(y_k + \rho_k \cdot)} e^{m u_k^2(y_k)(\overline{u}^2_{\ell+1,k}-1)} \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}^{2m}).$$

Then the rest of the proof of Lemma 3 applies without changes, and also (38) is proved. □

Still repeating the arguments of the preceding section with $x_{\ell+1,k}$ instead of x_k and $r_{\ell+1,k}$ instead of r_k, we define

$$\eta_{\ell+1,k}(x) := u_k(x_{\ell+1,k})[u_k(r_{\ell+1,k}x + x_{\ell+1,k}) - u_k(x_{\ell+1,k})],$$

and we have

Proposition 9 *Up to a subsequence*

$$\eta_{\ell+1,k}(x) \to \eta_0(x) = \log \frac{2}{1+|x|^2} \quad \textit{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$$

and

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{B_{Rr_{\ell+1,k}}(x_{\ell+1,k})} \lambda_k u_k^2 e^{m u_k^2} dx = \lim_{R\to\infty} \int_{B_R(0)} e^{2m\eta_0} dx = \Lambda_1. \tag{41}$$

Summarizing, we have proved that $(E^1_{\ell+1})$, $(E^2_{\ell+1})$ and (41) hold. These also imply that $(E^3_{\ell+1})$ holds, hence we have $(E_{\ell+1})$. Because of (2) and (E^3_ℓ), the procedure stops in a finite number I of steps, and we have (H_I).

Finally, we claim that $\lambda_k \to 0$ implies $u_k \rightharpoonup 0$ in $H^m(\Omega)$. This, (5) and elliptic estimates then imply that

$$u_k \to 0 \quad \text{in} \quad C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash\{x^{(1)},\dots,x^{(I)}\}).$$

To prove the claim, we observe that for any $\alpha > 0$

$$\begin{aligned}\int_\Omega |\Delta^m u_k| dx &= \int_\Omega \lambda_k u_k e^{mu_k^2} dx \\ &\le \frac{\lambda_k}{\alpha}\int_{\{x\in\Omega: u_k\ge\alpha\}} u_k^2 e^{mu_k^2}dx + \lambda_k \int_{\{x\in\Omega:u_k<\alpha\}} u_k e^{mu_k^2} dx \\ &\le \frac{C}{\alpha} + \lambda_k C_\alpha,\end{aligned}$$

where C_α depends only on α. Letting k and α go to infinity, we infer

$$\Delta^m u_k \to 0 \quad \text{in } L^1(\Omega). \tag{42}$$

Thanks to (12), we infer that up to a subsequence $u_k \rightharpoonup u_0$ in $H^m(\Omega)$. Then (42) and the boundary condition imply that $u_0 \equiv 0$, in particular the full sequence converges to 0 weakly in $H^m(\Omega)$. This completes the proof of the theorem.

Appendix

An elliptic estimate for Zygmund and Lorentz spaces

Theorem 10 *Let u solve $\Delta^m u = f \in L(\log L)^\alpha$ in Ω with Dirichlet boundary condition, $0 \le \alpha \le 1$, $\Omega \subset \mathbb{R}^n$ bounded and with smooth boundary, $n \ge 2m$. Then $\nabla^{2m-\ell} u \in L^{\left(\frac{n}{n-\ell},\frac{1}{\alpha}\right)}(\Omega)$, $1 \le \ell \le 2m-1$ and*

$$\|\nabla^{2m-\ell}u\|_{\left(\frac{n}{n-\ell},\frac{1}{\alpha}\right)} \le C\|f\|_{L(\log L)^\alpha}. \tag{43}$$

Proof. Define

$$\hat f := \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n\backslash\Omega, \end{cases}$$

and let $w := K * \hat f$, where K is the fundamental solution of Δ^m. Then

$$|\nabla^{2m-1}w| = |(\nabla^{2m-1}K) * \hat f| \le CI_1 * |\hat f|,$$

where $I_1(x) = |x|^{1-n}$. According to [BS, Cor. 6.16], $|\nabla^{2m-1}w| \in L^{\left(\frac{n}{n-1},\frac{1}{\alpha}\right)}(\mathbb{R}^n)$ and

$$\|\nabla^{2m-1}w\|_{\left(\frac{n}{n-1},\frac{1}{\alpha}\right)} \le C\|\hat f\|_{L(\log L)^\alpha} = C\|f\|_{L(\log L)^\alpha}. \tag{44}$$

We now use (44) to prove (43), following a method that we learned from [Hél]. Given $g : \Omega \to \mathbb{R}^n$ measurable, let v_g be the solution to $\Delta^m v_g = \operatorname{div} g$ in Ω, with the same boundary condition as u, and set $P(g) := |\nabla^{2m-1}v_g|$. By L^p estimates (see e.g. [ADN]), P is bounded from $L^p(\Omega;\mathbb{R}^n)$ into $L^p(\Omega)$ for $1<p<\infty$. Then, thanks to the interpolation theory for Lorentz spaces, see e.g. [Hél, Thm. 3.3.3], P is bounded from $L^{(p,q)}(\Omega;\mathbb{R}^n)$ into $L^{(p,q)}(\Omega)$ for $1<p<\infty$ and $1\le q\le\infty$. Choosing now $g=\nabla\Delta^{m-1}w$, we get $v_g=u$, hence $|\nabla^{2m-1}u|=P(\nabla\Delta^{m-1}w)$, and from (44) we infer

$$\|\nabla^{2m-1}u\|_{\left(\frac{n}{n-1},\frac{1}{\alpha}\right)}\le C\|\nabla\Delta^{m-1}w\|_{\left(\frac{n}{n-1},\frac{1}{\alpha}\right)}\le C\|f\|_{L(\log L)^\alpha}.$$

For $1<\ell\le 2m-1$ (43) follows from the Sobolev embeddings, see [O'N]. □

Other useful results

A proof of the results below can be found in [Mar1]. The following Lemma can be considered a generalized mean value identity for polyharmonic function.

Lemma 11 (Pizzetti [Piz]) *Let $u\in C^{2m}(B_R(x_0))$, $B_R(x_0)\subset\mathbb{R}^n$, for some m,n positive integers. Then there are positive constants $c_i=c_i(n)$ such that*

$$\fint_{B_R(x_0)} u(x)dx=\sum_{i=0}^{m-1}c_iR^{2i}\Delta^i u(x_0)+c_mR^{2m}\Delta^m u(\xi), \tag{45}$$

for some $\xi\in B_R(x_0)$.

Proposition 12 *Let $\Delta^m h=0$ in $B_2\subset\mathbb{R}^n$. For every $0\le\alpha<1$, $p\in[1,\infty)$ and $\ell\ge 0$ there are constants $C(\ell,p)$ and $C(\ell,\alpha)$ independent of h such that*

$$\begin{aligned}\|h\|_{W^{\ell,p}(B_1)}&\le C(\ell,p)\|h\|_{L^1(B_2)}\\ \|h\|_{C^{\ell,\alpha}(B_1)}&\le C(\ell,\alpha)\|h\|_{L^1(B_2)}.\end{aligned}$$

A simple consequence of Lemma 11 and Proposition 12 is the following Liouville-type Theorem.

Theorem 13 *Consider $h:\mathbb{R}^n\to\mathbb{R}$ with $\Delta^m h=0$ and $h(x)\le C(1+|x|^\ell)$ for some $\ell\ge 0$. Then h is a polynomial of degree at most* $\max\{\ell,2m-2\}$.

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