Quantization for an elliptic equation of order $2m$ with critical exponential non-linearity

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Abstract

On a smoothly bounded domain $\Omega \subset \mathbb{R}^{2m}$ we consider a sequence of positive solutions $u_k \overset{w}{\rightharpoondown} 0$ in $H^m(\Omega)$ to the equation $(-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2}$ subject to Dirichlet boundary conditions, where $0 < \lambda_k \to 0$. Assuming that

$$\Lambda := \lim_{k\to\infty} \int_\Omega u_k(-\Delta)^m u_k dx < \infty,$$

we prove that Λ is an integer multiple of $\Lambda_1 := (2m-1)!\,\mathrm{vol}(S^{2m})$, the total Q-curvature of the standard $2m$-dimensional sphere.

1 Introduction

Given a smoothly bounded domain $\Omega \subset \mathbb{R}^{2m}$, suppose that for each $k \in \mathbb{N}$ we have a smooth function $u_k > 0$ satisfying the equation

$$(-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2} \text{ in } \Omega \tag{1}$$

with

$$u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 \text{ on } \partial\Omega, \tag{2}$$

where $0 < \lambda_k \to 0$ as $k \to \infty$. We assume that (u_k) is bounded in $H^m(\Omega)$. Hence, after passing to a subsequence and integrating by parts we may assume that as $k \to \infty$ we have

$$\int_\Omega |\nabla^m u_k|^2 dx = \int_\Omega u_k(-\Delta)^m u_k dx = \lambda_k \int_\Omega u_k^2 e^{mu_k^2} dx \to \Lambda > 0. \tag{3}$$

Note that by elliptic estimates the quantity

$$\|u\| := \Big(\int_\Omega |\nabla^m u_k|^2 dx\Big)^{1/2} = \Big(\int_\Omega \sum_{|\alpha|=m} |\partial^\alpha u_k|^2 dx\Big)^{1/2}$$

defines a norm on the Beppo-Levi space $H_0^m(\Omega)$ which is equivalent to the standard Sobolev norm.

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Generalising previous results by Adimurthi and Struwe [3], Adimurthi and Druet [1] and Robert and Struwe [11], the first author proved in [8] the following theorem.

Theorem 1 *Let (u_k) be a sequence of positive solutions to (1), (2) with $0 < \lambda_k \to 0$ as $k \to \infty$ and satisfying (3) for some $\Lambda > 0$. Then $\sup_\Omega u_k \to \infty$ as $k \to \infty$ and there exist a subsequence (u_k) and sequences of points $x_k^{(i)} \to x^{(i)} \in \overline{\Omega}$, $1 \le i \le I$, for some integer $I \le C\Lambda$, such that the following is true.*

For every $1 \le i \le I$, letting $r_k^{(i)} > 0$ be given by

$$\lambda_k (r_k^{(i)})^{2m} u_k^2(x_k^{(i)}) e^{m u_k^2(x_k^{(i)})} = 2^{2m}(2m-1)! \tag{4}$$

and setting

$$\eta_k^{(i)}(x) := u_k(x_k^{(i)})(u_k(x_k^{(i)} + r_k^{(i)} x) - u_k(x_k^{(i)})) + \log 2,$$

we have $r_k^{(i)} \to 0$, $\mathrm{dist}(x_k^{(i)}, \partial\Omega)/r_k^{(i)} \to \infty$ as $k \to \infty$, and

$$\eta_k^{(i)}(x) \to \eta_0(x) = \log \frac{2}{1+|x|^2} \text{ in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}) \text{ as } k \to \infty. \tag{5}$$

Moreover, for $i \neq j$ there holds

$$\frac{|x_k^{(i)} - x_k^{(j)}|}{r_k^{(i)}} \to \infty \text{ as } k \to \infty. \tag{6}$$

In addition, with $R_k(x) := \inf_{1 \le i \le I} |x - x_k^{(i)}|$ there exists a constant $C > 0$ such that there holds

$$\lambda_k R_k^{2m}(x) u_k^2(x) e^{m u_k^2(x)} \le C \tag{7}$$

uniformly for all $x \in \Omega$, $k \in \mathbb{N}$.

Finally $u_k \to 0$ in $C^{2m-1}_{\mathrm{loc}}(\overline{\Omega} \backslash S)$, where $S = \{x^{(1)}, \dots, x^{(I)}\}$.

We remark that the function η_0 given by (5) satisfies the Q-curvature equation

$$(-\Delta)^m \eta_0 = (2m-1)! e^{2m\eta_0} \tag{8}$$

and

$$(2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx = \int_{S^{2m}} Q_{S^{2m}} \mathrm{dvol}_{g_{S^{2m}}} = (2m-1)! |S^{2m}| =: \Lambda_1. \tag{9}$$

For a discussion of the geometric meaning of (8) we refer to [4] or to the introduction of [7].

The purpose of this paper is to prove the following quantization result.

Theorem 2 *Under the hypothesis of Theorem 1 we have $\Lambda = I^* \Lambda_1$ for some $I^* \in \mathbb{N} \backslash \{0\}$.*

The analogue of Theorem 2 was proven by O. Druet [5] in dimension 2 ($m = 1$) and by the second author [13] in dimension 4 ($m = 2$) in the case of the Navier boundary condition $u_k = \Delta u_k = 0$ on $\partial\Omega$. Note that in the latter case the maximum principle implies that $\Delta u_k \leq 0$ in Ω whereas such an estimate is not available in the case of the Dirichlet boundary condition.

Quantization results similar to Theorem 2 previously have also been obtained for concentrating sequences of solutions u_k to the Q-curvature equation

$$(-\Delta)^m u_k = \lambda_k e^{2mu_k} \quad \text{in } \Omega \subset \mathbb{R}^{2m}. \tag{10}$$

In the case of the Navier boundary condition, assuming that $\lambda_k \to 0$ and

$$\Lambda := \lim_{k\to\infty} \int_\Omega \lambda_k e^{2mu_k} dx < \infty,$$

J. Wei [14] proved that when $m = 2$ and when Ω is convex the quantity Λ is an integer multiple of Λ_1. Moreover concentration points are simple and isolated, in the sense that $x^{(i)} \neq x^{(j)}$ for $i \neq j$, and $I^* = I$ in the notation of Theorems 1 and 2 above. Robert and Wei [12] proved the analogous result for a general domain Ω and in the case of Dirichlet boundary conditions. In [9], the first author and Petrache generalized the result of Robert and Wei to arbitrary dimensions.

Equation (1) is more difficult to deal with analytically than equation (10); the analogous questions whether for a blowing up sequence of solutions to (1) the concentration points are isolated, simple and stay away from the boundary are still open, even in dimension 2.

Our paper is organized as follows. In the next section we present the proof of Theorem 2 in the case when $\Omega = B_R$ is a ball and each function u_k is radially symmetric. In Section 3 we prove the theorem in the general case. Some useful technical results are collected in the Appendix. The overall strategy of the proof is very similar to the approach followed in [13], and some of the results in [13] can be carried over almost literally to the present setting. Several key steps in the proof, however, require conceptually new ideas in the case when $m \geq 3$. These ideas also shed new light on the previous approaches in low dimensions and have a unifying feature.

Throughout the paper the letter C denotes a generic constant independent of k which can change from line to line, or even within the same line.

2 Proof of Theorem 2 in the radial case

Let $\Omega = B_R = B_R(0)$ and assume that each u_k is radially symmetric. By slight abuse of notation we write $u_k(x) = u_k(r)$ if $|x| = r$. In the notation of Theorem 1 we then have $I = 1$ and we can choose $x_k^{(1)} = 0$ for every $k > 0$. In fact, as shown in assertion (17) of Lemma 4 below, we have $u_k(0) = \max_\Omega u_k$.

2.1 Strategy of the proof

Set $e_k := \lambda_k u_k^2 e^{mu_k^2}$ and let

$$\Lambda_k(r) := \int_{B_r} e_k dx, \quad N_k(s,t) := \Lambda_k(t) - \Lambda_k(s) = \int_{B_t \setminus B_s} e_k dx$$

as in [13]. We shall say that the property (H_ℓ) is satisfied if there exist sequences

$$s_k^{(0)} := 0 < r_k^{(1)} < s_k^{(1)} < \ldots < r_k^{(\ell)} < s_k^{(\ell)} \le R, \quad k \in \mathbb{N},$$

such that the following holds:

$(H_{\ell,1})$ $\lim_{k\to\infty} \frac{r_k^{(j)}}{s_k^{(j)}} = \lim_{k\to\infty} \frac{s_k^{(j-1)}}{r_k^{(j)}} = 0$ for $1 \le j \le \ell$,

$(H_{\ell,2})$ $\lim_{k\to\infty} \frac{u_k(s_k^{(j)})}{u_k(Lr_k^{(j)})} = 0$ for $1 \le j \le \ell$, $L > 0$,

$(H_{\ell,3})$ $\lim_{k\to\infty} \Lambda_k(s_k^{(j)}) = j\Lambda_1$ for $1 \le j \le \ell$,

$(H_{\ell,4})$ $\lim_{L\to\infty} \lim_{k\to\infty} \left(N_k(s_k^{(j-1)}, r_k^{(j)}/L) + N_k(Lr_k^{(j)}, s_k^{(j)})\right) = 0$ for $1 \le j \le \ell$.

For the proof of Theorem 2 we proceed via induction from the following two claims: (H_1) holds, and if (H_ℓ) holds then either $(H_{\ell+1})$ holds as well, or

$$\lim_{k\to\infty} N_k(s_k^{(\ell)}, R) = 0. \tag{11}$$

By (3) and $(H_{\ell,3})$ the induction terminates when $\ell > \frac{\Lambda}{\Lambda_1}$. Letting ℓ_0 be the largest integer such that (H_{ℓ_0}) holds, $(H_{\ell_0,3})$ and (11) imply

$$\Lambda = \lim_{k\to\infty} \Lambda_k(s_k^{(\ell_0)}) + \lim_{k\to\infty} N_k(s_k^{(\ell_0)}, R) = \ell_0\Lambda_1,$$

and Theorem 2 in the radial case follows.

2.2 Proof of (H_1)

Let $r_k > 0$ be defined as in Theorem 1 such that

$$\lambda_k r_k^{2m} u_k^2(0) e^{mu_k^2(0)} = 2^{2m}(2m-1)!,$$

and set

$$w_k(x) := u_k(0)(u_k(x) - u_k(0)) \text{ in } B_R.$$

We have

$$\begin{aligned} (-\Delta)^m w_k &= \lambda_k u_k(0) u_k e^{mu_k^2} \\ &= \lambda_k u_k(0) u_k e^{mu_k^2(0)} e^{2m\left(1+\frac{w_k}{2u_k^2(0)}\right)w_k} =: f_k \text{ in } B_R. \end{aligned}$$

Letting also

$$\sigma_k(r) := \int_{B_r} f_k dx \ge \Lambda_k(r),$$

then by (5) of Theorem 1 and (9) clearly we have

$$\begin{aligned} \lim_{L\to\infty} \lim_{k\to\infty} \Lambda_k(Lr_k) &= \lim_{L\to\infty} \lim_{k\to\infty} \sigma_k(Lr_k) \\ &= \lim_{L\to\infty} \lim_{k\to\infty} (2m-1)! \int_{B_L} e^{2m\eta_k} dx = \Lambda_1. \end{aligned} \tag{12}$$

For $0 < t \le R$ let g_k solve the equation

$$\Delta^m g_k = \Delta^m w_k \text{ in } B_t$$

with homogeneous Dirichlet boundary data

$$g_k = \partial_\nu g_k = \ldots = \partial_\nu^{m-1} g_k = 0 \text{ on } \partial B_t.$$

Then Lemma 22 in the Appendix gives the identity

$$(-1)^m \partial_\nu^m g_k(t) = \frac{A_k(t)}{\omega_{2m-1} t^{3m-2}} \tag{13}$$

similar to (20) in [13], where

$$A_k(t) := \int_0^t t_2 \cdots \int_0^{t_{m-1}} t_m \sigma_k(t_m) dt_m \ldots dt_2, \tag{14}$$

and where ω_{2m-1} is the $(2m-1)$-dimensional volume of S^{2m-1}.

Lemma 3 *For every $b < 2$ we can find $L = L(b)$ and $k_0 = k_0(b)$ such that for $k \ge k_0$ we have*

$$(-\partial_\nu)^m g_k(t) \ge \frac{2^{m-1}(m-1)!b}{t^m} \quad \textit{for } Lr_k \le t \le R. \tag{15}$$

Proof. Noting that

$$\frac{\Lambda_1}{\omega_{2m-1} 2^{m-1}(m-1)!} = 2^m (m-1)!,$$

from (12) and (13) together with the identity

$$\int_0^t t_2 \cdots \int_0^{t_{m-1}} t_m dt_m \cdots dt_2 = \frac{t^{2m-2}}{2^{m-1}(m-1)!}$$

we obtain the claim. □

These estimates now yield the following result analogous to Lemma 2.1 in [13]. Note, however, that the statement (17) below in the present case no longer can simply be deduced from the maximum principle, as was the case in [13]. In addition, the higher order nature of equation (1) requires substantial technical modifications of the approach used in [13].

Lemma 4 *For any $b < 2$ there is $L = L(b)$ and $k_0 = k_0(b)$ such that for $k \ge k_0$ there holds*

$$\begin{aligned} w_k'(t) &\le -\frac{b}{t} + tP(t) \quad \textit{in } B_R \backslash B_{Lr_k}, &(16)\\ w_k'(t) &\le 0 \quad \textit{in } B_R, &(17)\\ w_k(t) &\le b \log\left(\frac{r_k}{t}\right) + C \quad \textit{in } B_R, &(18)\end{aligned}$$

where P is a polynomial independent of k. In particular u_k is monotone decreasing. For any $\varepsilon \in]0,1[$ let $T_k > 0$ be such that $u_k(T_k) = \varepsilon u_k(0)$. Then we have

$$\lim_{k\to\infty} \frac{r_k}{T_k} = 0 \tag{19}$$

and

$$\lim_{k\to\infty} \Lambda_k(T_k) = \lim_{k\to\infty} \sigma_k(T_k) = \Lambda_1. \tag{20}$$

Proof. Fix $t > 0$ and write $w_k = g_k + h_k$, where

$$\Delta^m h_k = 0 \text{ in } B_t, \quad \text{and } g_k = \partial_\nu g_k = \ldots = \partial_\nu^{m-1} g_k = 0 \;\text{ on } \partial B_t.$$

Step 1. We claim that

$$\partial_\nu^m g_k(t) = t^{m-1} \underbrace{(t^{-1}(t^{-1}\cdots(t^{-1}}_{m-1 \text{ times}} w_k'(t) \underbrace{)'\cdots)')'}_{m-1 \text{ times}} . \tag{21}$$

Indeed, subtracting $\partial_\nu^m w_k(t)$ from both sides of (21) we need to show

$$-\partial_\nu^m h_k(t) = t^{m-1} \underbrace{(t^{-1}(t^{-1}\cdots(t^{-1}}_{m-1 \text{ times}} w_k'(t) \underbrace{)'\cdots)')'}_{m-1 \text{ times}} - \partial_\nu^m w_k(t).$$

Using the boundary condition $\partial_\nu^j w_k(t) = \partial_\nu^j h_k(t)$ for $0 \le j \le m-1$, and observing that on the right-hand side the terms involving $\partial_\nu^m w_k(t)$ cancel, we can replace w_k by h_k and it suffices to prove

$$-\partial_\nu^m h_k(t) = t^{m-1} \underbrace{(t^{-1}(t^{-1}\cdots(t^{-1}}_{m-1 \text{ times}} h_k'(t) \underbrace{)'\cdots)')'}_{m-1 \text{ times}} - \partial_\nu^m h_k(t).$$

But $\Delta^m h_k = 0$ and radial symmetry imply that $h(r) = \sum_{i=0}^{m-1} \alpha_i r^{2i}$; so

$$\underbrace{(t^{-1}(t^{-1}\cdots(t^{-1}}_{m-1 \text{ times}} h_k'(t) \underbrace{)'\cdots)')'}_{m-1 \text{ times}} = 0,$$

and (21) follows.

Step 2. Inserting now (15) into (21), for any given $b < 2$ we infer

$$\begin{aligned} &(-1)^{m-1} t^{m-1} \underbrace{\Big(t^{-1}\Big(t^{-1}\cdots\Big(t^{-1}}_{m-1 \text{ times}} \Big(w_k'(t) + \frac{b}{t}\Big) \underbrace{\Big)'\cdots\Big)'\Big)'}_{m-1 \text{ times}} \\ &= (-1)^{m-1}\partial_\nu^m g_k(t) + \frac{2^{m-1}(m-1)!b}{t^m} \le 0 \;\text{ for } Lr_k \le t \le R, \end{aligned} \tag{22}$$

provided that we fix $L = L(b)$ sufficiently large and then also choose k large enough. We now prove by induction over $1 \le j \le m$ that

$$\varphi_{j,k}(t) := (-1)^{m-j} t^{-1} \underbrace{\Big(t^{-1}\Big(t^{-1}\cdots\Big(t^{-1}}_{m-j \text{ times}} \Big(w_k'(t) + \frac{b}{t}\Big) \underbrace{\Big)'\cdots\Big)'\Big)'}_{m-j \text{ times}} \le P_j(t), \tag{23}$$

for $Lr_k \le t \le R$, where $P_j(t) \ge 0$ is a polynomial in t independent of k. The case $j = 1$ follows at once from (22) with $P_1 \equiv 0$. Using the Dirichlet boundary condition (which implies $\partial_\nu^j w_k(R) = 0$ for $1 \le j \le m-1$) we get $\varphi_{j,k}(R) \le C_j$

for some constant $C_j \geq 0$, $2 \leq j \leq m$. Observing that $\varphi'_{j,k}(t) = -t\varphi_{j-1,k}(t)$ for $2 \leq j \leq m$, we then obtain

$$\begin{aligned}\varphi_{j,k}(t) &= \varphi_{j,k}(R) + \int_t^R r\varphi_{j-1,k}(r)dr \\ &\leq C_j + \int_t^R rP_{j-1}(r)dr =: P_j(t),\end{aligned}$$

that is, (23). For $j = m$ we get

$$w'_k(t) \leq -\frac{b}{t} + tP_m(t), \;\; Lr_k \leq t \leq R.$$

Integrating once more, recalling that L depends on b, and that $w_k(Lr_k) \to \eta_0(L)$ as $k \to \infty$, for sufficiently large k we find

$$w_k(t) \leq w_k(Lr_k) - b\log\left(\frac{t}{Lr_k}\right) + C \leq b\log\left(\frac{r_k}{t}\right) + C$$

for $Lr_k \leq t \leq R$. For $0 < t < Lr_k$ (18) already follows from Theorem 1.

In order to prove (17), observe that (13) implies

$$(-\partial_\nu)^m g_k(t) \geq 0 \;\text{ for } 0 < t \leq R, \; k \in \mathbb{N},$$

and (21) yields

$$(-1)^{m-1}t^{m-1}\underbrace{(t^{-1}(t^{-1}\cdots(t^{-1}}_{m-1\text{ times}}w'_k(t)\underbrace{)'\cdots)')'}_{m-1\text{ times}} \leq 0. \tag{24}$$

In analogy with (23), for $1 \leq j \leq m$ we can show by induction that

$$\psi_{j,k}(t) := (-1)^{m-j}t^{-1}\underbrace{(t^{-1}(t^{-1}\cdots(t^{-1}}_{m-j\text{ times}}w'_k(t)\underbrace{)'\cdots)')'}_{m-j\text{ times}} \leq 0 \;\text{ for all } 0 < t \leq R.$$

Indeed $\psi_{1,k}(t) \leq 0$ by (24), while for $2 \leq j \leq m$, we have $\psi_{j,k}(R) = 0$ thanks to the boundary condition. Hence

$$\psi_{j,k}(t) = \int_t^R r\psi_{j-1}(r)dr \leq 0 \;\text{ for all } 0 < t \leq R,$$

and the case $j = m$ implies (17).

Step 3. In order to prove (19), assume by contradiction that

$$\liminf_{k\to\infty}\frac{T_k}{r_k} = L \in [0, \infty[.$$

Then from Theorem 1 for a suitable subsequence on the one hand we have

$$u_k(0)(u_k(T_k) - u_k(0)) + \log 2 = \eta_k\left(\frac{T_k}{r_k}\right) \to \log\left(\frac{2}{1+L^2}\right) \text{ as } k \to \infty.$$

But on the other hand, since $u_k(0) \to \infty$ we also have that

$$u_k(0)(u_k(T_k) - u_k(0)) = u_k^2(0)(\varepsilon - 1) \to -\infty$$

as $k \to \infty$, a contradiction.

It thus remains to prove (20). Using (18) and observing that

$$(\varepsilon - 1)u_k^2(0) \le w_k(r) \le 0 \quad \text{for } 0 \le r \le T_k,$$

from (4) for $k \ge k_0$ we get

$$\begin{aligned} f_k(r) &\le \lambda_k u_k^2(0) e^{m u_k^2(0)} e^{2m\left(1+\frac{w_k(r)}{2u_k^2(0)}\right) w_k(r)} \\ &\le \lambda_k r_k^{2m} u_k^2(0) e^{m u_k^2(0)} r_k^{-2m} e^{m(\varepsilon+1) w_k(r)} \le C r_k^{-2m} \left(\frac{r_k}{r}\right)^{m(\varepsilon+1)b}. \end{aligned}$$

Choosing now $b < 2$ such that $m(\varepsilon+1)b = 2m + \varepsilon$, and integrating over B_{T_k}, we find

$$\begin{aligned} \Lambda_1 &\le \lim_{k\to\infty} \Lambda_k(T_k) \le \lim_{k\to\infty} \sigma_k(T_k) \\ &= \Lambda_1 + \lim_{L\to\infty} \lim_{k\to\infty} \int_{B_{T_k}\setminus B_{Lr_k}} f_k dx \\ &\le \Lambda_1 + C \lim_{L\to\infty} \lim_{k\to\infty} \frac{1}{r_k^{2m}} \int_{B_{T_k}\setminus B_{Lr_k}} \left(\frac{r_k}{r}\right)^{2m+\varepsilon} dx \\ &\le \Lambda_1 + \frac{C}{\varepsilon} \lim_{L\to\infty} \lim_{k\to\infty} \left(\frac{r_k}{Lr_k}\right)^{\varepsilon} = \Lambda_1, \end{aligned}$$

hence (20). □

According to Lemma 4 we can now choose a sequence $\varepsilon_k \to 0$ as $k \to \infty$ and corresponding numbers $s_k = T_k(\varepsilon_k)$ such that $u_k(s_k) \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{r_k}{s_k} = 0, \quad \lim_{k\to\infty} \Lambda_k(s_k) = \Lambda_1, \quad \lim_{L\to\infty} \lim_{k\to\infty} N_k(Lr_k, s_k) = 0. \tag{25}$$

Observing that Theorem 1 implies $\lim_{k\to\infty} \frac{u_k(Lr_k)}{u_k(0)} = 1$ for every $L \ge 0$, we get

$$\lim_{k\to\infty} \frac{u_k(s_k)}{u_k(Lr_k)} = \lim_{k\to\infty} \frac{u_k(s_k)}{u_k(0)} = 0, \quad \text{for all } L > 0. \tag{26}$$

We also claim that

$$\lim_{L\to\infty} \lim_{k\to\infty} N_k(s_k, Ls_k) = 0. \tag{27}$$

To see this, remember that for $0 < s < t < R$

$$N_k(s,t) = \int_{B_t\setminus B_s} e_k dx = \omega_{2m-1} \int_s^t \lambda_k r^{2m-1} u_k^2 e^{m u_k^2} dr.$$

Now set

$$P_k(t) := t \frac{\partial}{\partial t} N_k(s,t) = t \int_{\partial B_t} e_k d\sigma = \omega_{2m-1} \lambda_k t^{2m} u_k^2(t) e^{m u_k^2(t)}.$$

Using the monotonicity of u_k that we proved in Lemma 4 we immediately obtain the estimate

$$P_k(t) = C\omega_{2m-1} \lambda_k u_k^2(t) e^{m u_k^2(t)} \int_{t/2}^t r^{2m-1} dr \le C N_k(t/2, t) \le C P_k(t/2) \tag{28}$$

analogous to (26) in [13]; hence we also conclude that

$$N_k(t, 2t) \le C N_k(t/2, t) \quad \text{for } t \in [0, R/2]. \tag{29}$$

Now (25) and (29) imply that for any $M \in \mathbb{N}$

$$\begin{aligned}\lim_{k\to\infty} N_k(2^{M-1}s_k, 2^M s_k) &\le C \lim_{k\to\infty} N_k(2^{M-2}s_k, s^{M-1}s_k) \\ &\le \cdots \le C_M \lim_{k\to\infty} N_k(s_k/2, s_k) = 0.\end{aligned}$$

Therefore if $2^M \ge L$ we have

$$\lim_{k\to\infty} N_k(s_k, Ls_k) \le \sum_{j=1}^{M} N_k(2^{j-1}s_k, 2^j s_k) = 0,$$

as claimed.

Setting $r_k^{(1)} := r_k$, $s_k^{(1)} := s_k$ and taking into account (25) - (27) and Theorem 1 we see that the property (H_1) is satisfied.

2.3 The inductive step

We now assume that (H_ℓ) holds for some integer $\ell \ge 1$ and fix numbers

$$s_k^{(0)} = 0 < r_k^{(1)} < s_k^{(1)} < \ldots < r_k^{(\ell)} < s_k^{(\ell)}, \ k \in \mathbb{N}$$

such that $(H_{\ell,1})$, $(H_{\ell,2})$, $(H_{\ell,3})$ and $(H_{\ell,4})$ hold true. To complete the proof of Theorem 2 it suffices to show that either $(H_{\ell+1})$ or (11) holds. The proof requires the following analogue of (29) in [13].

Lemma 5 *There is a constant $C_0 = C_0(\Lambda)$ such that for $t_k > s_k^{(\ell)}$ there holds*

$$N_k(s_k^{(\ell)}, t_k) \le \frac{P_k(t_k)}{m} + C_0 N_k^2(s_k^{(\ell)}, t_k) + o(1), \tag{30}$$

with error $o(1) \to 0$ as $k \to \infty$

Proof. For $s = s_k^{(\ell)} < t$ we integrate by parts to obtain

$$\begin{aligned} N_k(s,t) &= \omega_{2m-1} \int_s^t r^{2m-1} \lambda_k u_k^2 e^{m u_k^2} dr \\ &= \lambda_k \frac{\omega_{2m-1}}{2m} \big(r^{2m} u_k^2 e^{m u_k^2}\big)\Big|_s^t - \frac{\omega_{2m-1}}{2m} \int_s^t \lambda_k r^{2m} (2u_k + 2m u_k^3) u_k' e^{m u_k^2} dr \\ &\le \frac{P_k(t)}{2m} - \omega_{2m-1} \int_s^t \lambda_k r^{2m} \Big(\frac{1}{m} + u_k^2\Big) \frac{u_k}{u_k(0)} w_k' e^{m u_k^2} dr. \end{aligned} \tag{31}$$

Define $g_k(t)$ as in the beginning of the proof of Lemma 4. Then (13) and (21) imply

$$\underbrace{(t^{-1}(t^{-1}\cdots(t^{-1}}_{m-1 \text{ times}} w_k'(t) \underbrace{)'\cdots)')'}_{m-1 \text{ times}} = (-1)^m \frac{A_k(t)}{\omega_{2m-1} t^{4m-3}},$$

where A_k is as in (14). Integrating this relation $m-1$ times from t to R, and using the Dirichlet boundary condition $\partial_\nu^j w_k(R) = 0$ for $1 \le j \le m-1$ we get

$$\frac{w_k'(t)}{t} = -\int_t^R t_1 \int_{t_1}^R t_2 \cdots \int_{t_{m-2}}^R \frac{A_k(t_{m-1})}{\omega_{2m-1} t_{m-1}^{4m-3}} dt_{m-1} \cdots dt_1;$$

hence

$$-tw_k'(t)\frac{u_k(t)}{u_k(0)} = t^2 \frac{u_k(t)}{u_k(0)} \int_t^R t_1 \int_{t_1}^R t_2 \cdots \int_{t_{m-2}}^R \frac{A_k(t_{m-1})}{\omega_{2m-1} t_{m-1}^{4m-3}} dt_{m-1} \cdots dt_1 =: I.$$

More explicitly,

$$\begin{aligned} I = t^2 &\int_t^R t_1 \int_{t_1}^R t_2 \cdots \int_{t_{m-2}}^R \frac{1}{\omega_{2m-1} t_{m-1}^{4m-3}} \\ &\times \int_0^{t_{m-1}} \rho_1 \cdots \int_0^{\rho_{m-2}} \rho_{m-1} \tau_k(\rho_{m-1}, t) d\rho_{m-1} \cdots d\rho_1 \; dt_{m-1} \cdots dt_1, \end{aligned}$$

where

$$\tau_k(\rho, t) = \frac{u_k(t)\sigma_k(\rho)}{u_k(0)} = \int_{B_\rho} \lambda_k u_k(t) u_k e^{m u_k^2} dx.$$

We now show that I can be bounded in terms of $N_k(s,t)$ up to a small error. From this the desired inequality (30) will be immediate. Split

$$I =: II + III,$$

where II corresponds to $\rho_{m-1} \le t$. Since $u_k' \le 0$, for $\rho \le t$ we have

$$\begin{aligned} \tau_k(\rho, t) &= \int_{B_\rho} \lambda_k u_k(t) u_k e^{m u_k^2} dx \le \int_{B_\rho} \lambda_k u_k(\rho) u_k e^{m u_k^2} dx \\ &\le \int_{B_s} \lambda_k u_k(s) u_k e^{m u_k^2} dx + N_k(s, \rho) \le N_k(s,t) + o(1) \end{aligned} \tag{32}$$

with error $o(1) \to 0$ as $k \to \infty$. Here we used that for arbitrary $L > 1$ we can bound

$$\int_{B_s} \lambda_k u_k(s) u_k e^{m u_k^2} dx \le N_k(L r_k^{(\ell)}, s) + \frac{u_k(s)}{u_k(L r_k^{(\ell)})} \Lambda_k(L r_k^{(\ell)}),$$

and by $(H_{\ell,2})$, $(H_{\ell,4})$ the latter tends to 0, if first $k \to \infty$ and then $L \to \infty$. Since

$$t^2 \int_t^\infty t_1 \cdots \int_{t_{m-2}}^\infty \frac{1}{\omega_{2m-1} t_{m-1}^{4m-3}} \int_0^{t_{m-1}} \rho_1 \cdots \int_0^{\rho_{m-2}} \rho_{m-1} d\rho_{m-1} \cdots dt_1 \le C$$

uniformly in t, we conclude that

$$II \le C N_k(s,t) + o(1).$$

In order to obtain a similar bound for III, for $t \le \rho$ we estimate

$$\tau_k(\rho, t) = \frac{u_k(t)\sigma_k(\rho)}{u_k(0)} = \frac{u_k(t)}{u_k(\rho)+1} \int_{B_\rho} \lambda_k (u_k(\rho)+1) u_k e^{m u_k^2} dx \; .$$

Recalling (32), we have

$$\int_{B_\rho} \lambda_k(u_k(\rho)+1)u_k e^{mu_k^2}dx \le \tau_k(\rho,\rho)+o(1) \le N_k(s,\rho)+o(1).$$

Also note that by Hölder's inequality we can estimate

$$|u_k(t)-u_k(\rho)| \le \int_t^\rho |u_k'(r)|dr \le \|\nabla u_k\|_{L^{2m}}\big(\log\frac{\rho}{t}\big)^{\frac{2m-1}{2m}}.$$

Thus, with a constant $C=C(\Lambda)$ for all $t\le\rho$ we obtain

$$\begin{aligned} \frac{tu_k(t)}{\rho(u_k(\rho)+1)} &= \frac{t}{\rho}\left(\frac{u_k(t)-u_k(\rho)}{u_k(\rho)+1}+\frac{u_k(\rho)}{u_k(\rho)+1}\right) \\ &\le \frac{t}{\rho}\left(C\big(\log\frac{\rho}{t}\big)^{\frac{2m-1}{2m}}+1\right) \le C \end{aligned} \tag{33}$$

and with $C_1=C_1(\Lambda)$ we can bound

$$\frac{t}{\rho}\tau_k(\rho,t)\le C_1N_k(s,\rho)+o(1).$$

It follows that

$$\begin{aligned} III &= t^2\int_t^R t_1\int_{t_1}^R t_2\cdots\int_{t_{m-2}}^R \frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}} \\ &\quad\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}\tau_k(\rho_{m-1},t)d\rho_{m-1}\cdots d\rho_1\, dt_{m-1}\cdots dt_1 \\ &= t\int_t^R t_1\int_{t_1}^R t_2\cdots\int_{t_{m-2}}^R \frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}} \\ &\quad\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2\frac{t}{\rho_{m-1}}\tau_k(\rho_{m-1},t)d\rho_{m-1}\cdots d\rho_1\, dt_{m-1}\cdots dt_1 \\ &\le C_1t\int_t^R t_1\int_{t_1}^R t_2\cdots\int_{t_{m-2}}^R \frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}} \\ &\quad\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2(N_k(s,\rho_{m-1})+o(1))d\rho_{m-1}\cdots dt_1. \end{aligned}$$

For any $L\ge1$ we split the integral with respect to t_1 and use the obvious inequality $N_k(s,\rho_{m-1})\le 2\Lambda$ for large k to estimate

$$\begin{aligned} III &\le C_1t\int_t^{Lt} t_1\int_{t_1}^R t_2\int_{t_2}^R t_3\cdots\int_{t_{m-2}}^R \frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}} \\ &\quad\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2(N_k(s,\rho_{m-1})+o(1))d\rho_{m-1}\cdots dt_1 \\ &\quad+2C_1\Lambda t\int_{Lt}^R t_1\int_{t_1}^R t_2\int_{t_2}^R t_3\cdots\int_{t_{m-2}}^R \frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}} \\ &\quad\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2 d\rho_{m-1}\cdots dt_1+o(1) \end{aligned}$$

Observing the uniform bound

$$Lt\int_{Lt}^{\infty} t_1\cdots\int_{t_{m-2}}^{\infty}\frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}}\int_0^{t_{m-1}}\rho_1\cdots\int_0^{\rho_{m-2}}\rho_{m-1}^2 d\rho_{m-1}\cdots dt_1\le C,$$

with a constant $C_2=C_2(\Lambda)$ we obtain

$$\begin{aligned}III\le{}& C_1t\int_t^{Lt}t_1\int_{t_1}^{R}t_2\int_{t_2}^{R}t_3\cdots\int_{t_{m-2}}^{R}\frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}}\\ &\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2(N_k(s,\rho_{m-1})+o(1))d\rho_{m-1}\cdots dt_1\\ &+\frac{C_2\Lambda}{L}+o(1)\end{aligned}$$

To proceed we successively split the integral also with respect to $t_2,\dots,t_{m-1}$ and use the uniform bounds

$$\begin{aligned}Lt\int_0^{Lt}t_1\cdots\int_0^{Lt}t_{j-1}\int_{Lt}^{\infty}t_j\cdots\int_{t_{m-2}}^{\infty}\frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}}&\\ \times\int_0^{t_{m-1}}\rho_1\cdots\int_0^{\rho_{m-2}}\rho_{m-1}^2d\rho_{m-1}\cdots dt_j\le C&\end{aligned}$$

for $2\le j<m$ to estimate

$$\begin{aligned}III\le{}& C_1t\int_t^{Lt}t_1\int_{t_1}^{Lt}t_2\int_{t_2}^{R}t_3\cdots\int_{t_{m-2}}^{R}\frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}}\\ &\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2(N_k(s,\rho_{m-1})+o(1))d\rho_{m-1}\cdots dt_1\\ &+2C_1\Lambda t\int_t^{Lt}t_1\int_{Lt}^{R}t_2\int_{t_2}^{R}t_3\cdots\int_{t_{m-2}}^{R}\frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}}\\ &\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2d\rho_{m-1}\cdots dt_1+\frac{C_2\Lambda}{L}+o(1)\\ \le\cdots\le{}& C_1t\int_t^{Lt}t_1\int_{t_1}^{Lt}t_2\int_{t_2}^{Lt}t_3\cdots\int_{t_{m-2}}^{Lt}\frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}}\\ &\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2\big(N_k(s,Lt)+o(1)\big)d\rho_{m-1}\cdots dt_1\\ &+2C_1\Lambda t\int_t^{Lt}t_1\int_{t_1}^{Lt}t_2\int_{t_2}^{Lt}t_3\cdots\int_{Lt}^{R}\frac{1}{\omega_{2m-1}t_{m-1}^{4m-3}}\\ &\times\int_t^{t_{m-1}}\rho_1\cdots\int_t^{\rho_{m-2}}\rho_{m-1}^2d\rho_{m-1}\cdots dt_1+\frac{C_{m-1}\Lambda}{L}+o(1)\\ \le{}& C_mN_k(s,Lt)+\frac{C_m\Lambda}{L}+o(1),\end{aligned}$$

with constants $C_j=C_j(\Lambda)$, $2\le j\le m$. Using (27) in case $t\le 2s$ and (28) in case $t>2s$ we get

$$N_k(s,Lt)\le C(L)N_k(s,t)+o(1),$$

and with the constant $C_{m+1} = C_m\Lambda = C_{m+1}(\Lambda)$ there results

$$-tw_k'(t)\frac{u_k(t)}{u_k(0)} \leq C(L,\Lambda)N_k(s,t) + \frac{C_{m+1}}{L} + o(1).$$

Inserting this into (31) we infer

$$\begin{aligned} N_k(s,t) &\leq \frac{P_k(t)}{2m} - \omega_{2m-1}\int_s^t \lambda_k r^{2m}\Big(\frac{1}{m} + u_k^2\Big)\frac{u_k}{u_k(0)} w_k' e^{mu_k^2} dr \\ &\leq \frac{P_k(t)}{2m} + \Big(C(L,\Lambda)N_k(s,t) + \frac{C_{m+1}}{L}\Big)N_k(s,t) + o(1). \end{aligned} \tag{34}$$

Choosing $L = 2C_{m+1}$ we finally get (30) for an appropriate $C_0 = C_0(\Lambda)$. □

Lemma 6 *Let $C_0 = C_0(\Lambda)$ be the constant appearing in* (30). *If for some $t_k \in]s_k^{(\ell)}, R]$ there holds*

$$0 < \lim_{k\to\infty} N_k(s_k^{(\ell)}, t_k) =: \alpha < \frac{1}{2C_0}, \tag{35}$$

then

$$\lim_{k\to\infty}\frac{s_k^{(\ell)}}{t_k} = 0, \ \liminf_{k\to\infty} P_k(t_k) \geq \frac{m\alpha}{2}, \ \textit{and} \ \lim_{L\to\infty}\lim_{k\to\infty} N_k(s_k^{(\ell)}, t_k/L) = 0.$$

Proof. Assume that for some $t_k \in]s_k^{(\ell)}, R]$ we have (35). Since the same reasoning as in the proof of (27) also yields that

$$\lim_{L\to\infty}\lim_{k\to\infty} N_k(s_k^{(\ell)}, Ls_k^{(\ell)}) = 0,$$

necessarily $s_k^{(\ell)}/t_k \to 0$ as $k\to\infty$. Moreover, (30) yields

$$\begin{aligned} \liminf_{k\to\infty}\frac{P_k(t_k)}{m} &\geq \lim_{k\to\infty}\Big(N_k(s_k^{(\ell)}, t_k) - C_0N_k^2(s_k^{(\ell)}, t_k)\Big) \\ &\geq \lim_{k\to\infty}\frac{N_k(s_k^{(\ell)}, t_k)}{2} = \frac{\alpha}{2}, \end{aligned} \tag{36}$$

as claimed. Now we show that

$$\lim_{L\to\infty}\lim_{k\to\infty} N_k(s_k^{(\ell)}, t_k/L) = 0. \tag{37}$$

Indeed, if we assume

$$\lim_{L\to\infty}\limsup_{k\to\infty} N_k(s_k^{(\ell)}, t_k/L) = \beta > 0,$$

we have

$$\frac{\beta}{2} \leq N_k(s_k^{(\ell)}, t_k/L) \leq N_k(s_k^{(\ell)}, t_k) < \frac{1}{2C_0}$$

for any $L \geq 1$ and sufficiently large k. Therefore we can apply (36) with t_k/L instead of t_k for any $L \geq 1$ to get

$$\lim_{k\to\infty} P_k(t_k/L) \geq \frac{m\beta}{2}.$$

Then (28) yields

$$C \lim_{k\to\infty} N_k(t_k/(2L), t_k/L) \geq \lim_{k\to\infty} P_k(t_k/L) \geq \frac{m\beta}{2}.$$

Choosing $L = 2^j$ and summing over j for $0 \leq j \leq M-1$, we get

$$C \lim_{k\to\infty} \Lambda_k(t_k) \geq C \lim_{k\to\infty} N_k(2^{-M}t_k, t_k) \geq \frac{mM\beta}{2} \to \infty \quad \text{as } M \to \infty,$$

which contradicts (3). Therefore (37) is proven. $\square$

Suppose now that for some $t_k \geq s_k^{(\ell)}$ there holds

$$\limsup_{k\to\infty} N_k(s_k^{(\ell)}, t_k) > 0.$$

We then want to show that $(H_{\ell+1})$ holds. We can choose numbers $r_k^{\ell+1} \in]s_k^{(\ell)}, t_k[$ such that for a subsequence there holds

$$0 < \lim_{k\to\infty} N_k(s_k^{(\ell)}, r_k^{(\ell+1)}) < \frac{1}{2C_0}, \tag{38}$$

where C_0 is as in Lemma 6. Observe that Lemma 6 then implies

$$\lim_{k\to\infty} s_k^{(\ell)}/r_k^{(\ell+1)} = \lim_{L\to\infty}\lim_{k\to\infty} N_k(s_k^{(\ell)}, r_k^{(\ell+1)}/L) = 0, \tag{39}$$

and

$$\lim_{k\to\infty} P_k(r_k^{(\ell+1)}) > 0. \tag{40}$$

Proposition 7 *We have*

$$\eta_k^{(\ell+1)}(x) := u_k(r_k^{(\ell+1)})\big(u_k(r_k^{(\ell+1)}x) - u_k(r_k^{(\ell+1)})\big) \to \eta^{(\ell+1)}$$

in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}\backslash\{0\})$. *Moreover, for a suitable constant* $c^{(\ell+1)}$ *the function* $\eta_0^{(\ell+1)} := \eta^{(\ell+1)} + c^{(\ell+1)}$ *satisfies*

$$(-\Delta)^m \eta_0^{(\ell+1)} = (2m-1)!e^{2m\eta_0^{(\ell+1)}}, \qquad \int_{\mathbb{R}^{2m}} (2m-1)!e^{2m\eta_0^{(\ell+1)}}dx = \Lambda_1.$$

The above proposition, which will be proven in the following section, implies that

$$\lim_{L\to\infty}\lim_{k\to\infty} N_k(r_k^{(\ell+1)}/L, Lr_k^{(\ell+1)}) = \Lambda_1;$$

hence (39) yields

$$\begin{aligned}
&\lim_{L\to\infty}\lim_{k\to\infty} N_k(s_k^{(\ell)}, Lr_k^{(\ell+1)}) \\
&\quad = \lim_{L\to\infty}\lim_{k\to\infty} N_k(s_k^{(\ell)}, r_k^{(\ell+1)}/L) + \lim_{L\to\infty}\lim_{k\to\infty} N_k(r_k^{(\ell+1)}/L, Lr_k^{(\ell+1)}) \\
&\quad = 0 + \Lambda_1 = \Lambda_1.
\end{aligned} \tag{41}$$

Then the inductive hypothesis $(H_{\ell,3})$ gives

$$\begin{aligned}
\lim_{L\to\infty}\lim_{k\to\infty} \Lambda_k(Lr_k^{(\ell+1)}) &= \lim_{L\to\infty}\lim_{k\to\infty} \Big(\Lambda_k(s_k^{(\ell)}) + N_k(s_k^{(\ell)}, Lr_k^{(\ell+1)})\Big) \\
&= (\ell+1)\Lambda_1.
\end{aligned}$$

Now set $w_k^{(\ell+1)}(x) = u_k(r_k^{(\ell+1)})(u_k(x) - u_k(r_k^{(\ell+1)}))$ so that

$$(-\Delta)^m w_k^{(\ell+1)} = \lambda_k u_k(r_k^{(\ell+1)}) u_k e^{m u_k^2} =: f_k^{(\ell+1)}.$$

Similar to Lemma 4 and with the same proof (except that instead of Theorem 1 one needs to use Proposition 7) we have

Lemma 8 *For any $0 < \varepsilon < 1$, letting $T_k^{(\ell+1)}(\varepsilon) > 0$ be such that $u_k(T_k^{(\ell+1)}) = \varepsilon u_k(r_k^{(\ell+1)})$, we have*

$$\lim_{k\to\infty} N_k(s_k^{(\ell)}, T_k^{(\ell+1)}) = \Lambda_1. \tag{42}$$

Moreover $r_k^{(\ell+1)}/T_k^{(\ell+1)} \to 0$ as $k \to \infty$.

According to Lemma 8 and (41) we can choose numbers $\varepsilon_k \to 0$ and a subsequence so that for $s_k^{(\ell+1)} := T_k^{(\ell+1)}(\varepsilon_k)$ we have $u_k(s_k^{(\ell+1)}) \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{r_k^{(\ell+1)}}{s_k^{(\ell+1)}} = 0,$$

while

$$\lim_{k\to\infty} \Lambda_k(s_k^{(\ell+1)}) = (\ell+1)\Lambda_1, \quad \lim_{L\to\infty}\lim_{k\to\infty} N_k(Lr_k^{(\ell+1)}, s_k^{(\ell+1)}) = 0.$$

Again reasoning as in the proof of (27) we also infer

$$\lim_{k\to\infty} N_k(s_k^{(\ell+1)}, Ls_k^{(\ell+1)}) = 0 \quad \text{for every } L \geq 1.$$

Finally, observe that the definition of $s_k^{(\ell+1)}$ implies that

$$\lim_{k\to\infty} \frac{u_k(s_k^{(\ell+1)})}{u_k(Lr_k^{(\ell+1)})} = 0 \quad \text{for every } L \geq 0.$$

Together with (39) this completes the proof of $(H_{\ell+1})$, and hence of Theorem 2 in the radially symmetric case.

2.4 Proof of Proposition 7

As preparation for the proof of Proposition 7 we need the following two lemmas.

Lemma 9 *For $r_k^{(\ell+1)}$ as above, we have*

$$v_k(x) := u_k\big(r_k^{(\ell+1)}x\big) - u_k\big(r_k^{(\ell+1)}\big) \to 0 \quad \text{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}\backslash\{0\}).$$

Proof. We write $r_k = r_k^{(\ell+1)}$. Moreover, we consider only the case $m > 1$, the case $m = 1$ being considerably easier. As in the proof of Lemma 3.2 in [13] we have

$$(-\Delta)^m v_k(x) = \lambda_k r_k^{2m} u_k(r_k x) e^{m u_k^2(r_k x)} =: g_k(x) \geq 0,$$

with $g_k \to 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^{2m}\backslash\{0\})$. By scaling and Sobolev's embedding we also have

$$\begin{aligned} \|\nabla^2 v_k\|_{L^m(B_{R/r_k})} &= \|\nabla^2 u_k\|_{L^m(B_R)} \le C, \\ \|\nabla^m v_k\|_{L^2(B_{R/r_k})} &= \|\nabla^m u_k\|_{L^2(B_R)} \le C. \end{aligned} \tag{43}$$

Set $w_k := \Delta v_k$. Then a subsequence $w_k \to w$ weakly in $H^{m-2}_{\mathrm{loc}}(\mathbb{R}^{2m})$ and in $C^{2m-3,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \setminus \{0\})$ for some function $w \in L^m(\mathbb{R}^{2m})$ with $\nabla^{m-2} w \in L^2(\mathbb{R}^{2m})$. Clearly $\Delta^{m-1} w = 0$ in $\mathbb{R}^{2m}\backslash\{0\}$. In fact, since the point $x = 0$ has vanishing H^m-capacity, as in [13] we have $\Delta^{m-1} w = 0$ in $\mathbb{R}^{2m}$. Recalling that $w \in L^m(\mathbb{R}^{2m})$ we conclude that $w \equiv 0$; see Lemmas 23 and 24 in the appendix.

Recalling that (Δv_k) is bounded in $L^m(\mathbb{R}^{2m})$ and noting the condition $v_k(1) = 0$, from standard elliptic estimates we infer that (v_k) is bounded in $W^{2,m}(B_1)$. Hence a subsequence $v_k \to v$ weakly in $W^{2,m}(B_1)$ and in $C^{2m-1,\alpha}$ away from $x = 0$. We then have $\Delta v = 0$ and $v(1) = 0$, therefore $v \equiv 0$ on B_1.

By elliptic estimates, from (43) and the condition $v_k(1) = 0$ we also infer that (v_k) is bounded in $W^{2,m}_{\mathrm{loc}}(\mathbb{R}^{2m})$. Therefore, we also have that $v_k \to v$ weakly in $W^{2,m}_{\mathrm{loc}}(\mathbb{R}^{2m})$ and in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \setminus \{0\})$, with $\Delta v = 0$. By unique continuation it follows that $v \equiv 0$. This completes the proof. ☐

Lemma 10 *For any $L > 0$ there exists $k_0 = k_0(L)$ such that for all $k \ge k_0$ and any $1 \le j \le 2m-1$ there holds*

$$u_k(r_k^{(\ell+1)}) \int_{B_{Lr_k^{(\ell+1)}} \backslash B_{r_k^{(\ell+1)}/L}} |\nabla^j u_k| dx \le C(Lr_k^{(\ell+1)})^{2m-j}.$$

Proof. The proof is identical to the proof of Lemma 6 in [8], using Lemma 9 above instead of Lemma 3 in [8]. ☐

Proof of Proposition 7. For simplicity of notation, we now drop the index $\ell + 1$.

Step 1. We claim that $\eta_k \to \eta$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}\backslash\{0\})$ for some smooth function η. For any $L > 1$ let $\Omega_L := B_L(0)\backslash B_{1/L}(0)$. Recall that by Lemma 9 we have $\overline{u}_k(x) := \frac{u_k(r_k x)}{u_k(r_k)} \to 1$ uniformly on Ω_L as $k \to \infty$. Thus by (7) with error $o(1) \to 0$ as $k \to \infty$ we have

$$\begin{aligned} 0 \le (-\Delta)^m \eta_k(x) &= \lambda_k r_k^{2m} u_k^2(r_k) \overline{u}_k(x) e^{m u_k^2(r_k x)} \\ &\le (L^{2m} + o(1)) \lambda_k (r_k|x|)^{2m} u_k^2(r_k x) e^{m u_k^2(r_k x)} \le C L^{2m} + o(1). \end{aligned} \tag{44}$$

Split $\eta_k = h_k + l_k$ on Ω_{2L}, where

$$\Delta^m h_k = 0 \ \text{ on } \Omega_{2L}, \ \text{ and } l_k = \Delta l_k = \ldots = \Delta^{m-1} l_k = 0 \ \text{ on } \partial\Omega_{2L}.$$

Since $\|\Delta^m \eta_k\|_{L^\infty(\Omega_{2L})} \le C = C(L)$, by elliptic estimates we get that $l_k \to l$ in $C^{2m-1,\alpha}(\Omega_{2L})$. Together with Lemma 10 this implies

$$\|\nabla h_k\|_{L^1(\Omega_{2L})} \le \|\nabla l_k\|_{L^1(\Omega_{2L})} + \|\nabla \eta_k\|_{L^1(\Omega_{2L})} \le C.$$

Moreover, since $\eta_k = 0$ on $\partial B_1(0)$, we have

$$|h_k| = |l_k| \le C \quad \text{ on } \partial B_1(0). \tag{45}$$

Then, from a Poincaré-type inequality, we easily get $\|h_k\|_{L^1(\Omega_{2L})} \le C$. By virtue of Proposition 21, we infer that

$$\|h_k\|_{C^j(\Omega_L)} \le C_j \quad \text{for every } j \in \mathbb{N}.$$

Hence a subsequence $h_k \to h$ smoothly on Ω_L, and

$$\eta_k \to \eta := h + l \quad \text{in } C^{2m-1,\alpha}(\Omega_L),$$

proving our claim.

Step 2. With $\overline{u}_k(x) := \frac{u_k(r_k x)}{u_k(r_k)}$ as above, from (44) we get

$$\begin{aligned} (-\Delta)^m \eta_k &= \lambda_k r_k^{2m} u_k^2(r_k) e^{m u_k^2(r_k)} \overline{u}_k(x) e^{m(u_k^2(r_k\,\cdot) - u_k^2(r_k))} \\ &= \mu_k \overline{u}_k e^{m(\overline{u}_k+1)\eta_k}, \end{aligned} \tag{46}$$

where by (40) we may assume

$$\mu_k := \lambda_k r_k^{2m} u_k^2(r_k) e^{m u_k^2(r_k)} = \omega_{2m-1}^{-1} P_k(r_k) \to \mu_0 > 0.$$

Since $\overline{u}_k \to 1$ locally uniformly on $\mathbb{R}^{2m}\backslash\{0\}$ we may pass to the limit $k \to \infty$ in (46) to see that η solves the equation

$$(-\Delta)^m \eta = \mu_0 e^{2m\eta} \quad \text{on } \mathbb{R}^{2m}\backslash\{0\} \tag{47}$$

in the distribution sense. In fact, we now show that (47) holds on all of $\mathbb{R}^{2m}$. Note that by Step 1 for any $L > 1$ we have

$$\begin{aligned} \int_{\Omega_L} e^{2m\eta} dx &= \lim_{k\to\infty} \int_{\Omega_L} \overline{u}_k^2 e^{m(\overline{u}_k+1)\eta_k} dx \\ &= \lim_{k\to\infty} \int_{\Omega_L} \mu_k^{-1} \overline{u}_k (-\Delta)^m \eta_k dx \\ &\le \mu_0^{-1} \liminf_{k\to\infty} \int_{B_{Lr_k}} u_k (-\Delta)^m u_k dx \le \mu_0^{-1}\Lambda. \end{aligned}$$

As $L \to \infty$, by Fatou's lemma, we get $e^{2m\eta} \in L^1(\mathbb{R}^{2m})$. Moreover $\eta \ge 0$ on B_1, hence $\eta \in L^p(B_1)$ for every $p \in [1, \infty[$. Also note that $(-\Delta)^m \eta_k \ge 0$ and that from (32) we can bound

$$\begin{aligned} &\limsup_{k\to\infty} \int_{B_{1/L}(0)} (-\Delta)^m \eta_k \, dx \\ &\quad = \limsup_{k\to\infty} \int_{B_{r_k/L}(0)} \lambda_k u_k(r_k) u_k e^{m u_k^2} dx \le \limsup_{k\to\infty} N_k(s_k^{(\ell)}, r_k/L) \to 0 \end{aligned} \tag{48}$$

as $L \to \infty$. Since by Lemma 4 we have $\overline{u}_k \ge 1$, $\eta_k \ge 0$ on B_1, from (46) and (48) we also find that

$$\limsup_{k\to\infty} \int_{B_{1/L}(0)} \eta_k \, dx \to 0 \text{ as } L \to \infty\,. \tag{49}$$

By (47) and (48) for any test function $\varphi \in C_0^\infty(\mathbb{R}^{2m})$ we now obtain

$$\begin{aligned}\int_{\mathbb{R}^{2m}} \big((-\Delta)^m \eta - \mu_0 e^{2m\eta}\big)\varphi\, dx &= \lim_{L\to\infty} \int_{\mathbb{R}^{2m}} (-\Delta)^m \eta \varphi \tau_L\, dx \\ &= \lim_{L\to\infty} \liminf_{k\to\infty} \int_{\mathbb{R}^{2m}} \big((-\Delta)^m \eta - (-\Delta)^m \eta_k\big)\varphi\tau_L\, dx,\end{aligned} \tag{50}$$

where for $L \in \mathbb{N}$ we let $\tau_L(x) = \tau(Lx)$ with a fixed cut-off function $\tau \in C_0^\infty(B_2)$ such that $0 \le \tau \le 1$ and $\tau \equiv 1$ in B_1. But by Step 1 for any $L \ge 1$ we have

$$\begin{aligned}&\liminf_{k\to\infty} \int_{\mathbb{R}^{2m}} \big((-\Delta)^m \eta - (-\Delta)^m \eta_k\big)\varphi\tau_L\, dx \\ &\qquad = \liminf_{k\to\infty} \int_{\mathbb{R}^{2m}} (\eta - \eta_k)\big((-\Delta)^m \varphi\big)\tau_L\, dx,\end{aligned}$$

and since $\eta \in L^1(B_1)$ and on account of (49) the latter converges to 0 as $L \to \infty$ for any fixed $\varphi \in C_0^\infty(\mathbb{R}^{2m})$. From (50) we thus see that η solves (47) in the distribution sense on $\mathbb{R}^{2m}$. By elliptic estimates, η is smooth on all of $\mathbb{R}^{2m}$; see for instance [7], Corollary 8. The function $\eta_0 := \eta + \frac{1}{2m}\log\frac{\mu_0}{(2m-1)!}$ then satisfies

$$(-\Delta)^m \eta_0 = (2m-1)! e^{2m\eta_0} \quad \text{in } \mathbb{R}^{2m}, \qquad \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx < \infty. \tag{51}$$

Solutions to (51) have been classified in [7], where it was shown that either

(i) $\eta_0(x) = \log\frac{2\sigma}{1+\sigma|x-x_0|^2}$ for some $\sigma > 0$, $x_0 \in \mathbb{R}^{2m}$, or

(ii) $m > 1$ and there exist $1 \le j \le m-1$ and $a \ne 0$ such that

$$\lim_{|x|\to\infty} \Delta^j \eta_0(x) = a,$$

and hence for sufficiently large L, with error $o(1) \to 0$ as $k \to \infty$,

$$\begin{aligned}(Lr_k)^{2j-2m} u_k(r_k) \int_{B_{Lr_k}\setminus B_{r_k/L}} |\nabla^{2j} u_k| dx &= L^{2j-2m} \int_{B_L \setminus B_{1/L}} |\nabla^{2j}\eta_k| dx \\ &= L^{2j-2m} \int_{B_L\setminus B_{1/L}} |\nabla^{2j}\eta_0| dx + o(1) \ge CL^{2j} + o(1)\end{aligned} \tag{52}$$

for some constant $C > 0$ independent of L.

But (52) is incompatible with the estimate of Lemma 10 when L and k are large. Hence case (i) occurs (with $x_0 = 0$, by radial symmetry). In particular, we have $\int_{\mathbb{R}^{2m}} (2m-1)! e^{2m\eta_0} dx = \Lambda_1$. □

3 The general case

The following gradient bound analogous to [5], Proposition 2, and generalizing [13], Proposition 4.1, will be crucial in the sequel. The proof will be given in the next section.

Proposition 11 *There exists a uniform constant C such that*

$$\sup_{x\in\Omega} \inf_{1\le j\le I} |x - x_k^{(j)}|^\ell u_k(x) |\nabla^\ell u_k(x)| \le C \quad \textit{for all } 1 \le \ell \le 2m-1,\ k \in \mathbb{N}.$$

Fix an index $i \in \{1, \dots, I\}$ and let $x_k = x_k^{(i)} \to x^{(i)}$, $r_k = r_k^{(i)} \to 0$ as given by Theorem 1. After a translation we may assume that $x^{(i)} = 0$. Set as before

$$e_k := \lambda_k u_k^2 e^{mu_k^2}, \quad f_k := \lambda_k u_k(0) u_k e^{mu_k^2},$$

and

$$\Lambda_k(r) := \int_{B_r} e_k dx.$$

In the following we will use the notation

$$\bar{f}(r) := \fint_{\partial B_r} f d\sigma,$$

for any function f. Set also

$$\tilde{e}_k := \lambda_k \bar{u}_k^2 e^{m\bar{u}_k^2} \leq \bar{e}_k.$$

(Here we used Jensen's inequality.) Again we let $w_k(x) := u_k(0)(u_k(x) - u_k(0))$, satisfying

$$(-\Delta)^m \bar{w}_k = \lambda_k u_k(0) \overline{u_k e^{mu_k^2}} = \bar{f}_k.$$

Finally set

$$\tilde{\Lambda}_k(r) := \int_{B_r} \tilde{e}_k dx \leq \Lambda_k(r), \ \sigma_k(r) := \int_{B_r} \bar{f}_k dx. \tag{53}$$

Again Theorem 1 implies

$$\lim_{L\to\infty} \lim_{k\to\infty} \tilde{\Lambda}_k(Lr_k) = \lim_{L\to\infty} \lim_{k\to\infty} \Lambda_k(Lr_k) = \lim_{L\to\infty} \lim_{k\to\infty} \sigma_k(Lr_k) = \Lambda_1. \tag{54}$$

Recalling that $x_k^{(i)} = 0$ we let

$$\rho_k = \rho_k^{(i)} := \min\big\{ \inf_{j\neq i} \frac{|x_k^{(j)}|}{2}, \mathrm{dist}(0, \partial\Omega_k) \big\};$$

that is, we set $\rho_k = \mathrm{dist}(0, \partial\Omega_k)$ if the $(x_k^{(i)})$ are the only concentration points. Observe that by Theorem 1 we have $r_k = o(\rho_k)$ as $k \to \infty$.

Note that Proposition 11 implies the uniform bound

$$0 \leq \sup_{r/2\leq|x|\leq r} u_k^2(x) - \inf_{r/2\leq|x|\leq r} u_k^2(x) \leq Cr \sup_{|x|=r} |\nabla u_k^2(x)| \leq C \tag{55}$$

for $0 \leq r \leq \rho_k$.

Lemma 12 *Let $0 < \varepsilon < 1$ and assume that for $k \geq k_0 = k_0(\varepsilon)$ there holds*

$$\inf_{0\leq r\leq\rho_k} \bar{u}_k(r) \leq \frac{\varepsilon \bar{u}_k(0)}{2}.$$

Let $T_k = T_k(\varepsilon) \leq S_k = S_k(\varepsilon) \in]0, \rho_k]$ be the smallest numbers such that $\bar{u}_k(T_k) = \varepsilon u_k(0)$, $\bar{u}_k(S_k) = \varepsilon u_k(0)/2$, respectively. Then

$$\lim_{k\to\infty} \frac{r_k}{T_k} = \lim_{k\to\infty} \frac{T_k}{S_k} = 0. \tag{56}$$

Moreover for any $b < 2$ and $k \geq k_0 = k_0(b)$ there holds

$$\bar{w}_k(r) \leq b \log\left(\frac{r_k}{r}\right) + C \quad \textit{for } 0 \leq r \leq T_k, \tag{57}$$

and we have

$$\lim_{k\to\infty} \tilde{\Lambda}_k(T_k) = \Lambda_1. \tag{58}$$

Proof. Property (56) follows from (55) and our choice of T_k and S_k.

As in the proof of Lemma 4 for a given $t \leq T_k$ we decompose $\bar{w}_k = g_k + h_k$ on B_t, with

$$\Delta^m h_k = 0 \text{ in } B_t, \quad \text{and } g_k = \partial_\nu g_k = \ldots = \partial_\nu^{m-1} g_k = 0 \text{ on } \partial B_t.$$

By (54), we get the analogues of Lemma 3 and of (22); that is, for $L \geq L_0 = L_0(b)$, $k \geq k_0 = k_0(L)$ there holds

$$(-1)^{m-1} t^{m-1} \underbrace{\Big(t^{-1}\Big(t^{-1}\cdots\Big(t^{-1}}_{m-1 \text{ times}}\Big(\bar{w}_k'(t) + \frac{b}{t}\Big)\underbrace{\Big)'\cdots\Big)'\Big)'}_{m-1 \text{ times}} \leq 0$$

for all $t \in [Lr_k, S_k]$. We now inductively integrate from t to S_k as in Lemma 4. Using Proposition 11 to bound

$$|\partial_r^j \bar{w}_k(S_k)| = \frac{u_k(0)}{\bar{u}_k(S_k)} \frac{S_k^j \bar{u}_k(S_k)|\partial_r^j \bar{u}_k(S_k)|}{S_k^j} \leq \frac{C}{\varepsilon S_k^j},$$

and recalling (56), for $L \geq L_0$ and $k \geq k_0$ we get

$$t\bar{w}_k'(t) \leq -b + \frac{C}{\varepsilon}\frac{t^2}{S_k^2} = -b + o(1) \quad \text{for all } Lr_k \leq t \leq T_k,$$

with error $o(1) \to 0$ as $k \to \infty$. Since $b < 2$ is arbitrary, (57) follows as before.

In order to prove (58) observe that the definition of r_k gives

$$\begin{aligned}
\tilde{e}_k(r) &\leq C\lambda_k u_k^2(0) e^{m u_k^2(0)} e^{2m\left(1+\frac{\bar{w}_k(r)}{2u_k^2(0)}\right)\bar{w}_k(r)} \\
&\leq C\lambda_k r_k^{2m} u_k^2(0) e^{m u_k^2(0)} r_k^{-2m} e^{m(\varepsilon+1)\bar{w}_k(r)} \leq C r_k^{-2m}\left(\frac{r_k}{r}\right)^{m(\varepsilon+1)b}
\end{aligned}$$

for $Lr_k \leq r \leq T_k$. We then complete the proof as in the radial case. $\square$

For $0 \leq s < t \leq \rho_k$ set

$$N_k(s,t) := \Lambda_k(t) - \Lambda_k(s) = \int_{B_t \setminus B_s} \lambda_k u_k^2 e^{m u_k^2} dx,$$

and let

$$\tilde{N}_k(s,t) := \tilde{\Lambda}(t) - \tilde{\Lambda}(s) = \int_s^t \omega_{2m-1} \lambda_k r^{2m-1} \bar{u}_k^2 e^{m\bar{u}_k^2} dr \leq N_k(s,t). \tag{59}$$

From (55) we infer

$$\sup_{|x|=r} e^{m u_k^2(x)} \leq C e^{m\bar{u}_k^2(r)} \quad \text{for } 0 \leq r \leq \rho_k; \tag{60}$$

hence we obtain

$$\sup_{|x|=r} u_k^2(x)e^{mu_k^2(x)} \le C(1+\bar{u}_k^2(r))e^{m\bar{u}_k^2(r)} \quad \text{for } 0 \le r \le \rho_k. \tag{61}$$

Then (61) implies

$$N_k(s,t) \le C\tilde{N}_k(s,t) + o(1) \quad \text{for } 0 \le s \le t \le \rho_k, \tag{62}$$

with $o(1) \to 0$ as $k \to \infty$. Similarly, setting

$$\tilde{P}_k(t) = t\int_{\partial B_t} \tilde{e}_k d\sigma = \omega_{2m-1}\lambda_k t^{2m}\bar{u}_k^2(t)e^{m\bar{u}_k^2(t)} \le P_k(t) := t\int_{\partial B_t} e_k d\sigma,$$

we can estimate

$$P_k(t) \le C\tilde{P}_k(t) + o(1) \quad \text{for } 0 \le t \le \rho_k, \tag{63}$$

with $o(1) \to 0$ as $k \to \infty$. Finally, from (61) we also obtain the analogue of (28); that is, we have

$$P_k(t) \le CN_k(t/2,t) + o(1) \le CP_k(t/2) + o(1), \tag{64}$$

with error $o(1) \to 0$ as $k \to \infty$.

In particular, we obtain the following improvement of Lemma 12.

Lemma 13 *For any $0 < \varepsilon < 1$, if $T_k = T_k(\varepsilon) \le \rho_k$ is as in Lemma 12, then we have*

$$\lim_{k\to\infty} \Lambda_k(T_k) = \Lambda_1.$$

Proof. Indeed (58) and (62) imply

$$\lim_{L\to\infty}\lim_{k\to\infty} N_k(Lr_k, T_k) \le C \lim_{L\to\infty}\lim_{k\to\infty} \tilde{N}_k(Lr_k, T_k) = 0,$$

which together with (54) implies the lemma. □

If the assumptions of Lemma 12 hold for any $0 < \varepsilon < 1$ we may proceed to resolve secondary concentrations at scales $o(\rho_k)$ as in the radially symmetric case. Indeed, by Lemmas 12 and 13 we may then choose a subsequence (u_k), numbers $\varepsilon_k \to 0$ as $k \to \infty$ and corresponding numbers $s_k = T_k(\varepsilon_k) \le \rho_k$ with $r_k/s_k \to 0$ as $k \to \infty$ and such that

$$\lim_{k\to\infty} \Lambda_k(s_k) = \Lambda_1, \quad \lim_{L\to\infty}\lim_{k\to\infty} N_k(Lr_k, s_k) = 0,$$

while in addition $\bar{u}_k(s_k) \to \infty$ and

$$\lim_{k\to\infty} \frac{\bar{u}_k(s_k)}{\bar{u}_k(Lr_k)} = 0 \quad \text{for every } L > 0.$$

As before, by slight abuse of notation, we set $r_k = r_k^{(1)}$, $s_k = s_k^{(1)}$, so that the analogue of (H_1) holds, and iterate. Suppose that for some integer $\ell \ge 1$ we already have determined numbers

$$s_k^{(0)} := 0 < r_k^{(1)} < s_k^{(1)} < \cdots < r_k^{(\ell)} < s_k^{(\ell)} = o(\rho_k)$$

satisfying the analogues of $(H_{\ell,1})$ up to $(H_{\ell,4})$. Similar to Lemma 5 we then have the following result.

Lemma 14 *There is a constant $C_0 = C_0(\Lambda)$ such that for $s_k^{(\ell)} \le t_k = o(\rho_k)$ there holds*

$$\tilde{N}_k(s_k^{(\ell)}, t_k) \le \frac{\tilde{P}_k(t_k)}{m} + C_0 \tilde{N}_k^2(s_k^{(\ell)}, t_k) + o(1), \tag{65}$$

with error $o(1) \to 0$ as $k \to \infty$.

Proof. For ease of notation we write $s = s_k^{(\ell)}$. Replacing w_k with $\bar{w}_k$ in the proof of Lemma 5, similar to (31) we find

$$\tilde{N}_k(s,t) \le \frac{\tilde{P}_k(t)}{2m} - \int_s^t \omega_{2m-1} r^{2m} \frac{\bar{u}_k(r)}{u_k(0)} \bar{w}_k'(r) \tilde{e}_k dr + o(1),$$

with error $o(1) \to 0$ as $k \to \infty$, uniformly in $s \le t$. Proceeding as in Lemma 5, from the equation

$$\underbrace{(t^{-1}(t^{-1}\cdots(t^{-1}}_{m-1 \text{ times}} \bar{w}_k'(t) \underbrace{)'\cdots)')'}_{m-1 \text{ times}} = (-1)^m \frac{A_k(t)}{\omega_{2m-1} t^{4m-3}},$$

where A_k is defined by (14), with σ_k now given by (53), we get

$$t\bar{w}_k'(t) = -t^2 \int_t^{\rho_k} t_1 \int_{t_1}^{\rho_k} t_2 \cdots \int_{t_{m-2}}^{\rho_k} \frac{A_k(t_{m-1})}{\omega_{2m-1} t_{m-1}^{4m-3}} dt_{m-1} \cdots dt_1 + B_k(t, \rho_k),$$

where $B_k(t, \rho_k)$ corresponds to the boundary terms. By arguing as in the proof of Lemma 4 we see that B_k is a linear combination of terms of the form

$$\frac{t^{2l+2}}{\rho_k^{2l+2}} \rho_k^j \partial_r^j \bar{w}_k(\rho_k), \ 0 \le l \le m-2, \ 1 \le j \le m-1.$$

After multiplication with $\frac{\bar{u}_k(t)}{u_k(0)}$, the resulting terms can be written as

$$\frac{t^{2l+2}}{\rho_k^{2l+2}} \bar{u}_k(t) \rho_k^j \partial_r^j \bar{u}_k(\rho_k) = \frac{t^{2l+1}}{\rho_k^{2l+1}} \frac{t\bar{u}_k(t)}{\rho_k(\bar{u}_k(\rho_k)+1)} \rho_k^j (\bar{u}_k(\rho_k)+1) \partial_r^j \bar{u}_k(\rho_k).$$

But by Proposition 11 and the analogue of (33) we have

$$\rho_k^j (\bar{u}_k(\rho_k)+1) |\partial_r^j \bar{u}_k(\rho_k)| \le C, \ \frac{t\bar{u}_k(t)}{\rho_k(\bar{u}_k(\rho_k)+1)} \le C.$$

Hence for $t = t_k = o(\rho_k)$ we have $\frac{\bar{u}_k(t)}{u_k(0)} B_k(t, \rho_k) \to 0$ as $k \to \infty$, and up to an error $o(1) \to 0$ as $k \to \infty$ we obtain the identity

$$-t\bar{w}_k'(t) \frac{\bar{u}_k(t)}{u_k(0)} = t^2 \frac{\bar{u}_k(t)}{u_k(0)} \int_t^{\rho_k} t_1 \int_{t_1}^{\rho_k} t_2 \cdots \int_{t_{m-2}}^{\rho_k} \frac{A_k(t_{m-1})}{\omega_{2m-1} t_{m-1}^{4m-3}} dt_{m-1} \cdots dt_1.$$

The rest of the proof is similar to the proof of Lemma 5. □

On account of (62) and (63) we now obtain the analogue of Lemma 6. The proof is the same as in the radially symmetric case.

Lemma 15 *Let $C_0 = C_0(\Lambda)$ be the constant appearing in* (65)*, and let $t_k > s_k^{(\ell)}$ be such that for a subsequence*

$$\lim_{k\to\infty} \frac{t_k}{\rho_k} = 0, \quad 0 < \lim_{k\to\infty} N_k(s_k^{(\ell)}, t_k) =: \alpha < \frac{1}{2C_0}. \tag{66}$$

Then

$$\lim_{k\to\infty} \frac{s_k^{(\ell)}}{t_k} = 0, \ \liminf_{k\to\infty} P_k(t_k) \geq \frac{m\alpha}{2}, \ \textit{and} \ \lim_{L\to\infty} \lim_{k\to\infty} N_k(s_k^{(\ell)}, t_k/L) = 0.$$

We now closely follow [6]. By the preceding result it suffices to consider the following two cases. In **Case A** for any sequence $t_k = o(\rho_k)$ we have

$$\sup_{s_k^{(\ell)} < t < t_k} P_k(t) \to 0 \text{ as } k \to \infty,$$

and then in view of Lemma 15 also

$$\lim_{L\to\infty} \lim_{k\to\infty} N_k(s_k^{(\ell)}, \rho_k/L) = 0, \tag{67}$$

thus completing the concentration analysis at scales up to $o(\rho_k)$.

In **Case B** for some $s_k^{(\ell)} < t_k \leq \rho_k$ there holds

$$\limsup_{k\to\infty} N_k(s_k^{(\ell)}, t_k) > 0, \quad \lim_{k\to\infty} \frac{t_k}{\rho_k} = 0.$$

Then, as in the radial case, from Lemma 15 we infer that for a subsequence (u_k) and suitable numbers $r_k^{(\ell+1)} \in]s_k^{(\ell)}, t_k[$ we have

$$\lim_{k\to\infty} \frac{s_k^{(\ell)}}{r_k^{(\ell+1)}} = 0, \ \lim_{k\to\infty} N_k(s_k^{(\ell)}, r_k^{(\ell+1)}) > 0, \ \liminf_{k\to\infty} P_k(r_k^{(\ell+1)}) > 0; \tag{68}$$

in particular, $\bar{u}_k(r_k^{(\ell+1)}) \to \infty$ as $k \to \infty$. Also note that

$$\lim_{L\to\infty} \limsup_{k\to\infty} N_k(s_k^{(\ell)}, r_k^{(\ell+1)}/L) = \lim_{k\to\infty} \frac{r_k^{(\ell+1)}}{\rho_k} = \lim_{k\to\infty} \frac{t_k}{\rho_k} = 0. \tag{69}$$

Moreover, analoguous to Proposition 7 we have the following result, which is a special case of Proposition 17 below.

Proposition 16 *There exists a subsequence (u_k) such that*

$$\eta_k^{(\ell+1)}(x) := \bar{u}_k(r_k^{(\ell+1)})(u_k(r_k^{(\ell+1)}x) - \bar{u}_k(r_k^{(\ell+1)})) \to \eta^{(\ell+1)}(x)$$

in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m} \setminus \{0\})$ as $k \to \infty$, where $\eta_0^{(\ell+1)} := \eta^{(\ell+1)} + c^{(\ell+1)}$ solves (8), (9) for a suitable constant $c^{(\ell+1)}$.

From Proposition 16 the desired energy quantization result at the scale $r_k^{(\ell+1)}$ follows as in the radial case.

If $\rho_k \geq \rho_0 > 0$ we can argue as in [13], p. 416, to obtain numbers $s_k^{(\ell+1)}$ satisfying

$$\lim_{L\to\infty} \lim_{k\to\infty} \Lambda_k(s_k^{(\ell+1)}) = (\ell+1)\Lambda_1, \tag{70}$$

and such that

$$\lim_{L\to\infty}\lim_{k\to\infty}(\Lambda_k(s_k^{(\ell+1)})-\Lambda_k(Lr_k^{(\ell+1)}))=\lim_{k\to\infty}\frac{r_k^{(\ell+1)}}{s_k^{(\ell+1)}}=\lim_{k\to\infty}s_k^{(\ell+1)}=0,$$

while $\bar{u}_k(s_k^{(\ell+1)})\to\infty$ as $k\to\infty$. Moreover, for any $L\geq 1$ we have

$$\lim_{k\to\infty}\frac{\bar{u}_k(s_k^{(\ell+1)})}{\bar{u}_k(Lr_k^{(\ell+1)})}=0. \tag{71}$$

By iteration we then establish (70), (71) up to $\ell+1=\ell_0$ for some maximal index $\ell_0\geq 1$ where Case A occurs and thus complete the concentration analysis near the point $x^{(i)}$, getting

$$\lim_{k\to\infty}\Lambda_k(\rho_0)=\ell_0\Lambda_1.$$

If $\rho_k\to 0$ as $k\to\infty$, we distinguish the following two cases. In **Case 1** for some $\varepsilon_0\in]0,1[$ and all $t\in[r_k^{(\ell+1)},\rho_k]$ there holds $\bar{u}_k(t)\geq\varepsilon_0\bar{u}_k(r_k^{(\ell+1)})$. The decay estimate that we established in Lemma 12 then remains valid throughout this range and (70) holds true for any choice $s_k^{(\ell+1)}=o(\rho_k)$. Again the concentration analysis at scales up to $o(\rho_k)$ is complete. In **Case 2**, for any $\varepsilon\in]0,1[$ there is a minimal $T_k=T_k(\varepsilon)\in[r_k^{(\ell+1)},\rho_k]$ as in Lemma 12 such that $\bar{u}_k(T_k)=\varepsilon\bar{u}_k(r_k^{(\ell+1)})$. Then as before we can define numbers $s_k^{(\ell+1)}<\rho_k$ with $\bar{u}_k(s_k^{(\ell+1)})\to\infty$ as $k\to\infty$ so that (70), (71) also hold true, and we proceed by iteration up to some maximal index $\ell_0\geq 1$ where either Case 1 or Case A holds with final radii $r_k^{(\ell_0)}$, $s_k^{(\ell_0)}$, respectively.

For the concentration analysis at the scale ρ_k first assume that for some number $L\geq 1$ there is a sequence (x_k) such that $\rho_k/L\leq R_k(x_k)\leq|x_k|\leq L\rho_k$ and

$$\lambda_k|x_k|^{2m}u_k^2(x_k)e^{mu_k^2(x_k)}\geq\nu_0>0. \tag{72}$$

By Proposition 11 we may assume that $|x_k|=\rho_k$. Moreover, (55) implies that $\mathrm{dist}(0,\partial\Omega_k)/\rho_k\to\infty$ as $k\to\infty$. As in [13], Lemma 4.6, we then have $\bar{u}_k(\rho_k)/\bar{u}_k(r_k^{(\ell_0)})\to 0$ as $k\to\infty$, ruling out Case 1; that is, at scales up to $o(\rho_k)$ we end with Case A. The desired quantization result at the scale ρ_k then is a consequence of the following result similar to [13], Proposition 4.7, whose proof may be easily carried over to the present situation.

Proposition 17 *Assuming* (72), *there exist a finite set $S_0\subset\mathbb{R}^{2m}$ and a subsequence (u_k) such that*

$$\eta_k(x):=u_k(x_k)(u_k(\rho_kx)-u_k(x_k))\to\eta(x)$$

in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}\setminus S_0)$ as $k\to\infty$, where for a suitable constant c_0 the function $\eta_0=\eta+c_0$ solves (8), (9).

By Proposition 17 in case of (72) there holds

$$\lim_{L\to\infty}\lim_{k\to\infty}\int_{\{x\in\Omega;\,\frac{\rho_k}{L}\leq R_k(x)\leq|x|\leq L\rho_k\}}e_k\,dx=\Lambda_1. \tag{73}$$

Letting

$$X_{k,1} = X_{k,1}^{(i)} = \{x_k^{(j)}; \exists C > 0 : |x_k^{(j)}| \le C\rho_k \text{ for all } k\}$$

and carrying out the above blow-up analysis up to scales of order $o(\rho_k)$ also on all balls of center $x_k^{(j)} \in X_{k,1}$, then from (71) and (73) we have

$$\lim_{L\to\infty} \lim_{k\to\infty} \Lambda_k(L\rho_k) = \Lambda_1(1 + I_1),$$

where I_1 is the total number of bubbles concentrating at the points $x_k^{(j)} \in X_{k,1}^{(i)}$ at scales $o(\rho_k)$.

On the other hand, if (72) fails to hold clearly we have

$$\lim_{L\to\infty} \limsup_{k\to\infty} \int_{\{x\in\Omega;\, \frac{\rho_k}{L} \le R_k(x) \le |x| \le L\rho_k\}} e_k dx = 0, \tag{74}$$

and the energy estimate at the scale ρ_k again is complete.

In order to deal with secondary concentrations around $x_k^{(i)} = 0$ at scales exceeding ρ_k, with $X_{k,1}$ defined as above we let

$$\rho_{k,1} = \rho_{k,1}^{(i)} = \min\big\{ \inf_{\{j;\, x_k^{(j)} \notin X_{k,1}\}} \frac{|x_k^{(j)}|}{2},\ \mathrm{dist}(0, \partial\Omega_k)\big\};$$

that is, we again set $\rho_{k,1} = \mathrm{dist}(0, \partial\Omega_k)$, if $\{j; x_k^{(j)} \notin X_{k,1}\} = \emptyset$. From this definition it follows that $\rho_{k,1}/\rho_k \to \infty$ as $k \to \infty$. Then, using the obvious analogue of Lemma 15, either we have

$$\lim_{L\to\infty} \limsup_{k\to\infty} N_k\Big(L\rho_k, \frac{\rho_{k,1}}{L}\Big) = 0,$$

and we iterate to the next scale; or there exist radii $t_k \le \rho_{k,1}$ such that $t_k/\rho_k \to \infty$, $t_k/\rho_{k,1} \to 0$ as $k \to \infty$ and a subsequence (u_k) such that

$$P_k(t_k) \ge \nu_0 > 0 \text{ for all } k. \tag{75}$$

The argument then depends on whether (72) or (74) holds. In case of (72), as in [13], Lemma 4.6, the bound (75) and Proposition 7 imply that $\bar{u}_k(t_k)/\bar{u}_k(\rho_k) \to 0$ as $k \to 0$. Then we can argue as in Case A for $r \in [L\rho_k, \rho_{k,1}]$ for sufficiently large L, and we can continue as before to resolve concentrations in this range of scales.

In case of (74) we further need to distinguish whether Case A or Case 1 holds at the final stage of our analysis at scales $o(\rho_k)$. In fact, for the following estimates we also consider all points $x_k^{(j)} \in X_{k,1}^{(i)}$ in place of $x_k^{(i)}$. Recalling that in Case A we have (71) (with index ℓ_0 instead of $\ell+1$) and (67), on account of (74) for a suitable sequence of numbers $s_{k,1}^{(0)}$ such that $s_{k,1}^{(0)}/\rho_k \to \infty$, $t_k/s_{k,1}^{(0)} \to \infty$ as $k \to \infty$ we find

$$\lim_{L\to\infty} \lim_{k\to\infty} \Big(\Lambda(s_{k,1}^{(0)}) - \sum_{x_k^{(j)} \in X_{k,1}^{(i)}} \Lambda_k^{(j)}(L r_k^{(\ell_0^{(j)})})\Big) = 0,$$

where $\Lambda_k^{(j)}(r)$ and $r_k^{(\ell_0^{(j)})}$ are computed as above with respect to the concentration point $x_k^{(j)}$. In particular, with such a choice of $s_{k,1}^{(0)}$ we find the intermediate quantization result

$$\lim_{k\to\infty} \Lambda_k(s_{k,1}^{(0)}) = \Lambda_1 I_1$$

analogous to (70), where I_1 is defined as above. In Case 1 we can obtain the same conclusion by our earlier reasoning. Moreover, in Case 1 we can argue as in [13], Lemma 4.8, to conclude that $\bar{u}_k(t_k)/\bar{u}_k(Lr_k^{(\ell_0^{(j)})}) \to 0$ for any $L \geq 1$ as $k \to 0$; therefore, similar to (71) in Case A, we can achieve that for any $L \geq 1$ we have

$$\lim_{k\to\infty} \frac{\bar{u}_k(s_{k,1}^{(0)})}{\bar{u}_k(Lr_k^{(\ell_0^{(j)})})} = \lim_{k\to\infty} \frac{r_k^{(\ell_0^{(j)})}}{s_{k,1}^{(0)}} = \lim_{k\to\infty} \frac{\rho_k}{s_{k,1}^{(0)}} = \lim_{k\to\infty} \frac{s_{k,1}^{(0)}}{t_k} = 0$$

for all $x_k^{(j)} \in X_{k,1}^{(i)}$ where Case 1 holds, similar to (H_ℓ).

We then finish the argument by iteration. For $\ell \geq 2$ we inductively define the sets

$$X_{k,\ell} = X_{k,\ell}^{(i)} = \{x_k^{(j)}; \exists C > 0 : |x_k^{(j)}| \leq C\rho_{k,\ell-1} \text{ for all } k\}$$

and we let

$$\rho_{k,\ell} = \rho_{k,\ell}^{(i)} = \min\big\{ \inf_{\{j; x_k^{(j)} \notin X_{k,1}\}} \frac{|x_k^{(j)}|}{2},\ \mathrm{dist}(0, \partial\Omega_k)\big\};$$

that is, as before, we set $\rho_{k,\ell} = \mathrm{dist}(0, \partial\Omega_k)$, if $\{j; x_k^{(j)} \notin X_{k,\ell}^{(i)}\} = \emptyset$. Iteratively performing the above analysis at all scales $\rho_{k,\ell}$, thereby exhausting all concentration points $x_k^{(j)}$, upon passing to further subsequences, we finish the proof of Theorem 2.

3.1 Proof of Proposition 11

Our proof of Proposition 11 is modelled on the proof of [5], Proposition 2. In fact, the first steps of the proof seem almost identical to the corresponding arguments in [5]. The special character of the present problem only enters at the last stage, where we also need to distinguish the cases $\ell = 1$ and $2 \leq \ell \leq 2m-1$.

Fix any index $1 \leq \ell \leq 2m - 1$. The following constructions will depend on this choice; however, for ease of notation we suppress the index ℓ in the sequel.

Set $R_k(x) := \inf_{1\leq j\leq I} |x - x_k^{(j)}|$ and choose points y_k such that

$$R_k^\ell(y_k)u_k(y_k)|\nabla^\ell u_k(y_k)| = \sup_\Omega R_k^\ell u_k|\nabla^\ell u_k| =: L_k.$$

Suppose by contradiction that $L_k \to \infty$ as $k \to \infty$. From Theorem 1 then it follows that $s_k := R_k(y_k) \to 0$ as $k \to \infty$. Set

$$\Omega_k := \{y; y_k + s_k y \in \Omega\}$$

and let

$$v_k(y) := u_k(y_k + s_k y), \quad y \in \Omega_k.$$

Observe that for $1 \leq j \leq m$ via Sobolev's embedding from (3) we obtain

$$\|\nabla^j v_k\|^2_{L^{\frac{2m}{j}}(\Omega_k)} \leq C\|\nabla^m v_k\|^2_{L^2(\Omega_k)} = C\int_{\Omega_k} v_k(-\Delta)^m v_k dx \leq C. \tag{76}$$

Also let

$$y_k^{(i)} := \frac{x_k^{(i)} - y_k}{s_k},\ 1 \le i \le I$$

and set

$$S_k := \{y_k^{(i)}; 1 \le i \le I\}.$$

Clearly then we have

$$\mathrm{dist}(0, S_k) = \inf_{1 \le i \le I} |y_k^{(i)}| = 1$$

and

$$\sup_{y \in \Omega_k} \left(\mathrm{dist}(y, S_k)^\ell v_k(y) |\nabla^\ell v_k(y)|\right) = v_k(0)|\nabla^\ell v_k(0)| = L_k \to \infty \tag{77}$$

as $k \to \infty$. Moreover (7) implies

$$0 \le v_k(-\Delta)^m v_k = \lambda_k s_k^{2m} v_k^2 e^{m v_k^2} \le \frac{C}{\mathrm{dist}(y, S_k)^{2m}}. \tag{78}$$

Since $\lim_{k\to\infty} s_k = 0$, we may assume that as $k \to \infty$ the domains Ω_k exhaust a half-space

$$\Omega_0 = \mathbb{R}^{2m-1} \times] - \infty, R_0[,$$

where $0 < R_0 \le \infty$. We may also assume that either $\lim_{k\to\infty} |y_k^{(i)}| = \infty$ or $\lim_{k\to\infty} y_k^{(i)} = y^{(i)}$, $1 \le i \le I$, and we let S_0 be the set of these accumulation points of S_k, satisfying $\mathrm{dist}(0, S_0) = 1$. For $R > 0$ denote

$$K_{k,R} := \Omega_k \cap B_R(0) \setminus \bigcup_{y \in S_0} \overline{B_{1/R}(y)}.$$

Observing that $\lambda_k s_k^{2m} \to 0$, from (78) we obtain that

$$\lim_{k\to\infty} \|\Delta^m v_k\|_{L^\infty(K_{k,R})} = 0 \quad \text{for every } R > 0. \tag{79}$$

Lemma 18 *We have $R_0 = \infty$, hence $\Omega_0 = \mathbb{R}^{2m}$.*

Proof. Suppose by contradiction that $R_0 < \infty$. Choosing $R = 2R_0$ and observing that by (2) for $0 \le j < \ell \le 2m - 1$ we have $\partial_\nu^j v_k^2 = 0$ on $\partial\Omega_k$, from Taylor's formula and (77) we conclude

$$\sup_{K_{k,R}} \frac{v_k^2}{v_k(0)|\nabla^\ell v_k(0)|} \le C = C(R).$$

Letting $w_k := \frac{v_k}{\sqrt{v_k(0)|\nabla v_k(0)|}}$, we then have $0 \le w_k \le C$ on $K_{k,R}$. Using (76), Sobolev's embedding, (77) and (79) we infer

$$\|\nabla w_k\|_{L^{2m}(\Omega_k)} + \|\nabla^2 w_k\|_{L^m(\Omega_k)} + \|\Delta^m w_k\|_{L^\infty(K_{k,R})} \to 0 \ \text{ as } k \to \infty.$$

Since $\partial_\nu^j w_k = 0$ on $\partial\Omega_k$ for $0 \le j \le m-1$, it follows from elliptic regularity that $w_k \to 0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(K_{k,R})$ for $0 < \alpha < 1$, contradicting the fact that $w_k(0)|\nabla^\ell w_k(0)| = 1$. $\square$

Lemma 19 *As $k \to \infty$ we have $v_k(0) \to \infty$ and*

$$\frac{v_k}{v_k(0)} \to 1 \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash S_0).$$

Proof. First observe that

$$c_k := \sup_{B_{1/2}} v_k \to \infty \quad \text{as } k \to \infty.$$

Indeed, otherwise (76), (79) and elliptic regularity would contradict (77). Letting $w_k := \frac{v_k}{c_k}$, from (76) and (79) for any $R > 0$ we have

$$\|\nabla w_k\|_{L^{2m}(\Omega_k)} + \|\nabla^2 w_k\|_{L^m(\Omega_k)} + \|\Delta^m w_k\|_{L^\infty(K_{k,R})} \to 0 \quad \text{as } k \to \infty,$$

whence $w_k \to w \equiv const$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash S_0)$. Recalling that $\mathrm{dist}(0, S_0) = 1$, we obtain

$$w \equiv \sup_{B_{1/2}} w = \lim_{k\to\infty} \sup_{B_{1/2}} w_k = 1.$$

In particular we conclude that $\frac{v_k(0)}{c_k} = w_k(0) \to 1$ as $k \to \infty$ and therefore $v_k(0) = c_k w_k(0) \to \infty$, $\frac{v_k}{v_k(0)} = \frac{w_k}{w_k(0)} \to 1$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash S_0)$, as claimed. $\square$

For the final argument now we need to distinguish the cases $\ell = 1$ and $2 \le \ell \le 2m-1$. Consider first the case $\ell = 1$. Set

$$\tilde{v}_k(y) := \frac{v_k(y) - v_k(0)}{|\nabla v_k(0)|}.$$

From (77) and Lemma 19 we infer

$$|\nabla \tilde{v}_k(y)| = \frac{v_k(0)}{v_k(y)} \frac{v_k(y)|\nabla v_k(y)|}{v_k(0)|\nabla v_k(0)|} \le \frac{1 + o(1)}{\mathrm{dist}(y, S_0)}, \tag{80}$$

with error $o(1) \to 0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash S_0)$ as $k \to \infty$. Since $\tilde{v}_k(0) = 0$, from (80) we conclude that $\tilde{v}_k$ is bounded in $C^1(K_{k,R})$ for every $R > 0$, uniformly in k. Moreover, (78) and Lemma 19 give

$$|\Delta^m \tilde{v}_k| = \frac{v_k(0)}{v_k} \frac{v_k|\Delta^m v_k|}{v_k(0)|\nabla v_k(0)|} \le C(R) \frac{v_k(0)}{L_k v_k} \to 0 \tag{81}$$

uniformly on $K_{k,R}$ as $k \to \infty$, for any $R > 0$. The sequence $\tilde{v}_k$ then is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash S_0)$ for any $\alpha < 1$, and by Arzelà-Ascoli's theorem we can assume that $\tilde{v}_k \to \tilde{v}$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash S_0)$, where $\tilde{v}$ satisfies

$$\Delta^m \tilde{v} = 0, \quad \tilde{v}(0) = 0, \quad |\nabla \tilde{v}(0)| = 1, \quad |\nabla \tilde{v}(y)| \le \frac{1}{\mathrm{dist}(y, S_0)}. \tag{82}$$

Fix a point $x_0 \in S_0$. For any $r \in]0, \mathrm{dist}(x_0, S_0 \backslash \{x_0\})/2[$ let $\varphi \in C_0^\infty(B_r(x_0))$ be a function $0 \le \varphi \le 1$ such that $\varphi \equiv 1$ in $B_{r/2}(x_0)$, and satisfying $|\nabla^j \varphi| \le Cr^{-j}$ for $0 \le j \le m$. Integration by parts yields

$$\begin{aligned}\int_{B_r(x_0)} &(\nabla\varphi v_k \cdot \nabla\Delta^{m-1} v_k + \varphi v_k \Delta^m v_k) dx \\ &= -\int_{B_r(x_0)} \varphi \nabla v_k \cdot \nabla \Delta^{m-1} v_k \, dx =: I.\end{aligned} \tag{83}$$

Again integrating by parts $m-1$ times, we obtain

$$I = (-1)^m \int_{B_r(x_0)} \sum_{|\alpha|=m-1} \partial^\alpha(\varphi \nabla v_k) \cdot \nabla \partial^\alpha v_k \, dx,$$

so that by Hölder's inequality and (76) this term may be bounded

$$\begin{aligned} |I| &\le C \sum_{1 \le j \le m} r^{j-m} \int_{B_r(x_0)} |\nabla^j v_k| |\nabla^m v_k| dx \\ &\le C \sum_{1 \le j \le m} \|\nabla^j v_k\|_{L^{\frac{2m}{j}}} \|\nabla^m v_k\|_{L^2} \le C. \end{aligned}$$

Similarly, we have

$$0 \le \int_{B_r(x_0)} \varphi v_k (-\Delta)^m v_k \, dx \le C,$$

and from (83) we conclude the bound

$$\Big| \int_{B_r(x_0)} \nabla \varphi v_k \cdot \nabla \Delta^{m-1} v_k dx \Big| \le C. \tag{84}$$

Observe that $\nabla \varphi = 0$ in $B_{r/2}(x_0)$. By Lemma 19 therefore the integral on the left-hand side equals

$$\begin{aligned} &\int_{B_r(x_0)} \nabla \varphi v_k \cdot \nabla \Delta^{m-1} v_k dx \\ &\qquad = (1 + o(1)) v_k(0) |\nabla v_k(0)| \int_{B_r(x_0)} \nabla \varphi \cdot \nabla \Delta^{m-1} \tilde{v}_k dx \\ &\qquad = -(1 + o(1)) v_k(0) |\nabla v_k(0)| \int_{B_r(x_0)} \varphi \Delta^m \tilde{v}_k dx. \end{aligned}$$

Since $(-\Delta)^m \tilde{v}_k \ge 0$, it follows that

$$\int_{B_{r/2}(x_0)} (-\Delta)^m \tilde{v}_k dx \le \frac{C}{v_k(0)|\nabla v_k(0)|} = C L_k^{-1} \to 0 \text{ as } k \to \infty.$$

Recalling (81), we infer that $\Delta^m \tilde{v}_k \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$. Therefore $\Delta^m \tilde{v} \equiv 0$ in $\mathbb{R}^{2m}$. Since from (82) we have $|\tilde{v}(y)| \le C(1 + |y|)$ for $y \in \mathbb{R}^{2m}$, we may now invoke a Liouville-type theorem as in [7], Theorem 5, to see that $\tilde{v}$ is a polynomial of degree at most $2m-2$ if $m > 1$ and of degree at most 1 if $m = 1$. But then (82) implies that $\tilde{v} \equiv 0$, contradicting the fact that $|\nabla \tilde{v}_k(0)| = 1$. This completes the proof in the case $\ell = 1$.

In the case when $2 \le \ell \le m-1$ we set

$$\tilde{v}_k(y) := \frac{v_k(y) - v_k(0)}{|\nabla^\ell v_k(0)|}.$$

As shown above we have

$$\mathrm{dist}(y, S_k) v_k(y) |\nabla v_k(y)| \le C \sup_{x \in \Omega} R_k(x) u_k(x) |\nabla u_k(x)| \le C;$$

hence Lemma 19 implies with error $o(1) \to 0$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}\backslash S_0)$ as $k \to \infty$ that

$$|\nabla \tilde{v}_k| \leq \frac{C(1+o(1))}{v_k(0)|\nabla^\ell v_k(0)|\,\mathrm{dist}(y,S_0)} = \frac{C(1+o(1))}{L_k\,\mathrm{dist}(y,S_0)} \to 0. \tag{85}$$

Notice that this is stronger than its analogue (80). As in the case $\ell = 1$ we have

$$\Delta^m \tilde{v}_k = \frac{v_k(0)}{v_k}\frac{v_k \Delta^m v_k}{v_k(0)|\nabla^\ell v_k(0)|} \leq \frac{C(R)}{L_k} \to 0 \tag{86}$$

uniformly on $K_{k,R}$ as $k \to \infty$, for any $R > 0$, hence $\tilde{v}_k \to \tilde{v}$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}\backslash S_0)$, where $\tilde{v}$ satisfies

$$\Delta^m \tilde{v} = 0, \quad \tilde{v}(0) = 0, \quad |\nabla^\ell \tilde{v}(0)| = 1.$$

On the other hand (85) implies $\nabla \tilde{v} \equiv 0$, contradiction. This completes the proof. $\square$

Appendix

We collect here some technical results used in the above sections. The proof of the following proposition can be found in [7], Prop. 4.

Proposition 20 *Let $\Delta^m h = 0$ in $B_2 \subset \mathbb{R}^n$. For every $0 \leq \alpha < 1$ and $\ell \geq 0$ there is a constant $C(\ell,\alpha)$ independent of h such that*

$$\|h\|_{C^{\ell,\alpha}(B_1)} \leq C(\ell,\alpha)\|h\|_{L^1(B_2)}.$$

By a simple covering argument Proposition 20 can be extended to the case of annuli.

Proposition 21 *Let $\Delta^m h = 0$ in $B_{2L}(0)\backslash B_{1/2L}(0) \subset \mathbb{R}^n$ for some $L \geq 1$. For every $0 \leq \alpha < 1$ and $\ell \geq 0$ there is a constant $C = C(\ell,\alpha,L)$ such that*

$$\|h\|_{C^{\ell,\alpha}(B_L(0)\backslash B_{1/L}(0))} \leq C\|h\|_{L^1(B_{2L}(0)\backslash B_{1/2L}(0))}.$$

Lemma 22 *Let $g \in C^\infty(\overline{B}_t)$, where $B_t = B_t(0) \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$, $t > 0$. Assume that g is radially symmetric and satisfies*

$$g = \partial_\nu g = \ldots = \partial_\nu^{m-1} g = 0 \quad \text{on } \partial B_t. \tag{87}$$

Then

$$\int_{\partial B_t} t^{m-1}\partial_\nu^m g\, d\sigma = \int_0^t t_2 \cdots \int_0^{t_{m-1}} t_m \left(\int_{B_{t_m}} \Delta^m g dx\right) dt_m \ldots dt_2. \tag{88}$$

Proof. For $m = 1$ equation (88) simply reduces to

$$\int_{\partial B_t} \partial_\nu g\, d\sigma = \int_{B_t} \Delta g\, dx. \tag{89}$$

For $m = 2$ consider the function $\varphi(x) = x \cdot \nabla g(x)$ with

$$\int_{\partial B_t} \partial_\nu \varphi\, d\sigma = \int_{B_t} \Delta\varphi\, dx = \int_{B_t} \Delta(x\cdot\nabla g)dx = \int_{B_t}(x\cdot\nabla\Delta g + 2\Delta g)dx$$

and note that the condition $\partial_\nu g = 0$ on ∂B_t and (89) imply

$$\int_{\partial B_t} \partial_\nu \varphi \, d\sigma = \int_{\partial B_t} \partial_\nu (x \cdot \nabla g) d\sigma = \int_{\partial B_t} t \partial_\nu^2 g \, d\sigma, \quad \text{and} \quad \int_{B_t} \Delta g \, dx = 0.$$

Thus from Fubini's theorem we obtain the desired identity

$$\begin{aligned} \int_{\partial B_t} t \partial_\nu^2 g \, d\sigma &= \int_{B_t} x \cdot \nabla \Delta g \, dx \\ &= \int_0^t t_2 \Big(\int_{\partial B_{t_2}} \partial_\nu \Delta g \, d\sigma \Big) dt_2 = \int_0^t t_2 \Big(\int_{B_{t_2}} \Delta^2 g \, dx \Big) dt_2. \end{aligned}$$

We now proceed by induction. Assume that the lemma is true for $m-1$. Choosing $\varphi(x) = x \cdot \nabla g(x)$ with

$$\varphi = \partial_\nu \varphi = \ldots = \partial_\nu^{m-2} \varphi = 0 \quad \text{on } \partial B_t$$

we get

$$\begin{aligned} &\int_{\partial B_t} t^{m-1} \partial_\nu^m g d\sigma = \int_{\partial B_t} t^{m-2} \partial_\nu^{m-1} (t \partial_\nu g) d\sigma = \int_{\partial B_t} t^{m-2} \partial_\nu^{m-1} (x \cdot \nabla g) d\sigma \\ &= \int_0^t t_2 \cdots \int_0^{t_{m-2}} t_{m-1} \int_{B_{t_{m-1}}} \Delta^{m-1} (x \cdot \nabla g) dx dt_{m-1} \ldots dt_2 =: I. \end{aligned}$$

Observe that $\Delta^{m-1}(x \cdot \nabla g) = x \cdot \nabla \Delta^{m-1} g + 2(m-1)\Delta^{m-1} g$, hence

$$\begin{aligned} I &= \int_0^t t_2 \cdots \int_0^{t_{m-2}} t_{m-1} \int_{B_{t_{m-1}}} (x \cdot \nabla \Delta^{m-1} g) dx dt_{m-1} \ldots dt_2 \\ &\quad + 2(m-1) \int_0^t t_2 \cdots \int_0^{t_{m-2}} t_{m-1} \int_{B_{t_{m-1}}} \Delta^{m-1} g dx dt_{m-1} \ldots dt_2 \\ &= II + III. \end{aligned}$$

By inductive hypothesis and (87) the contribution from the second term is

$$III = 2(m-1) \int_{\partial B_t} t^{m-2} \partial_\nu^{m-1} g d\sigma = 0,$$

and our claim follows from writing

$$\begin{aligned} \int_{B_{t_{m-1}}} x \cdot \nabla \Delta^{m-1} g dx &= \int_0^{t_{m-1}} t_m \int_{\partial B_{t_m}} \partial_\nu \Delta^{m-1} g d\sigma dt_m \\ &= \int_0^{t_{m-1}} t_m \int_{B_{t_m}} \Delta^m g dx dt_m. \end{aligned} \tag{90}$$

□

Lemma 23 *Let $u \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, for some $p \geq 1$, satisfy $\Delta^j u = 0$ for some integer $j > 0$. Then $u \equiv 0$.*

Proof. We first claim that

$$\lim_{R\to\infty} \fint_{B_R(\xi)} u dx = 0$$

for every $\xi \in \mathbb{R}^n$. Indeed by Jensen's inequality

$$\left| \fint_{B_R(\xi)} u dx \right| \le \fint_{B_R(\xi)} |u| dx \le \left(\fint_{B_R(\xi)} |u|^p dx \right)^{\frac{1}{p}} \le \frac{1}{R^{n/p}} \|u\|_{L^p(\mathbb{R}^n)} \to 0,$$

as $R \to \infty$. By Pizzetti's formula (see [10]) we have constants $c_1, \dots, c_{j-1}$ such that

$$\fint_{B_R(\xi)} u dx = u(\xi) + c_1 R^2 \Delta u(\xi) + \dots + c_{j-1} R^{2j-2} \Delta^{j-1} u(\xi) =: P(R).$$

Taking the limit as $R \to \infty$ we see at once that the polynomial $P(R)$ is identically 0, and in particular $u(\xi) = P(0) = 0$. Since ξ was arbitrary the proof is complete. $\square$

Lemma 24 *There holds*

$$\mathrm{cap}_{H^m}(\{0\}) = \inf\{\|\nabla^m \varphi\|_{L^2};\ \varphi \in X\} = 0,$$

where

$$X = \{\varphi \in C_0^\infty(B_1(0));\ 0 \le \varphi \le 1, \exists r > 0 : \varphi(x) = 1 \text{ for } |x| \le r\}.$$

Proof. Let $f(x) = \log\log\log(1/|x|)$ with $\nabla^m f \in L^2(B_{e^{-e}}(0))$ and fix $g \in C^\infty(\mathbb{R})$ with $0 \le g \le 1$ satisfying $g(s) = 0$ for $s \le 0$, $g(s) = 1$ for $s \ge 1$. Letting

$$\varphi_k(x) = g(f(x) - k),\ k \in \mathbb{N},$$

we find $\varphi_k \in X$ for all k and $\|\nabla^m \varphi_k\|_{L^2} \to 0$ as $k \to \infty$. $\square$

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