Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature

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Abstract

We study the conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature $Q \in \mathbb{R}$ having finite volume, particularly in the case $Q \le 0$. We show that when $Q < 0$ such metrics exist in $\mathbb{R}^{2m}$ if and only if $m > 1$. Moreover we study their asymptotic behavior at infinity, in analogy with the case $Q > 0$, which we treated in a recent paper. When $Q = 0$, we show that such metrics have the form $e^{2p} g_{\mathbb{R}^{2m}}$, where p is a polynomial such that $2 \le \deg p \le 2m-2$ and $\sup_{\mathbb{R}^{2m}} p < +\infty$. In dimension 4, such metrics are exactly the polynomials p of degree 2 with $\lim_{|x|\to+\infty} p(x) = -\infty$.

1 Introduction and statement of the main theorems

Given a constant $Q \in \mathbb{R}$, we consider the solutions to the equation

$$(-\Delta)^m u = Q e^{2mu} \quad \text{on } \mathbb{R}^{2m}, \tag{1}$$

satisfying

$$\alpha := \frac{1}{|S^{2m}|} \int_{\mathbb{R}^{2m}} e^{2mu(x)} dx < +\infty. \tag{2}$$

Geometrically, if u solves (1) and (2), then the conformal metric $g := e^{2u} g_{\mathbb{R}^{2m}}$ has Q-curvature $Q_g^{2m} \equiv Q$ and volume $\alpha |S^{2m}|$. For the definition of the Q-curvature and related remarks, we refer to [Mar1]. Notice that given a solution u to (1) and $\lambda > 0$, the function $v := u - \frac{1}{2m}\log\lambda$ solves

$$(-\Delta)^m v = \lambda Q e^{2mv} \quad \text{in } \mathbb{R}^{2m},$$

hence what matters is just the sign of Q, and we can assume without loss of generality that $Q \in \{0, \pm(2m-1)!\}$.

Every solution to (1) is smooth. When $Q = 0$, that follows from standard elliptic estimates; when $Q \neq 0$ the proof is a bit more subtle, see [Mar1, Corollary 8].

For $Q \ge 0$, some explicit solutions to (1) are known. For instance every polynomial of degree at most $2m-2$ satisfies (1) with $Q = 0$, and the function

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$u(x) = \log \frac{2}{1+|x|^2}$ satisfies (1) with $Q = (2m-1)!$ and $\alpha = 1$. This latter solution has the property that $e^{2u} g_{\mathbb{R}^{2m}} = (\pi^{-1})^* g_{S^{2m}}$, where $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection.

For the negative case, we notice that the function $w(x) = \log \frac{2}{1-|x|^2}$ solves $(-\Delta)^m w = -(2m-1)! e^{2mw}$ on the unit ball $B_1 \subset \mathbb{R}^{2m}$ (in dimension 2 this corresponds to the Poincaré metric on the disk). However, no explicit entire solution to (1) with $Q < 0$ is known, hence one can ask whether such solutions actually exist. In dimension 2 ($m = 1$) it is easy to see that the answer is negative, but quite surprisingly the situation is different in dimension 4 and higher and we have:

Theorem 1 *Fix $Q < 0$. For $m = 1$ there is no solution to* (1)-(2)*. For every $m \geq 2$, there exist (several) radially symmetric solutions to* (1)-(2)*.*

Having now an existence result, we turn to the study of the asymptotic behavior at infinity of solutions to (1)-(2) when $m \geq 2$, $Q < 0$, having in mind applications to concentration-compactness problems in conformal geometry. To this end, given a solution u to (1)-(2), we define the auxiliary function

$$v(x) := -\frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y|}{|x-y|}\right) e^{2mu(y)} dy, \tag{3}$$

where $\gamma_m := \omega_{2m} 2^{2m-2} [(m-1)!]^2$ is characterized by the following property:

$$(-\Delta)^m \Big(\frac{1}{\gamma_m} \log \frac{1}{|x|}\Big) = \delta_0 \quad \text{in } \mathbb{R}^{2m}.$$

Then $(-\Delta)^m v = -(2m-1)! e^{2mu}$. We prove

Theorem 2 *Let u be a solution of* (1)-(2) *with $Q = -(2m-1)!$. Then*

$$u(x) = v(x) + p(x), \tag{4}$$

where p is a non-constant polynomial of even degree at most $2m-2$. Moreover there exist a constant $a \neq 0$, an integer $1 \leq j \leq m-1$ and a closed set $Z \subset S^{2m-1}$ of Hausdorff dimension at most $2m-2$ such that for every compact subset $K \subset S^{2m-1} \backslash Z$ we have

$$\begin{aligned} \lim_{t \to +\infty} \Delta^\ell v(t\xi) &= 0, \quad \ell = 1, \dots, m-1, \\ v(t\xi) &= 2\alpha \log t + o(\log t), \ \text{as } t \to +\infty, \\ \lim_{t \to +\infty} \Delta^j u(t\xi) &= a, \end{aligned} \tag{5}$$

for every $\xi \in K$ uniformly in ξ. If $m = 2$, then $Z = \emptyset$ and $\sup_{\mathbb{R}^{2m}} u < +\infty$. Finally

$$\liminf_{|x| \to +\infty} R_{g_u}(x) = -\infty, \tag{6}$$

where R_{g_u} is the scalar curvature of $g_u := e^{2u} g_{\mathbb{R}^{2m}}$.

Following the proof of Theorem 1, it can be shown that the estimate on the degree of the polynomial is sharp. Recently J. Wei and D. Ye [WY] showed the existence of solutions to $\Delta^2 u = 6e^{4u}$ in $\mathbb{R}^4$ with $\int_{\mathbb{R}^4} e^{4u} dx < +\infty$ which are not radially symmetric. It is plausible that also in the negative case non-radially symmetric solutions exist.

For the case $Q = 0$ we have

Theorem 3 *When $Q = 0$, any solution to (1)-(2) is a polynomial p with $2 \le \deg p \le 2m-2$ and with*
$$\sup_{\mathbb{R}^{2m}} p < +\infty.$$
In particular in dimension 2 (case $m = 1$), there are no solutions. In dimension 4 the solutions are exactly the polynomials of degree 2 with $\lim_{|x|\to\infty} p(x) = -\infty$. Finally, there exist $1 \le j \le m-1$ and $a < 0$ such that
$$\lim_{|x|\to\infty} \Delta^j p(x) = a. \tag{7}$$

The case when $Q > 0$, say $Q = (2m-1)!$, has been exhaustively treated. The problem
$$(-\Delta)^m u = (2m-1)! e^{2mu} \quad \text{on } \mathbb{R}^{2m}, \quad \int_{\mathbb{R}^{2m}} e^{2mu} dx < +\infty \tag{8}$$
admits standard solutions, i.e. solutions of the form $u(x) := \log \frac{2\lambda}{1+\lambda^2|x-x_0|^2}$, $\lambda > 0$, $x_0 \in \mathbb{R}^{2m}$ that arise from the stereographic projection and the action of the Möbius group of conformal diffeomorphisms on S^{2m}. In dimension 2 W. Chen and C. Li [CL] showed that every solution to (8) is standard. Already in dimension 4, however, as shown by A. Chang and W. Chen [CC], (8) admits non-standard solutions. In dimension 4 C-S. Lin [Lin] classified all solutions u to (8) and gave precise conditions in order for u to be a standard solution in terms of its asymptotic behavior at infinity.

In arbitrary even dimension, A. Chang and P. Yang [CY] proved that solutions of the form
$$u(x) = \log \frac{2}{1+|x|^2} + \xi(\pi^{-1}(x))$$
are standard, where $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection and ξ is a smooth function on S^{2m}. J. Wei and X. Xu [WX] showed that any solution u to (8) is standard under the weaker assumption that $u(x) = o(|x|^2)$ as $|x| \to \infty$, see also [Xu]. We recently treated the general case, see [Mar1], generalizing the work of C-S. Lin. In particular we proved a decomposition $u = p + v$ as in Theorem 2 and gave various analytic and geometric conditions which are equivalent to u being standard.

The classification of the solutions to (8) has been applied in concentration-compactness problems, see e.g. [LS], [RS], [Mal], [MS], [DR], [Str1], [Str2], [Ndi]. There is an interesting geometric consequence of Theorems 2 and 3, with applications in concentration-compactness: In the case of a closed manifold, metrics of equibounded volumes and prescribed Q-curvatures *of possibly varying sign* cannot concentrate at points of negative or zero Q-curvature. For instance we shall prove in a forthcoming paper [Mar2]

Theorem 4 *Let (M, g) be a $2m$-dimensional closed Riemannian manifold with Paneitz operator P_g^{2m} satisfying $\ker P_g^{2m} = \{const\}$, and let $u_k : M \to \mathbb{R}$ be a sequence of solutions of*
$$P_g^{2m} u_k + Q_g^{2m} = Q_k e^{2mu_k}, \tag{9}$$

where Q_g^{2m} is the Q-curvature of g (see e.g. [Cha]), and where the Q_k's are given continuous functions with $Q_k \to Q_0$ in C^0. Assume also that there is a $\Lambda > 0$ such that

$$\int_M e^{2mu_k} \mathrm{dvol}_g \leq \Lambda, \tag{10}$$

for all k. Then one of the following is true.

(i) For every $0 \leq \alpha < 1$, a subsequence is converging in $C^{2m-1,\alpha}(M)$.

(ii) There exists a finite set $S = \{x^{(i)} : 1 \leq i \leq I\}$ such that $u_k \to -\infty$ in $L^\infty_{\mathrm{loc}}(M \backslash S)$. Moreover

$$\int_M Q_g \mathrm{dvol}_g = I(2m-1)!|S^{2m}|, \tag{11}$$

and

$$Q_k e^{2mu_k} \mathrm{dvol}_g \rightharpoonup \sum_{i=1}^{I} (2m-1)!|S^{2m}|\delta_{x^{(i)}}, \tag{12}$$

in the sense of measures. Finally $Q_0(x^{(i)}) > 0$ for $1 \leq i \leq I$.

In sharp contrast with Theorem 4, on an open domain $\Omega \subset \mathbb{R}^{2m}$ (or a manifold with boundary), $m > 1$, concentration is possible at points of negative or zero curvature. Indeed, take any solution u of (1)-(2) with $Q \leq 0$, whose existence is given by Theorem 1, and consider the sequence

$$u_k(x) := u(k(x - x_0)) + \log k, \quad \text{for } x \in \Omega$$

for some fixed $x_0 \in \Omega$. Then $(-\Delta)^m u_k = Qe^{2mu_k}$ and u_k concentrates at x_0 in the sense that as $k \to \infty$ we have $u_k(x_0) \to +\infty$, $u_k \to -\infty$ a.e. in Ω and $e^{2mu_k}dx \rightharpoonup \alpha|S^{2m}|\delta_{x_0}$ in the sense of measures.

The 2 dimensional case ($m = 1$) is different and concentration at points of non-positive curvature can be ruled out on open domains too, because otherwise a standard blowing-up procedure would yield a solution to (1)-(2) with $Q \leq 0$, contradicting with Theorem 1.

An immediate consequence of Theorem 4 and the Gauss-Bonnet-Chern formula, is the following compactness result (see [Mar2]):

Corollary 5 *In the hypothesis of Theorem 4 assume that either*

1. *$\chi(M) \leq 0$ and $\dim M \in \{2, 4\}$, or*

2. *$\chi(M) \leq 0$, $\dim M \geq 6$ and (M, g) is locally conformally flat,*

where $\chi(M)$ is the Euler-Poincaré characteristic of M. Then only case (i) in Theorem 4 occurs.

The paper is organized as follows. The proof of Theorems 1, 2 and 3 is given in the following three sections; in the last section we collect some open questions. In the following, the letter C denotes a generic constant, which may change from line to line and even within the same line.

2 Proof of Theorem 1

Theorem 1 follows from Propositions 6 and 8 below.

Proposition 6 *For $m = 1$, $Q < 0$ there are no solutions to* (1)-(2)*.*

Proof. Assume that such a solution u exists. Then, by the maximum principle, and Jensen's inequality,

$$\fint_{\partial B_R} u d\sigma \geq u(0), \qquad \int_{\partial B_R} e^{2u} d\sigma \geq 2\pi R e^{2u(0)}.$$

Integrating in R on $[1, +\infty)$, we get

$$\int_{\mathbb{R}^2} e^{2u} dx = +\infty,$$

contradiction. □

Lemma 7 *Let $u(r)$ be a smooth radial function on $\mathbb{R}^n$, $n \geq 1$. Then there are* positive *constants b_m depending only on n such that*

$$\Delta^m u(0) = b_m u^{(2m)}(0), \tag{13}$$

$u^{(2m)} := \frac{\partial^{2m} u}{\partial r^{2m}}$. *In particular $\Delta^m u(0)$ has the sign of $u^{(2m)}(0)$.*

For a proof see [Mar1].

Proposition 8 *For $m \geq 2$, $Q < 0$ there exist radial solutions to* (1)-(2)*.*

Proof. We consider separately the cases when m is even and when m is odd. *Case 1: m even.* Let $u = u(r)$ be the unique solution of the following ODE:

$$\begin{cases} \Delta^m u(r) = -(2m-1)! e^{2mu(r)} & \\ u^{(2j+1)}(0) = 0 & 0 \leq j \leq m-1 \\ u^{(2j)}(0) = \alpha_j \leq 0 & 0 \leq j \leq m-1, \end{cases}$$

where $\alpha_0 = 0$ and $\alpha_1 < 0$. We claim that the solution exists for all $r \geq 0$. To see that, we shall use barriers, compare [CC, Theorem 2]. Let us define

$$w_+(r) = \frac{\alpha_1}{2} r^2, \quad g_+ := w_+ - u.$$

Then $\Delta^m g_+ \geq 0$. By the divergence theorem,

$$\int_{B_R} \Delta^j g_+ dx = \int_{\partial B_R} \frac{d\Delta^{j-1} g_+}{dr} d\sigma.$$

Moreover, from Lemma 7, we infer

$$\Delta^j g_+(0) \geq 0 \quad \text{for } 0 \leq j \leq m-1,$$

hence we see inductively that $\Delta^j g_+(r) \geq 0$ for every r such that $g_+(r)$ is defined and for $0 \leq j \leq m-1$. In particular $g_+ \geq 0$ as long as it exists.

Let us now define

$$w_-(r) := \sum_{i=0}^{m-1} \beta_i r^{2i} - A\log\frac{2}{1+r^2}, \quad g_- := u - w_-,$$

where the β_i's and A will be chosen later. Notice that

$$\Delta^m w_-(r) = \Delta^m\Big(-A\log\frac{2}{1+r^2}\Big) = -(2m-1)!A\Big(\frac{2}{1+r^2}\Big)^{2m}.$$

Since $\alpha_1 < 0$,

$$\lim_{r\to+\infty}\frac{\big(\frac{2}{1+r^2}\big)^{2m}}{e^{m\alpha_1 r^2}} = +\infty,$$

and taking into account that $u \le w_+$, we can choose A large enough, so that

$$\begin{aligned}\Delta^m g_-(r) &= (2m-1)!\Big[A\Big(\frac{2}{1+r^2}\Big)^{2m} - e^{2mu(r)}\Big]\\ &\geq (2m-1)!\Big[A\Big(\frac{2}{1+r^2}\Big)^{2m} - e^{m\alpha_1 r^2}\Big] \geq 0.\end{aligned}$$

We now choose each β_i so that

$$\Delta^j g_-(0) \geq 0, \quad 0 \le j \le m-1,$$

and proceed by induction as above to prove that $g_- \geq 0$. Hence

$$w_-(r) \le u(r) \le w_+(r)$$

as long as u exists, and by standard ODE theory, that implies that $u(r)$ exists for all $r \geq 0$. Finally

$$\int_{\mathbb{R}^{2m}} e^{2mu(|x|)}dx \le \int_{\mathbb{R}^{2m}} e^{m\alpha_1|x|^2}dx < +\infty.$$

Case 2: $m \geq 3$ odd. Let $u = u(r)$ solve

$$\begin{cases}\Delta^m u(r) = (2m-1)!e^{2mu(r)} & \\ u^{(2j+1)}(0) = 0 & 0 \le j \le m-1\\ u^{(2j)}(0) = \alpha_j \le 0 & 0 \le j \le m-1,\end{cases}$$

where the α_i's have to be chosen. Set

$$w_+(r) := \beta - r^2 - \log\frac{2}{1+r^2}, \quad g_+ := w_+ - u,$$

where $\beta < 0$ is such that $e^{-r^2+\beta} \le \big(\frac{2}{1+r^2}\big)^2$, hence

$$\frac{2}{1+r^2} - \frac{1+r^2}{2}e^{-r^2+\beta} \geq 0 \quad \text{for all } r > 0.$$

Then, as long as $g_+ \geq 0$, we have

$$\begin{aligned}\Delta^m g_+(r) &= (2m-1)!\left[\left(\frac{2}{1+r^2}\right)^{2m} - e^{2mu(r)}\right] \\ &\geq (2m-1)!\left[\left(\frac{2}{1+r^2}\right)^{2m} - e^{2mw_+(r)}\right] \geq 0\end{aligned}$$

Choose now the α_i's so that, $u^{(2i)}(0) < w_+^{(2i)}(0)$, for $0 \leq i \leq m-1$. From Lemma 7, we infer that

$$\Delta^i g_+(0) \geq 0, \quad 0 \leq i \leq m-1,$$

and we see by induction that $g_+ \geq 0$ as long as it is defined. As lower barrier, define

$$w_-(r) = \sum_{i=0}^{m-1} \beta_i r^{2i}, \quad g_- := u - w_-,$$

where the β_i's are chosen so that $\Delta^i g_-(0) \geq 0$. Then, observing that

$$\Delta^m g_-(r) = (2m-1)! e^{2mu(r)} > 0,$$

as long as u is defined, we conclude as before that $g_- \geq 0$ as long as it is defined. Then u is defined for all times.

Let $R > 0$ be such that, for every $r \geq R$, $w_+(r) \leq -\frac{r^2}{2}$. Then

$$\int_{\mathbb{R}^{2m}} e^{2mu(|x|)} dx \leq \int_{B_R} e^{2mu(|x|)} dx + \int_{\mathbb{R}^{2m}\setminus B_R} e^{-m|x|^2} dx < +\infty.$$

□

3 Proof of Theorem 2

The proof of Theorem 2 is divided in several lemmas. The following Liouville-type theorem will prove very useful.

Theorem 9 *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h \leq u - v$, where $e^{pu} \in L^1(\mathbb{R}^n)$ for some $p > 0$, $(-v)^+ \in L^1(\mathbb{R}^n)$. Then h is a polynomial of degree at most $2m-2$.*

Proof. As in [Mar1, Theorem 5], for any $x \in \mathbb{R}^{2m}$ we have

$$\begin{aligned}|D^{2m-1}h(x)| &\leq \frac{C}{R^{2m-1}} \fint_{B_R(x)} |h(y)| dy \\ &= -\frac{C}{R^{2m-1}} \fint_{B_R(x)} h(y) dy + \frac{2C}{R^{2m-1}} \fint_{B_R(x)} h^+ dy \qquad (14)\end{aligned}$$

and

$$\fint_{B_R(x)} h(y) dy = O(R^{2m-2}), \quad \text{as } R \to \infty.$$

Then

$$\fint_{B_R(x)} h^+ dy \le \fint_{B_R(x)} u^+ dy + C \fint_{B_R(x)} (-v)^+ dy \le \frac{1}{p} \fint_{B_R(x)} e^{pu} dy + \frac{C}{R^{2m}},$$

and both terms in (14) divided by R^{2m-1} go to 0 as $R \to \infty$. □

Lemma 10 *Let u be a solution of (1)-(2). Then, for $|x| \ge 4$*

$$v(x) \le 2\alpha \log |x| + C. \tag{15}$$

Proof. As in [Mar1, Lemma 9], changing v with $-v$. □

Lemma 11 *For any $\varepsilon > 0$, there is $R > 0$ such that for $|x| \ge R$,*

$$v(x) \ge \left(2\alpha - \frac{\varepsilon}{2}\right) \log |x| + \frac{(2m-1)!}{\gamma_m} \int_{B_1(x)} \log |x-y| e^{2mu(y)} dy. \tag{16}$$

Moreover

$$(-v)^+ \in L^1(\mathbb{R}^{2m}). \tag{17}$$

Proof. To prove (16) we follow [Lin], Lemma 2.4. Choose $R_0 > 0$ such that

$$\frac{1}{|S^{2m}|} \int_{B_{R_0}} e^{2mu} dx \ge \alpha - \frac{\varepsilon}{16},$$

and decompose

$$\begin{aligned} \mathbb{R}^{2m} &= B_{R_0} \cup A_1 \cup A_2, \\ A_1 &:= \{y \in \mathbb{R}^{2m} : 2|x-y| \le |x|, |y| \ge R_0\}, \\ A_2 &:= \{y \in \mathbb{R}^{2m} : 2|x-y| > |x|, |y| \ge R_0\}. \end{aligned}$$

Next choose $R \ge 2$ such that for $|x| > R$ and $|y| \le R_0$, we have $\log \frac{|x-y|}{|y|} \ge \log |x| - \varepsilon$. Then, observing that $\frac{(2m-1)!|S^{2m}|}{\gamma_m} = 2$, we have for $|x| > R$

$$\begin{aligned} \frac{(2m-1)!}{\gamma_m} \int_{B_{R_0}} \log \frac{|x-y|}{|y|} e^{2mu(y)} dy &\ge \left(\log |x| - \frac{\varepsilon}{16}\right) \frac{(2m-1)!}{\gamma_m} \int_{B_{R_0}} e^{2mu} dy \\ &\ge \left(2\alpha - \frac{\varepsilon}{8}\right) \log |x| - C\varepsilon. \end{aligned} \tag{18}$$

Observing that $\log |x-y| \ge 0$ for $y \notin B_1(x)$, $\log |y| \le \log(2|x|)$ for $y \in A_1$, $\int_{A_1} e^{2mu} dy \le \frac{\varepsilon |S^{2m}|}{16}$ and $\log(2|x|) \le 2 \log |x|$ for $|x| \ge R$, we infer

$$\begin{aligned} \int_{A_1} \log \frac{|x-y|}{|y|} e^{2mu(y)} dy &= \int_{A_1} \log |x-y| e^{2mu(y)} dy - \int_{A_1} \log |y| e^{2mu(y)} dy \\ &\ge \int_{B_1(x)} \log |x-y| e^{2mu(y)} dy - \log(2|x|) \int_{A_1} e^{2mu} dy \\ &\ge \int_{B_1(x)} \log |x-y| e^{2mu(y)} dy - \log |x| \frac{\varepsilon |S^{2m}|}{8}. \end{aligned} \tag{19}$$

Finally, for $y \in A_2$, $|x| > R$ we have that $\frac{|x-y|}{|y|} \geq \frac{1}{4}$, hence

$$\int_{A_2} \log \frac{|x-y|}{|y|} e^{2mu(y)} dy \geq -\log(4) \int_{A_2} e^{2mu} dy \geq -C\varepsilon. \tag{20}$$

Putting together (18), (19) and (20), and possibly taking R even larger, we obtain (16). From (16) and Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}^{2m} \setminus B_R} (-v)^+ dx &\leq C \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \chi_{|x-y|<1} \log \frac{1}{|x-y|} e^{2mu(y)} dy dx \\ &= C \int_{\mathbb{R}^{2m}} e^{2mu(y)} \int_{B_1(y)} \log \frac{1}{|x-y|} dx dy \\ &\leq C \int_{R^{2m}} e^{2mu(y)} dy < \infty. \end{aligned}$$

Since $v \in C^\infty(\mathbb{R}^{2m})$, we conclude that $\int_{B_R} (-v)^+ dx < \infty$ and (17) follows. $\square$

Lemma 12 *Let u be a solution of* (1)-(2)*, with $m \geq 2$. Then $u = v + p$, where p is a polynomial of degree at most $2m-2$.*

Proof. Let $p := u - v$. Then $\Delta^m p = 0$. Apply (17) and Theorem 9. $\square$

Lemma 13 *Let p be the polynomial of Lemma 12. Then if $m = 2$, there exists $\delta > 0$ such that*

$$p(x) \leq -\delta |x|^2 + C. \tag{21}$$

In particular $\lim_{|x| \to \infty} p(x) = -\infty$ and $\deg p = 2$. For $m \geq 3$ there is a (possibly empty) closed set $Z \subset S^{2m-1}$ of Hausdorff dimension $\dim^{\mathcal{H}}(Z) \leq 2m-2$ such that for every $K \subset S^{2m-1} \setminus Z$ closed, there exists $\delta = \delta(K) > 0$ such that

$$p(x) \leq -\delta |x|^2 + C \quad \textit{for } \frac{x}{|x|} \in K. \tag{22}$$

Consequently $\deg p$ is even.

Proof. From (17), we infer that there is a set A_0 of finite measure such that

$$v(x) \geq -C \quad \text{in } \mathbb{R}^{2m} \setminus A_0. \tag{23}$$

Case $m = 2$. Up to a rotation, we can write

$$p(x) = \sum_{i=1}^{4} (b_i x_i^2 + c_i x_i) + b_0.$$

Assume that $b_{i_0} \geq 0$ for some $1 \leq i_0 \leq 4$. Then on the set

$$A_1 := \{x \in \mathbb{R}^4 : |x_i| \leq 1 \text{ for } i \neq i_0,\ c_{i_0} x_{i_0} \geq 0\}$$

we have $p(x) \geq -C$. Moreover $|A_1| = +\infty$. Then, from (23) we infer

$$\int_{\mathbb{R}^4} e^{4u} dx \geq \int_{A_1 \setminus A_0} e^{4(v+p)} dx \geq C |A_1 \setminus A_0| = +\infty, \tag{24}$$

contradicting (2). Therefore $b_i < 0$ for every i and (21) follows at once.

Case $m \geq 3$. From (2) and (23) we infer that p cannot be constant. Write

$$p(t\xi) = \sum_{i=0}^{d} a_i(\xi)t^i, \qquad d := \deg p,$$

where for each $0 \leq i \leq d$, a_i is a homogeneous polynomial of degree i or $a_i \equiv 0$. With a computation similar to (24), (2) and (23) imply that $a_d(\xi) \leq 0$ for each $\xi \in S^{2m-1}$. Moreover d is even, otherwise $a_d(\xi) = -a_d(-\xi) \leq 0$ for every $\xi \in S^{2m-1}$, which would imply $a_d \equiv 0$. Set

$$Z = \{\xi \subset S^{2m-1} : a_d(\xi) = 0\}.$$

We claim that $\dim^{\mathcal{H}}(Z) \leq 2m-2$. To see that, set

$$V := \{x \in \mathbb{R}^{2m} \ : \ a_d(x) = 0\} = \{t\xi \ : \ t \geq 0, \ \xi \in Z\}.$$

Since V is a cone and $Z = V \cap S^{2m-1}$, we only need to show that $\dim^{\mathcal{H}}(V) \leq 2m-1$. Set

$$V_i := \{x \in \mathbb{R}^{2m} \ : \ a_d(x) = \ldots = \nabla^i a_d(x) = 0, \ \nabla^{i+1} a_d(x) \neq 0\}.$$

Noticing that $V_i = \emptyset$ for $i \geq d$ (otherwise $a_d \equiv 0$), we find $V = \cup_{i=0}^{d-1} V_i$. By the implicit function theorem, $\dim^{\mathcal{H}}(V_i) \leq 2m-1$ for every $i \geq 0$ and the claim is proved.

Finally, for every compact set $K \subset S^{2m-1} \backslash Z$, there is a constant $\delta > 0$ such that $a_d(\xi) \leq -\frac{\delta}{2}$, and since $d \geq 2$, (22) follows. □

Corollary 14 *Any solution u of (1)-(2) with $m = 2$, $Q < 0$ is bounded from above.*

Proof. Indeed $u = v + p$ and, for some $\delta > 0$,

$$v(x) \leq 2\alpha \log|x| + C, \quad p(x) \leq -\delta|x|^2 + C.$$

□

Lemma 15 *Let $v : \mathbb{R}^{2m} \to \mathbb{R}$ be defined as in (3) and Z as in Lemma 13. Then for every $K \subset S^{2m-1} \backslash Z$ compact we have*

$$\lim_{t \to +\infty} \Delta^{m-j} v(t\xi) = 0, \quad j = 1, \ldots, m-1 \tag{25}$$

for every $\xi \in K$ uniformly in ξ; for every $\varepsilon > 0$ there is $R = R(\varepsilon, K) > 0$ such that, for $t > R$, $\xi \in K$,

$$v(t\xi) \geq (2\alpha - \varepsilon)\log t \tag{26}$$

Proof. Fix $K \in S^{2m-1} \backslash Z$ compact and set $\mathcal{C}_K := \{t\xi : t \geq 0, \xi \in K\}$. For any $\sigma > 0$, $1 \leq j \leq 2m-1$,

$$\int_{\mathbb{R}^{2m} \backslash B_\sigma(x)} \frac{e^{2mu(y)}}{|x-y|^{2j}} dy \to 0 \quad \text{as } |x| \to \infty \tag{27}$$

by dominated convergence. Choose a compact set $\widetilde{K} \subset S^{2m-1} \backslash Z$ such that $K \subset \mathrm{int}(\widetilde{K}) \subset S^{2m-1}$. Since $u \leq C(\widetilde{K})$ on $\mathcal{C}_{\widetilde{K}}$ by Lemma 10 and Lemma 13, we can choose $\sigma = \sigma(\varepsilon) > 0$ so small that

$$\int_{B_\sigma(x)} \frac{e^{2mu}}{|x-y|^{2j}} dy \leq C(\widetilde{K}) \int_{B_\sigma(x)} \frac{1}{|x-y|^{2j}} dy \leq C(\widetilde{K})\varepsilon, \quad \text{for } x \in \mathcal{C}_K,\ |x| \text{ large,}$$

where $|x|$ is so large that $B_\sigma(x) \subset \mathcal{C}_{\widetilde{K}}$. Therefore

$$(-1)^{j+1} \Delta^j v(x) = C \int_{\mathbb{R}^{2m}} \frac{e^{2mu}}{|x-y|^{2j}} dy \to 0, \quad \text{for } x \in \mathcal{C}_K,\ \text{as } |x| \to \infty,$$

We have seen in Lemma 11, that for any $\varepsilon > 0$ there is $R > 0$ such that for $|x| \geq R$

$$v(x) \geq \left(2\alpha - \frac{\varepsilon}{2}\right) \log|x| + \frac{(2m-1)!}{\gamma_m} \int_{B_1(x)} \log|x-y| e^{2mu(y)} dy, \tag{28}$$

and (26) follows easily by choosing $\widetilde{K}$ as above and observing that $u \leq C(\widetilde{K})$ on $\mathcal{C}_{\widetilde{K}}$, hence on $B_1(x)$ for $x \in \mathcal{C}_K$ with $|x|$ large enough. □

Proof of Theorem 2. The decomposition $u = v + p$ and the properties of v and p follow at once from Lemmas 10, 12, 13 and 15; (6) follow as in [Mar1, Theorem 2]. As for (5), let j be the largest integer such that $\Delta^j p \not\equiv 0$. Then $\Delta^{j+1} p \equiv 0$ and from Theorem 9 we infer that $\deg p = 2j$, hence $\Delta^j p \equiv a \neq 0$. □

4 The case $Q = 0$

Proof of Theorem 3. From Theorem 9, with $v \equiv 0$, we have that u is a polynomial of degree at most $2m-2$. Then, as in [Mar1, Lemma 11], we have

$$\sup_{\mathbb{R}^{2m}} u < +\infty,$$

and, since u cannot be constant, we infer that $\deg u \geq 2$ is even. The proof of (7) is analogous to the case $Q < 0$, as long as we do not care about the sign of a. To show that $a < 0$, one proceeds as in [Mar1, Theorem 2]. For the case $m = 2$ one proceeds as in Lemma 13, setting $v \equiv 0$ and $A_0 = \emptyset$. □

Example. One might believe that every polynomial p on $\mathbb{R}^{2m}$ of degree at most $2m-2$ with $\int_{\mathbb{R}^{2m}} e^{2mp} dx < \infty$ satisfies $\lim_{|x|\to\infty} p(x) = -\infty$, as in the case $m = 2$. Consider on $\mathbb{R}^{2m}$, $m \geq 3$ the polynomial $u(x) = -(1+x_1^2)|\widetilde{x}|^2$, where $\widetilde{x} = (x_2, \ldots, x_{2m})$. Then $\Delta^m u \equiv 0$ and

$$\begin{aligned} \int_{\mathbb{R}^{2m}} e^{2mu} dx &= \int_{\mathbb{R}} \int_{\mathbb{R}^{2m-1}} e^{-2m(1+x_1^2)|\widetilde{x}|^2} d\widetilde{x} dx_1 \\ &= \int_{\mathbb{R}} \frac{dx_1}{(1+x_1^2)^{\frac{2m-1}{2}}} \cdot \int_{\mathbb{R}^{2m-1}} e^{-2m|\widetilde{y}|^2} d\widetilde{y} < +\infty. \end{aligned}$$

On the other hand, $\limsup_{|x|\to\infty} u(x) = 0$.

5 Open questions

Open Question 1 *Does the claim of Corollary 14 hold for $m > 2$? In other words, is any solution u to* (1)-(2) *with $Q < 0$ bounded from above?*

This is an important regularity issue, in particular with regard to the behavior at infinity of the function v defined in (3). If $\sup_{\mathbb{R}^{2m}} u < +\infty$, then one can take $Z = \emptyset$ in Theorem 2, as in the case $Q > 0$, see [Mar1, Theorem 1].

Definition 16 *Let $\mathcal{P}_0^{2m}$ be the set of polynomials p of degree at most $2m-2$ on $\mathbb{R}^{2m}$ such that $e^{2mp} \in L^1(\mathbb{R}^{2m})$. Let $\mathcal{P}_+^{2m}$ be the set of polynomials p of degree at most $2m-2$ on $\mathbb{R}^{2m}$ such that there exists a solution $u = v + p$ to* (1)-(2) *with $Q > 0$. Similarly for $\mathcal{P}_-^{2m}$ with $Q < 0$.*

Related to the first question is the following

Open Question 2 *What are the sets $\mathcal{P}_0^{2m}$, $\mathcal{P}_\pm^{2m}$? Is it true that $\mathcal{P}_0^{2m} \subset \mathcal{P}_+^{2m}$ and $\mathcal{P}_0^{2m} \subset \mathcal{P}_-^{2m}$?*

J. Wei and D. Ye [WY] proved that $\mathcal{P}_0^4 \subset \mathcal{P}_+^4$ (and actually more). Consider now on $\mathbb{R}^{2m}$, $m \geq 3$, the polynomial

$$p(x) = -(1 + x_1^2)|\widetilde{x}|^2, \quad \widetilde{x} = (x_2, \ldots, x_{2m}).$$

As seen above, $e^{2mp} \in L^1(\mathbb{R}^{2m})$, hence $p \in \mathcal{P}_0^{2m}$. Assume that $p \in \mathcal{P}_-^{2m}$ as well, i.e. there is a function $u = v + p$ satisfying (1)-(2) and $Q < 0$. Then we claim that $\sup_{\mathbb{R}^{2m}} u = \infty$. Assume by contradiction that u is bounded from above. Then (15) and (16) imply that

$$v(x) = 2\alpha \log |x| + o(\log |x|), \quad \text{as } |x| \to \infty.$$

Therefore,

$$\lim_{x_1 \to \infty} u(x_1, 0, \ldots, 0) = \lim_{x_1 \to \infty} 2\alpha \log x_1 = \infty,$$

contradiction.

Open Question 3 *Even in the case that u is not bounded from above, is it true that one can take $Z = \emptyset$ in Theorem 2 for $m \geq 3$ also?*

For instance, in order to show that $v(x) = 2\alpha \log |x| + o(\log |x|)$ as $|x| \to +\infty$, thanks to (16), it is enough to show that

$$\int_{B_1(x)} \log |x - y| e^{2mu(y)} dy = o(\log |x|), \quad \text{as } |x| \to +\infty,$$

which is true if $\sup_{\mathbb{R}^{2m}} u < \infty$, but it might also be true if $\sup_{\mathbb{R}^{2m}} u = \infty$.

Open Question 4 *What values can the α given by* (1)-(2) *assume for a fixed Q?*

As usual, it is enough to consider $Q \in \{0, \pm(2m-1)!\}$. When $m = 1$, $Q = 1$, then $\alpha = 1$, see [CL]. When $m = 2$, $Q = 6$, then α can take any value in $(0, 1]$, as shown in [CC]. Moreover α cannot be greater than 1 and the case $\alpha = 1$ corresponds to standard solutions, as proved in [Lin]. For the trivial case $Q = 0$, α can take any positive value, and for the other cases we have no answer.

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