Concentration-compactness phenomena in the higher order Liouville's equation

Luca Martinazzi*
ETH Zurich
Rämistrasse 101, CH-8092
luca@math.ethz.ch

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Abstract

We investigate different concentration-compactness phenomena related to the Q-curvature in arbitrary even dimension. We first treat the case of an open domain in $\mathbb{R}^{2m}$, then that of a closed manifold and, finally, the particular case of the sphere S^{2m}. In all cases we allow the sign of the Q-curvature to vary, and show that in the case of a closed manifold, contrary to the case of open domains in $\mathbb{R}^{2m}$, concentration phenomena can occur only at points of positive Q-curvature. As a consequence, on a locally conformally flat manifold of non-positive Euler characteristic we always have compactness.

1 Introduction and statement of the main results

Before stating our results, we recall a few facts concerning the Paneitz operator P_g^{2m} and the Q-curvature Q_g^{2m} on a $2m$-dimensional smooth Riemannian manifold (M, g). Introduced in [BO], [Pan], [Bra] and [GJMS], the Paneitz operator and the Q-curvature are the higher order equivalents of the Laplace-Beltrami operator and the Gaussian curvature respectively ($P_g^2 = -\Delta_g$ and $Q_g^2 = K_g$), and they now play a central role in modern conformal geometry. For their definitions and more related information we refer to [Cha]. Here we only recall a few properties which shall be used later. First of all we have the Gauss formula, describing how the Q-curvature changes under a conformal change of metric:

$$P_g^{2m} u + Q_g^{2m} = Q_{g_u}^{2m} e^{2mu}, \tag{1}$$

where $g_u := e^{2u} g$, and $u \in C^\infty(M)$ is arbitrary. Then, we have the conformal invariance of the total Q-curvature, when M is closed:

$$\int_M Q_{g_u}^{2m} \mathrm{dvol}_{g_u} = \int_M Q_g^{2m} \mathrm{dvol}_g. \tag{2}$$

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Finally, assuming (M,g) closed and locally conformally flat , we have the Gauss-Bonnet-Chern formula (see e.g. [Che], [Cha]):

$$\int_M Q_g^{2m} \mathrm{dvol}_g = \frac{\Lambda_1}{2}\chi(M), \tag{3}$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M and

$$\Lambda_1 := \int_{S^{2m}} Q_{g_{S^{2m}}} \mathrm{dvol}_{g_{S^{2m}}} = (2m-1)!|S^{2m}| \tag{4}$$

is a constant which we shall meet often in the sequel. In the 4-dimensional case, if (M,g) is not locally conformally flat, we have

$$\int_M \left(Q_g^4 + \frac{|W_g|^2}{4} \right) \mathrm{dvol}_g = 8\pi^2 \chi(M), \tag{5}$$

where W_g is the Weyl tensor. Recently S. Alexakis [Ale2] (see also [Ale1]) proved an analogous to (5) for $m \geq 3$:

$$\int_M \left(Q_g^{2m} + W \right) \mathrm{dvol}_g = \frac{\Lambda_1}{2}\chi(M), \tag{6}$$

where W is a local conformal invariant involving the Weyl tensor and its covariant derivatives.

We can now state the main problem treated in this paper. Given a $2m$-dimensional Riemannian manifold (M,g), consider a converging sequence of functions $Q_k \to Q_0$ in $C^0(M)$, and let $g_k := e^{2u_k} g$ be conformal metrics satisfying $Q_{g_k}^{2m} = Q_k$. In view of (1), the u_k's satisfy the following elliptic equation of order $2m$ with critical exponential non-linearity

$$P_g^{2m} u_k + Q_g^{2m} = Q_k e^{2mu_k}. \tag{7}$$

Assume further that there is a constant $C > 0$ such that

$$\mathrm{vol}(g_k) = \int_M e^{2mu_k} \mathrm{dvol}_g \leq C \quad \text{for all } k. \tag{8}$$

What can be said about the compactness properties of the sequence (u_k)?

In general non-compactness has to be expected, at least as a consequence of the non-compactness of the Möbius group on $\mathbb{R}^{2m}$ or S^{2m}. For instance, for every $\lambda > 0$ and $x_0 \in \mathbb{R}^{2m}$, the metric on $\mathbb{R}^{2m}$ given by $g_u := e^{2u} g_{\mathbb{R}^{2m}}$, $u(x) := \log \frac{2\lambda}{1+\lambda^2|x-x_0|^2}$, satisfies $Q_{g_u}^{2m} \equiv (2m-1)!$.

We start by considering the case when (M,g) is an open domain $\Omega \subset \mathbb{R}^{2m}$ with Euclidean metric $g_{\mathbb{R}^{2m}}$. Since $P_{g_{\mathbb{R}^{2m}}} = (-\Delta)^m$ and $Q_{g_{\mathbb{R}^{2m}}} \equiv 0$, Equation (7) reduces to $(-\Delta)^m u_k = Q_k e^{2mu_k}$. The compactness properties of this equation were studied in dimension 2 by Brézis and Merle [BM]. They proved that if $Q_k \geq 0$, $\|Q_k\|_{L^\infty} \leq C$ and $\|e^{2u_k}\|_{L^1} \leq C$, then up to selecting a subsequence, one of the following is true:

(i) (u_k) is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$.

(ii) $u_k \to -\infty$ locally uniformly in Ω.

(iii) There is a finite set $S = \{x^{(i)}; i = 1, \ldots, I\} \subset \Omega$ such that $u_k \to -\infty$ locally uniformly in $\Omega \backslash S$. Moreover $Q_k e^{2u_k} \rightharpoonup \sum_{i=1}^I \beta_i \delta_{x^{(i)}}$ weakly in the sense of measures, where $\beta_i \geq 2\pi$ for every $1 \leq i \leq I$.

Subsequently, Li and Shafrir [LS] proved that in case (iii) $\beta_i \in 4\pi\mathbb{N}$ for every $1 \leq i \leq I$.

Adimurthi, Robert and Struwe [ARS] studied the case of dimension 4 ($m = 2$). As they showed, the situation is more subtle because the blow-up set (the set of points x such that $u_k(x) \to \infty$ as $k \to \infty$) can have dimension up to 3 (in contrast to the finite blow-up set S in dimension 2). Moreover, as a consequence of a result of Chang and Chen [CC], quantization in the sense of Li-Shafrir does not hold anymore, see also [Rob1], [Rob2].

In the following theorem we extend the result of [ARS] to arbitrary even dimension (see also Proposition 6 below). The function a_k in (9) has no geometric meaning, and one can take $a_k \equiv 1$ at first. On the other hand, one can also apply Theorem 1 to non-geometric situations, by allowing $a_k \not\equiv 1$, see [Mar3].

Theorem 1 *Let Ω be a domain in $\mathbb{R}^{2m}$, $m > 1$, and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions satisfying*

$$(-\Delta)^m u_k = Q_k e^{2ma_k u_k}, \tag{9}$$

where $a_k, Q_0 \in C^0(\Omega)$, Q_0 is bounded, and $Q_k \to Q_0$, $a_k \to 1$ locally uniformly. Assume that

$$\int_\Omega e^{2ma_k u_k} dx \leq C, \tag{10}$$

for all k and define the finite (possibly empty) set

$$S_1 := \left\{ x \in \Omega : \lim_{r\to 0^+} \limsup_{k\to\infty} \int_{B_r(x)} |Q_k| e^{2ma_k u_k} dy \geq \frac{\Lambda_1}{2} \right\} = \{x^{(i)} : 1 \leq i \leq I\},$$

where Λ_1 is as in (4). Then one of the following is true.

(i) For every $0 \leq \alpha < 1$, a subsequence converges in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega \backslash S_1)$.

(ii) There exist a subsequence, still denoted by (u_k), and a closed nowhere dense set S_0 of Hausdorff dimension at most $2m - 1$ such that, letting $S = S_0 \cup S_1$, we have $u_k \to -\infty$ locally uniformly in $\Omega \backslash S$ as $k \to \infty$. Moreover there is a sequence of numbers $\beta_k \to \infty$ such that

$$\frac{u_k}{\beta_k} \to \varphi \text{ in } C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega \backslash S), \quad 0 \leq \alpha < 1,$$

where $\varphi \in C^\infty(\Omega \backslash S_1)$, $S_0 = \{x \in \Omega : \varphi(x) = 0\}$, and

$$(-\Delta)^m \varphi \equiv 0, \quad \varphi \leq 0, \quad \varphi \not\equiv 0 \quad \text{in } \Omega \backslash S_1.$$

If $S_1 \neq \emptyset$ and $Q_0(x^{(i)}) > 0$ for some $1 \leq i \leq I$, then case (ii) occurs.

We recently proved (see [Mar2]) the existence of solutions to the equation $(-\Delta)^m u = Qe^{2mu}$ on $\mathbb{R}^{2m}$ with $Q < 0$ constant and $e^{2mu} \in L^1(\mathbb{R}^{2m})$, for $m > 1$. Scaling any such solution we find a sequence of solutions $u_k(x) := u(kx) + \log k$ concentrating at a point of negative Q-curvature. For $m = 1$ that is not possible.

On a closed manifold things are different in several respects. Under the assumption (which we always make) that $\ker P_g^{2m}$ contains only constant functions, quantization of the total Q-curvature in the sense of Li-Shafrir (see (12) below) holds, as proved in dimension 4 by Druet and Robert [DR] and Malchiodi [Mal], and in arbitrary dimension by Ndiaye [Ndi]. Moreover the concentration set is finite. In [DR], however, it is assumed that the Q-curvatures are positive, while in [Mal] and [Ndi], a slightly different equation is studied ($P_g^{2m} u_k + Q_k = h_k e^{2mu_k}$, with h_k constant and Q_k prescribed), for which the negative case is simpler. With the help of results from our recent work [Mar2] and a technique of Robert and Struwe [RS], we can allow the prescribed Q-curvatures to have varying signs and, contrary to the case of an open domain in $\mathbb{R}^{2m}$, we can rule out concentration at points of negative Q-curvature.

Theorem 2 *Let (M,g) be a $2m$-dimensional closed Riemannian manifold, such that $\ker P_g = \{constants\}$, and let (u_k) be a sequence of solutions to (7), (8) where the Q_k's and Q_0 are given C^1 functions and $Q_k \to Q_0$ in $C^1(M)$. Let Λ_1 be as in (4). Then one of the following is true.*

(i) For every $0 \leq \alpha < 1$, a subsequence converges in $C^{2m-1,\alpha}(M)$.

(ii) There exists a finite (possibly empty) set $S_1 = \{x^{(i)} : 1 \leq i \leq I\}$ such that $Q_0(x^{(i)}) > 0$ for $1 \leq i \leq I$ and, up to taking a subsequence, $u_k \to -\infty$ locally uniformly on $(M \backslash S_1)$. Moreover

$$Q_k e^{2mu_k} \,\mathrm{dvol}_g \rightharpoonup \sum_{i=1}^{I} \Lambda_1 \delta_{x^{(i)}} \tag{11}$$

in the sense of measures; then (2) gives

$$\int_M Q_g \mathrm{dvol}_g = I\Lambda_1. \tag{12}$$

Finally, $S_1 = \emptyset$ if and only if $\mathrm{vol}(g_k) \to 0$.

An immediate consequence of Theorem 2 (Identity (12) in particular) and the Gauss-Bonnet-Chern formulas (3) and (5), is the following compactness result:

Corollary 3 *Under the hypothesis of Theorem 2 assume that either*

1. $\chi(M) \leq 0$ and $\dim M \in \{2,4\}$, or

2. $\chi(M) \leq 0$, $\dim M \geq 6$ and (M,g) is locally conformally flat,

and that $\mathrm{vol}(g_k) \not\to 0$. Then (i) in Theorem 2 occurs.

It is not clear whether the hypothesis that (M,g) be locally conformally flat when $\dim M \geq 6$ is necessary in Corollary 3. For instance, we could drop it if we knew that $W \geq 0$ in (6), in analogy with (5).

Contrary to what happens for the Yamabe equation (see [Dru1], [Dru2], [DH] and [DHR]), the concentration points of S in Theorem 2 are isolated, as already proved in [DR] in dimension 4. In fact, a priori one could expect to have

$$Q_k e^{2mu_k} \,\mathrm{dvol}_g \rightharpoonup \sum_{i=1}^{I} L_i \Lambda_1 \delta_{x^{(i)}}, \quad \text{for some } L_i \in \mathbb{N}\backslash\{0\}, \tag{13}$$

instead of (11). The compactness of M is again a crucial ingredient here; indeed X. Chen [Ch] showed that on $\mathbb{R}^2$ (where quantization holds, as already discussed) one can have (13) with $L_i > 1$.

Theorems 1 and 2 will be proven in Sections 2 and 3 respectively. In Section 4 we also consider the special case when $M = S^{2m}$. In the proofs of the above theorems we use techniques and ideas from several of the cited papers, particularly from [ARS], [BM], [DR], [Mal], [MS] and [RS]. In the following, the letter C denotes a generic positive constant, which may change from line to line and even within the same line.

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2 The case of an open domain in $\mathbb{R}^{2m}$

An important tool in the proof of Theorem 1 is the following estimate, proved by Brézis and Merle [BM] in dimension 2. For the proof in arbitrary dimension see [Mar1]. Notice the role played by the constant $\gamma_m := \frac{\Lambda_1}{2}$, which satisfies

$$(-\Delta)^m\Big(-\frac{1}{\gamma_m}\log|x|\Big) = \delta_0 \quad \text{in } \mathbb{R}^{2m}. \tag{14}$$

Theorem 4 *Let $f \in L^1(B_R(x_0))$, $B_R(x_0) \subset \mathbb{R}^{2m}$, and let v solve*

$$\begin{cases} (-\Delta)^m v = f & \text{in } B_R(x_0), \\ v = \Delta v = \ldots = \Delta^{m-1} v = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

Then, for any $p \in \Big(0, \frac{\gamma_m}{\|f\|_{L^1(B_R(x_0))}}\Big)$, we have $e^{2mp|v|} \in L^1(B_R(x_0))$ and

$$\int_{B_R(x_0)} e^{2mp|v|}dx \leq C(p)R^{2m}.$$

Lemma 5 *Let $f \in L^1(\Omega) \cap L^p_{\text{loc}}(\Omega\backslash S_1)$ for some $p > 1$, where $\Omega \subset \mathbb{R}^{2m}$ and $S_1 \subset \Omega$ is a finite set. Assume that*

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega \\ \Delta^j u = 0 & \text{on } \partial\Omega \text{ for } 0 \leq j \leq m-1. \end{cases}$$

Then u is bounded in $W^{2m,p}_{\text{loc}}(\Omega\backslash S_1)$; more precisely, for any $\overline{B_{4R}(x_0)} \subset (\Omega\backslash S_1)$, there is a constant C independent of f such that

$$\|u\|_{W^{2m,p}(B_R(x_0))} \leq C(\|f\|_{L^p(B_{4R}(x_0))} + \|f\|_{L^1(\Omega)}). \tag{15}$$

The proof of Lemma 5 is given in the appendix.

Proof of Theorem 1. We closely follow [ARS]. Let S_1 be defined as in the statement of the Theorem. Clearly (10) implies that $S_1 = \{x^{(i)} \in \Omega : 1 \leq i \leq I\}$ is finite. Given $x_0 \in \Omega\backslash S_1$, we have, for some $0 < R < \text{dist}(x_0, \partial\Omega)$,

$$\alpha := \limsup_{k\to\infty} \int_{B_R(x_0)} |Q_k|e^{2ma_k u_k}dx < \gamma_m. \tag{16}$$

For such x_0 and R write $u_k = v_k + h_k$ in $B_R(x_0)$, where

$$\begin{cases} (-\Delta)^m v_k = Q_k e^{2ma_k u_k} & \text{in } B_R(x_0) \\ v_k = \Delta v_k = \ldots = \Delta^{m-1} v_k = 0 & \text{on } \partial B_R(x_0) \end{cases}$$

and $(-\Delta)^m h_k = 0$. Set $h_k^+ := \chi_{\{h_k \geq 0\}} h_k$, $h_k^- := h_k - h_k^+$. Since $h_k^+ \leq u_k^+ + |v_k|$, we have

$$\|h_k^+\|_{L^1(B_R(x_0))} \leq \|u_k^+\|_{L^1(B_R(x_0))} + \|v_k\|_{L^1(B_R(x_0))}.$$

Observe that, for k large enough $mu_k^+ \leq 2ma_k u_k^+ \leq e^{2ma_k u_k}$ on $B_R(x_0)$, hence (10) implies

$$\int_{B_R(x_0)} u_k^+ dx \leq C \int_{B_R(x_0)} e^{2ma_k u_k} dx \leq C.$$

As for v_k, observe that $1 < \frac{\gamma_m}{\alpha}$, hence by Theorem 4

$$\int_{B_R(x_0)} 2m|v_k| dx \leq \int_{B_R(x_0)} e^{2m|v_k|} dx \leq CR^{2m},$$

with C depending on α and not on k. Hence

$$\|h_k^+\|_{L^1(B_R(x_0))} \leq C. \tag{17}$$

We distinguish 2 cases.

Case 1. Suppose that $\|h_k\|_{L^1(B_{R/2}(x_0))} \leq C$ uniformly in k. Then by Proposition 11 we have that h_k is equibounded in $C^\ell(B_{R/8}(x_0))$ for every $\ell \geq 0$. Moreover, by Pizzetti's formula (Identity (79) in the appendix) and (17),

$$\begin{aligned} \fint_{B_R(x_0)} |h_k(x)| dx &= 2 \fint_{B_R(x_0)} h_k^+(x) dx - \fint_{B_R(x_0)} h_k(x) dx \\ &\leq C - \fint_{B_R(x_0)} h_k(x) dx \\ &= C - \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i h_k(x_0) \leq C. \end{aligned}$$

Hence we can apply Proposition 11 locally on all of $B_R(x_0)$ and obtain bounds for (h_k) in $C^\ell_{\mathrm{loc}}(B_R(x_0))$ for any $\ell \geq 0$.

Fix $p \in (1, \gamma_m/\alpha)$. By Theorem 4 $\|e^{2m|v_k|}\|_{L^p(B_R(x_0))} \leq C(p)$, hence, using that $a_k \to 1$ uniformly on $B_R(x_0)$, we infer

$$\|(-\Delta)^m v_k\|_{L^p(B)} = \|(Q_k e^{2ma_k h_k}) e^{2ma_k v_k}\|_{L^p(B)} \leq C(B, p) \tag{18}$$

for every ball $B \subset\subset B_R(x_0)$ and for k large enough. In addition $\|v_k\|_{L^1(B_R(x_0))} \leq C$, hence by elliptic estimates,

$$\|v_k\|_{W^{2m,p}(B)} \leq C(B, p) \quad \text{for every ball } B \subset\subset B_R(x_0).$$

By the immersion $W^{2m,p} \hookrightarrow C^{0,\alpha}$, (v_k), is bounded in $C^{0,\alpha}_{\mathrm{loc}}(B_R(x_0))$, for some $\alpha > 0$. Going back to (18), we now see that $\Delta^m v_k$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(B_R(x_0))$, hence

$$\|v_k\|_{W^{2m,p}(B)} \leq C(B, p)$$

for every $p > 1$, $B \subset\subset B_R(x_0)$, and by the immersion $W^{2m,p} \hookrightarrow C^{2m-1,\alpha}$ we obtain that (v_k), hence (u_k), is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_R(x_0))$.

Case 2. Assume that $\|h_k\|_{L^1(B_{R/2}(x_0))} =: \beta_k \to \infty$ as $k \to \infty$. Set $\varphi_k := \frac{h_k}{\beta_k}$, so that

1. $\Delta^m \varphi_k = 0$,
2. $\|\varphi_k\|_{L^1(B_{R/2}(x_0))} = 1$,
3. $\|\varphi_k^+\|_{L^1(B_R(x_0))} \to 0$ by (17).

As above we have that φ_k is bounded in $C^{\ell}_{\mathrm{loc}}(B_R(x_0))$ for every $\ell \geq 0$, hence a subsequence converges in $C^{2m}_{\mathrm{loc}}(B_R(x_0))$ to a function φ, with

1. $\Delta^m \varphi = 0$,
2. $\|\varphi\|_{L^1(B_{R/2}(x_0))} = 1$,
3. $\|\varphi^+\|_{L^1(B_R(x_0))} = 0$, hence $\varphi \leq 0$.

Let us define $S_0 = \{x \in B_R(x_0) : \varphi(x) = 0\}$. Take $x \in S_0$; then by (79), $\Delta\varphi(x), \ldots, \Delta^{m-1}\varphi(x)$ cannot all vanish, unless $\varphi \equiv 0$ on $B_\rho(x) \subset B_R(x_0)$ for some $\rho > 0$, but then by analyticity, we would have $\varphi \equiv 0$, contradiction. Hence there exists j with $1 \leq j \leq 2m-3$ such that

$$D^j \varphi(x) = 0, \quad D^{j+1}\varphi(x) \neq 0,$$

i.e.

$$S_0 \subset \bigcup_{j=1}^{2m-3} \{x \in B_R(x_0) : D^j\varphi(x) = 0, D^{j+1}\varphi(x) \neq 0\}.$$

Therefore S_0 is $(2m-1)$-rectifiable. Since $\varphi < 0$ on $B_R(x_0) \backslash S_0$, we infer

$$h_k = \beta_k \varphi_k \to -\infty, \quad e^{2ma_k h_k} \to 0$$

locally uniformly on $B_R(x_0)\backslash S_0$. Then, as before, from

$$(-\Delta)^m v_k = (Q_k e^{2ma_k h_k})(e^{2ma_k v_k}),$$

we have that v_k is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash S_0)$. Then $u_k = h_k + v_k \to -\infty$ uniformly locally away from S_0.

Since Case 1 and Case 2 are mutually exclusive, covering $\Omega\backslash S_1$ with balls, we obtain that either a subsequence u_k is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash S_1)$, or a subsequence $u_k \to -\infty$ locally uniformly on $\Omega\backslash(S_0 \cup S_1)$. In this latter case, the behavior described in case (ii) of the theorem occurs. Indeed fix any $B_R(x_0) \subset \Omega\backslash S_1$ and take β_k as above. Then, on a ball $B_\rho(y_0) \subset \Omega\backslash S_1$, we can write $u_k = \tilde{v}_k + \tilde{h}_k$ as above, where $\tilde{h}_k \to -\infty$ locally uniformly away from a rectifiable set S_0 of dimension at most $(2m-1)$, $\frac{\tilde{h}_k}{\tilde{\beta}_k} \to \tilde{\varphi}$, where $\tilde{\beta}_k = \|\tilde{h}_k\|_{L^1(B_{\rho/2}(y))}$, and $\tilde{v}_k$ is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_\rho(y_0))$. Then $\frac{\tilde{v}_k}{\beta_k} \to 0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_\rho(y_0))$, and we have that either

(a) $\frac{\tilde{h}_k}{\beta_k}$ and $\frac{u_k}{\beta_k}$ are bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_\rho(y_0))$, or

(b) $\frac{\tilde{h}_k}{\beta_k}$ and $\frac{u_k}{\beta_k}$ go to $-\infty$ locally uniformly away from S_0.

Since the 2 cases are mutually exclusive, and on $B_R(x_0)$ case (a) occurs, upon covering $\Omega\backslash S_1$ with a sequence of balls, we obtain the desired behavior for $\frac{u_k}{\beta_k}$.

We now show that if $I \geq 1$ and $Q_0(x^{(i)}) > 0$ for some $1 \leq i \leq I$, then Case 2 occurs. Assume by contradiction that $Q_0(x_0) > 0$ for some $x_0 \in S_1$ and Case 1 occurs, i.e. (u_k) is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash S_1)$, so that $f_k := Q_k e^{2ma_k u_k}$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega\backslash S_1)$. Then there exists a finite signed measure μ on Ω, with $\mu \in L^\infty_{\mathrm{loc}}(\Omega\backslash S_1)$ such that

$$\begin{aligned} f_k &\rightharpoonup \mu \quad \text{as measures} \\ f_k &\rightharpoonup \mu \quad \text{in } L^p_{\mathrm{loc}}(\Omega\backslash S_1) \text{ for } 1 \leq p < \infty. \end{aligned}$$

Let us take $R > 0$ such that $\overline{B_R(x_0)} \subset \Omega$, $B_R(x_0) \cap S_1 = \{x_0\}$ and $Q_0 > 0$ on $B_R(x_0)$. By our assumption,

$$(-\Delta)^j u_k \geq -C, \quad \text{on } \partial B_R(x_0) \text{ for } 0 \leq j \leq m-1. \tag{19}$$

Let z_k be the solution to

$$\begin{cases} (-\Delta)^m z_k = Q_k e^{2ma_k u_k} & \text{in } B_R(x_0) \\ z_k = \Delta z_k = \ldots = \Delta^{m-1} z_k = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

By Proposition 13 and (19)

$$u_k \geq z_k - C. \tag{20}$$

By Lemma 5, up to a subsequence, $z_k \to z$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_R(x_0)\backslash\{x_0\})$, where

$$\begin{cases} (-\Delta)^m z = \mu & \text{in } B_R(x_0) \\ z = \Delta z = \ldots = \Delta^{m-1} z = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

Since $Q_0(x_0) > 0$, we have $\mu \geq \gamma_m \delta_{x_0} = (-\Delta)^m \ln \frac{1}{|x-x_0|}$, and Proposition 13 applied to the function $z(x) - \ln \frac{1}{|x-x_0|}$ implies

$$z(x) \geq \ln \frac{1}{|x-x_0|} - C,$$

hence

$$\int_{B_R(x_0)} e^{2mz} dx \geq \frac{1}{C} \int_{B_R(x_0)} \frac{1}{|x-x_0|^{2m}} dx = +\infty.$$

Then (20) and Fatou's lemma imply

$$\begin{aligned} \liminf_{k\to\infty} \int_{B_R(x_0)} e^{2ma_k u_k} dx &\geq \int_{B_R(x_0)} \liminf_{k\to\infty} e^{2ma_k u_k} dx \\ &\geq \frac{1}{C} \int_{B_R(x_0)} \liminf_{k\to\infty} e^{2ma_k z_k} dx \\ &\geq \frac{1}{C} \int_{B_R(x_0)} e^{2mz} dx = +\infty, \end{aligned} \tag{21}$$

contradicting (10). □

The following proposition gives a general procedure to rescale at points where u_k goes to infinity.

Proposition 6 *In the hypothesis of Theorem 1, assume that $a_k \equiv 1$ for every k and that case (ii) occurs. Then, for every $x_0 \in S$ such that $\sup_{B_R(x_0)} u_k \to \infty$ for every $0 < R < \mathrm{dist}(x_0, \partial\Omega)$ as $k \to \infty$, there exist points $x_k \to x_0$ and positive numbers $r_k \to 0$ such that*

$$v_k(x) := u_k(x_k + r_k x) + \ln r_k \le 0 \le \ln 2 + v_k(0), \tag{22}$$

and as $k \to \infty$ either a subsequence $v_k \to v$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where

$$(-\Delta)^m v = Q_0(x_0) e^{2mv},$$

or $v_k \to -\infty$ almost everywhere and there are positive numbers $\gamma_k \to +\infty$ such that

$$\frac{v_k}{\gamma_k} \to p \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}),$$

where p is a polynomial on even degree at most $2m-2$.

Proof. Following [ARS], take x_0 such that $\sup_{B_R(x_0)} u_k \to +\infty$ for every R and select, for $R < \mathrm{dist}(x_0, \partial\Omega)$, $0 \le r_k < R$ and $x_k \in \overline{B_{r_k}(x_0)}$ such that

$$(R - r_k) e^{u_k(x_k)} = (R - r_k) \sup_{B_{r_k}(x_0)} e^{u_k} = \max_{0 \le r < R} \Big((R - r) \sup_{B_r(x_0)} e^{u_k} \Big) =: L_k.$$

Then $L_k \to +\infty$ and $s_k := \frac{R - r_k}{2L_k} \to 0$ as $k \to \infty$, and

$$v_k(x) := u_k(x_k + s_k x) + \ln s_k \le 0 \quad \text{in } B_{L_k}(0)$$

satisfies

$$(-\Delta)^m v_k = \widetilde{Q}_k e^{2mv_k}, \quad \widetilde{Q}_k(x) := Q_k(x_k + s_k x),$$

and

$$\int_{B_{L_k}(0)} \widetilde{Q}_k e^{2mv_k} dx = \int_{B_{\frac{1}{2}(R - r_k)}(x_k)} Q_k e^{2mu_k} dx \le C.$$

We can now apply the first part of the theorem to the functions v_k, observing that there are no concentration points ($S_1 = \emptyset$), since $v_k \le 0$, and using Theorem 12 to characterize the function p. □

3 The case of a closed manifold

To prove Theorem 2 we assume that $\sup_M u_k \to \infty$ and we blow up at I suitably chosen sequences of points $x_{i,k} \to x^{(i)}$ with $u_k(x_{i,k}) \to \infty$ as $k \to \infty$, $1 \le i \le I$. We call the $x^{(i)}$'s concentration points. Then we show the following:

(i) If $x^{(i)}$ is a concentration point, then $Q_0(x^{(i)}) > 0$.

(ii) The profile of the u_k's at any concentration point is the function η_0 defined in (27) below, hence it carries the fixed amount of energy Λ_1, see (29).

(iii) $u_k \to -\infty$ locally uniformly in $M \backslash \{x^{(i)} : 1 \le i \le I\}$.

(iv) The *neck energy* vanishes in the sense of (47) below, hence in the limit only the energy of the profiles at the concentration points appears.

Parts (i) and (ii) (Proposition 8) follow from Lemma 7 below and the classification results of [Mar1] (or [Xu]) and [Mar2]. For parts (iii) and (iv) we adapt a technique of [DR], see also [Mal], [Ndi] for a different approach.

The following lemma (compare [Mal, Lemma 2.3]) is important, because its failure in the non-compact case is responsible for the rich concentration-compactness behavior in Theorem 1. Its proof relies on the existence and on basic properties of the Green function for the Paneitz operator P_g^{2m}, as proven in [Ndi, Lemma 2.1] (here we need the hypothesis $\ker P_g^{2m} = \{constants\}$).

Lemma 7 *Let (u_k) be a sequence of functions on (M,g) satisfying (7) and (8). Then for $\ell = 1,\dots,2m-1$, we have*

$$\int_{B_r(x)} |\nabla^\ell u_k|^p \,\mathrm{dvol}_g \le C(p) r^{2m-\ell p}, \quad 1 \le p < \frac{2m}{\ell},$$

for every $x \in M$, $0 < r < r_{\mathrm{inj}}$ and for every k, where r_{inj} is the injectivity radius of (M,g).

Proof. Set $f_k := Q_k e^{2mu_k} - Q_g^{2m}$, which is bounded in $L^1(M)$ thanks to (8). Let G_ξ be the Green's function for P_g^{2m} on (M,g) such that

$$u_k(\xi) = \fint_M u_k \,\mathrm{dvol}_g + \int_M G_\xi(y) f_k(y) \,\mathrm{dvol}_g(y). \tag{23}$$

For $x, \xi \in M$, $x \neq \xi$, [Ndi, Lemma 2.1] implies

$$|\nabla_\xi^\ell G_\xi(x)| \le \frac{C}{\mathrm{dist}(x,\xi)^\ell}, \quad 1 \le \ell \le 2m-1. \tag{24}$$

Then, differentiating (23) and using (24) and Jensen's inequality, we get

$$\begin{aligned} |\nabla^\ell u_k(\xi)|^p &\le C\bigg(\int_M \frac{1}{\mathrm{dist}(\xi,y)^\ell} |f_k(y)| \,\mathrm{dvol}_g(y)\bigg)^p \\ &\le C\int_M \bigg(\frac{\|f_k\|_{L^1(M)}}{\mathrm{dist}(\xi,y)^\ell}\bigg)^p \frac{|f_k(y)|}{\|f_k\|_{L^1(M)}} \,\mathrm{dvol}_g(y). \end{aligned}$$

From Fubini's theorem we then conclude

$$\begin{aligned} \int_{B_r(x)} |\nabla^\ell u_k(\xi)|^p \,\mathrm{dvol}_g(\xi) &\le C\|f_k\|^p_{L^1(M)} \sup_{y\in M} \int_{B_r(x)} \frac{1}{\mathrm{dist}(\xi,y)^{\ell p}} \,\mathrm{dvol}_g(\xi) \\ &\le Cr^{2m-\ell p}. \end{aligned}$$

□

Let $\exp_x : T_xM \cong \mathbb{R}^{2m} \to M$ denote the exponential map at x.

Proposition 8 *Let (u_k) be a sequence of solutions to (7), (8) with $\max u_k \to \infty$ as $k \to \infty$. Choose points $x_k \to x_0 \in M$ (up to a subsequence) such that $u_k(x_k) = \max_M u_k$. Then $Q_0(x_0) > 0$ and, setting*

$$\mu_k := 2\bigg(\frac{(2m-1)!}{Q_0(x_0)}\bigg)^{\frac{1}{2m}} e^{-u_k(x_k)} \tag{25}$$

we find that the functions $\eta_k : B_{\frac{r_{\mathrm{inj}}}{\mu_k}} \subset \mathbb{R}^{2m} \to \mathbb{R}$*, given by*

$$\eta_k(y) := u_k(\exp_{x_k}(\mu_k y)) + \log \mu_k - \frac{1}{2m} \log \frac{(2m-1)!}{Q_0(x_0)},$$

converge up to a subsequence to $\eta_0(y) = \ln \frac{2}{1+|y|^2}$ *in* $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$*. Moreover*

$$\lim_{R\to+\infty} \lim_{k\to\infty} \int_{B_{R\mu_k}(x_k)} Q_k e^{2mu_k} \,\mathrm{dvol}_g = \Lambda_1. \tag{26}$$

Remark. The function

$$\eta_0(x) := \log \frac{2}{1+|x|^2} \tag{27}$$

satisfies $(-\Delta)^m \eta_0 = (2m-1)! e^{2m\eta_0}$, which is (9) with $Q_k \equiv (2m-1)!$ and $a_k \equiv 1$. In fact η_0 has a remarkable geometric interpretation: If $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection, then

$$e^{2\eta_0} g_{\mathbb{R}^{2m}} = (\pi^{-1})^* g_{S^{2m}}, \tag{28}$$

where $g_{S^{2m}}$ is the round metric on S^{2m}. Then (28) implies

$$(2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx = \int_{S^{2m}} Q_{S^{2m}} \mathrm{dvol}_{g_{S^{2m}}} = (2m-1)! |S^{2m}| = \Lambda_1. \tag{29}$$

Proof of Proposition 8. Step 1. Set $\sigma_k = e^{-u_k(x_k)}$, and consider on $B_{\frac{r_{\mathrm{inj}}}{\sigma_k}} \subset \mathbb{R}^{2m}$ the functions

$$z_k(y) := u_k(\exp_{x_k}(\sigma_k y)) + \log(\sigma_k) \le 0, \tag{30}$$

and the metrics

$$\tilde{g}_k := (\exp_{x_k} \circ T_k)^* g,$$

where $T_k : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, $T_k y = \sigma_k y$. Then, setting $\hat{Q}_k(y) := Q_k(\exp_{x_k}(\sigma_k y))$, and pulling back (7) via $\exp_{x_k} \circ T_k$, we get

$$P^{2m}_{\tilde{g}_k} z_k + Q^{2m}_{\tilde{g}_k} = \sigma_k^{-2m} \hat{Q}_k e^{2mz_k}. \tag{31}$$

Setting now $\hat{g}_k := \sigma_k^{-2} \tilde{g}_k$, we have $P^{2m}_{\hat{g}_k} = \sigma_k^{2m} P^{2m}_{\tilde{g}_k}$, $Q^{2m}_{\hat{g}_k} = \sigma_k^{2m} Q^{2m}_{\tilde{g}_k}$, and from (31) we infer

$$P^{2m}_{\hat{g}_k} z_k + Q^{2m}_{\hat{g}_k} = \hat{Q}_k e^{2mz_k}. \tag{32}$$

Then, since the principal part of the Paneitz operator is $(-\Delta_g)^m$, we can write

$$P_{\hat{g}_k} = (-\Delta_{\hat{g}_k})^m + A_k,$$

where A_k is a linear differential operator of order at most $2m-1$; moreover the coefficients of A_k are going to 0 in $C^k_{\mathrm{loc}}(\mathbb{R}^{2m})$ for all $k \ge 0$, since $\hat{g}_k \to g_{\mathbb{R}^{2m}}$ in $C^k_{\mathrm{loc}}(\mathbb{R}^{2m})$ for all $k \ge 0$, and $P_{g_{\mathbb{R}^{2m}}} = (-\Delta)^m$. Then (32) can be written as

$$(-\Delta_{\hat{g}_k})^m z_k + A_k z_k + Q^{2m}_{\hat{g}_k} = \hat{Q}_k e^{2mz_k}. \tag{33}$$

Step 2. We now claim that $z_k \to z_0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where

$$(-\Delta)^m z_0 = Q_0(x_0) e^{2mz_0}, \quad \int_{\mathbb{R}^{2m}} e^{2mz_0} dx < \infty. \tag{34}$$

We first assume $m > 1$. Fix $R > 0$ and write $z_k = h_k + w_k$ on $B_R = B_R(0)$, where $\Delta^m_{\hat{g}_k} h_k = 0$ and

$$\begin{cases} (-\Delta_{\hat{g}_k})^m w_k = (-\Delta_{\hat{g}_k})^m z_k. & \text{in } B_R \\ w_k = \Delta w_k = \ldots = \Delta^{m-1} w_k = 0 & \text{on } \partial B_R \end{cases} \tag{35}$$

From $z_k \leq 0$ we infer $\|\hat{Q}_k e^{2mz_k}\|_{L^\infty(B_R)} \leq C$, and clearly $Q^{2m}_{\hat{g}_k} = \sigma_k^{2m} Q^{2m}_{\tilde{g}_k} \to 0$ in $L^\infty_{\text{loc}}(\mathbb{R}^{2m})$. Lemma 7 implies that $(A_k z_k)$ is bounded in $L^p(B_R)$, $1 \leq p < \frac{2m}{2m-1}$, hence from (35) and elliptic estimates we get uniform bounds for (w_k) in $W^{2m,p}(B_R)$, $1 \leq p < \frac{2m}{2m-1}$, hence in $C^0(B_R)$. Again using Lemma 7, we get

$$\|\Delta_{\hat{g}_k} h_k\|_{L^1(B_R)} \leq C(\|z_k\|_{W^{2,1}(B_R)} + \|w_k\|_{W^{2,1}(B_R)}) \leq C.$$

Since $\Delta^{m-1}_{\hat{g}_k}(\Delta_{\hat{g}_k} h_k) = 0$, elliptic estimates (compare Proposition 11) give

$$\|\Delta_{\hat{g}_k} h_k\|_{C^\ell(B_{R/2})} \leq C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{36}$$

This, together with $|h_k(0)| = |w_k(0)| \leq C$, and $h_k \leq -w_k \leq C$ and elliptic estimates (e.g. [GT, Thm. 8.18]), implies that $\|h_k\|_{L^1(B_{R/2})} \leq C$, hence, again using elliptic estimates,

$$\|h_k\|_{C^\ell(B_{R/4})} \leq C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{37}$$

Therefore (z_k) is bounded in $W^{2m,p}(B_{R/4})$, $1 \leq p < \frac{2m}{2m-1}$. We now go back to (35), replacing R with $R/4$ and redefining h_k and w_k accordingly on $B_{R/4}$. We now have that $(A_k z_k)$ is bounded in $L^p(B_{R/4})$ for $1 \leq p < \frac{2m}{2m-2}$ by Sobolev's embedding, and we infer as above that (w_k) is bounded in $W^{2m,p}(B_{R/4})$, $1 \leq p < \frac{2m}{2m-2}$, and h_k is bounded in $C^\ell(B_{R/16})$, $\ell \geq 0$. Iterating, we find that (z_k) is bounded in $W^{2m,p}(B_{R/4^{2m}})$ for every $p \in [1, \infty[$. By letting $R \to \infty$ and extracting a diagonal subsequence, we infer that (z_k) converges in $C^{2m-1,\alpha}_{\text{loc}}(\mathbb{R}^{2m})$. Then (34) follows from Fatou's lemma, letting $R \to \infty$, and the claim is proven.

When $m = 1$, since $P^2_g = -\Delta_g$, (32) implies at once that $(\Delta_{\hat{g}_k} z_k)$ is locally bounded in L^∞. Then, since $z_k \leq 0$ and $z_k(0) = 0$, the claim follows from elliptic estimates (e.g. [GT, Thm. 8.18]).

Step 3. We shall now rule out the possibility that $Q_0(x_0) \leq 0$.

Case $Q_0(x_0) = 0$. By the maximum principle one sees that, for $m = 1$, (34) has no solution (see e.g. [Mar2, Thm. 3]), contradiction. If $m \geq 2$, still by [Mar2, Thm. 3], any solution z_0 to (34) is a non-constant polynomial of degree at most $2m - 2$, and there are $1 \leq j \leq m - 1$ and $a < 0$ such that $\Delta^j z_0 \equiv a$. Following an argument of [RS], see also [Mal], we shall find a contradiction. Indeed we have

$$\lim_{k\to\infty} \int_{B_R} |\Delta^j z_k| dx = \int_{B_R} |\Delta^j z_0| dx = \frac{|a|\omega_{2m}}{2m} R^{2m} + o(R^{2m}), \quad \text{as } R \to +\infty.$$

Scaling back to u_k, we find

$$\lim_{k\to\infty} \left(\sigma_k^{2j-2m} \int_{B_{R\sigma_k}(x_k)} |\nabla^{2j} u_k| \,\mathrm{dvol}_g \right) \geq C^{-1} R^{2m} + o(R^{2m}), \quad \text{as } R \to +\infty,$$

while, from Lemma 7,

$$\int_{B_{R\sigma_k}(x_k)} |\nabla^{2j} u_k| \, \mathrm{dvol}_g \leq C(R\sigma_k)^{2m-2j}. \tag{38}$$

This yields the desired contradiction as $k, R \to +\infty$.

Case $Q_0(x_0) < 0$. By [Mar2, Thm. 1] there exists no solution to (34) for $m = 1$, a contradiction. If $m \geq 2$, from [Mar2, Thm. 2] we infer that there are a constant $a \neq 0$ and $1 \leq j \leq m-1$ such that

$$\lim_{\substack{|x| \to +\infty \\ x \in \mathcal{C}}} \Delta^j z_0(x) = a,$$

where $\mathcal{C} := \{t\xi \in \mathbb{R}^{2m} : t \geq 0, \ \xi \in K\}$ and $K \subset S^{2m-1}$ is a compact set with $\mathcal{H}^{2m-1}(K) > 0$. Then, as above,

$$\begin{aligned} \lim_{k\to\infty} \left(\sigma_k^{2j-2m} \int_{B_{R\sigma_k}(x_k)} |\nabla^{2j} u_k| \, \mathrm{dvol}_g \right) &\geq C^{-1} \int_{B_R \cap \mathcal{C}} |\Delta^j z_0| dx \\ &\geq C^{-1} R^{2m} + o(R^{2m}), \end{aligned}$$

again contradicting (38). Then we have shown that $Q_0(x_0) > 0$.

Step 4. Since $Q_k(x_0) > 0$, μ_k and η_k are well-defined. Repeating the procedure of Step 2, we find a function $\overline{\eta} \in C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$ such that $\eta_k \to \overline{\eta}$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where (compare (34))

$$(-\Delta)^m \overline{\eta} = (2m-1)! e^{2m\overline{\eta}}, \quad \int_{\mathbb{R}^{2m}} e^{2m\overline{\eta}} dx < +\infty.$$

By [Mar1, Thm. 2], either $\overline{\eta}$ is a standard solution, i.e. there are $x_0 \in \mathbb{R}^{2m}$, $\lambda > 0$ such that

$$\overline{\eta}(y) = \log \frac{2\lambda}{1 + \lambda^2 |y - y_0|^2}, \tag{39}$$

or $\Delta^j \overline{\eta}(x) \to a$ as $|x| \to \infty$ for some constant $a < 0$ and for some $1 \leq j \leq m-1$. In the latter case, as in Step 3, we reach a contradiction. Hence (39) is satisfied. Since $\max_M \eta_k = \eta_k(0) = \log 2$ for every k, we have $y_0 = 0$, $\lambda = 1$, i.e. $\overline{\eta} = \eta_0$. Since, by Fatou's lemma

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{R\mu_k(x_k)} Q_k e^{2mu_k} \, \mathrm{dvol}_g = (2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx,$$

(26) follows from (29). □

Proof of Theorem 2. Assume first that $u_k \leq C$. Then $P^{2m}_g u_k$ is bounded in $L^\infty(M)$ and Lemma 7 and by elliptic estimates $u_k - \overline{u}_k$ is bounded in $W^{2m,p}(M)$ for every $1 \leq p < \infty$, hence in $C^{2m-1,\alpha}(M)$ for every $\alpha \in [0, 1[$, where $\overline{u}_k := \fint_M u_k \, \mathrm{dvol}_g$. Observe that by Jensen's inequality and (8), $\overline{u}_k \leq C$.

If $\overline{u}_k$ remains bounded (up to a subsequence), then by Ascoli-Arzelà's theorem, for every $\alpha \in [0, 1[$, u_k is convergent (up to a subsequence) in $C^{2m-1,\alpha}(M)$, and we are in case (i) of Theorem 2.

If $\overline{u}_k \to -\infty$, we have that $u_k \to -\infty$ uniformly on M and we are in case (ii) of the theorem, with $S_1 = \emptyset$.

From now on we shall assume that $\max_M u_k \to \infty$ as $k \to \infty$, and closely follow the argument of [DR].

Step 1. There are $I > 0$ converging sequences $x_{i,k} \to x^{(i)} \in M$ with $u_k(x_{i,k}) \to \infty$ as $k \to \infty$, such that

(A_1) $Q_0(x^{(i)}) > 0$, $1 \le i \le I$.

(A_2) $\frac{\mathrm{dist}(x_{i,k}, x_{j,k})}{\mu_{i,k}} \to +\infty$ as $k \to +\infty$ for all $1 \le i, j \le I$, $i \neq j$, where

$$\mu_{i,k} := 2\left(\frac{(2m-1)!}{Q_0(x^{(i)})}\right)^{\frac{1}{2m}} e^{-u_k(x_{i,k})}.$$

(A_3) Set $\eta_{i,k}(y) := u_k(\exp_{x_{i,k}}(\mu_{i,k} y)) - u_k(x_{i,k})$. Then for $1 \le i \le I$

$$\eta_{i,k}(y) \to \eta_0(y) = \log \frac{2}{1+|y|^2} \quad \text{in } C^{2m}_{\mathrm{loc}}(\mathbb{R}^{2m}) \quad (k \to \infty). \tag{40}$$

(A_4) For $1 \le i \le I$

$$\lim_{R \to +\infty} \lim_{k \to +\infty} \int_{B_{R\mu_{i,k}}(x_{i,k})} Q_k e^{2mu_k} dx \to \Lambda_1. \tag{41}$$

(A_5) There exists $C > 0$ such that for all k

$$\sup_{x \in M} \left[e^{u_k(x)} R_k(x) \right] \le C, \quad R_k(x) := \min_{1 \le i \le I} \mathrm{dist}(x, x_{i,k}).$$

Step 1 follows from Proposition 8 and induction as follows. Define $x_{1,k} = x_k$ as in Proposition 8. Then (A_1), (A_3) and (A_4) are satisfied with $i = 1$. If $\sup_{x \in M} \left[e^{u_k(x)} \mathrm{dist}(x_{1,k}, x) \right] \le C$, then $I = 1$ and also (A_5) is satisfied, so we are done. Otherwise we choose $x_{2,k}$ such that

$$R_{1,k}(x_{2,k}) e^{u_k(x_{2,k})} = \max_{x \in M} R_{1,k}(x) e^{u_k(x)} \to \infty, \quad R_{1,k}(x) := \mathrm{dist}(x, x_{1,k}). \tag{42}$$

Then (A_2) with $i = 2$, $j = 1$ follows at once from (42), while (A_2) with $i = 1$, $j = 2$ follows from (A_3), as in [DR]. A slight modification of Proposition 8 shows that $(x_{2,k}, \mu_{2,k})$ satisfies (A_1), (A_3) and (A_4), and we continue so, until also property (A_5) is satisfied. The procedure stops after finitely many steps, thanks to (A_2), (A_4) and (8).

Step 2. With the same proof as in Step 2 of [DR, Thm. 1]:

$$\sup_{x \in M} R_k(x)^\ell |\nabla^\ell u_k(x)| \le C, \quad \ell = 1, 2, \dots, 2m-1. \tag{43}$$

Step 3. $u_k \to -\infty$ locally uniformly in $M \backslash S_1$, $S_1 := \{x^{(i)} : 1 \le i \le I\}$. This follows easily from (43) above and (46) below (which implies that $u_k \to -\infty$ locally uniformly in $B_{\delta_\nu}(x^{(i)}) \backslash \{x^{(i)}\}$ for any $1 \le i \le I$, $\nu \in [1, 2[$ and δ_ν as in Step 4), but we also sketch an instructive alternative proof, which does not make use of (46).

Our Theorem 1 can be reproduced on a closed manifold, with a similar proof and using Proposition 3.1 from [Mal] instead of Theorem 4 above. Then either

(a) u_k is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(M\backslash S_1)$, or

(b) $u_k \to -\infty$ locally uniformly in $M\backslash S_1$, or

(c) there exists a closed set $S_0 \subset M\backslash S_1$ of Hausdorff dimension at most $2m-1$ and numbers $\beta_k \to +\infty$ such that

$$\frac{u_k}{\beta_k} \to \varphi \text{ in } C^{2m-1,\alpha}_{\mathrm{loc}}(M\backslash(S_0 \cup S)), \tag{44}$$

where

$$\Delta_g^m \varphi \equiv 0, \quad \varphi \le 0, \quad \varphi \not\equiv 0 \quad \text{on } M\backslash S_1, \quad \varphi \equiv 0 \text{ on } S_0. \tag{45}$$

Case (a) can be ruled out using (8) as in (21) at the end of the proof of Theorem 1. Case (c) contradicts Lemma 7, by considering any ball $B_R(x_0) \subset\subset \Omega\backslash S_1$ with $\int_{B_R(x_0)} |\nabla\varphi| \, \mathrm{dvol}_g > 0$ and using (44). Hence Case (b) occurs, as claimed.

Step 4. We claim that for every $1 \le \nu < 2$, there exist $\delta_\nu > 0$ and $C_\nu > 0$ such that for $1 \le i \le I$

$$\mathrm{dist}(x, x_{i,k})^{2m\nu} e^{2mu_k(x)} \le C_\nu \mu_{i,k}^{2m(\nu-1)}, \quad \text{for } x \in B_{\delta_\nu}(x_{i,k}). \tag{46}$$

Then on the *necks* $\Sigma_{i,k} := B_{\delta_\nu}(x_{i,k})\backslash B_{R\mu_{i,k}}(x_{i,k})$ we have

$$\begin{aligned} \int_{\Sigma_{i,k}} e^{2mu_k} \, \mathrm{dvol}_g &\le C_\nu \mu_{i,k}^{2m(\nu-1)} \int_{\Sigma_{i,k}} \mathrm{dist}(x, x_{i,k})^{-2m\nu} \, \mathrm{dvol}_g(x) \\ &\le C_\nu \mu_{i,k}^{2m(\nu-1)} \int_{R\mu_{i,k}}^{\delta_\nu} r^{2m-1-2m\nu} dr \\ &= C_\nu R^{2m(1-\nu)} - C_\nu \mu_{i,k}^{2m(\nu-1)} \delta_\nu^{2m(1-\nu)}, \end{aligned}$$

whence

$$\lim_{R\to+\infty} \lim_{k\to+\infty} \int_{\Sigma_{i,k}} Q_k e^{2mu_k} \, \mathrm{dvol}_g = 0. \tag{47}$$

This, together with (26) and Step 3 implies (11), assuming that $x^{(i)} \neq x^{(j)}$ for $i \neq j$. This we be shown in Step 4c below. Then (12) follows at once from (2).

Let us prove (46). Fix $1 \le \nu < 2$ and set for $1 \le i \le I$

$$\tilde{R}_{i,k} := \min_{j\neq i} \mathrm{dist}(x_{i,k}, x_{j,k}).$$

Step 4a. Let $i \in \{1, \dots, I\}$ be such that for some $\theta > 0$ we have

$$\tilde{R}_{i,k} \le \theta \tilde{R}_{j,k} \quad \text{for } 1 \le j \le I, \ k \ge 1. \tag{48}$$

Set

$$\varphi_{i,k}(r) := r^{2m\nu} \exp\left(\fint_{\partial B_r(x_{i,k})} 2mu_k d\sigma_g \right), \tag{49}$$

for $0 < r < r_{\mathrm{inj}}$, where $d\sigma_g$ is the measure on $\partial B_r(x_{i,k})$ induced by g. Observe that

$$\varphi'_{i,k}(r\mu_{i,k}) < 0 \quad \text{if and only if} \quad r\mu_{i,k} < -\nu \left(\fint_{\partial B_{r\mu_{i,k}}(x_{i,k})} \frac{\partial u_k}{\partial n} d\sigma_g \right)^{-1}. \tag{50}$$

From (40) we infer

$$\mu_{i,k}\frac{\partial u_k}{\partial n}\bigg|_{\partial B_{\mu_{i,k}r}(x_{i,k})} \to \frac{\partial}{\partial r}\log\frac{2}{1+r^2} = \frac{-2r}{1+r^2},$$

hence

$$\mu_{i,k}\fint_{\partial B_{\mu_{i,k}r}(x_{i,k})}\frac{\partial u_k}{\partial n}d\sigma_g \to -\frac{2r}{1+r^2}, \quad \text{for } r>0 \text{ as } k\to\infty,$$

and (50) implies that for any $R \geq 2R_\nu := 2\sqrt{\frac{\nu}{2-\nu}}$, there exists $k_0(R)$ such that

$$\varphi'_{i,k}(r\mu_{i,k}) < 0 \quad \text{for } k \geq k_0(R),\ r \in [2R_\nu, R]. \tag{51}$$

Define

$$r_{i,k} := \sup\Big\{r \in [2R_\nu\mu_{i,k}, \tilde{R}_{i,k}/2] : \varphi'_{i,k}(\rho) < 0 \text{ for } \rho \in [2R_\nu\mu_{i,k}, r)\Big\}. \tag{52}$$

From (51) we infer that

$$\lim_{k\to+\infty}\frac{r_{i,k}}{\mu_{i,k}} = +\infty. \tag{53}$$

Let us assume that

$$\lim_{k\to\infty} r_{i,k} = 0. \tag{54}$$

Consider

$$v_{i,k}(y) := u_k(\exp_{x_{i,k}}(r_{i,k}y)) - C_{i,k}, \quad C_{i,k} := \fint_{\partial B_{r_{i,k}}(x_{i,k})} u_k d\sigma_g, \tag{55}$$

and let

$$\hat{g}_{i,k} := r_{i,k}^{-2}(\exp_{x_{i,k}} \circ T_{i,k})^* g, \quad \hat{Q}_{i,k}(y) := Q_k(\exp_{x_{i,k}}(r_{i,k}y)),$$

where

$$T_{i,k}(y) := r_{i,k}y \quad \text{for } y \in \mathbb{R}^{2m}.$$

Then

$$\begin{aligned} P^{2m}_{\hat{g}_{i,k}} v_{i,k} + r_{i,k}^{2m} Q_{\hat{g}_{i,k}} &= r_{i,k}^{2m}\hat{Q}_{i,k}e^{2m(v_{i,k}+C_{i,k})} \\ &= r_{i,k}^{2m(1-\nu)}\varphi_{i,k}(r_{i,k})\hat{Q}_{i,k}e^{2mv_{i,k}}. \end{aligned} \tag{56}$$

We also set

$$\mathcal{J}_i = \{j \neq i : \operatorname{dist}(x_{i,k}, x_{j,k}) = O(r_{i,k}) \text{ as } k\to\infty\}, \tag{57}$$

and

$$\tilde{x}^{(i)}_{j,k} := \frac{1}{r_{i,k}}\exp^{-1}_{x_{i,k}}(x_{j,k}), \quad \tilde{x}^{(i)}_j = \lim_{k\to\infty}\tilde{x}_{j,k}, \tag{58}$$

after passing to a subsequence, if necessary. Thanks to (48) and (52), we have that $|\tilde{x}^{(i)}_j| \geq 2$ for all $j \in \mathcal{J}_i$ and that

$$|\tilde{x}^{(i)}_j - \tilde{x}^{(i)}_\ell| \geq \frac{2}{\theta} \quad \text{for all } j,\ell \in \mathcal{J}_i,\ j \neq \ell.$$

By (43) and the choice of $C_{i,k}$ in (55), $v_{i,k}$ is uniformly bounded in

$$C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}\setminus\{0,\tilde{x}^{(i)}_j : j\in\mathcal{J}_i\}).$$

Thanks to (52) and (53), given $R > 2R_\nu$, there exists $k_0(R)$ such that $\varphi_{i,k}(r_{i,k}) < \varphi_{i,k}(R\mu_{i,k})$ for all $k \geq k_0$. From (40), we infer

$$\begin{aligned} \mu_{i,k}^{2m}\exp\left(\fint_{\partial B_{R\mu_{i,k}(x_{i,k})}} 2mu_k d\sigma\right) &= \exp\left(\fint_{\partial B_{R\mu_{i,k}}(x_{i,k})} 2m(u_k+\log\mu_{i,k})d\sigma\right) \\ &= C(R)+o(1), \quad \text{as } k\to\infty, \end{aligned} \tag{59}$$

where

$$C(R)\to 0, \quad \text{as } R\to\infty. \tag{60}$$

Then, together with (53), letting $k\to+\infty$ we get

$$\begin{aligned} r_{i,k}^{2m(1-\nu)}\varphi_{i,k}(r_{i,k}) &\leq r_{i,k}^{2m(1-\nu)}\varphi_{i,k}(R\mu_{i,k}) \\ &= \mu_{i,k}^{2m}\exp\left(\fint_{\partial B_{R\mu_{i,k}(x_{i,k})}} 2mu_k d\sigma\right)R^{2m\nu}\left(\frac{\mu_{i,k}}{r_{i,k}}\right)^{2m(\nu-1)} \\ &\to 0. \end{aligned} \tag{61}$$

Therefore the right-hand side of (56) goes to 0 locally uniformly in

$$\mathbb{R}^{2m}\setminus\{0,\tilde{x}^{(i)}_j : j\in\mathcal{J}_i\};$$

moreover

$$\hat{g}_{i,k}\to g_{\mathbb{R}^{2m}} \text{ in } C^k_{\mathrm{loc}}(\mathbb{R}^{2m}) \text{ for every } k\geq 0, \quad r_{i,k}^{2m}\hat{Q}_{i,k}\to 0 \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}^{2m}). \tag{62}$$

It follows that, up to a subsequence,

$$v_{i,k}\to h_i \text{ in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}\setminus\{0,\tilde{x}^{(i)}_j : j\in\mathcal{J}_i\}), \tag{63}$$

where, taking (43) into account,

$$\Delta^m h_i(x) = 0, \quad x\in\mathbb{R}^{2m}\setminus\{0,\tilde{x}^{(i)}_j : j\in\mathcal{J}_i\}$$

and

$$\tilde{R}(x)^\ell|\nabla^\ell h_i(x)|\leq C_\ell, \quad \text{for } \ell=1,\dots,2m-1,\ x\in\mathbb{R}^{2m}\setminus\{0,\tilde{x}^{(i)}_j : j\in\mathcal{J}_i\},$$

with $\tilde{R}(x) := \min\{|x|, |x-\tilde{x}^{(i)}_j| : j\in\mathcal{J}_i\}$. Then Proposition 15 from the appendix implies that

$$h_i(x) = -\lambda\log|x| - \sum_{j\in\mathcal{J}_i}\lambda_j\log|x-\tilde{x}^{(i)}_j| + \beta, \tag{64}$$

for some $\lambda,\beta,\lambda_j\in\mathbb{R}$. We now recall that the Paneitz operator is in divergence form, hence we can write

$$P^{2m}_{\hat{g}_{i,k}}v_{i,k} = \mathrm{div}_{\hat{g}_{i,k}}(A_{\hat{g}_{i,k}}v_{i,k}) \tag{65}$$

for some differential operator $A_{\hat{g}_{i,k}}$ of order $2m-1$, with coefficients converging to the coefficient of $(-1)^m\nabla\Delta^{m-1}$ uniformly in B_1, thanks to (62). Then integrating (56), using (62), (63) and (65), we get

$$\begin{aligned}\lim_{k\to\infty}\int_{B_{r_{i,k}}(x_{i,k})} Q_k e^{2mu_k}\,\mathrm{dvol}_g &= \lim_{k\to\infty}\varphi_{i,k}(r_{i,k})r_{i,k}^{2m(1-\nu)}\int_{B_1}\hat{Q}_{i,k}e^{2mv_{i,k}}\,\mathrm{dvol}_{\hat{g}_{i,k}}\\ &= \lim_{k\to\infty}\int_{B_1}\left(\mathrm{div}_{\hat{g}_{i,k}}(A_{\hat{g}_{i,k}}v_{i,k})+r_{i,k}^{2m}Q_{\hat{g}_{i,k}}\right)\mathrm{dvol}_{\hat{g}_{i,k}}\\ &= \lim_{k\to\infty}\int_{\partial B_1} n\cdot(A_{\hat{g}_{i,k}}v_{i,k})d\sigma_{\hat{g}_{i,k}}\\ &= (-1)^m\int_{\partial B_1}\frac{\partial\Delta^{m-1}h_i}{\partial n}d\sigma=\lambda\frac{\Lambda_1}{2},\end{aligned}\tag{66}$$

where here n denotes the exterior unit normal to ∂B_1 and the last identity can be inferred using (14) and the following:

$$\begin{aligned}\int_{\partial B_1}\frac{\partial\Delta^{m-1}h_i}{\partial n}d\sigma &= \lambda\int_{\partial B_1}\frac{\partial\Delta^{m-1}\log\frac{1}{|x|}}{\partial n}d\sigma\\ &\quad+\sum_{j\in\mathcal{J}_i}\lambda_j\int_{B_1}\underbrace{\Delta^m\log\frac{1}{|x-\tilde{x}_j^{(i)}|}}_{\equiv 0\ on\ B_1}dx\end{aligned}$$

From (43) with $\ell=1$, we get

$$|u_k(\exp_{x_{i,k}}(r_{i,k}y_1))-u_k(\exp_{x_{i,k}}(r_{i,k}y_2))|\leq Cr_{i,k}r\sup_{\partial B_{r_{i,k}r}(x_{i,k})}|\nabla u_k|\leq C,\tag{67}$$

for $0\leq r\leq\frac{3}{2}$, $|y_1|=|y_2|=r$. For $2R_\nu\mu_{i,k}\leq R\mu_{i,k}\leq r\leq r_{i,k}$, we infer from (59)

$$\varphi_{i,k}(r)\leq\varphi_{i,k}(R\mu_{i,k})\leq C(R)\mu_{i,k}^{2m(\nu-1)}+o(\mu_{i,k}^{2m(\nu-1)}).$$

This, (49), (59), (60) and (67) imply that for any $\eta>0$ there exist $R_\eta\geq 2R_\nu$ and $k_\eta\in\mathbb{N}$ such that

$$\mathrm{dist}(x,x_{i,k})^{2m\nu}e^{2mu_k}\leq\eta\mu_{i,k}^{2m(\nu-1)}\quad\text{for } x\in B_{r_{i,k}}(x_{i,k})\backslash B_{R_\eta\mu_{i,k}}(x_{i,k}),\ k\geq k_\eta.\tag{68}$$

It now follows easily that

$$\lim_{R\to+\infty}\lim_{k\to\infty}\int_{B_{r_{i,k}}(x_{i,k})\backslash B_{R\mu_{i,k}}(x_{i,k})}Q_ke^{2mu_k}dx=0,$$

and from (41)

$$\lim_{k\to+\infty}\int_{B_{r_{i,k}}(x_{i,k})}Q_ke^{2mu_k}dx=\Lambda_1.$$

That implies that $\lambda=2$. With a similar computation, integrating on $B_\delta(\tilde{x}_j^{(i)})$ for δ small instead of $B_1(0)$, one proves that $\lambda_j\geq 2$ for all $j\in\mathcal{J}_i$. Now set

$$\overline{h}_i(r):=\fint_{\partial B_r(0)}h_i d\sigma.$$

Then

$$\frac{d}{dr}\big(r^{2m\nu}e^{2m\overline{h}_i(r)}\big) = 2m\Bigg(\nu - 2 - \Bigg(\sum_{j\in\mathcal{J}_i}\frac{\lambda_j}{2|\tilde{x}_j^{(i)}|^2}\Bigg)r^2\Bigg)r^{2m\nu-1}e^{2m\overline{h}_i(r)},$$

for $0 < r < \frac{3}{2}$. In particular

$$\frac{d}{dr}\big(r^{2m\nu}e^{2m\overline{h}_i(r)}\big)\big|_{r=1} < 0$$

hence, for k large enough, $\varphi'_{i,k}(r_{i,k}) < 0$. This implies that

$$r_{i,k} = \frac{\tilde{R}_{i,k}}{2} \quad \text{for } k \text{ large.} \tag{69}$$

This in turn implies $\lim_{k\to\infty}\tilde{R}_{i,k} = 0$, when i satisfies (48) and $\lim_{k\to\infty} r_{i,k} = 0$. For i satisfying (48) and $\limsup_{k\to\infty}\tilde{R}_{i,k} > 0$, we infer, instead, that $\limsup_{k\to\infty} r_{i,k} > 0$. In both cases (68) holds.

Step 4b. Now assume that

$$\limsup_{k\to\infty}\tilde{R}_{i,k} > 0, \quad \text{for every } 1 \le i \le I. \tag{70}$$

Then (48) is satisfied for every $1 \le i \le I$, hence $\limsup_{k\to\infty} r_{i,k} > 0$, $1 \le i \le I$. Up to selecting a subsequence, we can set

$$\delta_\nu := \inf_{1\le i\le I}\frac{1}{2}\lim_{k\to\infty} r_{i,k} > 0.$$

Take now $\eta = 1$ in (68), and let R_1 be the corresponding R_η. Then (46) is true for $x \in B_{\delta_\nu}(x_{i,k})\backslash B_{R_1\mu_{i,k}}(x_{i,k})$. On the other hand, thanks to (A_3), we have $u_k(x) \le u_k(x_{i,k}) + C$ on $B_{R_1\mu_{i,k}}(x)$. Then, using (25), we get

$$\begin{aligned}\operatorname{dist}(x, x_{i,k})^{2m\nu}e^{2mu_k(x)} &\le C(R_1\mu_{i,k})^{2m\nu}e^{2mu_k(x_{i,k})}\\ &\le CR_1^{2m\nu}\mu_{i,k}^{2m(\nu-1)} \quad \text{for } x \in B_{R_1\mu_{i,k}}(x_{i,k}).\end{aligned}$$

This completes the proof of (46), under the assumption that (70) holds.

Step 4c. We now prove that in fact (70) holds true. Choose $1 \le i_0 \le I$ so that, up to a subsequence,

$$\tilde{R}_{i_0,k} = \min_{1\le i\le I}\tilde{R}_{i,k} \quad \text{for every } k \in \mathbb{N},$$

and assume by contradiction that $\lim_{k\to\infty}\tilde{R}_{i_0,k} = 0$. Clearly (48) holds for $i = i_0$, hence also (69) holds for $i = i_0$, by Step 4a. Then, setting $\mathcal{J}_{i_0}$ as is (57), we claim that, for any $i \in \mathcal{J}_{i_0}$, there exists $\theta(i) > 0$ such that

$$\tilde{R}_{i,k} \le \theta(i)\tilde{R}_{j,k} \quad \text{for } 1 \le j \le I.$$

Indeed

$$\tilde{R}_{i,k} = O(r_{i_0,k}) = O(\tilde{R}_{i_0,k}) \quad \text{as } k \to \infty.$$

It then follows that (48) holds for all $i \in \mathcal{J}_{i_0}$, and that Step 4a applies to them. Observing that $\mathcal{J}_{i_0} \neq \emptyset$ thanks to Step 4a (Identity (69) with i_0 instead of i), we can pick $i \in \mathcal{J}_{i_0}$ such that, up to a subsequence,

$$\mathrm{dist}(x_{i,k}, x_{i_0,k}) \geq \mathrm{dist}(x_{j,k}, x_{i_0,k}) \quad \text{for all } j \in \mathcal{J}_{i_0},\ k > 0.$$

Recalling the definition of $\tilde{x}_j^{(i)}$ for $j \in \mathcal{J}_i$, we get $|\tilde{x}_{i_0}^{(i)}| \geq |\tilde{x}_j^{(i)} - \tilde{x}_{i_0}^{(i)}|$ for all $j \in \mathcal{J}_i$. A consequence of this inequality is that the scalar product

$$\tilde{x}_{i_0}^{(i)} \cdot \tilde{x}_j^{(i)} > 0 \tag{71}$$

for all $j \in \mathcal{J}_i$. In other words all the $\tilde{x}_j^{(i)}$'s with $j \in \mathcal{J}_i$ lie in the same half space orthogonal to $\tilde{x}_{i_0}^{(i)}$ and whose boundary contains $0 = \tilde{x}_i^{(i)}$. Multiplying (56) by $\nabla v_{i,k}$ and integrating over $B_\delta = B_\delta(0)$ ($\delta > 0$ small), we get

$$\begin{aligned} \int_{B_\delta} P^{2m}_{\hat{g}_{i,k}} v_{i,k} \nabla v_{i,k}\, \mathrm{dvol}_{\hat{g}_{i,k}} &= -\int_{B_\delta} r_{i,k}^{2m} \hat{Q}_{i,k} \nabla v_{i,k}\, \mathrm{dvol}_{\hat{g}_{i,k}} \\ &\quad + \frac{r_{i,k}^{2m(1-\nu)}}{2m} \varphi_{i,k}(r_{i,k}) \int_{B_\delta(0)} \hat{Q}_{i,k} \nabla e^{2mv_{i,k}}\, \mathrm{dvol}_{\hat{g}_{i,k}} \\ &=: (I)_k + (II)_k. \end{aligned} \tag{72}$$

Recalling (62) and (63), we see at once that $\lim_{k\to\infty} (I)_k = 0$. Integrating by parts, we also see that

$$\begin{aligned} |(II)_k| &\leq C \frac{r_{i,k}^{2m(1-\nu)}}{2m} \varphi_{i,k}(r_{i,k}) \int_{B_\delta(0)} \frac{\nabla \hat{Q}_{i,k}}{\hat{Q}_{i,k}} \hat{Q}_{i,k} e^{2mv_{i,k}} d\,\mathrm{vol}_{\hat{g}_{i,k}} \\ &\quad + \frac{r_{i,k}^{2m(1-\nu)}}{2m} \varphi_{i,k}(r_{i,k}) \int_{\partial B_\delta(0)} O(1) d\sigma_{\hat{g}_{i,k}} \\ &\to 0 \quad \text{as } k \to \infty, \end{aligned}$$

where the last term vanishes thanks to (61), and the first term on the right of $(II)_k$ vanishes thanks to (66) and the remark that

$$\frac{\nabla \hat{Q}_{i,k}}{\hat{Q}_{i,k}} \to 0 \quad \text{in } L^\infty(B_\delta). \tag{73}$$

Recalling (63), using (43) and (62), we arrive at

$$\int_{B_\delta} \nabla h_i (-\Delta)^m h_i dx = 0. \tag{74}$$

Let us assume m even. Then, integrating by parts, we get

$$\begin{aligned} 0 &= \frac{1}{2} \int_{\partial B_\delta} ((-\Delta)^{\frac{m}{2}} h_i)^2 n d\sigma \\ &\quad - \sum_{j=0}^{\frac{m}{2}-1} \int_{\partial B_\delta} (\nabla (-\Delta)^j h_i) \frac{\partial (-\Delta)^{m-1-j} h_i}{\partial n} d\sigma \\ &\quad + \sum_{j=0}^{\frac{m}{2}-1} \int_{\partial B_\delta} \nabla \left(\frac{\partial (-\Delta)^j h_i}{\partial n} \right) (-\Delta)^{m-1-j} h_i d\sigma. \end{aligned} \tag{75}$$

Then, taking the limit as $\delta \to 0$, and writing

$$h_i(x) = 2\log\frac{1}{|x|} + G_i(x)$$

we see that all terms in (75) vanish (G_i is regular in a neighborhood of 0 and the vector function $\nabla \log \frac{1}{|x|}$ is anti-symmetric), up to at most

$$\lim_{\delta\to 0}\int_{\partial B_\delta}(-\nabla G_i)\partial_\nu(-\Delta)^{m-1}\Big(2\log\frac{1}{|x|}\Big)d\sigma = 2\gamma_m \nabla G_i(0),$$

see (14). But then (75) gives

$$2\gamma_m \nabla G_i(0).$$

Also when m is odd, in a completely analogous way, we get $\nabla G_i(0) = 0$, a contradiction with (64) and (71). This ends the proof of Step 4.

Step 5. Finally, if case (ii) occurs and $S \neq \emptyset$, then (41) implies

$$\limsup_{k\to\infty} \mathrm{vol}(g_k) \geq Q_0(x^{(1)})^{-1}\Lambda_1 > 0.$$

This justifies the last claim of the theorem. □

4 The case $M = S^{2m}$

In the case of the $2m$-dimensional sphere, the concentration-compactness of Theorem 2 becomes quite explicit: only one concentration point can appear and, by composing with suitable Möbius transformations, we have a global understanding of the concentration behavior. This was already noticed in [Str] and [MS], in dimension 2 and 4 under the assumption, which we now drop, that the Q-curvatures are positive.

Theorem 9 *Let (S^{2m}, g) be the $2m$-dimensional round sphere, and let $u_k : M \to \mathbb{R}$ be a sequence of solutions of*

$$P_g u_k + (2m-1)! = Q_k e^{2mu_k}, \tag{76}$$

where $Q_k \to Q_0$ in C^0 for a given continuous function Q_0. Assume also that

$$\mathrm{vol}(g_k) = \int_{S^{2m}} e^{2mu_k}\mathrm{dvol}_g = |S^{2m}|, \tag{77}$$

where $g_k := e^{2mu_k}g$. Then one of the following is true.

(i) For every $0 \leq \alpha < 1$, a subsequence converges in $C^{2m-1,\alpha}(S^{2m})$.

(ii) There is a point $x_0 \in S^{2m}$ such that up to a subsequence $u_k \to -\infty$ locally uniformly in $S^{2m}\backslash\{x_0\}$. Moreover $Q_0(x_0) > 0$,

$$Q_k e^{2mu_k}\,\mathrm{dvol}_g \rightharpoonup \Lambda_1 \delta_{x_0}$$

and there exist Möbius diffeomorphisms Φ_k such that the metrics $h_k := \Phi_k^* g_k$ satisfy

$$h_k \to g \text{ in } H^{2m}(S^{2m}), \quad Q_{h_k} \to (2m-1)! \text{ in } L^2(S^{2m}). \tag{78}$$

Proof. On the round sphere $P_g = \prod_{i=0}^{m-1}(-\Delta_g + i(2m-i-1))$; moreover $\ker \Delta_g = \{constants\}$ and the non-zero eigenvalues of $-\Delta_g$ are all positive. That easily implies that $\ker P_g^{2m} = \{constants\}$. From Theorem 2, and the Gauss-Bonnet-Chern theorem, we infer that in case (ii) we have

$$\Lambda_1 = \int_M Q_g \,\mathrm{dvol}_g = I\Lambda_1,$$

hence $I = 1$, and $Q_k e^{2mu_k}\,\mathrm{dvol}_g \rightharpoonup \Lambda_1 \delta_{x_0}$. In fact, in order to apply Theorem 2, we would need $Q_k \to Q_0$ in $C^1(M)$, but this hypothesis is only used in (73) in the last part of the proof of Theorem 2, in order to show that the concentration points are isolated. Since in the case of the sphere only one concentration point appears, that part of the proof is superfluous, and the assumption $Q_k \to Q_0$ in $C^0(M)$ suffices.

To prove the second part of the theorem, for every k we define a Möbius transformation $\Phi_k : S^{2m} \to S^{2m}$ such that the *normalized metric* $h_k := \Phi_k^* g_k$ satisfies

$$\int_{S^{2m}} x \,\mathrm{dvol}_{h_k} = 0.$$

Then (78) follows by reasoning as in [MS, bottom of Page 16]. □

Appendix

A A few useful results

Here we collect a few results which have been used above. For the proofs of Lemma 10, Propositions 11 and 13, and Theorem 12, see e.g. [Mar1].

The following Lemma can be considered a generalized mean value identity for polyharmonic function.

Lemma 10 (Pizzetti [Piz]) *Let $\Delta^m h = 0$, in $B_R(x_0) \subset \mathbb{R}^n$, for some m, n positive integers. Then there are positive constants $c_i = c_i(n)$ such that*

$$\fint_{B_R(x_0)} h(z)dz = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i h(x_0). \tag{79}$$

Proposition 11 *Let $\Delta^m h = 0$ in $B_2 \subset \mathbb{R}^n$. For every $0 \le \alpha < 1$, $p \in [1,\infty)$ and $\ell \ge 0$ there are constants $C(\ell,p)$ and $C(\ell,\alpha)$ independent of h such that*

$$\begin{aligned} \|h\|_{W^{\ell,p}(B_1)} &\le C(\ell,p)\|h\|_{L^1(B_2)} \\ \|h\|_{C^{\ell,\alpha}(B_1)} &\le C(\ell,\alpha)\|h\|_{L^1(B_2)}. \end{aligned}$$

A simple consequence of Lemma 10 and Proposition 11 is the following Liouville-type Theorem.

Theorem 12 *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h(x) \le C(1+|x|^\ell)$ for some integer $\ell \ge 0$. Then h is a polynomial of degree at most* $\max\{\ell, 2m-2\}$*.*

Proposition 13 *Let $u \in C^{2m-1}(\overline{B}_1)$ such that*

$$\begin{cases} (-\Delta)^m u \leq C & \text{in } B_1 \\ (-\Delta)^j u \leq C & \text{on } \partial B_1 \text{ for } 0 \leq j < m. \end{cases} \tag{80}$$

Then there exists a constant C independent of u such that $u \leq C$ in B_1.

Lemma 14 *Let $\Delta u \in L^1(\Omega)$ and $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then for every $1 \leq p < \frac{n}{n-1}$ we have*

$$\|u\|_{W^{1,p}(\Omega)} \leq C(p)\|\Delta u\|_{L^1(\Omega)}$$

Proof. Let $u \in C^\infty(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$. If $1 \leq p < \frac{n}{n-1}$, then $q := \frac{p}{p-1} > n$. From L^p-theory (see e.g. [Sim, Pag. 91]) and the imbedding $W^{1,q} \hookrightarrow L^\infty$ we infer

$$\begin{aligned} \|\nabla u\|_{L^p(\Omega)} &\leq C \sup_{\substack{\varphi \in W_0^{1,q}(\Omega) \\ \|\nabla\varphi\|_{L^q(\Omega)} \leq 1}} \int_\Omega \nabla u \cdot \nabla\varphi dx = C \sup_{\substack{\varphi \in W_0^{1,q}(\Omega) \\ \|\nabla\varphi\|_{L^q(\Omega)} \leq 1}} \int_\Omega -\Delta u \varphi dx \\ &\leq C \sup_{\substack{\varphi \in L^\infty(\Omega) \\ \|\varphi\|_{L^\infty(\Omega)} \leq 1}} \int_\Omega -\Delta u \varphi dx \leq C\|\Delta u\|_{L^1}. \end{aligned}$$

To estimate $\|u\|_{L^p(\Omega)}$ we use Poincaré's inequality. For the general case one can use a standard mollifying procedure. □

Proof of Lemma 5. By Lemma 14, $\|\Delta^{m-1}u\|_{W^{1,r}(\Omega)} \leq C(r)\|f\|_{L^1(\Omega)}$ for $1 \leq r < \frac{2m}{2m-1}$. Then, by L^p-theory, $\|u\|_{W^{2m-1,r}(\Omega)} \leq C(r)\|f\|_{L^1(\Omega)}$, and by Sobolev's embedding,

$$\|u\|_{L^s(\Omega)} \leq C(s)\|f\|_{L^1(\Omega)}, \quad \text{for all } 1 \leq s < \infty. \tag{81}$$

Now fix $B = B_{4R}(x_0) \subset\subset (\Omega \backslash S_1)$ and write $u = u_1 + u_2$, where

$$\begin{cases} (-\Delta)^m u_2 = f & \text{in } B_{4R}(x_0) \\ \Delta^j u_2 = 0 & \text{on } \partial B_{4R}(x_0) \text{ for } 0 \leq j \leq m-1. \end{cases}$$

By L^p-theory

$$\|u_2\|_{W^{2m,p}(B_{4R}(x_0))} \leq C(p,B)\|f\|_{L^p(B_{4R}(x_0))}, \tag{82}$$

with $C(p,B)$ depending on p and the chosen ball B. Together with (81), we find

$$\|u_1\|_{L^1(B_{4R}(x_0))} \leq C(p,B)(\|f\|_{L^p(B_{4R}(x_0))} + \|f\|_{L^1(\Omega)}).$$

By Proposition 11

$$\|u_1\|_{W^{2m,p}(B_R(x_0))} \leq C(p,B)(\|f\|_{L^p(B_{4R}(x_0))} + \|f\|_{L^1(\Omega)}),$$

and (15) follows. □

Proposition 15 *Let $S = \{x_1, \ldots, x_I\} \subset \mathbb{R}^{2m}$ be a finite set and let $h \in C^\infty(\mathbb{R}^{2m} \backslash S)$ satisfy $\Delta^m h = 0$ and*

$$\operatorname{dist}(x,S)|\nabla h(x)| \leq C, \quad \text{for } x \in \mathbb{R}^{2m} \backslash S. \tag{83}$$

Then there are constants β and λ_i, $1 \leq i \leq I$, such that

$$h(x) = \sum_{i=1}^{I} \lambda_i \log \frac{1}{|x - x_i|} + \beta. \tag{84}$$

Proof. Thanks to (83), $h \in L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$, so that $\Delta^m h$ is well defined in the sense of distributions and it is supported in S. Therefore

$$\Delta^m h = \sum_{i=1}^{I} \beta_i \delta_{x_i},$$

for some constants β_i. Then, recalling (14), if we set

$$v(x) := h(x) - \sum_{i=1}^{I} \lambda_i \log \frac{1}{|x - x_i|}, \quad \lambda_i := (-1)^m \frac{\beta_i}{\gamma_m},$$

we get $\Delta^m v \equiv 0$ in $\mathbb{R}^{2m}$ in the sense of distributions (hence v is smooth) and

$$|\nabla v(x)||x| \leq C \quad \text{in } \mathbb{R}^{2m}. \tag{85}$$

Then $|v(x)| \leq C(\log(1+|x|)+1)$. By Theorem 12 v is a polynomial, which (85) forces to be constant, say $v \equiv -\beta$. Now (84) follows at once. □

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