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An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs

Second edition

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Preface to the first edition

Initially thought as lecture notes of a course given by the first author at the Scuola Normale Superiore in the academic year 2003-2004, this volume grew into the present form thanks to the constant enthusiasm of the second author.

Our aim here is to illustrate some of the relevant ideas in the theory of regularity of linear and nonlinear elliptic systems, looking in particular at the context and the specific situation in which they generate. Therefore this is not a reference volume: we always refrain from generalizations and extensions. For reasons of space we did not treat regularity questions in the linear and nonlinear Hodge theory, in Stokes and Navier-Stokes theory of fluids, in linear and nonlinear elasticity; other topics that should be treated, we are sure, were not treated because of our limited knowledge. Finally, we avoided to discuss more recent and technical contributions, in particular, we never entered regularity questions related to variational integrals or systems with general growth p.

In preparing this volume we particularly took advantage from the references [6] [37] [39] [52], from a series of unpublished notes by Giuseppe Modica, whom we want to thank particularly, from [98] and from the papers [109] [110] [111].

We would like to thank also Valentino Tosatti and Davide Vittone, who attended the course, made comments and remarks and read part of the manuscript.

Part of the work was carried out while the second author was a graduate student at Stanford, supported by a Stanford Graduate Fellowship.

Preface to the second edition

This second edition is a deeply revised version of the first edition, in which several typos were corrected, details to the proofs, exercises and examples were added, and new material was covered. In particular we added the recent results of T. Rivière [88] on the regularity of critical points of conformally invariant functionals in dimension 2 (especially 2-dimensional harmonic maps), and the partial regularity of stationary harmonic maps following the new approach of T. Rivière and M. Struwe [90], which avoids the use of the moving-frame technique of F. Hélein. This gave us the motivation to briefly discuss the limiting case $p = 1$ of the L^p-estimates for the Laplacian, introducing the Hardy space $\mathcal{H}^1$ and presenting the celebrated results of Wente [112] and of Coifman-Lions-Meyer-Semmes [22].

Part of the work was completed while the second author was visiting the Centro di Ricerca Matematica Ennio De Giorgi in Pisa, whose warm hospitality is gratefully acknowledged.

Contents

Chapter 1

Harmonic functions

We begin by illustrating some aspects of the classical model problem in the theory of elliptic regularity: the Dirichlet problem for the Laplace operator.

1.1 Introduction

From now on Ω will be a bounded, connected and open subset of $\mathbb{R}^n$.

Definition 1.1 *Given a function $u \in C^2(\Omega)$ we say that u is*

- *harmonic if $\Delta u = 0$*
- *subharmonic if $\Delta u \geq 0$*
- *superharmonic if $\Delta u \leq 0$,*

where

$$\Delta u(x) := \sum_{\alpha=1}^{n} D_\alpha^2 u(x), \qquad D_\alpha := \frac{\partial}{\partial x^\alpha}$$

is the Laplacian operator.

Exercise 1.2 Prove that if $f \in C^2(\mathbb{R})$ is convex and $u \in C^2(\Omega)$ is harmonic, then $f \circ u$ is subharmonic.

Throughout this chapter we shall study some important properties of harmonic functions and we shall be concerned with the problem of the existence of harmonic functions with prescribed boundary value, namely with the solution of the following *Dirichlet problem*:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in $C^2(\Omega) \cap C^0(\overline{\Omega})$, for a given function $g \in C^0(\partial\Omega)$.

1.2 The variational method

The problem of finding a harmonic function with prescribed boundary value $g \in C^0(\partial\Omega)$ is tied, though not equivalent (see section 1.2.2), to the following one: find a minimizer u for the functional $\mathcal{D}$

$$\mathcal{D}(u) = \frac{1}{2}\int_\Omega |Du|^2 dx \tag{1.2}$$

in the class

$$\mathcal{A} = \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \ : \ u = g \text{ on } \partial\Omega\}.$$

The functional $\mathcal{D}$ is called *Dirichlet integral*.

In fact, formally, if a minimizer u exists, then *the first variation of the Dirichlet integral* vanishes:

$$\frac{d}{dt}\mathcal{D}(u + t\varphi)\Big|_{t=0} = 0$$

for all smooth compactly supported functions φ in Ω; an integration by parts then yields

$$\begin{aligned} 0 &= \frac{d}{dt}\mathcal{D}(u + t\varphi)\Big|_{t=0} \\ &= \int_\Omega \nabla u \cdot \nabla \varphi dx \\ &= -\int_\Omega \Delta u \varphi dx, \quad \forall \varphi \in C_0^\infty(\Omega), \end{aligned}$$

and by the arbitrariness of φ we conclude $\Delta u = 0$, which is the *Euler-Lagrange equation for the Dirichlet integral: minimizers of the Dirichlet integral are harmonic.*

This was stated as an equivalence by Dirichlet and used by Riemann in his geometric theory of functions.

Dirichlet's principle: A minimizer u of the Dirichlet integral in Ω with prescribed boundary value g always exists, is unique and is a harmonic function; it solves

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Conversely, any solution of (1.3) is a minimizer of the Dirichlet integral in the class of functions with boundary value g.

Dirichlet saw no need to prove this principle; however, as we shall see, in general Dirichlet's principle does not hold and, in the circumstances in which it holds, it is not trivial.

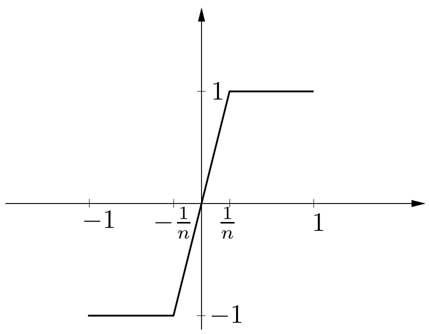

Figure 1.1: The function u_n as defined in (1.4)

1.2.1 Non-existence of minimizers of variational integrals

The following examples, the first being a classical example of Weierstrass, show that minimizers to a variational integral need not exist.

1. Consider the functional

$$\mathcal{F}(u) = \int_{-1}^{1} (x\dot{u})^2 dx$$

defined on the class of Lipschitz functions

$$\mathcal{A} = \{u \in \text{Lip}([-1,1]) \ : \ u(-1) = -1, \ u(1) = 1\}.$$

The following sequence of functions in $\mathcal{A}$

$$u_n(x) := \begin{cases} -1 & \text{for } x \in [-1, -\frac{1}{n}] \\ 1 & \text{for } x \in [\frac{1}{n}, 1] \\ nx & \text{for } x \in [-\frac{1}{n}, \frac{1}{n}] \end{cases} \tag{1.4}$$

shows that $\inf_{\mathcal{A}} \mathcal{F} = 0$, but evidently $\mathcal{F}$ cannot attain the value 0 on $\mathcal{A}$.

2. Consider

$$\mathcal{F}(u) = \int_{0}^{1} (1 + \dot{u}^2)^{\frac{1}{4}} dx,$$

defined on

$$\mathcal{A} = \{u \in \text{Lip}([0,1]) \ : \ u(0) = 1, \ u(1) = 0\}.$$

The sequence of functions

$$u(x) = \begin{cases} 1 - nx & \text{for } x \in [0, \frac{1}{n}] \\ 0 & \text{for } x \in [\frac{1}{n}, 1] \end{cases}$$

shows that $\inf_{\mathcal{A}} \mathcal{F} = 1$. On the other hand, if $\mathcal{F}(u) = 1$, then u is constant, thus cannot belong to $\mathcal{A}$.

3. Consider the *area functional* defined on the unit ball $B_1 \subset \mathbb{R}^2$

$$\mathcal{F}(u) = \int_{B_1} \sqrt{1 + |Du|^2} dx,$$

defined on

$$\mathcal{A} = \{u \in \mathrm{Lip}(B_1) \ : \ u = 0 \text{ on } \partial B_1, \ u(0) = 1\}.$$

As $\mathcal{F}(u) \geq \pi$ for every $u \in \mathcal{A}$, the sequence of functions

$$u(x) = \begin{cases} 1 - n|x| & \text{for } |x| \in [0, \frac{1}{n}] \\ 0 & \text{for } |x| \in [\frac{1}{n}, 1] \end{cases}$$

shows that $\inf_{\mathcal{A}} \mathcal{F} = \pi$. On the other hand if $\mathcal{F}(u) = \pi$ for some $u \in \mathcal{A}$, then u is constant, thus cannot belong to $\mathcal{A}$.

1.2.2 Non-finiteness of the Dirichlet integral

We have seen that a minimizer of the Dirichlet integral is a harmonic function. In some sense the converse is not true: we exhibit a harmonic function with *infinite* Dirichlet integral.

The Laplacian in polar coordinates on $\mathbb{R}^2$ is

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

and it is easily seen that $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic functions. Now define on the unit ball $B_1 \subset \mathbb{R}^2$

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta).$$

Provided

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty,$$

the series converges uniformly, while its derivatives converge uniformly on compact subsets of the ball, so that u belongs to $C^\infty(B_1) \cap C^0(\overline{B}_1)$ and is harmonic.

The Dirichlet integral of u is

$$\mathcal{D}(u) = \frac{1}{2} \int_0^{2\pi} d\theta \int_0^1 (|\partial_r u|^2 + \frac{1}{r^2} |\partial_\theta u|^2) r dr = \frac{\pi}{2} \sum_{n=1}^{\infty} n(a_n^2 + b_n^2).$$

Thus, if we choose $a_n = 0$ for all $n \geq 0$, $b_n = 0$ for all $n \geq 1$, with the exception of $b_{n!} = n^{-2}$, we obtain

$$u(r,\theta) = \sum_{n=1}^{\infty} r^{n!} n^{-2} \sin(n!\theta),$$

and we conclude that $u \in C^\infty(B_1) \cap C^0(\overline{B}_1)$, it is harmonic, yet

$$\mathcal{D}(u) = \frac{\pi}{2} \sum_{n=1}^{\infty} n^{-4} n! = \infty.$$

In fact, every function $v \in C^\infty(B_1) \cap C^0(\overline{B}_1)$ that agrees with the function u defined above on ∂B_1 has *infinite* Dirichlet integral.

1.3 Some properties of harmonic functions

Proposition 1.3 (Weak maximum principle) *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is subharmonic, then*

$$\sup_\Omega u = \max_{\partial\Omega} u;$$

If u is superharmonic, then

$$\inf_\Omega u = \min_{\partial\Omega} u.$$

Proof. We prove the proposition for u subharmonic, since for a superharmonic u it is enough to consider $-u$. Suppose first that $\Delta u > 0$ in Ω. Were $x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u$, we would have $u_{x^i x^i}(x_0) \leq 0$ for every $1 \leq i \leq n$. Summing over i we would obtain $\Delta u(x_0) \leq 0$, contradiction.

For the general case $\Delta u \geq 0$ consider the function $v(x) = u(x) + \varepsilon |x|^2$. Then $\Delta v > 0$ and, by what we have just proved, $\sup_\Omega v = \max_{\partial\Omega} v$. On the other hand, as $\varepsilon \to 0$, we have $\sup_\Omega v \to \sup_\Omega u$ and $\max_{\partial\Omega} v \to \max_{\partial\Omega} u$. □

Exercise 1.4 Similarly, prove the following generalization of Proposition 1.3: let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\sum_{\alpha,\beta=1}^{n} A^{\alpha\beta} D_{\alpha\beta} u + \sum_{\alpha=1}^{n} b^\alpha D_\alpha u \geq 0,$$

where $A^{\alpha\beta}, b^\alpha \in C^0(\overline{\Omega})$ and $A^{\alpha\beta}$ is *elliptic*: $\sum_{\alpha,\beta=1}^n A^{\alpha\beta} \xi_\alpha \xi_\beta \geq \lambda |\xi|^2$, for some $\lambda > 0$ and every $\xi \in \mathbb{R}^n$. Then

$$\sup_\Omega u = \max_{\partial\Omega} u.$$

Remark 1.5 The continuity of the coefficients in Exercise 1.4 is necessary. Indeed Nadirashvili gave a counterexample to the maximum principle with $A^{\alpha\beta}$ elliptic and bounded, but discontinuous, see [82].

Proposition 1.6 (Comparison principle) *Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be such that u is subharmonic, v is superharmonic and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.*

Proof. Since $u - v$ is subharmonic with $u - v \leq 0$ on $\partial\Omega$, from the weak maximum principle, Proposition 1.3, we get $u - v \leq 0$ in Ω. □

Clearly

$$u \leq v + \max_{\partial\Omega} |u - v| \qquad \text{on } \partial\Omega,$$

consequently:

Corollary 1.7 (Maximum estimate) *Let u and v be two harmonic functions in Ω. Then*

$$\sup_{\Omega} |u - v| \leq \max_{\partial\Omega} |u - v|.$$

Corollary 1.8 (Uniqueness) *Two harmonic functions on Ω that agree on $\partial\Omega$ are equal.*

Proposition 1.9 (Mean value inequalities) *Suppose that $u \in C^2(\Omega)$ is subharmonic. Then for every ball $B_r(x) \Subset \Omega$*

$$u(x) \leq \fint_{\partial B_r(x)} u(y) d\mathcal{H}^{n-1}(y), \text{[1]} \tag{1.5}$$

$$u(x) \leq \fint_{B_r(x)} u(y) dy. \tag{1.6}$$

If u is superharmonic, the reverse inequalities hold; consequently for u harmonic equalities are true.

Proof. Let u be subharmonic. From the divergence theorem, for each

[1] by $\fint_A f(x)dx$ we denote the average of f on A i.e., $\frac{1}{|A|}\int_A f(x)dx$. Similarly $\fint_A f d\mathcal{H}^{n-1} = \frac{1}{\mathcal{H}^{n-1}(A)} \int_A f d\mathcal{H}^{n-1}$.

$\rho \in (0, r]$ we have

$$\begin{aligned} 0 &\le \int_{B_\rho(x)} \Delta u(y)dy \\ &= \int_{\partial B_\rho(x)} \frac{\partial u}{\partial \nu}(y) d\mathcal{H}^{n-1}(y) \\ &= \int_{\partial B_1(0)} \frac{\partial u}{\partial \rho}(x+\rho y)\rho^{n-1} d\mathcal{H}^{n-1}(y) \\ &= \rho^{n-1}\frac{d}{d\rho}\int_{\partial B_1(0)} u(x+\rho y) d\mathcal{H}^{n-1}(y) \\ &= \rho^{n-1}\frac{d}{d\rho}\left(\frac{1}{\rho^{n-1}}\int_{\partial B_\rho(x)} u(y) d\mathcal{H}^{n-1}(y)\right) \\ &= n\omega_n \rho^{n-1}\frac{d}{d\rho} ⨍_{\partial B_\rho(x)} u(y) d\mathcal{H}^{n-1}(y), \end{aligned} \tag{1.7}$$

where $\omega_n := |B_1|$. This implies that the last integral is non-decreasing and, since

$$\lim_{\rho \to 0} ⨍_{\partial B_\rho(x)} u(y) d\mathcal{H}^{n-1}(y) = u(x),$$

(1.5) follows. We leave the rest of the proof for the reader. □

Corollary 1.10 (Strong maximum principle) *If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is subharmonic (resp. superharmonic), then it cannot attain its maximum (resp. minimum) in Ω unless it is constant.*

Proof. Assume u is subharmonic and let $x_0 \in \Omega$ be such that $u(x_0) = \sup_\Omega u$. Then the set

$$S := \{x \in \Omega : u(x) = u(x_0)\}$$

is closed because u is continuous and is open thanks to (1.6). Since Ω is connected we have $S = \Omega$. □

Remark 1.11 If u is harmonic, the mean value inequality is also a direct consequence of the representation formula (1.11) below.

Exercise 1.12 Prove that if $u \in C^2(\Omega)$ satisfies one of the mean value properties, then it is correspondigly harmonic, subharmonic or superharmonic.

Exercise 1.13 Prove that if $u \in C^0(\Omega)$ satisfies the mean value equality

$$u(x) = ⨍_{B_r(x)} u(y)dy, \quad \forall B_r(x) \subset \Omega$$

then $u \in C^\infty(\Omega)$ and it is harmonic.
[Hint: Regularize u with a family $\varphi_\varepsilon = \rho_\varepsilon(|x|)$ of mollifiers with radial simmetry and use the mean value property to prove that $u * \rho_\varepsilon = u$ in any $\Omega_0 \Subset \Omega$ for ε small enough.]

Proposition 1.14 *Consider a sequence of harmonic functions u_j that converge locally uniformly in Ω to a function $u \in C^0(\Omega)$. Then u is harmonic.*

Proof. The mean value property is stable under uniform convergence, thus holds true for u, which is therefore harmonic thanks to Exercise 1.13. □

Remark 1.15 Being harmonic is preserved under the weaker hypothesis of weak L^p convergence, $1 \le p < \infty$, or even of the convergence is the sense of distributions. This follows at once from the so-called Weyl's lemma.

Lemma 1.16 (Weyl) *A function $u \in L^1_{\mathrm{loc}}(\Omega)$ is harmonic if and only if*

$$\int_\Omega u \Delta\varphi dx = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Proof. Consider a family of radial mollifiers ρ_ε, i.e. $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\varepsilon^{-1}x)$, where $\rho \in C^\infty(\mathbb{R}^n)$ is radially symmetric, $\mathrm{supp}(\rho) \subset B_1$ and $\int_{B_1}\rho(x)dx = 1$. Define $u_\varepsilon = u * \rho_\varepsilon$. Then, from the standard properties of convolution we find

$$\begin{aligned}\int_\Omega u_\varepsilon \Delta\varphi dx &= \int_\Omega u(\Delta\varphi * \rho_\varepsilon)dx \\ &= \int_\Omega u\Delta(\varphi * \rho_\varepsilon)dx \\ &= 0, \qquad \text{for every } \varphi \in C_c^\infty(\Omega_\varepsilon),\end{aligned}$$

where

$$\Omega_\varepsilon := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\}.$$

In particular $\Delta u_\varepsilon = 0$ on Ω_ε. Now fix $R > 0$ and let $0 < \varepsilon \le \frac{1}{2}R$. We have by Fubini's theorem

$$\begin{aligned}\int_{\Omega_\varepsilon} |u_\varepsilon(y)|dy &\le \int_{\Omega_\varepsilon} \frac{1}{\varepsilon^n}\int_\Omega \rho\left(\frac{|x-y|}{\varepsilon}\right)|u(x)|dxdy \\ &\le \int_\Omega |u(x)|dx.\end{aligned} \tag{1.8}$$

Here we may assume that $u \in L^1(\Omega)$, since being harmonic is a local property. By the mean value property applied with balls of radius $\frac{R}{2}$ and (1.8), we obtain that the u_ε are uniformly bounded in $\Omega_{R/2}$. They are also locally equicontinuous in Ω_R because for $x_0 \in \Omega_R$ and $x_1, x_2 \in B_{\frac{R}{2}}(x_0)$, still by the mean-value property,

$$\begin{aligned} |u_\varepsilon(x_1) - u_\varepsilon(x_2)| &\leq \frac{2^n}{\omega_n R^n} \int_{B_{\frac{R}{2}}(x_1)\Delta B_{\frac{R}{2}}(x_2)} |u_\varepsilon(x)|dx \\ &\leq \frac{2^n}{\omega_n R^n} \sup_{B_R(x_0)} |u_\varepsilon| \cdot \operatorname{meas}\left(B_{\frac{R}{2}}(x_2)\Delta B_{\frac{R}{2}}(x_1)\right), \end{aligned}$$

where

$$B_{\frac{R}{2}}(x_1)\Delta B_{\frac{R}{2}}(x_2) := \Big(B_{\frac{R}{2}}(x_1)\backslash B_{\frac{R}{2}}(x_2)\Big) \cup \Big(B_{\frac{R}{2}}(x_2)\backslash B_{\frac{R}{2}}(x_1)\Big).$$

By Ascoli-Arzelà's theorem (Theorem 2.3 below), we can extract a sequence u_{ε_k} which converges uniformly in Ω_R to a continuous function v as $k \to \infty$ and $\varepsilon_k \to 0$, which is harmonic thanks to Exercise 1.13. But $u = v$ almost everywhere in Ω_R by the properties of convolutions, hence u is harmonic in Ω_R. Letting $R \to 0$ we conclude. □

Proposition 1.17 *Given $u \in C^0(\Omega)$, the following facts are equivalent:*

(i) *For every ball $B_R(x) \Subset \Omega$ we have*

$$u(x) \leq \fint_{\partial B_R(x)} u(y)d\mathcal{H}^{n-1}(y);$$

(ii) *for every ball $B_R(x) \Subset \Omega$ we have*

$$u(x) \leq \fint_{B_R(x)} u(y)dy;$$

(iii) *for every $x \in \Omega$, $R_0 > 0$, there exist $R \in (0, R_0)$ such that $B_R(x) \Subset \Omega$ and*

$$u(x) \leq \fint_{B_R(x)} u(y)dy; \tag{1.9}$$

(iv) *for each $h \in C^0(\Omega)$ harmonic in $\Omega' \Subset \Omega$ with $u \leq h$ on $\partial\Omega'$, we have $u \leq h$ in Ω';*

(v) $\int_\Omega u(x)\Delta\varphi(x)dx \geq 0,\ \forall \varphi \in C^\infty_c(\Omega),\ \varphi \geq 0.$

Proof. Clearly (i) implies (ii) and (ii) implies (iii).
(iii)⇒(iv): Since h satisfies the mean value property the function $w := u - h$ satisfies

$$w(x) \leq \fint_{B_R(x)} w(y)dy \quad \text{for all balls } B_R(x) \subset \Omega' \text{ s.t. (1.9) holds.}$$

Then

$$\sup_{\Omega'} w = \max_{\partial\Omega'} w \leq 0,$$

the first identity following exactly as in the proof of Corollary 1.10.
(iv)⇒(i): Let $B_R(x) \Subset \Omega$, and choose h harmonic in $B_R(x)$ and $h = u$ in $\Omega \backslash B_R(x)$. This can be done by Proposition 1.24 below. Then

$$u(x) \leq h(x) = \fint_{\partial B_R(x)} h d\mathcal{H}^{n-1} = \fint_{\partial B_R(x)} u d\mathcal{H}^{n-1}.$$

The equivalence of (v) to (ii) can be proved by mollifying u, compare Exercise 1.13. □

Often a continuous function satisfying one of the conditions in Proposition 1.17 is called *subharmonic*.

Exercise 1.18 Use Proposition 1.17 to prove the following:

1. A finite linear combination of harmonic functions is harmonic.
2. A positive finite linear combination of subharmonic (resp. superharmonic) functions is a subharmonic (resp. superharmonic) function.
3. The supremum (resp. infimum) of a finite number of subharmonic (resp. superharmonic) functions is a subharmonic (resp. superharmonic) function.

Theorem 1.19 (Harnack inequality) *Given a non-negative harmonic function $u \in C^2(\Omega)$, for every ball $B_{3r}(x_0) \Subset \Omega$ we have*

$$\sup_{B_r(x_0)} u \leq 3^n \inf_{B_r(x_0)} u.$$

Proof. By the mean value property, Proposition 1.9, and from $u \geq 0$ we get that for $y_1, y_2 \in B_r(x_0)$

$$\begin{aligned} u(y_1) &= \frac{1}{\omega_n r^n} \int_{B_r(y_1)} u dx \\ &\leq \frac{1}{\omega_n r^n} \int_{B_{2r}(x_0)} u dx \\ &= \frac{3^n}{\omega_n (3r)^n} \int_{B_{2r}(x_0)} u dx \\ &\leq \frac{3^n}{\omega_n (3r)^n} \int_{B_{3r}(y_2)} u dx \\ &= 3^n u(y_2). \end{aligned}$$

□

Theorem 1.20 (Liouville) *A bounded harmonic function* $u : \mathbb{R}^n \to \mathbb{R}$ *is constant.*

Proof. Define $m = \inf_{\mathbb{R}^n} u$. Then $u - m \geq 0$ and by Harnack's inequality, Theorem 1.19,

$$\sup_{B_R}(u - m) \leq 3^n \inf_{B_R}(u - m), \quad \forall R > 0.$$

Letting $R \to \infty$, the term on the right tends to 0 and we conclude that $\sup_{\mathbb{R}^n} u = m$. □

Proposition 1.21 *Let* u *be harmonic (hence smooth by Exercise 1.13) and bounded in* $B_R(x_0)$. *For* $r < R$ *we may find constants* $c(k, n)$ *such that*

$$\sup_{B_r(x_0)} |\nabla^k u| \leq \frac{c(k,n)}{(R-r)^k} \sup_{B_R(x_0)} |u|. \tag{1.10}$$

Exercise 1.22 Prove Proposition 1.21.
[Hint: First prove (1.10) for $k = 1$ using the mean-value identity (it might be easier to start with the case $r = R/2$ and then use a covering or a scaling argument). Then notice that each derivative of u is harmonic and use an inductive procedure.]

Proposition 1.23 *Let* (u_k) *be an equibounded sequence of harmonic functions in* Ω, *i.e. assume that* $\sup_\Omega |u_k| \leq c$ *for a constant* c *independent of* k. *Then up to extracting a subsequence* $u_k \to u$ *in* $C^\ell_{\mathrm{loc}}(\Omega)$ *for every* ℓ, *where* u *is a harmonic function on* Ω.

Proof. This follows easily from Proposition 1.21 and the Ascoli-Arzelà theorem (Theorem 2.3 below), with a simple covering argument. □

1.4 Existence in general bounded domains

Before dealing with the existence of harmonic functions is general domains we state a classical representation formula providing us with the solution of the Dirichlet problem (1.1) on a ball.

1.4.1 Solvability of the Dirichlet problem on balls: Poisson's formula

Proposition 1.24 (H.A. Schwarz or S.D. Poisson) *Let $a \in \mathbb{R}^n$, $r > 0$ and $g \in C^0(\partial B_r(a))$ be given and define the function u by*

$$u(x) := \begin{cases} \dfrac{r^2 - |x-a|^2}{n\omega_n r} \displaystyle\int_{\partial B_r(a)} \frac{g(y)}{|x-y|^n} d\mathcal{H}^{n-1}(y) & x \in B_r(a) \\ g(x) & x \in \partial B_r(a). \end{cases} \tag{1.11}$$

Then $u \in C^\infty(B_r(a)) \cap C^0(\overline{B_r(a)})$ and solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } B_r(a) \\ u = g & \text{on } \partial B_r(a) \end{cases}$$

Proof. We only sketch it. By direct computation we see that u is harmonic. For the continuity on the boundary assume, without loss of generality, that $a = 0$ and define

$$K(x,y) := \frac{r^2 - |x|^2}{n\omega_n r|x-y|^n}, \quad x \in B_r(0),\ y \in \partial B_r(0).$$

One can prove that

$$\int_{\partial B_r(0)} K(x,y) d\mathcal{H}^{n-1}(y) = 1, \quad \text{for every } x \in B_r(0).$$

Let $x_0 \in \partial B_r(0)$ and for any $\varepsilon > 0$ choose δ such that $|g(x) - g(x_0)| < \varepsilon$ if $x \in \partial B_r(0) \cap B_\delta(x_0)$. Then, for $x \in B_r(0) \cap B_{\delta/2}(x_0)$,

$$\begin{aligned} |u(x) - g(x_0)| &\leq \left| \int_{\partial B_r(0)} K(x,y)[g(y) - g(x_0)] d\mathcal{H}^{n-1}(y) \right| \\ &\leq \int_{\partial B_r(0) \cap B_\delta(x_0)} K(x,y)|g(y) - g(x_0)| d\mathcal{H}^{n-1}(y) \\ &\quad + \int_{\partial B_r(0) \setminus B_\delta(x_0)} K(x,y)|g(y) - g(x_0)| d\mathcal{H}^{n-1}(y) \\ &\leq \varepsilon + \frac{(r^2 - |x|^2)r^{n-2}}{\left(\frac{\delta}{2}\right)^n} 2 \sup_{\partial B_r(0)} |g|. \end{aligned}$$

Hence $|u(x) - g(x_0)| \to 0$ as $x \to x_0$. □

1.4.2 Perron's method

We now present a method for solving the Dirichlet problem (1.1).

Given an open bounded domain $\Omega \subset \mathbb{R}^n$ and $g \in C^0(\partial\Omega)$ define

$$S_- := \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : \Delta u \geq 0 \text{ in } \Omega,\ u \leq g \text{ on } \partial\Omega\};$$

$$S_+ := \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : \Delta u \leq 0 \text{ in } \Omega,\ u \geq g \text{ on } \partial\Omega\}.$$

These sets are non-empty, since g is bounded and constant functions are harmonic: $u \equiv \sup_\Omega g$ and $v \equiv \inf_\Omega g$ belong to S_+ and S_- respectively. We also observe that, by the comparison principle, $v \leq u$ for each $v \in S_-$ and $u \in S_+$. We define

$$u_*(x) = \sup_{u \in S_-} u(x), \quad u^*(x) = \inf_{u \in S_+} u(x).$$

and shall

1. prove that both u_* and u^* are harmonic;
2. find conditions on Ω in order to have $u_*, u^* \in C^0(\overline{\Omega})$ and $u_* = u^* = g$ on $\partial\Omega$.

This is referred to as *Perron's method*.

Step 1. It is enough to prove that u_* is harmonic in a generic ball $B \subset \Omega$. Fix $x_0 \in B$. By the definition of u_* we may find a sequence $v_j \in S_-$ such that $v_j(x_0) \to u_*(x_0)$. Define

$$v'_j := \max(v_1, \dots, v_j) \in S_-,$$
$$v''_j := P_B v'_j,$$

where $P_B v'_j$ is obtained by (1.11) as the harmonic extention of v'_j on B matching v'_j on ∂B. Observe that by definition (v'_j) is an increasing sequence and, by the maximum principle, (v''_j) is increasing as well. Since the sequence (v''_j) is equibounded and increasing it converges locally uniformly in B to a harmonic function h thanks to Proposition 1.23.

Observe that $h \leq u_*$ and $h(x_0) = u_*(x_0)$. We claim that $h = u_*$ in B. If $h(z) < u_*(z)$ for some $z \in B$, choose $w \in S_-$ such that $w(z) > h(z)$ and define $w_j = \max\{v''_j, w\}$. Also define w'_j and w''_j as done before with v'_j and v''_j. Again we have that $w''_j \to \tilde{h}$ for some harmonic function $\tilde{h}$. From the definition it is easy to prove that $v''_j \leq w''_j$, thus $h \leq \tilde{h}$ and $h(x_0) = \tilde{h}(x_0)$. By the strong maximum principle, this implies $h = \tilde{h}$ on all of B. This is a contradiction because

$$\tilde{h}(z) = \lim w''_j(z) \geq w(z) > h(z) = \tilde{h}(z).$$

This proves that $h = u_*$ and then u_* is harmonic in B, hence in all of Ω since B was arbitrary. Clearly the same proof applies to u^*.

Step 2. The functions u^* and u_* need not achieve the boundary data g, and in general they don't.

Definition 1.25 *A point* $x_0 \in \partial\Omega$ *is called* regular *if for every* $g \in C^0(\partial\Omega)$ *and every* $\varepsilon > 0$ *there exist* $v \in S_-$ *and* $w \in S^+$ *such that* $g(x_0) - v(x_0) \le \varepsilon$ *and* $w(x_0) - g(x_0) \le \varepsilon$.

Exercise 1.26 The Dirichlet problem (1.1) has solution for every $g \in C^0(\partial\Omega)$ if and only if each point of $\partial\Omega$ is regular.
[Hint: Use Perron's method and prove that $u_* \in C^0(\overline{\Omega})$ and $u_* = g$ on $\partial\Omega$.]

Definition 1.27 *Given* $x_0 \in \partial\Omega$*, an* upper barrier *at* x_0 *is a superharmonic function* $b \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *such that* $b(x_0) = 0$ *and* $b > 0$ *on* $\overline{\Omega} \setminus \{x_0\}$. *We say that* b *is a* lower barrier *if* $-b$ *is an upper barrier.*

Proposition 1.28 *Suppose that* $x_0 \in \Omega$ *admits upper and lower barriers. Then* x_0 *is a regular point.*

Proof. Define $M = \max_{\partial\Omega} |g|$ and, for each $\varepsilon > 0$, choose $\delta > 0$ such that for $x \in \Omega$ with $|x - x_0| < \delta$ we have $|g(x) - g(x_0)| < \varepsilon$. Let b be an upper barrier and choose $k > 0$ such that $kb(x) \ge 2M$ if $|x - x_0| \ge \delta$ (by compactness $\inf_{\overline{\Omega} \setminus B_\delta(x_0)} b > 0$). Then define

$$w(x) := g(x_0) + \varepsilon + kb(x);$$

$$v(x) := g(x_0) - \varepsilon - kb(x)$$

and observe that $w \in S_+$ and $v \in S_-$. Moreover $w(x_0) - g(x_0) = \varepsilon$ and $g(x_0) - v(x_0) = \varepsilon$. □

In the following proposition we see that, under suitable hypotheses on the geometry of Ω, the existence of barriers, and therefore of a solution to the Dirichlet problem, is guaranteed.

Proposition 1.29 *Suppose that for each* $x_0 \in \partial\Omega$ *there exists a ball* $B_R(y)$ *in the complement of* Ω *such that* $\overline{B}_R(y) \cap \overline{\Omega} = \{x_0\}$ *(see Figure 1.2). Then every point of* $\partial\Omega$ *is regular, hence the Dirichlet problem* (1.1) *is solvable on* Ω *for arbitrary continuous boundary data.*

Proof. For any $x_0 \in \partial\Omega$ and a ball $B_R(y)$ as in the statement of the proposition, consider the upper barrier $b(x) := R^{2-n} - |x - y|^{2-n}$ for $n > 2$ and $b(x) := \log \frac{|x-y|}{R}$ for $n = 2$, and the lower barrier $-b(x)$. One can easily verify that $\Delta b = 0$ in $\mathbb{R}^n \setminus \{y\}$. □

Exercise 1.30 The hypotesis of Proposition 1.29 is called *exterior sphere condition*. Show that convex domains and C^2 domains satisfy the exterior sphere condition.

Remark 1.31 The Perron method is non-constructive because it doesn't provide any way to find approximate solutions.

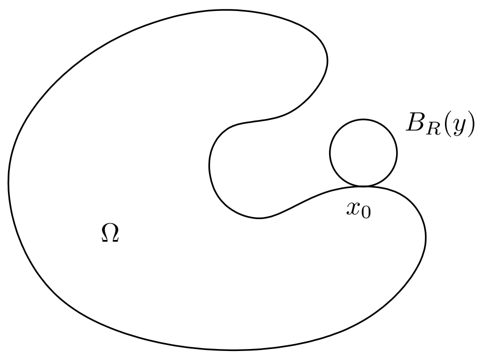

Figure 1.2: The exterior sphere condition.

1.4.3 Poincaré's method

We now present a different method of solving the Dirichlet problem (1.1).

Cover Ω with a sequence B_i of balls, i.e. choose balls $B_i \subset \Omega$, $i = 1, 2, 3, \dots$ such that $\Omega = \bigcup_{i=1}^{\infty} B_i$. Now define the sequence of integers

$$i_k = 1, 2, 1, 2, 3, 1, 2, 3, 4, \dots, 1, \dots, n, \dots$$

Given $g \in C^0(\overline{\Omega})$, define the sequence (u_k) by $u_1 := g$ and for $k > 1$

$$u_k(x) := \begin{cases} u_{k-1}(x) & \text{for } x \in \overline{\Omega} \setminus B_{i_k} \\ P_{i_k} u_{k-1}(x) & \text{for } x \in B_{i_k}, \end{cases}$$

where $P_{i_k} u_{k-1}$ is the harmonic extention on B_{i_k} of $u_{k-1}\big|_{\partial B_{i_k}}$, given by (1.11).

Proposition 1.32 *If each point of $\partial\Omega$ is regular, then u_k converges to the solution u of the Dirichlet problem* (1.1).

Proof. Suppose first $g \in C^0(\overline{\Omega})$ subharmonic, meaning that it satisfies the properties of Proposition 1.17. We can inductively prove that u_k is subharmonic and

$$g = u_1 \leq u_2 \leq \dots u_k \leq \dots \leq \sup_{\Omega} g.$$

Suppose indeed that u_k is subharmonic (this is true for $k = 1$ by assumption). Then by the comparison principle $u_{k+1} \geq u_k$, and it is not difficult to prove that u_{k+1} satisfies for instance (iii) or (iv) of Proposition 1.17, hence is subharmonic.

Since, for each i, u_k is harmonic in B_i for infinitely many k, increasing and uniformly bounded with respect to k, by Proposition 1.23 we see that its limit u is a harmonic functions in each ball B_i, hence in Ω. Using barriers it is not difficult to show that $u = g$ on the boundary.

Now suppose that g, not necessarily subharmonic, belongs to $C^2(\mathbb{R}^n)$ and $\Delta g \geq -\lambda$. Then $g_0(x) = g(x) + \frac{\lambda}{2n}|x|^2$ is subharmonic and we may solve the Dirichlet problem with boundary data g_0. We may also solve the Dirichlet problem with data $\frac{\lambda}{2n}|x|^2$ (that is subharmonic) and by linearity we may solve the Dirichlet problem with data g.

Finally, suppose $g \in C^0(\overline{\Omega})$, which we can think of as continuosly extended to $\mathbb{R}^n$, and regularize it by convolution. For each convoluted function $g_\varepsilon \in C^\infty(\overline{\Omega})$ we find a harmonic map u_ε with $u_\varepsilon = g_\varepsilon \to g$ uniformly on $\partial\Omega$. Then by the maximum principle, for any sequence $\varepsilon_k \to 0$ we have that (u_{ε_k}) is a Cauchy sequence in $C^0(\overline{\Omega})$, hence it uniformly converges to a harmonic function u which equals g on $\partial\Omega$. □

Remark 1.33 The method of Poincaré decreases the Dirichlet integral:

$$\mathcal{D}(g) \geq \mathcal{D}(u_2) \geq \ldots \geq \mathcal{D}(u_k) \geq \ldots \geq \mathcal{D}(u).$$

Consequently if g has a $W^{1,2}$ extension i.e., an extension with finite Dirichlet integral, then the harmonic extension u lies in $W^{1,2}(\Omega)$ (for the definition of $W^{1,2}(\Omega)$ see Section 3.2 below).

On the other hand one can also have

$$\mathcal{D}(g) = \mathcal{D}(u_k) = \infty \quad \text{for every } k = 1, 2, \ldots,$$

compare section 1.2.2.

Remark 1.34 By Riemann's mapping theorem one can show that, if $\Omega \subset \mathbb{R}^2$ is the interior of a closed Jordan curve Γ, then all boundary points of Ω are regular. Lebesgue has instead exhibited a Jordan domain Ω in $\mathbb{R}^3$ (i.e. the interior of a homeomorphic image of S^2) where the problem $\Delta u = 0$ in Ω, $u = g$ on $\partial\Omega$ cannot be solved for every $g \in C^0(\partial\Omega)$.