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CONCENTRATION-COMPACTNESS PHENOMENA IN CONFORMAL GEOMETRY

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Abstract

Consider a smooth Riemannian manifold (M, g) of arbitrary even dimension $2m$, and a sequence of conformal metrics $g_k = e^{2u_k}g$ on M, $u_k \in C^\infty(M)$. In this work we study the concentration-compactness behaviour of this sequence of metrics, under the assumption that their volumes are equibounded and their Q-curvatures $Q^{2m}_{g_k}$ converge uniformly or even in C^0 to a given continuous function Q_0.

We start by taking (M, g) to be $\mathbb{R}^{2m}$ with the Euclidean metric. Then, in analogy with a 4-dimensional result of Adimurthy, F. Robert and M. Struwe, we show that, in case of non-compactness and up to subsequences, the metrics vanish in the limit uniformly locally outside a rectifiable set of dimension at most $2m-1$.

We have a much stronger result, if (M, g) is a *closed* Riemannian manifold, satisfying a generic (hence not restrictive) condition which will be discussed. In this case, we either have compactness, or in the limit (up to subsequences) the metrics vanish outside a *finite* concentration set S. Moreover Q_0 is positive on S, and the measures $Q^{2m}_{g_k}\mathrm{dvol}_{g_k}$ converge weakly to $\sum_{x\in S}\Lambda_1\delta_x$, where $\Lambda_1 = (2m-1)!\mathrm{vol}(S^{2m})$ is the total Q-curvature of the sphere. In particular the total Q-curvature of the metrics g_k (it does not depend on k) is an integer multiple of the total Q-curvature of the sphere. Our approach generalizes the 4-dimensional argument of O. Druet and F. Robert to arbitrary dimension, also allowing for the Q-curvatures $Q^{2m}_{g_k}$ to have varying sign. Quantization results for similar equations, have also been obtained by A. Malchiodi and C. B. Ndiaye using other techniques.

In the case of the round sphere, $(M, g) = (S^{2m}, g_{S^{2m}})$, the concentration result is particularly explicit. We either have compactness, or we have concentration at a single point, and the pull-back metrics $\Phi_k^* g_k$ converge up to a subsequence towards the round metric $g_{S^{2m}}$, if the Φ_k's are suitably chosen Möbius diffeomorphisms. This generalizes to arbitrary dimension previous results in dimension 2 by M. Struwe and in dimension 4 by A. Malchiodi and M. Struwe. We also allow the $Q^{2m}_{g_k}$'s to have varying sign, and show that concentration only occurs at places of positive Q-curvature.

These concentration-compactness results rely heavily on a blow-up technique and on the classification and study of the asymptotic behaviour at infinity of the conformal metrics on $\mathbb{R}^{2m}$ of constant Q-curvature $Q \in \mathbb{R}$ and finite volume. When $Q > 0$, we do this in arbitrary dimension, building upon several previous partial results. For $Q \leq 0$, we first show the existence of such metrics if $m > 1$,

which was previously unknown, and then develop an analysis analogous to the one done for the positive case.

Quite remarkably, the above geometrical results, can be used to give an elegant proof of a concentration-compactness result for the equation

$$(-\Delta)^m u_k = \lambda_k u_k e^{m u_k^2},$$

which arises from the Adams-Moser-Trudinger inequality:

$$\sup_{u \in H_0^m(\Omega),\, \|\nabla^m u\|^2_{L^2(\Omega)} \leq \Lambda_1} \fint_\Omega e^{mu^2} dx = c_0(m) < +\infty.$$

This generalizes previous works of Adimurthy, O. Druet, F. Robert and M. Struwe. The proof we give, shows a clean relation between the geometric problem of prescribing the Q-curvature and an apparently unrelated imbedding problem in functional analysis. Here we use some sharp Lorentz-space estimates, which allow a more transparent approach.

Zusammenfassung

Sei (M, g) eine glatte Riemannsche Mannigfaltigkeit beliebiger gerader Dimension $2m$, und sei $g_k = e^{2u_k} g$, mit $u_k \in C^\infty(M)$, eine Folge konformer Metriken auf M. In dieser Arbeit studieren wir das Verhalten dieser Folge von Metriken im Hinblick auf Konzentrations-Kompaktheit unter der Annahme, daß ihre Volumina gleichmässig beschränkt sind und ihre Q-Krümmungen $Q^{2m}_{g_k}$ gleichmäßig, oder sogar in C^0, gegen eine gegebene stetige Funktion Q_0 konvergieren.

Zu Beginn wählen wir für (M, g) den euklidischen $\mathbb{R}^{2m}$. In Analogie zu einem Resultat von Adimurthy, F. Robert und M. Struwe in Dimension 4 zeigen wir im nicht-kompakten Fall zunächst, daß eine Teilfolge der Folge der Metriken g_k im Limes lokal gleichmäßig außerhalb einer rektifizierbaren Menge der Dimension höchstens $2m - 1$ verschwindet.

Eine viel stärkere Aussage erhalten wir, wenn (M, g) eine *geschlossene* Riemannsche Mannigfaltigkeit ist, welche eine generische (also nicht restriktive) Bedingung erfüllt, die wir später erklären. In diesem Fall liegt entweder Kompaktheit vor, oder eine Teilfolge der Metriken verschwindet im Limes außerhalb einer endlichen Konzentrationsmenge S. Desweiteren ist Q_0 positiv auf S, und die Maße $Q^{2m}_{g_k} \mathrm{dvol}_{g_k}$ konvergieren schwach gegen $\sum_{x \in S} \Lambda_1 \delta_x$, wobei wir mit $\Lambda_1 = (2m - 1)!\mathrm{vol}(S^{2m})$ die totale Q-Krümmung der Sphäre beschreiben. Insbesondere ist die totale Q-Krümmung der Metriken g_k (welche nicht von k abhängt) ein ganzzahliges Vielfaches der totalen Q-Krümmung der Sphäre. Unser Zugang verallgemeinert ein Argument, welches O. Druet and F. Robert für den vierdimensionalen Fall gegeben haben, auf beliebige Dimensionen, wobei das Vorzeichen der Q-Krümmungen $Q^{2m}_{g_k}$ nun springen darf. Mit anderen Methoden A. Malchiodi und C. B. Ndiaye haben Quantisierungsresultate für ähnliche Gleichungen erzielt.

Im Falle der runden Sphäre, $(M, g) = (S^{2m}, g_{S^{2m}})$, können wir unser Resultat über das Konzentrationsverhalten der Folge g_k noch verfeinern: Entweder liegt Kompaktheit vor, oder es kommt zur Konzentration an einem einzigen Punkt und eine Teilfolge der zurückgezogenen Metriken $\Phi_k^* g_k$ konvergiert gegen die runde Metrik $g_{S^{2m}}$, unter der Voraussetzung, daß die Φ_k geeignet gewählte Möbiusdiffeomorphismen sind. Dies verallgemeinert bekannte Resultate von M. Struwe für Dimension 2 und von A. Malchiodi und M. Struwe für Dimension 4 auf den Fall beliebiger Dimension. Darüberhinaus lassen wir zu, daß das Vorzeichen von $Q^{2m}_{g_k}$ springt, und zeigen, daß die Konzentration nur an Stellen positiver Q-Krümmung auftritt.

Unsere Resultate zur Konzentrations-Kompaktheit fußen zum einen auf einer blow-up-Technik, und zum anderen auf der Klassifikation und dem Studium des

asymptotischen Verhaltens der konformen Metriken auf $\mathbb{R}^{2m}$ von konstanter Q-Krümmung $Q \in \mathbb{R}$, und endlichem Volumen. Im Falle $Q > 0$ zeigen wir dies unter Benutzung verschiedener früherer Teilresultate in beliebiger Dimension. Im Falle $Q \leq 0$ zeigen wir zunächst die Existenz solcher Metriken für $m > 1$ — ein bis dahin unbekanntes Resultat — und führen sodann analoge Untersuchungen wie im Falle $Q > 0$ durch.

Eine bemerkenswerte Tatsache ist, daß obige geometrische Ergebnisse einen eleganten Beweis eines Konzentrations-Kompaktheits-Resultates für die Gleichung

$$(-\Delta)^m u_k = \lambda_k u_k e^{m u_k^2}$$

liefern, welche mit der Adams-Moser-Trudinger Ungleichung

$$\sup_{u \in H_0^m(\Omega),\, \|\nabla^m u\|_{L^2(\Omega)}^2 \leq \Lambda_1} \fint_\Omega e^{mu^2} dx = c_0(m) < +\infty$$

verknüpft ist. Dies verallgemeinert frühere Arbeiten von Adimurthy, O. Druet, F. Robert und M. Struwe. Unser Beweis deckt einen Zusammenhang zwischen dem geometrischen Problem, die Q-Krümmung vorzuschreiben, und einem offensichtlich damit nicht verwandten funktionalanalytischen Einbettungsproblem auf. Wir benützen hierbei einige scharfe Abschätzungen für Lorentzräume, welche eine besonders klare Argumentation ermöglichen.

Riassunto

Consideriamo una varietà riemanniana liscia (M, g) di dimensione pari $2m$ ed una successione di metriche conformi $g_k = e^{2u_k} g$ su M, $u_k \in C^\infty(M)$. In questo lavoro studiamo i fenomeni di concentrazione-compattezza di questa successione di metriche, nell'ipotesi che i loro volumi siano equilimitati e che le loro Q-curvature $Q^{2m}_{g_k}$ convergano uniformemente o addirittura in C^0 verso una data funzione Q_0.

Iniziamo con il prendere (M, g) uguale a $\mathbb{R}^{2m}$ con metrica euclidea. Quindi, in analogia con un risultato di Adimurthy, F. Robert e M. Struwe in dimensione 4, mostriamo che in caso di non compattezza e a meno di una sottosuccessione, le metriche vanno a zero nel limite per $k \to \infty$ localmente uniformemente al di fuori di un insieme rettificabile di dimensione al più $2m - 1$.

Abbiamo un risultato molto più forte se (M, g) è una varietà riemanniana *chiusa*, che soddisfa una certa condizione generica (quindi ben poco restrittiva) che discuteremo. In questo caso, o abbiamo compattezza, oppure nel limite (a meno di sottosuccessioni) le metriche vanno a zero al di fuori di un insieme di concetrazione *finito* S. Inoltre Q_0 è positivo su S, e le misure $Q^{2m}_{g_k} \mathrm{dvol}_{g_k}$ convergono debolmente verso $\sum_{x \in S} \Lambda_1 \delta_x$, laddove $\Lambda_1 = (2m-1)!\mathrm{vol}(S^{2m})$ è la Q-curvatura totale della sfera. In particolare, la Q-curvatura totale delle metriche g_k (non dipende da k) è un multiplo intero della Q-curvatura totale della sfera. Il nostro approccio generalizza a dimensione arbitraria un metodo che O. Druet e F. Robert hanno sviluppato in dimensione 4. Inoltre permettiamo alle Q-curvature $Q^{2m}_{g_k}$ di cambiare segno. Risultati di quantizzazione per equazioni simili sono anche stati ottenuti da A. Malchiodi e C. B. Ndiaye usando tecniche diverse.

Nel caso della sfera standard, $(M, g) = (S^{2m}, g_{S^{2m}})$, il risultato di concentrazione-compattezza risulta particolarmente esplicito. O abbiamo compattezza, o abbiamo concentrazione in un singolo punto, e i pull-back $\Phi_k^* g_k$ delle metriche g_k convergono, a meno di una sottosuccessione, verso la metrica standard $g_{S^{2m}}$, se le Φ_k sono diffeomorfismi di Möbius opportunamente scelti. Questo generalizza a dimensione arbitraria precedenti risultati di M. Struwe in dimensione 2 e di A. Malchiodi e M. Struwe in dimensione 4. Inoltre permettiamo alle $Q^{2m}_{g_k}$ di cambiare segno, e mostriamo che la concentrazione può avvenire solo nei punti di Q-curvatura positiva.

Questi risultati di concentrazione-compattezza dipendono fortemente da una certa tecnica di blow-up e dalla classificazione e studio asintotico ad infinito delle metriche conformi su $\mathbb{R}^{2m}$ di Q-curvatura costante $Q \in \mathbb{R}$ e volume finito. Per $Q > 0$, facciamo ciò in dimensione arbitraria, migliorando vari lavori precedenti.

Per $Q \leq 0$, dapprima mostriamo l'esistenza di tali metriche quando $m > 1$, cosa precedentemente non nota, poi sviluppiamo un'analisi analoga a quella prodotta nel caso positivo.

È molto interessante notare che i risultati geometrici sopra descritti possono essere usati per dare un'elegante dimostrazione di un risultato di concentrazione-compattezza per l'equazione

$$(-\Delta)^m u_k = \lambda_k u_k e^{m u_k^2},$$

che nasce nell'ambito della disuguaglianza di Adams-Moser-Trudinger:

$$\sup_{u \in H_0^m(\Omega),\, \|\nabla^m u\|_{L^2(\Omega)}^2 \leq \Lambda_1} \fint_\Omega e^{mu^2} dx = c_0(m) < +\infty.$$

Così facendo, generalizziamo precedenti lavori di Adimurthy, O. Druet, F. Robert e M. Struwe. La dimostrazione che diamo mostra una relazione molto chiara tra il problema geometrico della Q-curvatura prescritta e un problema di immersione in analisi funzionale che, in apparenza, è completamente scollegato. Qui facciamo uso di alcune stime in spazi di Lorentz, che permettono un approccio più trasparente.

Contents

Chapter 1

Introduction

In this short introduction, we attempt to clarify how the various theorems proved in this work happen to be tightly intertwined, though at a first sight it might not seem so. In Section 1.1 we recall some facts about the Paneitz operators and the Q-curvatures, as these objects will be largely used in what follows. In Section 1.2 we provide the reader with a map to navigate through the following chapters.

1.1 The Paneitz operators and the Q-curvatures

The study of the Paneitz operators moved into the center of conformal geometry in the last decades, in part with regard to the problem of prescribing the Q-curvature. Given a 4-dimensional Riemannian manifold (M, g), the Q-curvature Q_g^4 and the Paneitz operator P_g^4 were introduced by Branson-Oersted [BO] and Paneitz [Pan]:

$$\begin{aligned} Q_g^4 &:= -\frac{1}{6}\left(\Delta_g R_g - R_g^2 + 3|\operatorname{Ric}_g|^2\right) \\ P_g^4(f) &:= \Delta_g^2 f + \operatorname{div}\left(\frac{2}{3}R_g g - 2\operatorname{Ric}_g\right)df, \quad \text{for } f \in C^\infty(M), \end{aligned}$$

where R_g and Ric_g denote the scalar and Ricci curvatures of g, and Δ_g is the Laplace-Beltrami operator of g with the analysts' sign. Higher order Q-curvatures Q^{2m} and Paneitz operators P^{2m} on a $2m$-dimensional manifold (actually more in general) were introduced in [Bra] and [GJMS].

One can think about the Paneitz operator as a higher order analog of the Laplace-Beltrami operator, and the Q-curvature can be thought of as a higher order analog of the Gaussian curvautre. In fact, in dimension 2 we simply have $P_g^2 = -\Delta_g$ and $Q_g^2 = K_g$. The expression *higher order* is justified, since in general Q^{2m} involves derivatives of the metric up to order $2m$, and

$$P_g^{2m} = (-\Delta_g)^m + A_g, \tag{1.1}$$

where A_g is a differential operator of order at most $2m-1$. In particular P_g^{2m} has order $2m$.

The interest of these objects lies in their covariant nature. If we consider in dimension $2m$ the conformal metric $g_u := e^{2u}g$, we have

$$P^{2m}_{g_u} = e^{-2mu}P^{2m}_g, \tag{1.2}$$

and

$$P^{2m}_g u + Q^{2m}_g = Q^{2m}_{g_u} e^{2mu}, \tag{1.3}$$

see for instance [Cha] Chapter 4. Notice that (1.3) is a generalized version of Gauß's identity, which states that in dimension 2

$$-\Delta_g u + K_g = K_{g_u} e^{2u},$$

where K_g is the Gaussian curvature. The nice transformation (1.2) enjoyed by the Paneitz operator is the reason why we do not simply take $P^{2m}_g = (-\Delta_g)^m$, and it explain the purpose of the correction term A_g in (1.1).

Although for the case $m > 2$ there are no explicit formulas for P^{2m}_g and Q^{2m}_g (we will not need them, anyway), we know that on flat $\mathbb{R}^{2m}$, $P^{2m}_g = (-\Delta)^m$ and $Q^{2m}_g \equiv 0$. Then (1.2) and (1.3) can be used to define the Paneitz operator and the Q-curvature for any locally conformally flat manifold. For instance, in the model case of the round sphere (S^{2m}, g), we have

$$P^{2m}_g = \prod_{i=0}^{m-1}(-\Delta_g + i(2m-i-1)), \quad Q^{2m}_g \equiv (2m-1)!,$$

which is consistent with (1.1). A formal definition of the Paneitz operator P^k_g of order k on an arbitrary Riemannian manifold (M, g) of dimension n can be given by prescribing that P^k_g satisfies the properties listed in the following theorem, of which we shall only need that case $n = k = 2m$.

Theorem 1.1 ([GJMS]) *Let k be a positive integer. Suppose n is odd, or $k \leq n$. Then for any Riemannian manifold of dimension n there is a linear differential operator P^k_g on scalar functions satisfying the following:*

(i) If $g_u = e^{2u}g$, then

$$P^k_{g_u}\varphi = e^{-\frac{n+k}{2}}P^k_g\left(e^{\frac{n-k}{2}}\varphi\right) \quad \text{for every } \varphi \in C^\infty(M).$$

(ii) The leading symbol of P^k_g is $(-\Delta_g)^{\frac{k}{2}}$, and on Euclidean $(\mathbb{R}^n, g_{\mathbb{R}^n})$ we have $P^k_{g_{\mathbb{R}^n}} = (-\Delta)^{\frac{k}{2}}$.

(iii) $P^k_g = \tilde{P}^k_g + \frac{n-k}{2}Q^k_g$, where Q^k_g is a local scalar invariant, and $\tilde{P}^k_g = \delta S^{k-2}_g d$, where δ is the divergence operator on 1-forms and S^{k-2}_g is a differential operator on 1-forms.

(iv) P^k_g is self-adjoint.

The above theorem does not imply uniqueness of the operators P^k_g, although it follows easily that, in the locally conformally flat case, we do have uniqueness. For example, given P^4_g as in the theorem above for $n = 4$, the operator $P^4_g + |W_g|^2$, where W_g is the Weyl tensor, satisfies the properties of the theorem as well. Since in dimension 2 local conformal flatness is for free, we can also easily see that $P_g^2 = -\Delta_g$ is the only Paneitz operator of order 2 in dimension 2. For $k = 2$ and $n > 2$, Q_g^2 is a multiple of the scalar curvature R_g and P_g^k is nothing else than the well-known conformal Laplacian L_g.

Now the Q-curvature Q_g^k in dimension $n \neq k$ is defined by property (iii) in the theorem above. In [Bra], T. Branson extended this definition to the case $n = k = 2m$ as follows. If we fix $k \in \mathbb{N}$ even, and call $P_g^{k,n}$ a Paneitz operator of order k in dimension n, as given in Theorem 1.1. Then $P_g^{k,n}$ has coefficients given by universal formulae in curvature and its derivatives which are rational in n, and the zeroth order term of $P_g^{n,k}$ is of the form $\frac{n-k}{2} Q_g^{n,k}$ with coefficients rational in n and regular at $n = k$. Then $Q_g^{2m} = Q_g^{k,k}$ can be defined, roughly speaking, as

$$Q_g^{2m} = \lim_{\substack{n \in \mathbb{C} \\ n \to 2m}} Q_g^{n,2m}.$$

An analytic continuation argument in the dimension n also show that the Q-curvature so defined satisfies the transformation law (1.3). Equivalent definitions of the Q-curvature have been provided by [FG], [FH] and [GH].

Let us from now on focus on the case $n = k = 2m$. A geometrically interesting fact is that the total Q-curvature is a global conformal invariant, that is, if M is closed and $2m$-dimensional, then

$$\int_M Q_{g_u}^{2m} \mathrm{dvol}_{g_u} = \int_M Q_g^{2m} \mathrm{dvol}_g, \quad g_u = e^{2u} g.$$

Further evidence of the geometric relevance of the Q-curvatures is given by the Gauss-Bonnet-Chern's theorem [Che]: On a locally conformally flat closed manifold of dimension $2m$, since Q_g^{2m} is a multiple of the Pfaffian plus a divergence term (see [BGP]), we have

$$\int_M Q_g^{2m} \mathrm{dvol}_g = (2m-1)! \,\mathrm{vol}(S^{2m}) \frac{\chi(M)}{2}, \tag{1.4}$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M. Recently S. Alexakis [Ale1], [Ale2] proved that

$$Q_g^{2m} = W_g + \mathrm{div}\, T_g + C_{2m} \mathrm{Pfaff}_g,$$

where W_g is a local conformal invariant, T_g is a Riemannian vector field and Pfaff_g denotes the Pfaffian of g. Then, the Gauss-Bonnet-Chern formula can be expressed in terms of the Q-curvature on any close manifold (M, g), without the assumption that g be locally conformally flat. We then have

$$\int_M (Q_g^{2m} - W_g) \,\mathrm{dvol}_g = (2m-1)! \,\mathrm{vol}(S^{2m}) \frac{\chi(M)}{2}.$$

1.2 Structure of the chapters

We start in Chapter 2 by addressing the special case when (M, g) is $\mathbb{R}^{2m}$ with the Euclidean metric $g_{\mathbb{R}^{2m}}$. Remember that $P_{g_{\mathbb{R}^{2m}}}^{2m} = (-\Delta)^m$ and $Q_{g_{\mathbb{R}^{2m}}}^{2m} \equiv 0$. Our purpose now is to classify all conformal metrics $e^{2u} g_{\mathbb{R}^{2m}}$ on $\mathbb{R}^{2m}$ having constant positive Q-curvature and finite volume. Thanks to (1.3), this is equivalent to classify the solutions u to

$$(-\Delta)^m u = Qe^{2mu} \quad \text{on } \mathbb{R}^{2m}, \tag{1.5}$$

satisfying $\int_{\mathbb{R}^{2m}} e^{2mu}dx < \infty$, where $Q > 0$ is constant. Equation (1.5) is also called Liouville equation (also when Q is not constant). We will see that there exist *standard solutions*. These correspond to metrics which arise from the pull-back of the round metric of S^{2m} via the stereographic projection and possibly a Möbius transformation. Such solutions have a well-controlled behaviour at infinity. Then we develop some criteria to characterize the non-standard solutions, with the purpose of ruling out their appearance in the blow-up theory which will be discussed later. Of particular importance is the following criterium: If a solution u is non standard, then there exist $1 \leq j \leq m-1$ and a constant $a \neq 0$ such that

$$\lim_{|x|\to\infty} \Delta^j u(x) = a. \tag{1.6}$$

In Chapter 3 we discuss the existence and classification of conformal metrics on $\mathbb{R}^{2m}$ with non-positive constant Q-curvature $Q \leq 0$ and finite volume. A simple computation, based on the maximum principle shows that for $m = 1$ such metrics do not exist at all. Then one might be led to believe that this is the case also in higher dimension, but it actually turns out that such metrics do exist for $m > 1$, as we show in Section 3.2. Then we study the asymptotic behaviour at infinity, as in the case of positive Q-curvature. An important difference between the two cases is that for $Q \leq 0$ there are no standard solutions and there are no solutions presenting a "nice" behaviour at infinity. In particular a property similar to (1.6) can be shown for *every* solution in the negative case. This will be crucial in proving that concentration phenomena on closed manifolds can only occur at points of positive curvature.

In Chapter 4 we turn our attention to the concentration-compactness results. We are given a Riemannian manifold (M, g) and a sequence of conformal metrics $g_k = e^{2u_k}g$, with $\mathrm{vol}(g_k)$ equi-bounded and $Q^{2m}_{g_k} \to Q_0$ uniformly for a continuous function Q_0. The first result we prove concerns the case when (M, g) is $\mathbb{R}^{2m}$ endowed with the Euclidean metric. Its proof is solely based on linear elliptic estimates and the generalization of a non-linear estimate of Brézis and Merle, Theorem 2.7.

Then we consider the case when (M, g) is a closed manifold with the property that $\ker P^{2m}_g$ contains only the constant functions. This is a generic assumptions and is needed in order for P^{2m}_g to have a Green's function. Working with the Green's representation formula we can show integral gradient estimates (Lemma 4.7) which, combined to the classification results of Chapters 2 and 3 imply that in the concentration case

(i) concentration can only occur at finitely many (sequences of) points where $Q_0 > 0$;

(ii) if we scale the metrics at such (sequences of) concentration points, we obtain a sequence of metrics converging, up to a subsequence, to the round metric on S^{2m}.

Still working with the gradient estimates, we can show that away from the finitely many concentration points, the metrics are vanishing locally uniformly. We then prove that the total Q-curvatures of the metrics g_k of the concentrating subsequence is an integer multiple of the total Q-curvature of the round S^{2m}. This also allows us to estimate the number of concentration points in terms of the total Q-curvature of the initial metric g. An immediate consequence is Corollary 4.3, which gives some compactness criteria in terms of the Euler characteristic of M. Finally we consider the model case when (M, g) is the round sphere S^{2m}.

We conclude by studying in Chapter 5 the concentration-compactness behaviour of sequences (u_k) of solutions to the following elliptic equation, related to the Adams-Moser-Trudinger inequality

$$\begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2} & \text{in } \Omega \\ u_k > 0 & \text{in } \Omega \\ u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Here Ω is a bounded domain with smooth boundary. Assuming $\lambda_k \to 0$ and

$$\int_\Omega u_k(-\Delta)^m u_k dx = \lambda_k \int_\Omega u_k^2 e^{mu_k^2} dx \to \Lambda \geq 0 \quad \text{as } k \to \infty, \tag{1.8}$$

we will see that in the case of non-compactness, we have concentration on a finite set S, with $u_k \to 0$ locally uniformly on $\Omega\backslash S$. Moreover we will show that $\Lambda \geq \Lambda_1$, where Λ_1 is the total Q-curvature of the round sphere S^{2m}. The reason for such an unexpected relation with the Q-curvature, is that, when we scale u_k at a blow-up sequence, we find a new sequence of functions η_k which solve an approximate Liouville-type equation (1.5), see Corollary 5.4. Then in the limit we find a solution to (1.5), which can be classified, thanks to the results of Chapter 2. This requires, though, some *a priori* gradient bounds. We prove them by observing that (1.8) implies that the right-hand side of (1.7) is slightly more than integrable, as it belongs to a so-called Zygmund space:

$$(-\Delta)^m u_k = \lambda_k u_k e^{mu_k^2} \in L(\log L)^{\frac{1}{2}}$$

(see the proof of Lemma 5.5 for the definition of $L(\log L)^{\frac{1}{2}}$). Using some sharp elliptic estimates for such Zygmund spaces, we obtain uniform bounds for $\nabla^\ell u_k$ in the Lorentz space $L^{(2m/\ell,2)}(\Omega)$, $1 \leq \ell \leq 2m-1$. The use of Lorentz spaces is very natural here. In fact, the estimates that we would obtain, if we used the usual Sobolev spaces instead, would not fully exploit the integrability hypotesis (1.8), and they would be too weak for our purposes.

The content of Chapters 2 and 3 corresponds to the papers [Mar1] and [Mar2] respectively. The content of Chapters 4 and 5 corresponds to the material in [Mar3] and [Mar4] respectively. The list of authors from whom we borrowed ideas is quite long, and is discussed chapter by chapter.

Chapter 2

Conformal metrics on $\mathbb{R}^{2m}$ with positive constant Q-curvature

In this chapter we classify the solutions to the equation $(-\Delta)^m u = (2m-1)!e^{2mu}$ on $\mathbb{R}^{2m}$ giving rise to a conformal metric $g = e^{2u}g_{\mathbb{R}^{2m}}$ with finite volume in terms of analytic and geometric properties. The analytic conditions involve the growth rate of u and the asymptotic behaviour of Δu at infinity. As a consequence we give a geometric characterization in terms of the scalar curvature of the metric $e^{2u}g_{\mathbb{R}^{2m}}$ at infinity, and we observe that the pull-back of this metric to S^{2m} via the stereographic projection can be extended to a smooth Riemannian metric if and only if it is round.

2.1 Introduction and statement of the main theorems

Let $g_u = e^{2u}g_{\mathbb{R}^{2m}}$ be a conformal metric on $\mathbb{R}^{2m}$ of Q-curvature identically equal to $(2m-1)!$. Since $P^{2m}_{g_{\mathbb{R}^{2m}}} = (-\Delta)^m$ and $Q^{2m}_{g_{\mathbb{R}^{2m}}} \equiv 0$, we infer from the generalized Gauss identity (1.3) that u satisfies

$$(-\Delta)^m u = (2m-1)!e^{2mu} \quad \text{on } \mathbb{R}^{2m}, \tag{2.1}$$

and, conversely, solutions to (2.1) yield metrics of Q-curvature $(2m-1)!$. Actually, we can replace $(2m-1)!$ by any other positive constant Q, in that we simply consider the function $v := u + \frac{1}{2m}\log\frac{(2m-1)!}{Q}$, i.e. the metric

$$g_v := e^{2v}g_{\mathbb{R}^{2m}} = \left(\frac{(2m-1)!}{Q}\right)^{\frac{1}{m}} g_u.$$

Assuming that the volume of g_u is finite, is equivalent to imposing $\int_{\mathbb{R}^{2m}} e^{2mu}dx < \infty$.

As we shall see, regularity is not an issue, since every solution to (2.1) with $e^{2mu} \in L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$ is smooth (Corollary 2.8).

Now given a solution u to (2.1) with $e^{2mu} \in L^1$, define the auxiliary function

$$v(x) := \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y|}{|x-y|}\right) e^{2mu(y)} dy, \tag{2.2}$$

where γ_m is defined by the following property: $(-\Delta)^m\left(\frac{1}{\gamma_m}\log\frac{1}{|x|}\right) = \delta_0$ in $\mathbb{R}^{2m}$ (compare Proposition 2.22 below). Then $(-\Delta)^m v = (2m-1)!e^{2mu}$. We prove

Theorem 2.1 *Let u be a solution of* (2.1) *with*

$$\alpha := \frac{1}{|S^{2m}|}\int_{\mathbb{R}^{2m}} e^{2mu(x)} dx < +\infty. \tag{2.3}$$

Then

$$u(x) = v(x) + p(x), \tag{2.4}$$

where p is a polynomial of even degree at most $2m-2$, v is as in (2.2) *and*

$$\begin{aligned} \sup_{x\in\mathbb{R}^{2m}} p(x) &< +\infty, \\ \lim_{|x|\to\infty} \Delta^j v(x) &= 0, \quad j = 1,\dots,m-1, \\ v(x) &= -2\alpha\log|x| + o(\log|x|), \;\; \textit{as } |x| \to +\infty. \end{aligned}$$

It is well known that the function

$$u(x) := \log\frac{2\lambda}{1+\lambda^2|x-x_0|^2} \tag{2.5}$$

solves (2.1) and (2.3) with $\alpha = 1$ for any $\lambda > 0$, $x_0 \in \mathbb{R}^{2m}$. We call the functions of the form (2.5) *standard solutions.* They all arise as pull-back under the stereographic projection of metrics on S^{2m} which are round, i.e. conformally diffeomorphic to the standard metric. A. Chang and P. Yang [CY] proved that the round metrics are the only metrics on S^{2m} having Q-curvature identically equal to $(2m-1)!$.

In the next theorem we give conditions under which an entire solution of Liouville's equation satisfying (2.3) is necessarily a standard solution.

Theorem 2.2 *Let u be a solution of* (2.2) *satisfying* (2.3). *Then the following are equivalent:*

(i) *u is a standard solution,*

(ii) $\lim_{|x|\to\infty} \Delta u(x) = 0$

(ii') $\lim_{|x|\to\infty} \Delta^j u(x) = 0$ *for* $j = 1,\dots,m-1$,

(iii) $u(x) = o(|x|^2)$ *as* $|x| \to \infty$,

(iv) $\deg p = 0$, *where p is the polynomial in* (2.4).

(v) $\liminf_{|x|\to+\infty} R_{g_u} > -\infty$, *where* $g_u = e^{2u}g_{\mathbb{R}^{2m}}$.

(vi) $\pi^* g_u$ *can be extended to a Riemannian metric on* S^{2m}, *where* $\pi : S^{2m} \to \mathbb{R}^{2m}$ *is the stereographic projection.*

Moreover, if u is not a standard solution, there exist $1 \leq j \leq m-1$ and a constant $a < 0$ such that

$$\Delta^j u(x) \to a \quad \text{as } |x| \to +\infty. \tag{2.6}$$

The 2-dimensional case ($m = 1$) of Theorem 2.2 was treated by W. Chen and C. Li [CL], who proved that *every* solution with finite total Gaussian curvature is a standard one. The 4-dimensional case was treated by C-S. Lin [Lin], with a classification of u in terms of its growth, or of the behaviour of Δu at ∞. The classification of C-S. Lin in terms of Δu was used by F. Robert and M. Struwe [RS] to study the blow-up behaviour of sequences of solutions u_k to

$$\begin{cases} \Delta^2 u_k = \lambda u_k e^{32\pi^2 u_k^2} & \text{in } \Omega \subset \mathbb{R}^4 \\ u_k = \frac{\partial u_k}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and by A. Malchiodi [Mal] to show a compactness criterion for sequences of solutions u_k to the equation

$$P_g^4 u_k + Q_k^4 = h_k e^{4u_k}, \quad h_k \text{ constant}$$

on a closed 4-manifold. The same criterion could be used in higher dimension in the proof of an analogous compactness result. This was observed by C. B. Ndiaye [Ndi], who then used a different technique to show compactness. We will discuss this in Chapters 4 and 5, where the above theorems will be used.

In higher dimension ($m > 2$), J. Wei and X. Xu [WX] (see also [Xu]) treated a special case of Theorem 2.2: if $u(x) = o(|x|^2)$ at infinity, then u is always a standard solution. This result is not sufficient to prove compactness. Moreover, the proof appears to be overly simplified. For instance, in their Lemma 2.2 the argument for showing that $u \leq C$ is not conclusive, and in the crucial Lemma 2.4 they simply refer to [Lin] for details. This latter lemma corresponds to Lemma 2.13 here and it is the main regularity result, as it implies that $u \leq C$, hence that the right-hand side of (2.1) belongs to $L^\infty(\mathbb{R}^{2m})$. Its generalization is a major issue, because Lin's analysis is focused on the function Δu, and it makes use of the Harnack's inequality and of the fact that $\Delta(u - v) \equiv C$. In the general case, Harnack's inequality does not work and there are no uniform bounds for $\Delta^{(m-2)}(u - v)$ (while it is still true that $\Delta^{(m-1)}(u - v) \equiv C$). To overcome this difficulties, we spend a few pages in the following section to study polyharmonic functions. As a reward we obtain a Liouville-type theorem for polyharmonic functions (Theorem 2.6) which allows us to make the proof of [Lin] more direct and transparent.

The characterization in terms of the scalar curvature at infinity is new and quite interesting, as it shows that non-standard solutions have a geometry essentially different from standard solutions, and it also shows that the Q-curvature and the scalar curvature are independent of each other in dimension 4 and higher. On the other hand, since in dimension 2 we have $2Q_g = R_g$, our characterization (v) is consistent with the result of [CL].

The characterization in (vi) implies the result of A. Chang and P. Yang [CY] described above, which here follows from the general case.

The chapter is organized as follows. In Section 2.2 we collect some relevant results about polyharmonic functions which will be needed later. Section 2.3

contains the proof of Theorems 2.1 and 2.2; in Section 2.4 we give examples to show that the hypothesis of Theorem 2.2 are sharp in terms of the growth at infinity and of the degree of p. Recently J. C. Wei and D. Ye [WY] proved that already in dimension 4 there is a great abundance of non-radially symmetric solutions. In the last section we collect some useful results, which were needed in the previous sections.

In the following, the letter C denotes a generic constant, which may change from line to line and even within the same line.

2.2 A few remarks on polyharmonic functions

We briefly recall some properties of polyharmonic functions, which will be used in the sequel. For the standard elliptic estimates for the Laplace operator, we refer to [GT] or [GM]. The next lemma can be considered a generalized mean value inequality. We give the short proof for the convenience of the reader, and because identity (2.11) will be used in the next section.

Lemma 2.3 (Pizzetti [Piz]) *Let $\Delta^m h = 0$ in $B_R(x_0) \subset \mathbb{R}^n$, for some m, n positive integers. Then*

$$\fint_{B_R(x_0)} h(z)dz = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i h(x_0), \tag{2.7}$$

where

$$c_0 = 1, \quad c_i = \frac{n}{n+2i} \frac{(n-2)!!}{(2i)!!(2i+n-2)!!}, \quad i \geq 1. \tag{2.8}$$

Proof. We can translate and assume that $x_0 = 0$. We first prove by induction on m that there are constants $b_0^{(m)}, \ldots, b_{m-1}^{(m)}$ such that

$$\fint_{\partial B_r} h(z)dS = \sum_{i=0}^{m-1} b_i^{(m)} r^{2i} \Delta^i h(0), \quad 0 < r < R, \ B_r := B_r(0). \tag{2.9}$$

For $m = 1$ this reduces to the mean value theorem for harmonic functions. Assume now that the assertion has been proved up to $m-1$, and that $\Delta^m h = 0$. Let G_r be the Green function of Δ^m in B_r:

$$\Delta^m G_r = \delta_0 \text{ in } B_r, \quad G_r = \Delta G_r = \ldots = \Delta^{m-1} G_r = 0 \text{ on } \partial B_r. \tag{2.10}$$

For simplicity, let us only consider the case $n = 2m$. Then $G_r(x) = G_1\left(\frac{x}{r}\right)$,

$$G_1(x) = \beta + \alpha_0 \log|x| + \alpha_1 |x|^2 + \ldots + \alpha_{m-1}|x|^{2m-2},$$

where the constants can be computed inductively starting with α_0 up to α_{m-1} in order to satisfy (2.10). Notice that G_1 is radial. Integrating by parts

$$\begin{aligned} 0 &= \int_{B_r} G_r \Delta^m h dx \\ &= h(0) - \sum_{i=0}^{m-1} \int_{\partial B_r} \frac{\partial \Delta^{m-1-i} G_r}{\partial n} \Delta^i h dS \\ &= h(0) - \sum_{i=0}^{m-1} \fint_{\partial B_r} a_i r^{2i} \Delta^i h dS, \end{aligned} \tag{2.11}$$

where each a_i depends only on n and m. For each term on the right-hand side with $i \geq 1$, we can use the inductive hypothesis

$$r^{2i} \fint_{\partial B_r} \Delta^i h dS = r^{2i} \sum_{j=0}^{m-i-1} b_j^{(m-1)} r^{2j} \Delta^{j+i} h(0), \quad 0 \leq i \leq m-1,$$

and substituting we obtain (2.9). To conclude the induction it is enough to multiply (2.9) by r^{n-1}, integrate with respect to r from 0 to R and divide by $\frac{R^n}{n}$.

To compute the c_i's, we test with the functions $h(x) = r^{2i} := |x|^{2i}$, $i \geq 1$ (for the case $i = 0$ use the function $h(x) \equiv 1$). Since $\Delta r^{2i} = 2i(2i + n - 2) r^{2i-2}$, we have that $\Delta^k h(0) = 0$ for $k \neq i$ and $\Delta^i h(0) = \frac{(2i)!!(2i+n-2)!!}{(n-2)!!}$. Hence Pizzetti's formula reduces to

$$c_i R^{2i} \frac{(2i)!!(2i+n-2)!!}{(n-2)!!} = \fint_{B_R} r^{2i} dx = \frac{n}{n+2i} R^{2i},$$

whence (2.8). □

Remark. From (2.11), moreover, for an arbitrary C^{2m}-function u it follows that

$$\fint_{B_R(x_0)} u(z) dz = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i u(x_0) + c_m R^{2m} \Delta^m u(\xi), \tag{2.12}$$

for some $\xi \in B_R(x_0)$.

Proposition 2.4 *Let $\Delta^m h = 0$ in $B_4 \subset \mathbb{R}^n$. For every $0 \leq \alpha < 1$, $p \in [1, \infty)$ and $k \geq 0$ there are constants $C(k,p), C(k,\alpha)$ independent of h such that*

$$\begin{aligned} \|h\|_{W^{k,p}(B_1)} &\leq C(k,p)\|h\|_{L^1(B_4)} \\ \|h\|_{C^{k,\alpha}(B_1)} &\leq C(k,\alpha)\|h\|_{L^1(B_4)}. \end{aligned}$$

The proof of Proposition 2.4 is given in Section 2.5. As a consequence of Proposition 2.4 and Pizzetti's formula we have the following Liouville-type theorem, compare [ARS].

Theorem 2.5 *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h(x) \leq C(1 + |x|^\ell)$, for some $\ell \geq 2m - 2$. Then $h(x)$ is a polynomial of degree at most ℓ.*

Proof. Thanks to Proposition 2.4, we have for any $x \in \mathbb{R}^n$

$$|D^{\ell+1}h(x)| \leq \frac{C}{R^{\ell+1}} \fint_{B_R(x)} |h(y)|dy = -\frac{C}{R^{\ell+1}} \fint_{B_R(x)} h(y)dy + O(R^{-1}), \quad \text{as } R \to \infty. \tag{2.13}$$

On the other hand, Pizzetti's formula implies that

$$\fint_{B_R(x)} h(y)dy = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i h(x) = O(R^{2m-2}),$$

and letting $R \to \infty$, we obtain $D^{\ell+1}h = 0$. ☐

A variant of the above theorem, which will be used later is the following

Theorem 2.6 *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h(x) \leq u - v$, where $e^{pu} \in L^1(\mathbb{R}^n)$ for some $p > 0$, $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $-v(x) \leq C(\log(1 + |x|) + 1)$. Then h is a polynomial of degree at most $2m - 2$.*

Proof. The only thing to change in the proof of Theorem 2.5, is the estimate of the term $\frac{2C}{R^{2m-1}} \fint_{B_R(x)} h^+ dy$, corresponding to the $O(R^{-1})$ in (2.13). We have

$$\begin{aligned} \fint_{B_R(x)} h^+ dy &\leq \fint_{B_R(x)} u^+ dy + C \fint_{B_R(x)} \log(1 + |y|)dy + C \\ &\leq \frac{1}{p} \fint_{B_R(x)} e^{pu} dy + C \log R + C, \end{aligned}$$

and all terms go to 0 when divided by R^{2m-1} and for $R \to \infty$. ☐

The following estimate has been obtained by Brézis and Merle [BM] in dimension 2 and by C.S. Lin [Lin] and J. Wei [Wei] in dimension 4. Notice that the constant γ_m, defined by the relation

$$(-\Delta)^m \left(\frac{1}{\gamma_m} \log \frac{1}{|x|} \right) = \delta_0, \quad \text{in } \mathbb{R}^{2m}$$

(see Proposition 2.22 in Section 2.5), plays an important role.

Theorem 2.7 *Let $f \in L^1(B_R(x_0))$ and let v solve*

$$\begin{cases} (-\Delta)^m v = f & \text{in } B_R(x_0) \subset \mathbb{R}^{2m}, \\ v = \Delta v = \ldots = \Delta^{m-1} v = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

Then, for any $p \in \left(0, \frac{\gamma_m}{\|f\|_{L^1(B_R(x_0))}}\right)$, we have $e^{2mp|v|} \in L^1(B_R(x_0))$ and

$$\int_{B_R(x_0)} e^{2mp|v|} dx \leq C(p) R^{2m},$$

where γ_m is given by (2.50).

Proof. We can assume $x_0 = 0$ and, up to rescaling, that $\|f\|_{L^1(B_R)} = 1$. Define

$$w(x) := \frac{1}{\gamma_m} \int_{B_R} \log \frac{2R}{|x-y|} |f(y)| dy, \quad x \in \mathbb{R}^{2m}.$$

Extend f to be zero outside $B_R(x_0)$; then

$$(-\Delta)^m w = |f| \quad \text{in } \mathbb{R}^{2m}.$$

We claim that $w \geq |v|$ in B_R. Indeed by (2.51) below and from $|x-y| \leq 2R$ for $x, y \in B_R$, we immediately see that

$$(-\Delta)^j w \geq 0, \quad j = 0, 1, 2, \dots$$

In particular the function $z := w - v$ satisfies

$$\begin{cases} (-\Delta)^m z \geq 0 & \text{in } B_R \\ (-\Delta)^j z \geq 0 & \text{on } \partial B_R \text{ for } 0 \leq j \leq m-1. \end{cases}$$

By the maximum principle (see Proposition 2.21 below), $(-\Delta)^j z \geq 0$ in B_R, $0 \leq j \leq m-1$ and the case $j = 0$ corresponds $w \geq v$. Working also with $-v$ we complete the proof of our claim.

Now it suffices to show that for $p \in (0, \gamma_m)$ we have $\|e^{2mpw}\|_{L^1(B_R)} \leq C(p)R^{2m}$. By Jensen's inequality we have

$$\begin{aligned} \int_{B_R} e^{2mpw} dx &= \int_{B_R} e^{\frac{2mp}{\gamma_m} \int_{B_R} \log \frac{2R}{|x-y|} |f(y)| dy} dx \\ &\leq \int_{B_R} \int_{B_R} |f(y)| e^{\frac{2mp}{\gamma_m} \log \frac{2R}{|x-y|}} dy dx \\ &= \int_{B_R} |f(y)| \left(\int_{B_R} \left(\frac{2R}{|x-y|} \right)^{\frac{2mp}{\gamma_m}} dx \right) dy \end{aligned}$$

On the other hand

$$\begin{aligned} \int_{B_R} \left(\frac{2R}{|x-y|} \right)^{\frac{2mp}{\gamma_m}} dx &\leq \int_{B_R} \left(\frac{2R}{|x|} \right)^{\frac{2mp}{\gamma_m}} dx \\ &= \omega_{2m} \int_0^R r^{2m-1-\frac{2mp}{\gamma_m}} (2R)^{\frac{2mp}{\gamma_m}} dr \\ &= \omega_{2m} \frac{\gamma_m}{2m\gamma_m - 2mp} R^{2m} 2^{\frac{2mp}{\gamma_m}}. \end{aligned}$$

We then conclude

$$\int_{B_R} e^{2mpw} dx \leq \frac{C(m)}{\gamma_m - p} R^{2m}.$$

□

Corollary 2.8 *Every solution u to (2.1) with $e^{2mu} \in L^1_{\text{loc}}(\mathbb{R}^{2m})$ is smooth.*

Proof. Given $B_4(x_0) \subset \mathbb{R}^{2m}$, write $(2m-1)!e^{2mu}\big|_{B_4(x_0)} = f_1 + f_2$ with

$$\|f_1\|_{L^1(B_4(x_0))} < \gamma_m, \quad f_2 \in L^\infty(B_4(x_0)),$$

and $u = u_1 + u_2 + u_3$, with

$$\begin{cases} (-\Delta)^m u_i = f_i & \text{in } B_4(x_0) \\ u_i = \Delta u_i = \ldots = \Delta^{m-1} u_i = 0 & \text{on } \partial B_4(x_0) \end{cases}$$

for $i = 1, 2$, and $\Delta^m u_3 = 0$. Then, by Theorem 2.7, $e^{2mu_1} \in L^p(B_4(x_0))$ for some $p > 1$, while, by standard elliptic estimates $u_2 \in L^\infty(B_4(x_0))$ and u_3 is smooth, hence $u_3 \in L^\infty(B_3(x_0))$. Then $e^{2mu} \in L^p(B_3(x_0))$. Write now $u\big|_{B_3(x_0)} = v_1 + v_2$, where

$$\begin{cases} (-\Delta)^m v_1 = (2m-1)!e^{2mu} & \text{in } B_3(x_0) \\ v_1 = \Delta v_1 = \ldots = \Delta^{m-1} v_1 = 0 & \text{on } \partial B_3(x_0) \end{cases}$$

and $\Delta^m v_2 = 0$. Then, by L^p-estimates and Sobolev's embedding theorem, $v_1 \in W^{2m,p}(B_3(x_0)) \hookrightarrow C^{0,\alpha}(B_3(x_0))$ for some $0 < \alpha < 1$, while v_2 is smooth. Then $u \in C^{0,\alpha}(B_2(x_0))$ and with the same procedure of writing u as the sum of a polyharmonic (hence smooth) function plus a function with vanishing Navier boundary condition, we can bootstrap and use Schauder's estimate to prove that $u \in C^\infty(B_1(x_0))$. $\square$

2.3 Proof of the main theorems

The proof of Theorems 2.1 and 2.2, which we give in this section, is divided into several lemmas. It consists of a careful study of the functions v, defined in (2.2), and $u - v$. In what follows the generic constant C may depend also on u.

Remark. In general $v \neq u$, even if u is a standard solution. To see that, rescale u by a factor $r > 0$ as follows:

$$\widetilde{u}(x) := u(rx) + \log r.$$

Then $\widetilde{u}$ is again a solution, with the same energy. On the other hand the corresponding $\widetilde{v}$ satisfies

$$\begin{aligned} \widetilde{v}(x) &= \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y|}{|x-y|}\right) e^{2mu(ry)} r^{2m} dy \\ &= \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y'|}{|rx-y'|}\right) e^{2mu(y')} dy' = v(rx). \end{aligned} \tag{2.14}$$

That shows that after rescaling, $u - v$ changes by a contant.

Lemma 2.9 *Let u be a solution of* (2.1), (2.3). *Then, for* $|x| \geq 4$,

$$v(x) \geq -2\alpha \log|x| + C. \tag{2.15}$$

Proof. The proof is similar to that in dimension 4, compare [Lin]. Fix x with $|x| \geq 4$, and decompose $\mathbb{R}^{2m} = A_1 \cup A_2 \cup B_2$, where $B_2 = B_2(0)$ and

$$A_1 := B_{|x|/2}(x), \quad A_2 := \mathbb{R}^{2m} \backslash (A_1 \cup B_2).$$

For $y \in A_1$ we have

$$|y| \geq |x| - |x-y| \geq \frac{|x|}{2} \geq |x-y|, \quad \log \frac{|y|}{|x-y|} \geq 0,$$

hence

$$\int_{A_1} \log \frac{|y|}{|x-y|} e^{2mu(y)} dy \geq 0. \tag{2.16}$$

For $y \in A_2$, since $|x|, |y| \geq 2$, we have

$$|x-y| \leq |x| + |y| \leq |x||y|, \quad \log \frac{|y|}{|x-y|} \geq \log \frac{1}{|x|},$$

hence

$$\int_{A_2} \log \frac{|y|}{|x-y|} e^{2mu(y)} dy \geq -\log |x| \int_{A_2} e^{2mu(y)} dy. \tag{2.17}$$

For $y \in B_2$, $\log |x-y| \leq \log |x| + C$ and, since u is smooth, we find

$$\begin{aligned} \int_{B_2} \log \frac{|y|}{|x-y|} e^{2mu(y)} dy &\geq \int_{B_2} \log |y| e^{2mu(y)} dy - \log |x| \int_{B_2} e^{2mu} dy \\ &\quad -C \int_{B_2} e^{2mu} dy \\ &\geq -\log |x| \int_{B_2} e^{2mu} dy + C. \end{aligned} \tag{2.18}$$

Putting together (2.16), (2.17) and (2.18) and observing that $\log \frac{1}{|x|} < 0$, we conclude that

$$\begin{aligned} v(x) &\geq \frac{(2m-1)!}{\gamma_m} \int_{A_2 \cup B_2} \log \left(\frac{|y|}{|x-y|} \right) e^{2mu(y)} dy \\ &\geq -\frac{(2m-1)!}{\gamma_m} \log |x| \int_{A_2 \cup B_2} e^{2mu} dy + C \\ &\geq -\frac{(2m-1)!|S^{2m}|}{\gamma_m} \alpha \log |x| + C. \end{aligned}$$

Finally, observing that $(2m-2)!! = 2^{m-1}(m-1)!$, we infer

$$\frac{(2m-1)!|S^{2m}|}{\gamma_m} = \frac{(2m-1)! 2 (2\pi)^m (2m-2)!!}{(2m-1)!! 2^{3m-2} [(m-1)!]^2 \pi^m} = 2.$$

□

Lemma 2.10 *Let u be a solution of* (2.1) *and* (2.3)*, with $m \geq 2$. Then $u = v + p$, where p is a polynomial of degree at most $2m-2$. Moreover*

$$\begin{aligned} \Delta^j u(x) &= \Delta^j v(x) + p_j \\ &= (-1)^j \frac{2^{2j}(j-1)!(m-1)!}{(m-j-1)!|S^{2m}|} \int_{\mathbb{R}^{2m}} \frac{e^{2mu(y)}}{|x-y|^{2j}} dy + p_j, \end{aligned}$$

where p_j is a polynomial of degree at most $2(m-1-j)$.

Proof. Let $p := u - v$. Then $\Delta^m p = 0$. By Lemma 2.9 we have

$$p(x) \leq u(x) + 2\alpha \log |x| + C,$$

and Theorem 2.6 implies that p is a polynomial of degree at most $2m-2$. To compute $\Delta^j v$, one can use (2.51) and the definition of γ_m. □

Lemma 2.11 *Let p be the polynomial of Lemma 2.10. Then*

$$\sup_{x\in\mathbb{R}^{2m}} p(x) < +\infty.$$

In particular $\deg p$ *is even.*

Proof. Define

$$f(r) := \sup_{\partial B_r} p.$$

If $\sup_{\mathbb{R}^{2m}} p = +\infty$, there exists $s > 0$ such that

$$\lim_{r\to+\infty} \frac{f(r)}{r^s} = +\infty, \tag{2.19}$$

see [Gor, Theorem 3.1].[1] Moreover $|\nabla p(x)| \leq C|x|^{2m-3}$ hence, also taking into account Lemma 2.9, there is $R > 0$ such that for every $r \geq R$, we can find x_r with $|x_r| = r$ such that

$$u(y) = v(y) + p(y) \geq r^s \quad \text{for } |y - x_r| \leq \frac{1}{r^{2m-3}}.$$

Then, using Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^{2m}} e^{2mu} dx &\geq \int_R^{+\infty} \int_{\partial B_r(0)\cap B_{r^{3-2m}}(x_r)} e^{2mr^s} d\sigma dr \\ &\geq C \int_R^{+\infty} \frac{\exp(2mr^s)}{r^{(2m-3)(2m-1)}} dr = +\infty, \end{aligned}$$

contradicting the hypothesis $e^{2mu} \in L^1(\mathbb{R}^{2m})$. □

The following lemma will be used in the proof of Lemma 2.13.

Lemma 2.12 *Let $G = G(|x|)$ be the Green's function for Δ^m in $B_1 \subset \mathbb{R}^n$ for n, m given positive integers. Then there are constants c_i depending on m and n such that for $|x| = 1$, and $0 \leq i \leq m-1$,*

$$(-1)^i \frac{\partial \Delta^{m-1-i} G(x)}{\partial r} = c_i > 0.$$

Proof. Since $G = G(|x|)$, we only need to show that $c_i > 0$. Fix i and let h solve

$$\begin{cases} \Delta^m h = 0 & \text{in } B_1 \\ (-\Delta)^i h = -1 & \text{on } \partial B_1 \\ (-\Delta)^j h = 0 & \text{on } \partial B_1 \text{ for } 0 \leq j \leq m-1,\ j \neq i. \end{cases}$$

[1] The statement of Theorem 3.1 in [Gor] is about $\mu(r) := \inf_{\partial B_r} |p|$, but the proof works in our case too.

By the maximum principle (see Proposition 2.21 below), $h(0) < 0$, hence (2.11) implies

$$0 < -h(0) = (-1)^i \int_{\partial B_1} \frac{\partial \Delta^{m-1-i} G}{\partial r} dS = c_i \omega_n.$$

□

Lemma 2.13 *Let $v : \mathbb{R}^{2m} \to \mathbb{R}$ be defined as in (2.2). Then*

$$\lim_{|x|\to\infty} \Delta^{m-j} v(x) = 0, \quad j = 1, \dots, m-1 \tag{2.20}$$

and for any $\varepsilon > 0$ there is $R > 0$ such that for $|x| > R$

$$v(x) \le (-2\alpha + \varepsilon) \log|x|. \tag{2.21}$$

Proof. We proceed by steps.

Step 1. We claim that for any $\varepsilon > 0$ there is $R > 0$ such that for $|x| \ge R$

$$v(x) \le \Big(-2\alpha + \frac{\varepsilon}{2} \Big) \log|x| - \frac{(2m-1)!}{\gamma_m} \int_{B_\tau(x)} \log|x-y| e^{2mu(y)} dy, \tag{2.22}$$

where $\tau \in (0,1)$ will be fixed later. Notice that the second term on the right-hand side may be very large. To prove the claim, set $\mathbb{R}^{2m} = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= \{ y \in \mathbb{R}^{2m} : |y| < R_0 \} \\ A_2 &= \Big\{ y \in \mathbb{R}^{2m} : |x-y| < \frac{|x|}{2},\ |y| \ge R_0 \Big\} \\ A_3 &= \Big\{ y \in \mathbb{R}^{2m} : |x-y| \ge \frac{|x|}{2},\ |y| \ge R_0 \Big\}, \end{aligned}$$

and where R_0 is chosen so large that

$$\frac{(2m-1)!}{\gamma_m} \int_{A_1} \log \frac{|y|}{|x-y|} e^{2mu(y)} dy \le \Big(-2\alpha + \frac{\varepsilon}{4} \Big) \log|x| \tag{2.23}$$

for $|x|$ large enough. As for A_2 we have

$$\begin{aligned} \int_{A_2} \log \frac{|y|}{|x-y|} e^{2mu} dy &= \int_{A_2} \log \frac{|y|}{|x-y|} e^{2mu} dy + \int_{A_2} \log \frac{|y|}{|x-y|} e^{2mu} dy \\ &\le -\int_{B_\tau(x)} \log|x-y| e^{2mu} dy - \log\tau \int_{A_2} e^{2mu} dy \\ &\quad + \int_{A_2} \log|y| e^{2mu} dy \qquad (2.24) \\ &\le -\int_{B_\tau(x)} \log|x-y| e^{2mu} dy + o(1)(1 + \log(2|x|)), \end{aligned}$$

where $o(1) \to 0$ as $R_0 \to \infty$ and we used that $\int_{A_2} e^{2mu} dy \to 0$ as $R_0 \to \infty$ and $\log|y| \le \log(2|x|)$ for $y \in A_2$. Finally, for $y \in A_3$, one easily verifies that $\frac{|x-y|}{|y|} \ge \frac{1}{4}$, hence

$$\int_{A_3} \log \frac{|y|}{|x-y|} e^{2mu} dy \le \log(4) \cdot \int_{A_3} e^{2mu} dy = o(1), \quad \text{as } R_0(\varepsilon) \to \infty. \tag{2.25}$$

Putting now (2.23), (2.24) and (2.25) together, choosing R_0 large enough (depending on τ and ε) and then $R > 0$ large enough, we get (2.22).

Together with Fubini's theorem, (2.22) implies

$$\begin{aligned}\int_{\mathbb{R}^{2m}\setminus B_R(0)} v^+ dx &\leq C\int_{\mathbb{R}^{2m}}\int_{\mathbb{R}^{2m}} \chi_{|x-y|\leq\tau}\log\frac{1}{|x-y|}e^{2mu(y)}dydx \\ &= C\int_{\mathbb{R}^{2m}} e^{2mu(y)}\int_{B_\tau(y)}\log\frac{1}{|x-y|}dxdy \\ &\leq C\int_{R^{2m}} e^{2mu(y)}dy \leq C. \end{aligned} \tag{2.26}$$

Step 2. From now on, x will be a point in $\mathbb{R}^{2m}$ with $|x| > R$, where R is as in Step 1. Fix $p > 1$ such that $p(2m-2) < 2m$, and $p' = \frac{p}{p-1}$. By Theorem 2.7, there is $\delta > 0$ such that if

$$\int_{B_4(x)} e^{2mu}dy < \delta, \tag{2.27}$$

then

$$\int_{B_4(x)} e^{2mp'|z|}dy \leq C, \tag{2.28}$$

with C independent of x, where z solves

$$\begin{cases} (-\Delta)^m z = (2m-1)!e^{2mu} & \text{in } B_4(x) \\ \Delta^j z = 0 & \text{on } \partial B_4(x) \text{ for } 0 \leq j \leq m-1. \end{cases}$$

We now choose $R > 0$ such that (2.27) is satisfied whenever $|x| \geq R$, and claim that for such x,

$$\int_{B_\tau(x)} e^{2mp'u}dy \leq C\int_{B_\tau(x)} e^{2mp'|z|}dy \leq C\varepsilon. \tag{2.29}$$

We now observe that for any $\sigma > 0$,

$$\int_{\mathbb{R}^{2m}\setminus B_\sigma(x)} \frac{e^{2mu(y)}}{|x-y|^{2j}}dy \to 0 \quad \text{as } |x| \to \infty \tag{2.30}$$

by dominated convergence; by Hölder's inequality and (2.29), if σ is small enough,

$$\int_{B_\sigma(x)} \frac{e^{2mu}}{|x-y|^{2j}}dy \leq \left(\int_{B_\sigma(x)} e^{2mp'u}dy\right)^{\frac{1}{p'}}\left(\int_{B_\sigma(x)} \frac{1}{|x-y|^{2jp}}dy\right)^{\frac{1}{p}} \leq C\varepsilon^{\frac{1}{p'}}.$$

Therefore

$$(-\Delta)^j v(x) = C\int_{\mathbb{R}^{2m}} \frac{e^{2mu}}{|x-y|^{2j}}dy \to 0, \quad \text{as } |x| \to \infty.$$

Finally (2.21) follows from (2.22), (2.29) and Hölder's inequality.

Step 3. It remains to prove (2.29). Set $h := v - z$, so that

$$\begin{cases} \Delta^m h = 0 & \text{in } B_4(x) \\ \Delta^j h = \Delta^j v & \text{on } \partial B_4(x) \text{ for } 0 \leq j \leq m-1, \end{cases}$$

Integrating $(-\Delta)^m v = (2m-1)!e^{2mu}$ and then integrating by parts we get

$$(-1)^m \int_{\partial B_\rho(x)} \frac{\partial}{\partial r}(\Delta^{m-1}v)dS = (2m-1)! \int_{B_\rho(x)} e^{2mu}dy.$$

Dividing by $\omega_{2m}\rho^{2m-1}$, integrating on $[0,R]$ and using Fubini's, we find

$$\begin{aligned}\int_0^R \fint_{\partial B_\rho(x)} \frac{\partial}{\partial r}(\Delta^{m-1}v)d\sigma d\rho &= \int_0^R \fint_{\partial B_1(x)} \frac{\partial}{\partial r}(\Delta^{m-1}v(\rho,\theta))d\theta d\rho \\ &= \fint_{\partial B_1(x)} \int_0^R \frac{\partial}{\partial r}(\Delta^{m-1}v(\rho,\theta))d\rho d\theta = \fint_{\partial B_R(x)} \Delta^{m-1}v d\sigma - \Delta^{m-1}v(x).\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^R \frac{1}{\rho^{2m-1}} \int_{B_\rho(x)} e^{2mu(y)}dyd\rho &= \int_0^R \frac{1}{\rho^{2m-1}} \int_{B_R(x)} e^{2mu(y)}\chi_{|x-y|\le\rho}dyd\rho \\ &= \int_{B_R(x)} e^{2mu(y)} \int_{|x-y|}^R \frac{1}{\rho^{2m-1}}d\rho dy \\ &= \frac{1}{(2m-2)} \int_{B_R(x)} \left[\frac{1}{|x-y|^{2m-2}} - \frac{1}{R^{2m-2}}\right] e^{2mu(y)}dy.\end{aligned}$$

Hence, multiplying above by $\frac{(2m-1)!}{\omega_{2m}}$ and setting $C_{m-1} := \frac{(2m-1)!}{(2m-2)\omega_{2m}}$,

$$\begin{aligned}\fint_{\partial B_R} (-\Delta)^{m-1}v d\sigma &= (-\Delta)^{m-1}v(x) \\ &\quad -C_{m-1} \int_{B_R(x)} \left[\frac{1}{|x-y|^{2m-2}} - \frac{1}{R^{2m-2}}\right] e^{2mu(y)}dy \\ &= C_{m-1}\left[\int_{|x-y|\ge R} \frac{e^{2mu(y)}}{|x-y|^{2m-2}}dy + \int_{B_R(x)} \frac{e^{2mu(y)}}{R^{2m-2}}dy\right]\end{aligned}$$

which implies at once, setting $R=4$,

$$\fint_{\partial B_4(x)} (-\Delta)^{m-1}v dS \le C, \tag{2.31}$$

with C independent of x. Similarly, one can show that

$$\fint_{\partial B_4(x)} (-\Delta)^i v dS \le C, \quad 1 \le i \le m-1. \tag{2.32}$$

By Lemma 2.12 and by (2.11) rescaled and translated to $B_4(x)$ and with the function $-\Delta h$ instead of h, $m-1$ instead of m, we obtain

$$\begin{aligned}-\Delta h(x) &= -\sum_{i=0}^{m-2} \int_{\partial B_4(x)} \frac{\partial \Delta^{m-2-i} G}{\partial n} \Delta^i(\Delta h)dS \qquad (2.33) \\ &= \sum_{i=1}^{m-1} \int_{\partial B_4(x)} c_{i-1}(-\Delta)^i h dS \le C,\end{aligned}$$

where G is the Green function for Δ^{m-1} on $B_4(x)$:

$$\Delta^{m-1}G = \delta_x, \quad \Delta^i G = 0, \text{ on } \partial B_4(x), \text{ for } 0 \le i \le m-2.$$

On the other hand, since the $c_i > 0$, there is some $\tau > 0$ such that the following holds: if $\xi \in B_{2\tau}(x)$ and G_ξ is the Green's function defined by

$$\Delta^{m-1}G_\xi = \delta_\xi, \quad \Delta^i G_\xi = 0, \text{ on } \partial B_4(x), \text{ for } 0 \le i \le m-2,$$

then also

$$0 \le (-1)^i \frac{\partial \Delta^{m-2-i} G_\xi(\eta)}{\partial r} \le C, \quad \text{for } \eta \in \partial B_4(x),\ r := \frac{\eta - x}{4}.$$

Therefore, as in (2.33), we infer

$$-\Delta h \le C \quad \text{on } B_{2\tau}(x), \tag{2.34}$$

for some $\tau \in (0,2)$.

On the other hand, thanks to (2.26) and (2.28),

$$\int_{B_4(x)} h^+ dy \le \int_{B_4(x)} (v^+ + |z|)dy \le C.$$

By elliptic estimates,

$$\sup_{B_\tau(x)} h \le \fint_{B_4(x)} h^+ dy + C \sup_{B_{2\tau(x)}} (-\Delta h) \le C,$$

C independent of x, as usual. Since the polynomial p is bounded from above, we infer

$$u \le h + p + |z| \le C + |z|,$$

and (2.29) follows at once. □

Corollary 2.14 *Any solution u of* (2.1), (2.3) *is bounded from above.*

Proof. Indeed u is continuous, $u = v + p$, and

$$\lim_{|x|\to\infty} v(x) = -\infty, \quad \sup_{x \in \mathbb{R}^{2m}} p(x) < +\infty,$$

by Lemma 2.11. □

Lemma 2.15 *Assume that $|u(x)| = o(|x|^2)$ as $|x| \to \infty$. Then $u = v + C$. Furthermore, for any $\varepsilon > 0$ there exists $R > 0$ such that*

$$-2\alpha \log|x| - C \le u(x) \le (-2\alpha + \varepsilon)\log|x|, \tag{2.35}$$

for $|x| \ge R$.

Proof. Since $v(x) = -2\alpha \log|x| + o(\log|x|)$ at ∞, if $\deg p \ge 2$, we have that $u(x) = v(x) + p(x)$ cannot be $o(|x|^2)$. Hence, knowing that $\deg p$ is even, we get $u = v + C$ for some constant C. Then (2.35) follows at once from Lemma 2.9 and Lemma 2.13. □

Lemma 2.16 *Set $g_u = e^{2u} g_{\mathbb{R}^{2m}}$. If u is a standard solution, then*

$$R_{g_u} \equiv 2m(2m-1).$$

If u is not a standard solution, then

$$\liminf_{|x|\to+\infty} R_{g_u}(x) = -\infty. \tag{2.36}$$

Proof. Assume that u is a standard solution and set

$$u_\lambda(x) := \log \frac{2\lambda}{1+\lambda^2|x|^2}, \quad g_\lambda := e^{2u_\lambda} g_{\mathbb{R}^{2m}}. \tag{2.37}$$

Then, up to translation, $u = u_\lambda$ for some $\lambda > 0$. Since $g_1 = (\pi^{-1})^* g_{S^{2m}}$, where π is the stereographic projection, we have $R_{g_1} \equiv 2m(2m-1)$. Then consider the diffeomorphism of $\mathbb{R}^{2m}$ defined by $\varphi_\lambda(x) := \lambda x$. Then $g_\lambda = \varphi_\lambda^* g_1$, hence $R_{g_\lambda} = R_{g_1} \circ \varphi_\lambda \equiv 2m(2m-1)$.

Assume now that $u = v + p$ is not a standard solution. Since $g_{\mathbb{R}^{2m}}$ is flat, the formula for the conformal change of scalar curvature, in the case $m > 1$, reduces to

$$R_{g_u} = -2(2m-1)e^{-2u}\Big(\Delta u + (m-1)|\nabla u|^2\Big), \tag{2.38}$$

see for instance [SY] pag 184. Then differentiating the expression (2.2) for v and using that $u \le C$, we find that $|\nabla v(x)| \to 0$ as $|x| \to \infty$. We have already seen that $\Delta v(x) \to 0$ as $|x| \to \infty$; since $\deg p \ge 2$ implies

$$\deg \Delta p < \deg |\nabla p|^2,$$

we then have

$$\limsup_{|x|\to\infty} \Big(\Delta u + (m-1)|\nabla u|^2\Big) = \limsup_{|x|\to\infty} \Big(\Delta p + (m-1)|\nabla p|^2\Big) = +\infty.$$

Observing that $e^{-2u} \ge \frac{1}{C} > 0$, u being bounded from above, we easily obtain (2.36). □

Proof of Theorem 2.1. Put together Lemmas 2.9, 2.10, 2.11 and 2.13. □

Proof of Theorem 2.2. (i) ⇒ (iii) is obvious, while (iii) ⇒ (i) follows from the argument of [WX].

(iii) ⇔ (iv) follows from Theorem 2.1.

(iv) ⇒ (ii') ⇒ (ii). Assume that $\deg p = 0$. Then by Theorem 2.1,

$$\lim_{|x|\to\infty} \Delta^j u(x) = \lim_{|x|\to|\infty} \Delta^j p(x) = 0, \quad 1 \le j \le m-1.$$

(ii) ⇒ (iv). By Theorem 2.1, $\sup_{\mathbb{R}^{2m}} p < \infty$ and

$$\lim_{|x|\to\infty} \Delta p(x) = \lim_{|x|\to\infty} \Delta u = 0,$$

hence $\Delta p \equiv 0$ and, by Liouville's theorem, p is constant.

(i) $\Leftrightarrow$ (v) follows from Lemma 2.16.

(i) $\Rightarrow$ (vi) Given a conformal diffeomorphism φ of $\mathbb{R}^{2m}$, $\widetilde{\varphi} := \pi^{-1}\circ\varphi\circ\pi$ is a conformal diffeomorphism of S^{2m}. Any metric of the form $g_u = e^{2u}g_{\mathbb{R}^{2m}}$, with u standard solution of (2.1), can be easily written as $\varphi^* g_1$, for some conformal diffeomorphism φ of $\mathbb{R}^{2m}$,where g_1 is as in (2.37). Then

$$\pi^* g_u = \pi^*\varphi^* g_1 = (\varphi\circ\pi)^* g_1 = (\pi\circ\widetilde{\varphi})^* g_1 = \widetilde{\varphi}^*\pi^* g_1 = \widetilde{\varphi}^* g_{S^2},$$

and clearly $\widetilde{\varphi}^* g_{S^2}$ is a smooth Riemannian metric on S^{2m}.

(vi) $\Rightarrow$ (i). Assume u is non-standard. Then $u = v + p$, $\deg p \geq 2$. Considering that $\sup_{\mathbb{R}^{2m}} p < +\infty$, we infer that p goes to $-\infty$ at least quadratically in some directions. Let $S = (0, \dots, 0, 1) \in S^{2m}$ be the South Pole, and

$$\pi : S^{2m}\backslash\{S\} \to \mathbb{R}^{2m}, \quad \pi(\xi) := \frac{(\xi_1, \dots, \xi_{2m})}{1+\xi_{2m+1}}$$

be the stereographic projection from S. Then

$$(\pi^{-1})^* g_{S^{2m}} = \rho_0 g_{\mathbb{R}^{2m}}, \quad \rho_0(x) := \frac{4}{(1+|x|^2)^2},$$

and

$$\pi^* g_u = \rho_1 g_{S^{2m}}, \quad \rho_1 := \frac{e^{2u}}{\rho_0}\circ\pi \in C^\infty(S^{2m}\backslash\{S\}).$$

Since $e^{2u(x)} \to 0$ more rapidly than $|x|^{-4}$ in some directions, we have

$$\liminf_{\xi\to S}\rho_1(\xi) = \liminf_{|x|\to\infty}\frac{e^{2u(x)}}{\rho_0(x)} = 0,$$

hence $\rho_1 g_{S^{2m}}$ does not extend to a Riemannian metric on S^{2m}.

To prove (2.6), let j be the largest integer such that $\Delta^j p \neq 0$. Then $\Delta^{j+1}p \equiv 0$ and from Theorem 2.6 we infer that $\deg p \leq 2j$. In fact $\deg p = 2j$ and $\Delta^j p \equiv C_0 \neq 0$. From Pizzetti's formula (2.9), we have

$$2m\sum_{i=0}^{j} b_i R^{2i}\Delta^i p(0) = \fint_{\partial B_R} 2mp dS$$

Exponentiating and using Jensen's inequality and Lemma 2.9, we infer

$$\exp\Big(2m\sum_{i=0}^{j} b_i R^{2i}\Delta^i p(0)\Big) \leq \fint_{\partial B_R} e^{2mp} dS \leq CR^{4m\alpha}\fint_{\partial B_R} e^{2mu} dS,$$

for $R \geq 4$. Therefore

$$\varphi(R) := R^{-4m\alpha+2m-1}\exp\Big(2m\sum_{i=0}^{j} b_i R^{2i}\Delta^i p(0)\Big) \in L^1([4,+\infty)),$$

and this is not possible if $C_0 = \Delta^j p > 0$, hence $C_0 < 0$.

$\square$

2.4 Examples

Generalizing an argument of [CC], we now see that solutions of the kind $v+p$ actually exist, even among radially symmetric functions, with $\deg p = 2m-2$, and with $\deg p = 2$. For simplicity, we only treat the case when m is even; if m is odd, the proof is similar. We need the following lemma.

Lemma 2.17 *Let $u(r)$ be a smooth radially symmetric function on $\mathbb{R}^n$, $n \geq 1$. Then for $m \geq 0$ we have*

$$\Delta^m u(0) = \frac{n}{c_m(n+2m)(2m)!} u^{(2m)}(0), \tag{2.39}$$

where the c_i's are the constants in Pizzetti's formula, and $u^{(2m)} := \frac{\partial^{2m} u}{\partial r^{2m}}$. In particular $\Delta^m u(0)$ has the sign of $u^{(2m)}(0)$.

Proof. We first prove that

$$c_m \Delta^m u(0) = \frac{1}{R^{2m}} \fint_{B_R(0)} \frac{r^{2m}}{(2m)!} u^{(2m)}(0) dx. \tag{2.40}$$

Then, observing that

$$\fint_{B_R(0)} \frac{r^{2m}}{(2m)!} dx = \frac{nR^{2m}}{(n+2m)(2m)!}, \tag{2.41}$$

(2.39) follows at once. We prove (2.40) by induction. The case $m=0$ reduces to $u(0)=u(0)$. Let us now assume that (2.40) has been proven for $i=0,\dots,m-1$ and let us prove it for m. Since u is smooth, we have $u^{(i)}(0)=0$ for any odd i, hence Taylor's formula reduces to

$$u(r) = \sum_{i=0}^{m} \frac{r^{2i}}{(2i)!} u^{(2i)}(0) + o(r^{2m+1}).$$

We now divide by R^{2m} in (2.12), take the limit as $R \to 0$ and, observing that $\Delta^{m+1} u(\xi)$ remains bounded as $R \to 0$, we find

$$\lim_{R\to 0} \frac{\fint_{B_R} \left(u - \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i u(0) \right) dx}{R^{2m}} \quad = \quad c_m \Delta^m u(0).$$

Substituting Taylor's formula and using the inductive hypothesis, we see that most of the terms on the left-hand side cancel out (before taking the limit) and we are left with

$$\lim_{R\to 0} \frac{1}{R^{2m}} \fint_{B_R} \left(\frac{r^{2m} u^{(2m)}(0)}{(2m)!} + o(r^{2m+1}) \right) dx = c_m \Delta^m u(0).$$

Finally, to deduce (2.40), observe that, $\frac{1}{R^{2m}} \fint_{B_R(0)} o(r^{2m+1}) dx \to 0$ as $R \to 0$, while $\frac{1}{R^{2m}} \fint_{B_R} \frac{r^{2m} u^{(2m)}(0)}{(2m)!} dx$ does not depend on R thanks to (2.41). □

Proposition 2.18 *For every* $m \geq 2$ even*, there exists a radially symmetric function* u *solving* (2.1), (2.3) *with* $u(x) = -C|x|^{2m-2} + O(|x|^{2m-4})$.

Proof. Set $w_0 = \log \frac{2}{1+r^2}$. Then $\Delta^m w_0 = (2m-1)! e^{2mw_0}$. Define $u = u(r)$ to be the unique solution to the following ODE

$$\begin{cases} \Delta^m u = (2m-1)! e^{2mu} \\ u(0) = \log 2 \\ u^{(2j+1)}(0) = 0 & j = 0, \dots, m-1 \\ u^{(2j)}(0) = \alpha_j \leq w_0^{(2j)}(0) & j = 1, \dots, m-2 \\ u^{(2m-2)}(0) = \alpha_{m-1} < w_0^{(2m-2)}(0) \end{cases}$$

where the α_j's are fixed. We shall first see that $w_0 \geq u$. Set $g := w_0 - u$. Then $g(r) > 0$ for $r > 0$ small enough, hence also $\Delta^m g > 0$ for small $r > 0$. From Lemma 2.17 we get

$$\Delta^j g(0) \geq 0, \quad j = 1, \dots, m-2; \qquad \Delta^{m-1} g(0) > 0. \tag{2.42}$$

We can prove inductively that $\Delta^{m-j} g \geq 0$, $j = 0, \dots, m-1$ as long as $g(r) > 0$. Indeed

$$\int_{B_R(0)} \Delta^j g dx = \int_{\partial B_R(0)} \frac{\partial \Delta^{j-1} g}{\partial r} d\sigma, \tag{2.43}$$

hence, as long as $g(r) > 0$, we have $\frac{\partial \Delta^{j-1} g}{\partial r} > 0$, in particular $\frac{\partial g}{\partial r} > 0$, hence $g(r) > 0$ for all $r > 0$ for which it is defined. From (2.42) and (2.43) we inductively infer

$$\Delta^{m-j} g(r) \geq C r^{2j-2},$$

and, since $\Delta w_0(r) \to 0$ as $r \to \infty$, there is $r_0 > 0$ such that

$$\Delta u \leq -C r^{2m-4}, \quad \text{for } r \geq r_0,$$

integrating which, we find

$$u(r) \leq -C r^{2m-2} \quad \text{for } r \geq r_0. \tag{2.44}$$

To estimate u from below, we use the function

$$w_1(r) = \log 2 - C_1 r^2 - \ldots - C_{m-1} r^{2m-2},$$

where the constants C_i are chosen so that

$$\Delta^j u(0) \geq \Delta^j w_1(0).$$

Then we can proceed as above to prove that $u - w_1 \geq 0$. Hence the solution exists for all times and, thanks to (2.44) and Theorem 2.1, it has the asymptotic behaviour

$$u(r) = -C r^{2m-2} + O(r^{2m-4}).$$

□

Remark. Observe the abundance of solutions: we can choose the $(m-1)$-tuple of initial data $(\alpha_1, \dots, \alpha_{m-1})$ in a set containing an open subset of $\mathbb{R}^{m-1}$.

In the next example we show a radially symmetric solution in $\mathbb{R}^{2m}$, $m \geq 4$ even, of the form $u = v + p$, with $\deg p = 2$, thus showing that the hypothesis $u(x) = o(|x|^2)$ as $|x| \to \infty$ in Theorem 2.2 is sharp.

Proposition 2.19 *Let* $w_0(r) := \log \frac{2}{1+r^2}$ *and let* $u = u(r)$ *(*$r = |x|$*,* $x \in \mathbb{R}^{2m}$ *and* m *even) solve the following ODE:*

$$\begin{cases} \Delta^m u = (2m-1)! e^{2mu} \\ u(0) = \log 2 \\ u^{(2j+1)}(0) = 0 & j = 0, \dots, m-1 \\ u^{(2j)}(0) = w_0^{(2j)}(0) & j = 2, 3, \dots, m-1 \\ u''(0) = w_0''(0) - 1. \end{cases}$$

Then $u(r)$ *is defined for all* $r \geq 0$ *and* $u(r) = -Cr^2 + O(\log r)$ *as* $r \to +\infty$*.*

Proof. As in the proof of Proposition 2.18, we can show that $g := w_0 - u \geq 0$ and $u(r) \leq -Cr^2$. To control u from below, we use the function $w_1(r) = w_0(r) - r^2$, so that redefining $g := u - w_1$, we have

$$g''(0) = 1, \quad g^{(j)}(0) = 0, \quad j = 0, 1, 3, 4, \dots, 2m-1.$$

and we can prove that $g \geq 0$ as before. Hence $u(r)$ exists for all $r \geq 0$, it is non-standard and $u(r) = -Cr^2 + O(\log r)$ as $r \to \infty$, as w_1 bounds it from below. □

Remark. Using (2.38), we can easily compute that in the above examples

$$\lim_{|x| \to \infty} R_g(x) \to -\infty,$$

where $g = e^{2u} g_{\mathbb{R}^{2m}}$.

2.5 Some useful results

We prove here a few results used above.

Lemma 2.20 *Assume that* $u : B_4 \to \mathbb{R}$ *satisfies*

$$\begin{aligned} \|\Delta u\|_{W^{k,p}(B_4)} &\leq C \\ \|u\|_{L^1(B_4)} &\leq C, \end{aligned}$$

for some $p \in (1, \infty)$*. Then*

$$\|u\|_{W^{k+2,p}(B_1)} \leq C.$$

Proof. By Fubini's theorem we can choose $r > 0$ with $2 \leq r \leq 4$ such that

$$\|u\|_{L^1(\partial B_r)} \leq C \|u\|_{L^1(B_4)}.$$

Let's now write $u = u_1 + u_2$, where

$$\begin{cases} \Delta u_1 = 0 & \text{in } B_r \\ u_1 = u & \text{on } \partial B_r \end{cases} \qquad \begin{cases} \Delta u_2 = \Delta u & \text{in } B_r \\ u_2 = 0 & \text{on } \partial B_r \end{cases}$$

By standard L^p-estimates we have $\|u_2\|_{W^{k+2,p}(B_r)} \le C\|\Delta u\|_{W^{k,p}(B_r)}$. From the representation formula of Poisson

$$u_1(x) = \int_{\partial B_r} u_1(y)\Gamma(x-y)dS(y),$$

we obtain $\|u_1\|_{C^k(B_1)} \le C_k\|u_1\|_{L^1(\partial B_r)}$ for every $k \ge 0$. □

Proof of Proposition 2.4. Let $\|h\|_{L^1(B_4)} \le C$, and let us assume $n > 2$. We proceed by steps.

Step 1. We show by induction on j that

$$\|\Delta^{m-j}h\|_{L^\infty(B_2)} \le C. \tag{2.45}$$

The step $j = 0$ is obvious, as $\Delta^m h \equiv 0$. Let us prove the step $j \ge 1$. Let

$$G_{2r}(x) := \frac{1}{(2-n)\omega_n}\left(\frac{1}{|x|^{n-2}} - \frac{1}{(2r)^{n-2}}\right)$$

be the Green function for the Laplace operator on B_{2r} with singularity at 0. Then

$$\Delta^{m-j}h(0) = \fint_{\partial B_{2r}} \Delta^{m-j}hdx + \int_{B_{2r}} G_{2r}\Delta^{m-j+1}hdx.$$

By inductive hypothesis and the scaling property of G_{2r}, the last term is bounded by Cr^2, hence

$$\Delta^{m-j}h(0) \le \fint_{\partial B_{2r}} \Delta^{m-j}hdx + Cr^2,$$

and integrating with respect to r on $[1/2, 1]$, we obtain

$$\Delta^{m-j}h(0) \le \fint_{B_2} \Delta^{m-j}hdx + C. \tag{2.46}$$

To estimate $\fint_{B_2} \Delta^{m-j}hdx$, we use Pizzetti's formula for h at $x \in B_2$,

$$c_{m-j}\Delta^{m-j}h(x) = -\sum_{i=0}^{m-j-1} c_i\Delta^i h(x) \underbrace{- \sum_{i=m-j+1}^{m} c_i\Delta^i h(x) + \fint_{B_1(x)} hdy}_{\le C}$$

by the inductive hypothesis again, and the L^1-bound on h and get

$$c_{m-j}\Delta^{m-j}h(x) \le -\sum_{i=0}^{m-j-1} c_i\Delta^i h(x) + C. \tag{2.47}$$

Averaging in (2.47) over B_2 and using (2.46), we find

$$c_{m-j}\Delta^{m-j}h(0) \le -\sum_{i=0}^{m-j-1}\left(c_i \fint_{B_2} \Delta^i h(x)dx\right) + C.$$

and its scaled version

$$c_{m-j}\Delta^{m-j}h(0) \leq -\sum_{i=0}^{m-j-1}\left(c_i r^{2(i-m+j)} \fint_{B_{2r}} \Delta^i h(x)dx\right) + Cr^{2(j-m)}. \quad (2.48)$$

Consider now a non-negative function $\varphi \in C_c^\infty((1,2))$, with $\int_1^2 \varphi(r)dr = 1$. From (2.48), we find

$$c_{m-j}\Delta^{m-j}h(0) \leq -\sum_{i=0}^{m-j-1} c_i \int_1^2 \left(r^{2(i-m+j)} \fint_{B_{2r}} \Delta^i h(x)dx\ \varphi(r)\right)dr + C.$$

Each term in the sum on the right-hand side can be written as

$$\begin{aligned}
&\left|C\int_1^2 r^{2(i-m+j)-n}\int_{\partial B_{2r}} \frac{\partial \Delta^{i-1}h}{\partial\nu} dS\varphi(r)dr\right| \\
\leq\ & C\left|\int_{B_2\setminus B_1} r^{2(i-m+j)-n}\frac{\partial\Delta^{i-1}h(x)}{\partial\nu}\varphi(|x|)dx\right| \\
=\ & C\int_{B_2\setminus B_1} |h(x)|\left|\frac{\partial}{\partial\nu}\Delta^{i-1}\left(r^{2(i-m+j)-n}\varphi(|x|)\right)\right|dx \\
\leq\ & C\fint_{B_2}|h(x)|dx.
\end{aligned}$$

Working with $-h$ and observing the local character of the above estimates, we obtain (2.45).

Step 2. Fix $\ell \geq m$. We can prove inductively that

$$\|\Delta^{\ell-j}h\|_{W^{2j,p}(B_2)} \leq C(p).$$

The step $j = 0$ is obvious, as $\Delta^\ell h \equiv 0$. For the inductive step, we see that by Lemma 2.20 applied to $\Delta^{\ell-j}h$ (and a simple covering argument to fix the radii), we have

$$\|\Delta^{\ell-j}h\|_{W^{2j,p}(B_1)} \leq C\|\Delta(\Delta^{\ell-j}h)\|_{W^{2j-2,p}(B_2)} + C\underbrace{\|\Delta^{\ell-j}h\|_{L^1(B_2)}}_{\leq C \text{ by Step 1}} \leq C,$$

for every $1 < p < \infty$, and the usual covering argument extends the estimate to B_2. Therefore $\|h\|_{W^{2\ell,p}(B_1)} \leq C(p,\ell)$, and we conclude applying Sobolev's theorem. □

Proposition 2.21 *Let $u \in C^{2m}(\overline{B}_1)$ such that*

$$\begin{cases} (-\Delta)^m u \leq C_1 & \text{in } B_1 \\ (-\Delta)^j u \leq C_1 & \text{on } \partial B_1 \text{ for } 0 \leq j \leq m-1 \end{cases} \quad (2.49)$$

Then there exists a constant C independent of u such that

$$u \leq C \quad \text{in } B_1.$$

If $C_1 = 0$ in (2.49), then $u < 0$ in B_1, unless $u \equiv 0$.

Proof. By induction on m. The case $m = 1$ follows from the maximum principle, applied to the function $v(x) := u(x) - C|x|^2$, which is subharmonic for C large enough. Assume now that the case $m - 1$ has been dealt with and let us consider u satisfying (2.49). Then $v := -\Delta u$ satisfies $v \leq C$ in B_1 by inductive hypothesis. Applying the case $m = 1$ again we conclude. Similarly if $C_1 = 0$. $\square$

Proposition 2.22 (Fundamental solution) *For $m \geq 1$, set*

$$\gamma_m := \omega_{2m} 2^{2m-2}[(m-1)!]^2, \tag{2.50}$$

where $\omega_{2m} := |S^{2m-1}| = \frac{(2\pi)^m}{(2m-2)!!}$. Then the function

$$K(x) := \frac{1}{\gamma_m} \log \frac{1}{|x|}$$

is a fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^{2m}$, i.e. $(-\Delta)^m K = \delta_0$.

Proof. The case $m = 1$ is well-known, so we shall assume $m \geq 2$. Set $r := |x|$. For radial functions we have $\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$, hence for $j \geq 1$

$$-\Delta \log \frac{1}{r} = \frac{2(m-1)}{r^2}, \qquad -\Delta \frac{1}{r^{2j}} = \frac{4j(m-1-j)}{r^{2j+2}}.$$

Then

$$(-\Delta)^j \log \frac{1}{r} = 2^{2j-1} \frac{(j-1)!(m-1)!}{(m-j-1)!} \frac{1}{r^{2j}} \tag{2.51}$$

$$(-\Delta)^{m-1} \log \frac{1}{r} = 2^{2m-3}(m-2)!(m-1)! \frac{1}{r^{2m-2}}. \tag{2.52}$$

Given a function $\varphi \in C_c^\infty(\mathbb{R}^{2m})$, we can apply the usual procedure of integrating by parts in $\mathbb{R}^{2m} \backslash B_\varepsilon(0)$ using

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(0)} |D^k K| dS = 0, \quad 0 \leq k \leq 2m-2,$$

to obtain

$$\begin{aligned} \int_{\mathbb{R}^{2m}} (-\Delta)^m \varphi K dx &= \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(0)} -\varphi \frac{\partial (-\Delta)^{m-1} K}{\partial \nu} dS \\ &= \fint_{\partial B_\varepsilon(0)} \varphi dS \to \varphi(0). \end{aligned}$$

$\square$

Chapter 3

Conformal metrics on $\mathbb{R}^{2m}$ with non-positive Q-curvature

We now turn to the study the conformal metrics on $\mathbb{R}^{2m}$ with constant non-positive Q-curvature $Q \leq 0$ having finite volume. We show that when $Q < 0$ such metrics exist in $\mathbb{R}^{2m}$ if and only if $m > 1$. Moreover we study their asymptotic behavior at infinity, in analogy with the case $Q > 0$, which we treated in Chapter 2. When $Q = 0$, we show that such metrics have the form $e^{2p} g_{\mathbb{R}^{2m}}$, where p is a polynomial such that $2 \leq \deg p \leq 2m - 2$ and $\sup_{\mathbb{R}^{2m}} p < +\infty$. In dimension 4, such metrics are exactly the polynomials p of degree 2 with $\lim_{|x| \to +\infty} p(x) = -\infty$.

3.1 Introduction and statement of the main theorems

Given a constant $Q \in \mathbb{R}$, we consider the solutions to the equation

$$(-\Delta)^m u = Q e^{2mu} \quad \text{on } \mathbb{R}^{2m}, \tag{3.1}$$

satisfying

$$\alpha := \frac{1}{|S^{2m}|} \int_{\mathbb{R}^{2m}} e^{2mu(x)} dx < +\infty. \tag{3.2}$$

Geometrically, if u solves (3.1) and (3.2), then the conformal metric $g := e^{2u} g_{\mathbb{R}^{2m}}$ has Q-curvature $Q_g^{2m} \equiv Q$ and volume $\alpha |S^{2m}|$. Notice that given a solution u to (3.1) and $\lambda > 0$, the function $v := u - \frac{1}{2m} \log \lambda$ solves

$$(-\Delta)^m v = \lambda Q e^{2mv} \quad \text{in } \mathbb{R}^{2m},$$

hence what matters is just the sign of Q, and we can assume without loss of generality that $Q \in \{0, \pm(2m-1)!\}$.

As for the positive case, every solution to (3.1) is smooth, see Corollary 2.8 (the proof does not depend on the sign of Q.).

For $Q \geq 0$, some explicit solutions to (3.1) are known. For instance every polynomial of degree at most $2m-2$ satisfies (3.1) with $Q = 0$, and the function $u(x) = \log \frac{2}{1+|x|^2}$, which we already encountered in Chapter 2, satisfies (3.1) with $Q = (2m-1)!$ and $\alpha = 1$. This latter solution has the property that $e^{2u} g_{\mathbb{R}^{2m}} = (\pi^{-1})^* g_{S^{2m}}$, where $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection.

For the negative case, we notice that the function $w(x) = \log \frac{2}{1-|x|^2}$ solves $(-\Delta)^m w = -(2m-1)! e^{2mw}$ on the unit ball $B_1 \subset \mathbb{R}^{2m}$ (in dimension 2 this corresponds to the Poincaré metric on the disk). However, no explicit entire solution to (3.1) with $Q < 0$ is known, hence one can ask whether such solutions actually exist. In dimension 2 ($m = 1$) it is easy to see that the answer is negative, but quite surprisingly the situation is different in dimension 4 and higher and we have:

Theorem 3.1 *Fix $Q < 0$. For $m = 1$ there is no solution to* (3.1)-(3.2). *For every $m \geq 2$, there exist (several) radially symmetric solutions to* (3.1)-(3.2).

Having now an existence result, we turn to the study of the asymptotic behavior at infinity of solutions to (3.1)-(3.2) when $m \geq 2$, $Q < 0$, having in mind applications to concentration-compactness problems in conformal geometry. To this end, given a solution u to (3.1)-(3.2), we define the auxiliary function

$$v(x) := -\frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y|}{|x-y|}\right) e^{2mu(y)} dy, \tag{3.3}$$

where $\gamma_m := \omega_{2m} 2^{2m-2} [(m-1)!]^2$ is characterized by the following property:

$$(-\Delta)^m \Big(\frac{1}{\gamma_m} \log \frac{1}{|x|}\Big) = \delta_0 \quad \text{in } \mathbb{R}^{2m}.$$

Then $(-\Delta)^m v = -(2m-1)! e^{2mu}$. We prove

Theorem 3.2 *Let u be a solution of* (3.1)-(3.2) *with $Q = -(2m-1)!$. Then*

$$u(x) = v(x) + p(x), \tag{3.4}$$

where p is a non-constant polynomial of even degree at most $2m-2$. Moreover there exist a constant $a \neq 0$, an integer $1 \leq j \leq m-1$ and a closed set $Z \subset S^{2m-1}$ of Hausdorff dimension at most $2m-2$ such that for every compact subset $K \subset S^{2m-1} \backslash Z$ we have

$$\begin{aligned} \lim_{t\to+\infty} \Delta^\ell v(t\xi) &= 0, \quad \ell = 1, \dots, m-1, \\ v(t\xi) &= 2\alpha \log t + o(\log t), \ \text{as } t \to +\infty, \\ \lim_{t\to+\infty} \Delta^j u(t\xi) &= a, \end{aligned} \tag{3.5}$$

for every $\xi \in K$ uniformly in ξ. If $m = 2$, then $Z = \emptyset$ and $\sup_{\mathbb{R}^{2m}} u < +\infty$. Finally

$$\liminf_{|x|\to+\infty} R_{g_u}(x) = -\infty, \tag{3.6}$$

where R_{g_u} is the scalar curvature of $g_u := e^{2u} g_{\mathbb{R}^{2m}}$.

Following the proof of Theorem 3.1, it can be shown that the estimate on the degree of the polynomial is sharp. Recently J. Wei and D. Ye [WY] showed the existence of solutions to $\Delta^2 u = 6e^{4u}$ in $\mathbb{R}^4$ with $\int_{\mathbb{R}^4} e^{4u}dx < +\infty$ which are not radially symmetric. It is plausible that also in the negative case non-radially symmetric solutions exist.

For the case $Q = 0$ we have

Theorem 3.3 *When $Q = 0$, any solution to* (3.1)-(3.2) *is a polynomial p with $2 \le \deg p \le 2m-2$ and with*

$$\sup_{\mathbb{R}^{2m}} p < +\infty.$$

In particular in dimension 2 (case $m = 1$), there are no solutions. In dimension 4 the solutions are exactly the polynomials of degree 2 with $\lim_{|x|\to\infty} p(x) = -\infty$. Finally, there exist $1 \le j \le m-1$ and $a < 0$ such that

$$\lim_{|x|\to\infty} \Delta^j p(x) = a. \tag{3.7}$$

There is an interesting geometric consequence of Theorems 3.2 and 3.3, with applications in concentration-compactness: In the case of a closed manifold, metrics of equibounded volumes and prescribed Q-curvatures *of possibly varying sign* cannot concentrate at points of negative or zero Q-curvature, as we shall see in Chapter 4 (Theorem 4.2).

In sharp contrast with the case of a closed manifold, on an open domain $\Omega \subset \mathbb{R}^{2m}$ (or a manifold with boundary), $m > 1$, concentration is possible at points of negative or zero curvature. Indeed, take any solution u of (3.1)-(3.2) with $Q \le 0$, whose existence is given by Theorem 3.1, and consider the sequence

$$u_k(x) := u(k(x - x_0)) + \log k, \quad \text{for } x \in \Omega$$

for some fixed $x_0 \in \Omega$. Then $(-\Delta)^m u_k = Qe^{2mu_k}$ and u_k concentrates at x_0 in the sense that as $k \to \infty$ we have $u_k(x_0) \to +\infty$, $u_k \to -\infty$ a.e. in Ω and $e^{2mu_k}dx \rightharpoonup \alpha|S^{2m}|\delta_{x_0}$ in the sense of measures.

The 2 dimensional case ($m = 1$) is different and concentration at points of non-positive curvature can be ruled out on open domains too, because otherwise a standard blowing-up procedure would yield a solution to (3.1)-(3.2) with $Q \le 0$, contradicting with Theorem 3.1.

This chapter is organized as follows. The proof of Theorems 3.1, 3.2 and 3.3 is given in the following three sections; in the last section we collect some open questions. In the following, the letter C denotes a generic constant, which may change from line to line and even within the same line.

3.2 Existence theory

Theorem 3.1 follows from Propositions 3.4 and 3.5 below.

Proposition 3.4 *For $m = 1$, $Q < 0$ there are no solutions to* (3.1)-(3.2).

Proof. Assume that such a solution u exists. Then, by the maximum principle, and Jensen's inequality,

$$\fint_{\partial B_R} u d\sigma \geq u(0), \qquad \int_{\partial B_R} e^{2u} d\sigma \geq 2\pi R e^{2u(0)}.$$

Integrating in R on $[1, +\infty)$, we get

$$\int_{\mathbb{R}^2} e^{2u} dx = +\infty,$$

contradiction. □

Proposition 3.5 *For $m \geq 2$, $Q < 0$ there exist radial solutions to* (3.1)-(3.2).

Proof. We consider separately the cases when m is even and when m is odd. *Case 1: m even.* Let $u = u(r)$ be the unique solution of the following ODE:

$$\begin{cases} \Delta^m u(r) = -(2m-1)! e^{2mu(r)} & \\ u^{(2j+1)}(0) = 0 & 0 \leq j \leq m-1 \\ u^{(2j)}(0) = \alpha_j \leq 0 & 0 \leq j \leq m-1, \end{cases}$$

where $\alpha_0 = 0$ and $\alpha_1 < 0$. We claim that the solution exists for all $r \geq 0$. To see that, we shall use barriers, compare [CC, Theorem 2]. Let us define

$$w_+(r) = \frac{\alpha_1}{2} r^2, \quad g_+ := w_+ - u.$$

Then $\Delta^m g_+ \geq 0$. By the divergence theorem,

$$\int_{B_R} \Delta^j g_+ dx = \int_{\partial B_R} \frac{d\Delta^{j-1} g_+}{dr} d\sigma.$$

Moreover, from Lemma 2.17, we infer

$$\Delta^j g_+(0) \geq 0 \quad \text{ for } 0 \leq j \leq m-1,$$

hence we see inductively that $\Delta^j g_+(r) \geq 0$ for every r such that $g_+(r)$ is defined and for $0 \leq j \leq m-1$. In particular $g_+ \geq 0$ as long as it exists.

Let us now define

$$w_-(r) := \sum_{i=0}^{m-1} \beta_i r^{2i} - A \log \frac{2}{1+r^2}, \quad g_- := u - w_-,$$

where the β_i's and A will be chosen later. Notice that

$$\Delta^m w_-(r) = \Delta^m \Big(- A \log \frac{2}{1+r^2} \Big) = -(2m-1)! A \Big(\frac{2}{1+r^2} \Big)^{2m}.$$

Since $\alpha_1 < 0$,

$$\lim_{r \to +\infty} \frac{\left(\frac{2}{1+r^2}\right)^{2m}}{e^{m\alpha_1 r^2}} = +\infty,$$

and taking into account that $u \le w_+$, we can choose A large enough, so that

$$\begin{aligned}\Delta^m g_-(r) &= (2m-1)!\left[A\left(\frac{2}{1+r^2}\right)^{2m} - e^{2mu(r)}\right]\\ &\ge (2m-1)!\left[A\left(\frac{2}{1+r^2}\right)^{2m} - e^{m\alpha_1 r^2}\right] \ge 0.\end{aligned}$$

We now choose each β_i so that

$$\Delta^j g_-(0) \ge 0, \quad 0 \le j \le m-1,$$

and proceed by induction as above to prove that $g_- \ge 0$. Hence

$$w_-(r) \le u(r) \le w_+(r)$$

as long as u exists, and by standard ODE theory, that implies that $u(r)$ exists for all $r \ge 0$. Finally

$$\int_{\mathbb{R}^{2m}} e^{2mu(|x|)}dx \le \int_{\mathbb{R}^{2m}} e^{m\alpha_1|x|^2}dx < +\infty.$$

Case 2: $m \ge 3$ odd. Let $u = u(r)$ solve

$$\begin{cases}\Delta^m u(r) = (2m-1)!e^{2mu(r)} & \\ u^{(2j+1)}(0) = 0 & 0 \le j \le m-1\\ u^{(2j)}(0) = \alpha_j \le 0 & 0 \le j \le m-1,\end{cases}$$

where the α_i's have to be chosen. Set

$$w_+(r) := \beta - r^2 - \log\frac{2}{1+r^2}, \quad g_+ := w_+ - u,$$

where $\beta < 0$ is such that $e^{-r^2+\beta} \le \left(\frac{2}{1+r^2}\right)^2$, hence

$$\frac{2}{1+r^2} - \frac{1+r^2}{2}e^{-r^2+\beta} \ge 0 \quad \text{for all } r > 0.$$

Then, as long as $g_+ \ge 0$, we have

$$\begin{aligned}\Delta^m g_+(r) &= (2m-1)!\left[\left(\frac{2}{1+r^2}\right)^{2m} - e^{2mu(r)}\right]\\ &\ge (2m-1)!\left[\left(\frac{2}{1+r^2}\right)^{2m} - e^{2mw_+(r)}\right] \ge 0\end{aligned}$$

Choose now the α_i's so that, $u^{(2i)}(0) < w_+^{(2i)}(0)$, for $0 \le i \le m-1$. From Lemma 2.17, we infer that

$$\Delta^i g_+(0) \ge 0, \quad 0 \le i \le m-1,$$

and we see by induction that $g_+ \ge 0$ as long as it is defined. As lower barrier, define

$$w_-(r) = \sum_{i=0}^{m-1}\beta_i r^{2i}, \quad g_- := u - w_-,$$

where the β_i's are chosen so that $\Delta^i g_-(0) \geq 0$. Then, observing that

$$\Delta^m g_-(r) = (2m-1)! e^{2mu(r)} > 0,$$

as long as u is defined, we conclude as before that $g_- \geq 0$ as long as it is defined. Then u is defined for all times.

Let $R > 0$ be such that, for every $r \geq R$, $w_+(r) \leq -\frac{r^2}{2}$. Then

$$\int_{\mathbb{R}^{2m}} e^{2mu(|x|)} dx \leq \int_{B_R} e^{2mu(|x|)} dx + \int_{\mathbb{R}^{2m} \setminus B_R} e^{-m|x|^2} dx < +\infty.$$

□

3.3 Asymptotic behaviour in the negative case

The proof of Theorem 3.2, to which this section is devoted, is divided in several lemmas. The following Liouville-type theorem will prove very useful.

Theorem 3.6 *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h \leq u - v$, where $e^{pu} \in L^1(\mathbb{R}^n)$ for some $p > 0$, $(-v)^+ \in L^1(\mathbb{R}^n)$. Then h is a polynomial of degree at most $2m-2$.*

Proof. As in the proof of Theorem 2.5, for any $x \in \mathbb{R}^{2m}$ we have

$$\begin{aligned} |D^{2m-1}h(x)| &\leq \frac{C}{R^{2m-1}} \fint_{B_R(x)} |h(y)| dy \\ &= -\frac{C}{R^{2m-1}} \fint_{B_R(x)} h(y) dy + \frac{2C}{R^{2m-1}} \fint_{B_R(x)} h^+ dy \end{aligned} \tag{3.8}$$

and

$$\fint_{B_R(x)} h(y) dy = O(R^{2m-2}), \quad \text{as } R \to \infty.$$

Then

$$\fint_{B_R(x)} h^+ dy \leq \fint_{B_R(x)} u^+ dy + C \fint_{B_R(x)} (-v)^+ dy \leq \frac{1}{p} \fint_{B_R(x)} e^{pu} dy + \frac{C}{R^{2m}},$$

and both terms in (3.8) divided by R^{2m-1} go to 0 as $R \to \infty$. □

Lemma 3.7 *Let u be a solution of* (3.1)-(3.2). *Then, for $|x| \geq 4$*

$$v(x) \leq 2\alpha \log |x| + C. \tag{3.9}$$

Proof. As in Lemma 2.9, changing v with $-v$. □

Lemma 3.8 *For any $\varepsilon > 0$, there is $R > 0$ such that for $|x| \geq R$,*

$$v(x) \geq \left(2\alpha - \frac{\varepsilon}{2}\right) \log|x| + \frac{(2m-1)!}{\gamma_m} \int_{B_1(x)} \log|x-y| e^{2mu(y)} dy. \tag{3.10}$$

Moreover

$$(-v)^+ \in L^1(\mathbb{R}^{2m}). \tag{3.11}$$

Proof. To prove (3.10) we follow [Lin], Lemma 2.4. Choose $R_0 > 0$ such that

$$\frac{1}{|S^{2m}|} \int_{B_{R_0}} e^{2mu} dx \geq \alpha - \frac{\varepsilon}{16},$$

and decompose

$$\begin{aligned} \mathbb{R}^{2m} &= B_{R_0} \cup A_1 \cup A_2, \\ A_1 &:= \{y \in \mathbb{R}^{2m} : 2|x-y| \leq |x|, |y| \geq R_0\}, \\ A_2 &:= \{y \in \mathbb{R}^{2m} : 2|x-y| > |x|, |y| \geq R_0\}. \end{aligned}$$

Next choose $R \geq 2$ such that for $|x| > R$ and $|y| \leq R_0$, we have $\log \frac{|x-y|}{|y|} \geq \log|x| - \varepsilon$. Then, observing that $\frac{(2m-1)!|S^{2m}|}{\gamma_m} = 2$, we have for $|x| > R$

$$\begin{aligned} \frac{(2m-1)!}{\gamma_m} \int_{B_{R_0}} \log \frac{|x-y|}{|y|} e^{2mu(y)} dy &\geq \left(\log|x| - \frac{\varepsilon}{16}\right) \frac{(2m-1)!}{\gamma_m} \int_{B_{R_0}} e^{2mu} dy \\ &\geq \left(2\alpha - \frac{\varepsilon}{8}\right) \log|x| - C\varepsilon. \end{aligned} \tag{3.12}$$

Observing that $\log|x-y| \geq 0$ for $y \notin B_1(x)$, $\log|y| \leq \log(2|x|)$ for $y \in A_1$, $\int_{A_1} e^{2mu} dy \leq \frac{\varepsilon|S^{2m}|}{16}$ and $\log(2|x|) \leq 2\log|x|$ for $|x| \geq R$, we infer

$$\begin{aligned} \int_{A_1} \log \frac{|x-y|}{|y|} e^{2mu(y)} dy &= \int_{A_1} \log|x-y| e^{2mu(y)} dy - \int_{A_1} \log|y| e^{2mu(y)} dy \\ &\geq \int_{B_1(x)} \log|x-y| e^{2mu(y)} dy - \log(2|x|) \int_{A_1} e^{2mu} dy \\ &\geq \int_{B_1(x)} \log|x-y| e^{2mu(y)} dy - \log|x| \frac{\varepsilon|S^{2m}|}{8}. \end{aligned} \tag{3.13}$$

Finally, for $y \in A_2$, $|x| > R$ we have that $\frac{|x-y|}{|y|} \geq \frac{1}{4}$, hence

$$\int_{A_2} \log \frac{|x-y|}{|y|} e^{2mu(y)} dy \geq -\log(4) \int_{A_2} e^{2mu} dy \geq -C\varepsilon. \tag{3.14}$$

Putting together (3.12), (3.13) and (3.14), and possibly taking R even larger, we obtain (3.10). From (3.10) and Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}^{2m} \setminus B_R} (-v)^+ dx &\leq C \int_{\mathbb{R}^{2m}} \int_{\mathbb{R}^{2m}} \chi_{|x-y|<1} \log \frac{1}{|x-y|} e^{2mu(y)} dy dx \\ &= C \int_{\mathbb{R}^{2m}} e^{2mu(y)} \int_{B_1(y)} \log \frac{1}{|x-y|} dx dy \\ &\leq C \int_{R^{2m}} e^{2mu(y)} dy < \infty. \end{aligned}$$

Since $v \in C^\infty(\mathbb{R}^{2m})$, we conclude that $\int_{B_R} (-v)^+ dx < \infty$ and (3.11) follows. □

Lemma 3.9 *Let u be a solution of* (3.1)-(3.2)*, with $m \geq 2$. Then $u = v + p$, where p is a polynomial of degree at most $2m - 2$.*

Proof. Let $p := u - v$. Then $\Delta^m p = 0$. Apply (3.11) and Theorem 3.6. □

Lemma 3.10 *Let p be the polynomial of Lemma 3.9. Then if $m = 2$, there exists $\delta > 0$ such that*

$$p(x) \leq -\delta|x|^2 + C. \tag{3.15}$$

In particular $\lim_{|x|\to\infty} p(x) = -\infty$ *and* $\deg p = 2$*. For $m \geq 3$ there is a (possibly empty) closed set $Z \subset S^{2m-1}$ of Hausdorff dimension $\dim^{\mathcal{H}}(Z) \leq 2m - 2$ such that for every $K \subset S^{2m-1} \backslash Z$ closed, there exists $\delta = \delta(K) > 0$ such that*

$$p(x) \leq -\delta|x|^2 + C \quad \text{for } \frac{x}{|x|} \in K. \tag{3.16}$$

Consequently $\deg p$ *is even.*

Proof. From (3.11), we infer that there is a set A_0 of finite measure such that

$$v(x) \geq -C \quad \text{in } \mathbb{R}^{2m} \backslash A_0. \tag{3.17}$$

Case $m = 2$. Up to a rotation, we can write

$$p(x) = \sum_{i=1}^{4} (b_i x_i^2 + c_i x_i) + b_0.$$

Assume that $b_{i_0} \geq 0$ for some $1 \leq i_0 \leq 4$. Then on the set

$$A_1 := \{x \in \mathbb{R}^4 : |x_i| \leq 1 \text{ for } i \neq i_0,\ c_{i_0} x_{i_0} \geq 0\}$$

we have $p(x) \geq -C$. Moreover $|A_1| = +\infty$. Then, from (3.17) we infer

$$\int_{\mathbb{R}^4} e^{4u} dx \geq \int_{A_1 \backslash A_0} e^{4(v+p)} dx \geq C|A_1 \backslash A_0| = +\infty, \tag{3.18}$$

contradicting (3.2). Therefore $b_i < 0$ for every i and (3.15) follows at once.

Case $m \geq 3$. From (3.2) and (3.17) we infer that p cannot be constant. Write

$$p(t\xi) = \sum_{i=0}^{d} a_i(\xi) t^i, \qquad d := \deg p,$$

where for each $0 \leq i \leq d$, a_i is a homogeneous polynomial of degree i or $a_i \equiv 0$. With a computation similar to (3.18), (3.2) and (3.17) imply that $a_d(\xi) \leq 0$ for each $\xi \in S^{2m-1}$. Moreover d is even, otherwise $a_d(\xi) = -a_d(-\xi) \leq 0$ for every $\xi \in S^{2m-1}$, which would imply $a_d \equiv 0$. Set

$$Z = \{\xi \subset S^{2m-1} : a_d(\xi) = 0\}.$$

We claim that $\dim^{\mathcal{H}}(Z) \leq 2m - 2$. To see that, set

$$V := \{x \in \mathbb{R}^{2m} \ : \ a_d(x) = 0\} = \{t\xi \ : \ t \geq 0,\ \xi \in Z\}.$$

Since V is a cone and $Z = V \cap S^{2m-1}$, we only need to show that $\dim^{\mathcal{H}}(V) \leq 2m-1$. Set

$$V_i := \{x \in \mathbb{R}^{2m} \ : \ a_d(x) = \ldots = \nabla^i a_d(x) = 0, \ \nabla^{i+1} a_d(x) \neq 0\}.$$

Noticing that $V_i = \emptyset$ for $i \geq d$ (otherwise $a_d \equiv 0$), we find $V = \cup_{i=0}^{d-1} V_i$. By the implicit function theorem, $\dim^{\mathcal{H}}(V_i) \leq 2m-1$ for every $i \geq 0$ and the claim is proved.

Finally, for every compact set $K \subset S^{2m-1} \backslash Z$, there is a constant $\delta > 0$ such that $a_d(\xi) \leq -\frac{\delta}{2}$, and since $d \geq 2$, (3.16) follows. □

Corollary 3.11 *Any solution u of* (3.1)-(3.2) *with $m = 2$, $Q < 0$ is bounded from above.*

Proof. Indeed $u = v + p$ and, for some $\delta > 0$,

$$v(x) \leq 2\alpha \log|x| + C, \quad p(x) \leq -\delta|x|^2 + C.$$

□

Lemma 3.12 *Let $v : \mathbb{R}^{2m} \to \mathbb{R}$ be defined as in* (3.3) *and Z as in Lemma 3.10. Then for every $K \subset S^{2m-1} \backslash Z$ compact we have*

$$\lim_{t \to +\infty} \Delta^{m-j} v(t\xi) = 0, \quad j = 1, \ldots, m-1 \tag{3.19}$$

for every $\xi \in K$ uniformly in ξ; for every $\varepsilon > 0$ there is $R = R(\varepsilon, K) > 0$ such that, for $t > R$, $\xi \in K$,

$$v(t\xi) \geq (2\alpha - \varepsilon) \log t \tag{3.20}$$

Proof. Fix $K \in S^{2m-1} \backslash Z$ compact and set $\mathcal{C}_K := \{t\xi : t \geq 0, \xi \in K\}$. For any $\sigma > 0$, $1 \leq j \leq 2m-1$,

$$\int_{\mathbb{R}^{2m} \backslash B_\sigma(x)} \frac{e^{2mu(y)}}{|x-y|^{2j}} dy \to 0 \quad \text{as } |x| \to \infty \tag{3.21}$$

by dominated convergence. Choose a compact set $\widetilde{K} \subset S^{2m-1} \backslash Z$ such that $K \subset \operatorname{int}(\widetilde{K}) \subset S^{2m-1}$. Since $u \leq C(\widetilde{K})$ on $\mathcal{C}_{\widetilde{K}}$ by Lemma 3.7 and Lemma 3.10, we can choose $\sigma = \sigma(\varepsilon) > 0$ so small that

$$\int_{B_\sigma(x)} \frac{e^{2mu}}{|x-y|^{2j}} dy \leq C(\widetilde{K}) \int_{B_\sigma(x)} \frac{1}{|x-y|^{2j}} dy \leq C(\widetilde{K})\varepsilon, \quad \text{for } x \in \mathcal{C}_K, \ |x| \text{ large},$$

where $|x|$ is so large that $B_\sigma(x) \subset \mathcal{C}_{\widetilde{K}}$. Therefore

$$(-1)^{j+1} \Delta^j v(x) = C \int_{\mathbb{R}^{2m}} \frac{e^{2mu}}{|x-y|^{2j}} dy \to 0, \quad \text{for } x \in \mathcal{C}_K, \text{ as } |x| \to \infty,$$

We have seen in Lemma 3.8, that for any $\varepsilon > 0$ there is $R > 0$ such that for $|x| \geq R$

$$v(x) \geq \left(2\alpha - \frac{\varepsilon}{2}\right) \log|x| + \frac{(2m-1)!}{\gamma_m} \int_{B_1(x)} \log|x-y| e^{2mu(y)} dy, \tag{3.22}$$

and (3.20) follows easily by choosing $\widetilde{K}$ as above and observing that $u \leq C(\widetilde{K})$ on $\mathcal{C}_{\widetilde{K}}$, hence on $B_1(x)$ for $x \in \mathcal{C}_K$ with $|x|$ large enough. $\square$

Proof of Theorem 3.2. The decomposition $u = v + p$ and the properties of v and p follow at once from Lemmas 3.7, 3.9, 3.10 and 3.12; (3.6) follow as in Theorem 2.2. As for (3.5), let j be the largest integer such that $\Delta^j p \not\equiv 0$. Then $\Delta^{j+1} p \equiv 0$ and from Theorem 3.6 we infer that $\deg p = 2j$, hence $\Delta^j p \equiv a \neq 0$. $\square$

3.4 The case $Q = 0$

Proof of Theorem 3.3. From Theorem 3.6, with $v \equiv 0$, we have that u is a polynomial of degree at most $2m - 2$. Then, as in [Mar1, Lemma 11], we have

$$\sup_{\mathbb{R}^{2m}} u < +\infty,$$

and, since u cannot be constant, we infer that $\deg u \geq 2$ is even. The proof of (3.7) is analogous to the case $Q < 0$, as long as we do not care about the sign of a. To show that $a < 0$, one proceeds as in [Mar1, Theorem 2]. For the case $m = 2$ one proceeds as in Lemma 3.10, setting $v \equiv 0$ and $A_0 = \emptyset$. $\square$

Example One might believe that every polynomial p on $\mathbb{R}^{2m}$ of degree at most $2m - 2$ with $\int_{\mathbb{R}^{2m}} e^{2mp} dx < \infty$ satisfies $\lim_{|x| \to \infty} p(x) = -\infty$, as in the case $m = 2$. Consider on $\mathbb{R}^{2m}$, $m \geq 3$ the polynomial $u(x) = -(1 + x_1^2)|\widetilde{x}|^2$, where $\widetilde{x} = (x_2, \ldots, x_{2m})$. Then $\Delta^m u \equiv 0$ and

$$\begin{aligned} \int_{\mathbb{R}^{2m}} e^{2mu} dx &= \int_{\mathbb{R}} \int_{\mathbb{R}^{2m-1}} e^{-2m(1+x_1^2)|\widetilde{x}|^2} d\widetilde{x} dx_1 \\ &= \int_{\mathbb{R}} \frac{dx_1}{(1 + x_1^2)^{\frac{2m-1}{2}}} \cdot \int_{\mathbb{R}^{2m-1}} e^{-2m|\widetilde{y}|^2} d\widetilde{y} < +\infty. \end{aligned}$$

On the other hand, $\limsup_{|x| \to \infty} u(x) = 0$.

3.5 Open questions

Open Question 1 *Does the claim of Corollary 3.11 hold for $m > 2$? In other words, is any solution u to (3.1)-(3.2) with $Q < 0$ bounded from above?*

This is an important regularity issue, in particular with regard to the behavior at infinity of the function v defined in (3.3). If $\sup_{\mathbb{R}^{2m}} u < +\infty$, then one can take $Z = \emptyset$ in Theorem 3.2, as in the case $Q > 0$, see Theorem 2.1.

Definition 3.13 *Let $\mathcal{P}_0^{2m}$ be the set of polynomials p of degree at most $2m - 2$ on $\mathbb{R}^{2m}$ such that $e^{2mp} \in L^1(\mathbb{R}^{2m})$. Let $\mathcal{P}_+^{2m}$ be the set of polynomials p of degree at most $2m - 2$ on $\mathbb{R}^{2m}$ such that there exists a solution $u = v + p$ to (3.1)-(3.2) with $Q > 0$. Similarly for $\mathcal{P}_-^{2m}$ with $Q < 0$.*

Related to the first question is the following

Open Question 2 *What are the sets* $\mathcal{P}_0^{2m}$, $\mathcal{P}_\pm^{2m}$*? Is it true that* $\mathcal{P}_0^{2m} \subset \mathcal{P}_+^{2m}$ *and* $\mathcal{P}_0^{2m} \subset \mathcal{P}_-^{2m}$*?*

J. Wei and D. Ye [WY] proved that $\mathcal{P}_0^4 \subset \mathcal{P}_+^4$ (and actually more). Consider now on $\mathbb{R}^{2m}$, $m \geq 3$, the polynomial

$$p(x) = -(1 + x_1^2)|\widetilde{x}|^2, \quad \widetilde{x} = (x_2, \dots, x_{2m}).$$

As seen above, $e^{2mp} \in L^1(\mathbb{R}^{2m})$, hence $p \in \mathcal{P}_0^{2m}$. Assume that $p \in \mathcal{P}_-^{2m}$ as well, i.e. there is a function $u = v + p$ satisfying (3.1)-(3.2) and $Q < 0$. Then we claim that $\sup_{\mathbb{R}^{2m}} u = \infty$. Assume by contradiction that u is bounded from above. Then (3.9) and (3.10) imply that

$$v(x) = 2\alpha \log|x| + o(\log|x|), \quad \text{as } |x| \to \infty.$$

Therefore,

$$\lim_{x_1 \to \infty} u(x_1, 0, \dots, 0) = \lim_{x_1 \to \infty} 2\alpha \log x_1 = \infty,$$

contradiction.

Open Question 3 *Even in the case that* u *is not bounded from above, is it true that one can take* $Z = \emptyset$ *in Theorem 3.2 for* $m \geq 3$ *also?*

For instance, in order to show that $v(x) = 2\alpha \log|x| + o(\log|x|)$ as $|x| \to +\infty$, thanks to (3.10), it is enough to show that

$$\int_{B_1(x)} \log|x - y| e^{2mu(y)} dy = o(\log|x|), \quad \text{as } |x| \to +\infty,$$

which is true if $\sup_{\mathbb{R}^{2m}} u < \infty$, but it might also be true if $\sup_{\mathbb{R}^{2m}} u = \infty$.

Open Question 4 *What values can the* α *given by* (3.1)-(3.2) *assume for a fixed* Q*?*

As usual, it is enough to consider $Q \in \{0, \pm(2m-1)!\}$. When $m = 1$, $Q = 1$, then $\alpha = 1$, see [CL]. When $m = 2$, $Q = 6$, then α can take any value in $(0, 1]$, as shown in [CC]. Moreover α cannot be greater than 1 and the case $\alpha = 1$ corresponds to standard solutions, as proved in [Lin]. For the trivial case $Q = 0$, α can take any positive value, and for the other cases we have no answer.

Chapter 4

Concentration-Compactness for the Liouville equation

Using the tools developed in Chapter 2 and 3, we now investigate different concentration-compactness phenomena related to the Q-curvature in arbitrary even dimension. We first treat the case of an open domain in $\mathbb{R}^{2m}$, then that of a closed manifold and, finally, the particular case of the sphere S^{2m}. In all cases we allow the sign of the Q-curvature to vary, and show that in the case of a closed manifold, contrary to the case of open domains in $\mathbb{R}^{2m}$, concentration phenomena can occur only at points of positive Q-curvature. As a consequence, on a locally conformally flat manifold of non-positive Euler characteristic we always have compactness. In the next chapter we shall apply some of these results to prove an energy quantization estimate for an equation related to the Adams-Moser-Trudinger inequality.

4.1 Introduction and statement of the main results

Before stating the main results of this chapter, we recall a few properties of the Paneitz operator P_g^{2m} and the Q-curvature Q_g^{2m} on a $2m$-dimensional smooth Riemannian manifold (M, g), which shall be used later. First of all we have the Gauss formula, describing how the Q-curvature changes under a conformal change of metric: If $g_u := e^{2u}g$, $u \in C^\infty(M)$, then

$$P_g^{2m}u + Q_g^{2m} = Q_{g_u}^{2m}e^{2mu}. \tag{4.1}$$

Then, we have the conformal invariance of the total Q-curvature, when M is closed:

$$\int_M Q_{g_u}^{2m}\mathrm{dvol}_{g_u} = \int_M Q_g^{2m}\mathrm{dvol}_g. \tag{4.2}$$

Finally, assuming (M, g) closed and locally conformally flat , we have the Gauss-Bonnet-Chern formula (see e.g. [Che], [Cha]):

$$\int_M Q_g^{2m}\mathrm{dvol}_g = \frac{\Lambda_1}{2}\chi(M), \tag{4.3}$$

where $\chi(M)$ is the Euler-Poincaré characteristic of M and

$$\Lambda_1 := \int_{S^{2m}} Q_{g_{S^{2m}}} \mathrm{dvol}_{g_{S^{2m}}} = (2m-1)!|S^{2m}| \tag{4.4}$$

is a constant which we shall meet often in the sequel. In the 4-dimensional case, if (M,g) is not locally conformally flat, we have

$$\int_M \left(Q_g^4 + \frac{|W_g|^2}{4} \right) \mathrm{dvol}_g = 8\pi^2 \chi(M), \tag{4.5}$$

where W_g is the Weyl tensor. Recently S. Alexakis [Ale2] (see also [Ale1]) proved an analogous to (4.5) for $m \geq 3$:

$$\int_M \left(Q_g^{2m} + W \right) \mathrm{dvol}_g = \frac{\Lambda_1}{2} \chi(M), \tag{4.6}$$

where W is a local conformal invariant involving the Weyl tensor and its covariant derivatives.

We can now state the main problem treated in this chapter. Given a $2m$-dimensional Riemannian manifold (M,g), consider a converging sequence of functions $Q_k \to Q_0$ in $C^0(M)$, and let $g_k := e^{2u_k} g$ be conformal metrics satisfying $Q_{g_k}^{2m} = Q_k$. In view of (4.1), the u_k's satisfy the following elliptic equation of order $2m$ with critical exponential non-linearity

$$P_g^{2m} u_k + Q_g^{2m} = Q_k e^{2mu_k}. \tag{4.7}$$

Assume further that there is a constant $C > 0$ such that

$$\mathrm{vol}(g_k) := \int_M e^{2mu_k} \mathrm{dvol}_g \leq C \quad \text{for all } k. \tag{4.8}$$

What can be said about the compactness properties of the sequence (u_k)?

In general non-compactness has to be expected, at least as a consequence of the non-compactness of the Möbius group on $\mathbb{R}^{2m}$ or S^{2m}. For instance, for every $\lambda > 0$ and $x_0 \in \mathbb{R}^{2m}$, the metric on $\mathbb{R}^{2m}$ given by $g_u := e^{2u} g_{\mathbb{R}^{2m}}$, $u(x) := \log \frac{2\lambda}{1+\lambda^2|x-x_0|^2}$, satisfies $Q_{g_u}^{2m} \equiv (2m-1)!$.

We start by considering the case when (M,g) is an open domain $\Omega \subset \mathbb{R}^{2m}$ with Euclidean metric $g_{\mathbb{R}^{2m}}$. Since $P_{g_{\mathbb{R}^{2m}}} = (-\Delta)^m$ and $Q_{g_{\mathbb{R}^{2m}}} \equiv 0$, Equation (4.7) reduces to $(-\Delta)^m u_k = Q_k e^{2mu_k}$. The compactness properties of this equation were studied in dimension 2 by Brézis and Merle [BM]. They proved that if $Q_k \geq 0$, $\|Q_k\|_{L^\infty} \leq C$ and $\|e^{2u_k}\|_{L^1} \leq C$, then up to selecting a subsequence, one of the following is true:

(i) (u_k) is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$.

(ii) $u_k \to -\infty$ locally uniformly in Ω.

(iii) There is a finite set $S = \{x^{(i)}; i = 1, \dots, I\} \subset \Omega$ such that $u_k \to -\infty$ locally uniformly in $\Omega \backslash S$. Moreover $Q_k e^{2u_k} \rightharpoonup \sum_{i=1}^I \beta_i \delta_{x^{(i)}}$ weakly in the sense of measures, where $\beta_i \geq 2\pi$ for every $1 \leq i \leq I$.

Subsequently, Li and Shafrir [LS] proved that in case (iii) $\beta_i \in 4\pi\mathbb{N}$ for every $1 \le i \le I$.

Adimurthi, Robert and Struwe [ARS] studied the case of dimension 4 ($m = 2$). As they showed, the situation is more subtle because the blow-up set (the set of points x such that $u_k(x) \to \infty$ as $k \to \infty$) can have dimension up to 3 (in contrast to the finite blow-up set S in dimension 2). Moreover, as a consequence of a result of Chang and Chen [CC], quantization in the sense of Li-Shafrir does not hold anymore, see also [Rob1], [Rob2].

In the following theorem we extend the result of [ARS] to arbitrary even dimension (see also Proposition 4.5 below). The function a_k in (4.9) has no geometric meaning, and one can take $a_k \equiv 1$ at first. But we shall need it for later applications (see Proposition 5.7).

Theorem 4.1 *Let Ω be a domain in $\mathbb{R}^{2m}$, $m > 1$, and let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions satisfying*

$$(-\Delta)^m u_k = Q_k e^{2ma_k u_k}, \tag{4.9}$$

where $a_k, Q_0 \in C^0(\Omega)$, Q_0 is bounded, and $Q_k \to Q_0$, $a_k \to 1$ locally uniformly. Assume that

$$\int_\Omega e^{2ma_k u_k} dx \le C, \tag{4.10}$$

for all k and define the finite (possibly empty) set

$$S_1 := \left\{ x \in \Omega : \lim_{r\to 0^+} \lim_{k\to\infty} \int_{B_r(x)} |Q_k| e^{2ma_k u_k} dy \ge \frac{\Lambda_1}{2} \right\} = \left\{ x^{(i)} : 1 \le i \le I \right\},$$

where Λ_1 is as in (4.4). Then one of the following is true.

(i) For every $0 \le \alpha < 1$, a subsequence converges in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash S_1)$.

(ii) There exist a subsequence, still denoted by (u_k), a closed nowhere dense set S_0 of Hausdorff dimension at most $2m-1$ such that, letting $S = S_0 \cup S_1$, we have $u_k \to -\infty$ locally uniformly in $\Omega\backslash S$ as $k \to \infty$. Moreover there is a sequence of numbers $\beta_k \to \infty$ such that

$$\frac{u_k}{\beta_k} \to \varphi \text{ in } C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash S), \quad 0 \le \alpha < 1,$$

where $\varphi \in C^\infty(\Omega\backslash S_1)$, $S_0 = \{x \in \Omega : \varphi(x) = 0\}$, and

$$(-\Delta)^m \varphi \equiv 0, \quad \varphi \le 0, \quad \varphi \not\equiv 0 \quad \text{in } \Omega\backslash S_1.$$

If $S_1 \ne \emptyset$ and $Q_0(x^{(i)}) > 0$ for some $1 \le i \le I$, then case (ii) occurs.

In Theorem 3.1 above we proved the existence of solutions to the equation $(-\Delta)^m u = Qe^{2mu}$ on $\mathbb{R}^{2m}$ with $Q < 0$ constant and $e^{2mu} \in L^1(\mathbb{R}^{2m})$, for $m > 1$. Scaling any such solution we find a sequence of solutions $u_k(x) := u(kx) + \log k$ concentrating at a point of negative Q-curvature. For $m = 1$ that is not possible.

On a closed manifold things are different in several respects. Under the assumption (which we always make) that $\ker P^{2m}_g$ contains only constant functions, quantization of the total Q-curvature in the sense of Li-Shafrir (see (4.12) below) holds, as proved in dimension 4 by Druet and Robert [DR] and Malchiodi [Mal], and in arbitrary dimension by Ndiaye [Ndi]. Moreover the concentration set is finite. In [DR], however, it is assumed that the Q-curvatures are positive, while in [Mal] and [Ndi], a slightly different equation is studied ($P_g^{2m}u_k + Q_k = h_k e^{2mu_k}$, with h_k constant and Q_k prescribed), for which the negative case is simpler. With the help of Theorem 3.2 and 3.3 from Chapter3 and a technique of Robert and Struwe [RS], we can allow the prescribed Q-curvatures to have varying signs and, contrary to the case of an open domain in $\mathbb{R}^{2m}$, we can rule out concentration at points of negative Q-curvature. Moreover, using Theorems 2.1 and 2.2 from Chapter 2, we can generalize the techniques of [DR] to prove quantization of the total Q-curvature.

Theorem 4.2 *Let (M,g) be a $2m$-dimensional closed Riemannian manifold, such that $\ker P_g = \{constants\}$, and let (u_k) be a sequence of solutions to (4.7), (4.8) where the Q_k's and Q_0 are given continuous functions and $Q_k \to Q_0$ in $C^0(M)$. Let Λ_1 be as in (4.4). Then one of the following is true.*

(i) For every $0 \le \alpha < 1$, a subsequence converges in $C^{2m-1,\alpha}(M)$.

(ii) There exists a finite (possibly empty) set $S = \{x^{(i)} : 1 \le i \le I\}$ such that $Q_0(x^{(i)}) > 0$ for $1 \le i \le I$ and, up to taking a subsequence, $u_k \to -\infty$ locally uniformly on $(M \backslash S)$. Moreover

$$Q_k e^{2mu_k} \,\mathrm{dvol}_g \rightharpoonup \sum_{i=1}^{I} \Lambda_1 \delta_{x^{(i)}} \tag{4.11}$$

in the sense of measures; then (4.2) gives

$$\int_M Q_g \mathrm{dvol}_g = I\Lambda_1. \tag{4.12}$$

Finally, $S = \emptyset$ if and only if $\mathrm{vol}(g_k) \to 0$.

An immediate consequence of Theorem 4.2 (Identity (4.12) in particular) and the Gauss-Bonnet-Chern formulas (4.3) and (4.5), is the following compactness result:

Corollary 4.3 *Under the hypothesis of Theorem 4.2 assume that either*

1. $\chi(M) \le 0$ and $\dim M \in \{2,4\}$, or

2. $\chi(M) \le 0$, $\dim M \ge 6$ and (M,g) is locally conformally flat,

and that $\mathrm{vol}(g_k) \not\to 0$. Then (i) in Theorem 4.2 occurs.

It is not clear whether the hypothesis that (M,g) be locally conformally flat when $\dim M \ge 6$ is necessary in Corollary 4.3. For instance, we could drop it if we knew that $W \ge 0$ in (4.6), in analogy with (4.5).

Theorems 4.1 and 4.2 will be proven in Sections 4.2 and 4.3 respectively. In Section 4.4 we also consider the special case when $M = S^{2m}$.

In the proofs of the above theorems we use techniques and ideas from several of the cited papers, particularly from [ARS], [BM], [DR], [Mal], [MS] and [RS].

As usual the letter C will denote a generic positive constant, which may change from line to line and even within the same line.

4.2 The case of an open domain in $\mathbb{R}^{2m}$

In this section we devote ourselves to the proof of Theorem 4.1. In the following the constants γ_m (defined in Proposition 2.22) and Λ_1 (see (4.4)) are often used, and it is useful to notice that $\gamma_m := \frac{\Lambda_1}{2}$.

Preliminary to the proof on Theorem 4.1 we need the following Lemma.

Lemma 4.4 *Let $f \in L^1(\Omega) \cap L^p_{\mathrm{loc}}(\Omega \backslash S_1)$ for some $p > 1$, where $\Omega \subset \mathbb{R}^{2m}$ and $S_1 \subset \Omega$ is a finite set. Assume that*

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega \\ \Delta^j u = 0 & \text{on } \partial\Omega \text{ for } 0 \le j \le m-1. \end{cases}$$

Then u is bounded in $W^{2m,p}_{\mathrm{loc}}(\Omega \backslash S_1)$; more precisely, for any $\overline{B_{4R}(x_0)} \subset (\Omega \backslash S_1)$, there is a constant C independent of f such that

$$\|u\|_{W^{2m,p}(B_R(x_0))} \le C(\|f\|_{L^p(B_{4R}(x_0))} + \|f\|_{L^1(\Omega)}). \tag{4.13}$$

The proof of Lemma 4.4 is given at the end of this Section.

Proof of Theorem 4.1. We closely follow [ARS]. Choose a subsequence (u_k) and a maximal set (finite by (4.10)) $S_1 = \{x^{(i)} \in \Omega : 1 \le i \le I\}$ such that for every i and $0 < R < \mathrm{dist}(x^{(i)}, \partial\Omega)$ we have

$$\liminf_{k \to \infty} \int_{B_R(x^{(i)})} |Q_k| e^{2ma_k u_k} dx \ge \gamma_m.$$

By maximality of S_1, given $x_0 \in \Omega \backslash S_1$, we have, for some $0 < R < \mathrm{dist}(x_0, \partial\Omega)$,

$$\alpha := \limsup_{k \to \infty} \int_{B_R(x_0)} |Q_k| e^{2ma_k u_k} dx < \gamma_m. \tag{4.14}$$

For such x_0 and R write $u_k = v_k + h_k$ in $B_R(x_0)$, where

$$\begin{cases} (-\Delta)^m v_k = Q_k e^{2ma_k u_k} & \text{in } B_R(x_0) \\ v_k = \Delta v_k = \ldots = \Delta^{m-1} v_k = 0 & \text{on } \partial B_R(x_0) \end{cases}$$

and $(-\Delta)^m h_k = 0$. Set $h_k^+ := \chi_{\{h_k \ge 0\}} h_k$, $h_k^- := h_k - h_k^+$. Since $h_k^+ \le u_k^+ + |v_k|$, we have

$$\|h_k^+\|_{L^1(B_R(x_0))} \le \|u_k^+\|_{L^1(B_R(x_0))} + \|v_k\|_{L^1(B_R(x_0))}.$$

Observe that, for k large enough $mu_k^+ \le 2ma_k u_k^+ \le e^{2ma_k u_k}$, on $B_R(x_0)$, hence by (4.10)

$$\int_{B_R(x_0)} u_k^+ dx \le C \int_{B_R(x_0)} e^{2ma_k u_k} dx \le C.$$

As for v_k, observe that $1 < \frac{\gamma_m}{\alpha}$, hence by Theorem 2.7

$$\int_{B_R(x_0)} 2m|v_k| dx \le \int_{B_R(x_0)} e^{2m|v_k|} dx \le CR^{2m},$$

with C depending on α and not on k. Hence

$$\|h_k^+\|_{L^1(B_R(x_0))} \le C. \tag{4.15}$$

We distinguish 2 cases.

Case 1. Suppose that $\|h_k\|_{L^1(B_{R/2}(x_0))} \le C$ uniformly in k. Then by Proposition 2.4 we have that h_k is equibounded in $C^\ell(B_{R/8}(x_0))$ for every $\ell \ge 0$. Moreover, by Pizzetti's formula (Identity (2.7) in the appendix) and (4.15),

$$\begin{aligned}\fint_{B_R(x_0)} |h_k(x)|dx &= \fint_{B_R(x_0)} h_k^+(x)dx - \fint_{B_R(x_0)} h_k^-(x)dx \le C - \fint_{B_R(x_0)} h_k(x)dx \\ &= C - h_k(x_0) + \sum_{i=0}^{m-1} c_i R^{2i}\Delta^i h_k(x_0) \le C.\end{aligned}$$

Hence we can apply Proposition 2.4 locally on all of $B_R(x_0)$ and obtain bounds for (h_k) in $C^\ell_{\mathrm{loc}}(B_R(x_0))$ for any $\ell \ge 0$.

Fix $p \in (1, \gamma_m/\alpha)$. By Theorem 2.7 $\|e^{2m|v_k|}\|_{L^p(B_R(x_0))} \le C(p)$, hence, using that $a_k \to 1$ uniformly on $B_R(x_0)$, we infer

$$\|(-\Delta^m)v_k\|_{L^p(B)} = \|(Q_k e^{2ma_k h_k})e^{2ma_k v_k}\|_{L^p(B)} \le C(B,p) \tag{4.16}$$

for every ball $B \subset\subset B_R(x_0)$ and for k large enough. In addition $\|v_k\|_{L^p(B_R(x_0))} \le C$, hence by elliptic estimates, $\|v_k\|_{W^{2m,p}(B)} \le C(B,p)$ for every ball $B \subset\subset B_R(x_0)$. By the immersion $W^{2m,p} \hookrightarrow C^{0,\alpha}$, (v_k), is bounded in $C^{0,\alpha}_{\mathrm{loc}}(B_R(x_0))$.

Going back to (4.16), we now see that $\Delta^m v_k$ is locally bounded, hence

$$\|v_k\|_{W^{2m,p}(B)} \le C(B,p)$$

for every $p > 1$, $B \subset\subset B_R(x_0)$, and by the immersion $W^{2m,p} \hookrightarrow C^{2m-1,\alpha}$ we obtain that (v_k), hence (u_k), is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_R(x_0))$.

Case 2. Assume that $\|h_k\|_{L^1(B_{R/2}(x_0))} =: \beta_k \to \infty$ as $k \to \infty$. Set $\varphi_k := \frac{h_k}{\beta_k}$, so that

1. $\Delta^m \varphi_k = 0$,
2. $\|\varphi_k\|_{L^1(B_{R/2}(x_0))} = 1$,
3. $\|\varphi_k^+\|_{L^1(B_R(x_0))} \to 0$ by (4.15).

As above we have that φ_k is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_R(x_0))$ for every $\alpha \in [0,1)$, hence a subsequence converges in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_R(x_0))$ to a function φ, with

1. $\Delta^m \varphi = 0$,
2. $\|\varphi\|_{L^1(B_{R/2}(x_0))} = 1$,
3. $\|\varphi^+\|_{L^1(B_R(x_0))} = 0$, hence $\varphi \le 0$.

Let us define $S_0 = \{x \in B_R(x_0) : \varphi(x) = 0\}$. Take $x \in S_0$; then by (2.7), $\Delta\varphi(x), \dots, \Delta^{m-1}\varphi(x)$ cannot all vanish, unless $\varphi \equiv 0$ on $B_\rho(x) \subset B_R(x_0)$ for some $\rho > 0$, but then by analyticity, we would have $\varphi \equiv 0$, contradiction. Hence there exists j with $1 \le j \le 2m-3$ such that

$$D^j\varphi(x) = 0, \quad D^{j+1}\varphi(x) \ne 0,$$

i.e.

$$S_0 \subset \bigcup_{j=1}^{2m-3} \{x \in B_R(x_0) : D^j\varphi(x) = 0, D^{j+1}\varphi(x) \neq 0\}.$$

Therefore S_0 is $(2m-1)$-rectifiable. Then $\varphi < 0$ almost everywhere and by continuity

$$h_k = \beta_k \varphi_k \to -\infty, \quad e^{2ma_k h_k} \to 0$$

locally uniformly on $B_R(x_0)\backslash S_0$. Then, as before, from

$$(-\Delta)^m v_k = (Q_k e^{2ma_k h_k})(e^{2ma_k v_k}),$$

we have that v_k is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash S_0)$. Then $u_k = h_k + v_k \to -\infty$ uniformly locally away from S_0.

Since Case 1 and Case 2 are mutually exclusive, we obtain that away from S_1 we have that either a subsequence u_k is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega)$, or a subsequence $u_k \to -\infty$ locally uniformly.

We now show that if $I \geq 1$ and $Q_0(x^{(i)}) > 0$ for some $1 \leq i \leq I$, then Case 2 occurs. Assume by contradiction that $Q_0(x_0) > 0$ for some $x_0 \in S_1$ and Case 1 occurs, i.e. (u_k) is bounded in $C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega\backslash S_1)$, so that $f_k := Q_k e^{2ma_k u_k}$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega\backslash S_1)$. Then there exists a finite signed measure μ on Ω, with $\mu \in L^\infty_{\mathrm{loc}}(\Omega\backslash S_1)$ such that

$$\begin{array}{lll} f_k & \rightharpoonup & \mu \quad \text{as measures} \\ f_k & \rightharpoonup & \mu \quad \text{in } L^p_{\mathrm{loc}}(\Omega\backslash S_1) \text{ for } 1 \leq p < \infty. \end{array}$$

Let us take $R > 0$ such that $\overline{B_R(x_0)} \subset \Omega$, $B_R(x_0) \cap S_1 = \{x_0\}$ and $Q_0 > 0$ on $B_R(x_0)$. By our assumption,

$$(-\Delta)^j u_k \geq C, \quad \text{on } \partial B_R(x_0) \text{ for } 0 \leq j \leq m-1. \tag{4.17}$$

Let z_k be the solution to

$$\begin{cases} (-\Delta)^m z_k = Q_k e^{2ma_k u_k} & \text{in } B_R(x_0) \\ z_k = \Delta z_k = \ldots = \Delta^{m-1} z_k = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

By Proposition 2.21 and (4.17)

$$u_k \geq z_k - C. \tag{4.18}$$

By Lemma 4.4 $z_k \to z$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(B_R(x_0)\backslash\{x_0\})$, where

$$\begin{cases} (-\Delta)^m z = \mu & \text{in } B_R(x_0) \\ z = \Delta z = \ldots = \Delta^{m-1} z = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

Since $Q_0(x_0) > 0$, we have $\mu \geq \gamma_m \delta_{x_0} = (-\Delta)^m \ln \frac{1}{|x-x_0|}$, and Proposition 2.21 applied to the function $z(x) - \ln \frac{1}{|x-x_0|}$ implies

$$z(x) \geq \ln \frac{1}{|x-x_0|} - C,$$

hence

$$\int_{B_R(x_0)} e^{2mz} dx \geq C \int_{B_R(x_0)} \frac{1}{|x-x_0|^{2m}} dx = +\infty.$$

Then (4.18) and Fatou's lemma imply

$$\begin{aligned}\liminf_{k\to\infty}\int_{B_R(x_0)} e^{2ma_k u_k}dx &\geq \int_{B_R(x_0)} \liminf_{k\to\infty} e^{2ma_k u_k}dx \\ &\geq C\int_{B_R(x_0)} \liminf_{k\to\infty} e^{2ma_k z_k}dx \\ &\geq C\int_{B_R(x_0)} e^{2mz}dx = +\infty,\end{aligned} \tag{4.19}$$

contradicting (4.10). □

The following proposition gives a general procedure to blow up at points where u_k goes to infinity.

Proposition 4.5 *In the hypothesis of Theorem 4.1, assume that $a_k \equiv 1$ for every k and that case (ii) occurs. Then, for every $x_0 \in S$ such that $\sup_{B_R(x_0)} u_k \to \infty$ for every $0 < R < \mathrm{dist}(x_0, \partial\Omega)$ as $k \to \infty$, there exist points $x_k \to x_0$ and positive numbers $r_k \to 0$ such that*

$$v_k(x) := u_k(x_k + r_k x) + \ln r_k \leq 0 \leq \ln 2 + v_k(0), \tag{4.20}$$

and as $k \to \infty$ either a subsequence $v_k \to v$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where

$$(-\Delta)^m v = Q_0(x_0)e^{2mv},$$

or $v_k \to -\infty$ almost everywhere and there are positive numbers $\gamma_k \to +\infty$ such that

$$\frac{v_k}{\gamma_k} \to p \quad \textit{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}),$$

where p is a polynomial on even degree at most $2m-2$.

Proof. Following [ARS], take x_0 such that $\sup_{B_R(x_0)} u_k \to +\infty$ for every R and select, for $R < \mathrm{dist}(x_0, \partial\Omega)$, $0 \leq r_k < R$ and $x_k \in \overline{B_{r_k}(x_0)}$ such that

$$(R - r_k)e^{u_k(x_k)} = (R - r_k) \sup_{B_{r_k}(x_0)} e^{u_k} = \max_{0\leq r<R}\Big((R-r)\sup_{B_r(x_0)} e^{u_k}\Big) =: L_k.$$

Then $L_k \to +\infty$ and $s_k := \frac{R-r_k}{2L_k} \to 0$ as $k \to \infty$, and

$$v_k(x) := u_k(x_k + s_k x) + \ln s_k \leq 0 \quad \text{in } B_{L_k}(0)$$

satisfies

$$(-\Delta)^m v_k = \widetilde{Q}_k e^{2mv_k}, \quad \widetilde{Q}_k(x) := Q_k(x_k + s_k x),$$

and

$$\int_{B_{L_k}(0)} \widetilde{Q}_k e^{2mv_k}dx = \int_{B_{\frac{1}{2}(R-r_k)}(x_k)} Q_k e^{2mu_k}dx \leq C.$$

We can now apply the first part of the theorem to the functions v_k, observing that there are no concentration points ($S_1 = \emptyset$), since $v_k \leq 0$, and using Theorem 2.5 to characterize the function p. □

We now give a proof of Lemma 4.4. Preliminary to that, we need the following lemma.

Lemma 4.6 *Let $\Delta u \in L^1(\Omega)$ and $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Then for every $1 \le p < \frac{n}{n-1}$ we have*

$$\|u\|_{W^{1,p}(\Omega)} \le C(p)\|\Delta u\|_{L^1(\Omega)}$$

Proof. Let $u \in C^\infty(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$. If $1 \le p < \frac{n}{n-1}$, then $q := \frac{p}{p-1} > n$. From L^p-theory (see e.g. [Sim, Pag. 91]) and the imbedding $W^{1,q} \hookrightarrow L^\infty$ we infer

$$\begin{aligned}\|\nabla u\|_{L^p(\Omega)} &\le C \sup_{\substack{\varphi \in W_0^{1,q}(\Omega)\\ \|\nabla\varphi\|_{L^q(\Omega)} \le 1}} \int_\Omega \nabla u \cdot \nabla\varphi dx = C \sup_{\substack{\varphi \in W_0^{1,q}(\Omega)\\ \|\nabla\varphi\|_{L^q(\Omega)} \le 1}} \int_\Omega -\Delta u\varphi dx \\ &\le C \sup_{\substack{\varphi \in L^\infty(\Omega)\\ \|\varphi\|_{L^\infty(\Omega)} \le 1}} \int_\Omega -\Delta u\varphi dx \le C\|\Delta u\|_{L^1}.\end{aligned}$$

To estimate $\|u\|_{L^p(\Omega)}$ we use Poincaré's inequality. For the general case one can use a standard mollifying procedure. □

Proof of Lemma 4.4. By Lemma 4.6, $\|\Delta^{m-1}u\|_{W^{1,r}(\Omega)} \le C(r)\|f\|_{L^1(\Omega)}$ for $1 \le r < \frac{2m}{2m-1}$. Then, by L^p-theory, $\|u\|_{W^{2m-1,r}(\Omega)} \le C(r)\|f\|_{L^1(\Omega)}$, and by Sobolev's embedding,

$$\|u\|_{L^s(\Omega)} \le C(s)\|f\|_{L^1(\Omega)}, \quad \text{for all } 1 \le s < \infty. \tag{4.21}$$

Now fix $B = B_{4R}(x_0) \subset\subset (\Omega \backslash S_1)$ and write $u = u_1 + u_2$, where

$$\begin{cases} (-\Delta)^m u_2 = f & \text{in } B_{4R}(x_0) \\ \Delta^j u_2 = 0 & \text{on } \partial B_{4R}(x_0) \text{ for } 0 \le j \le m-1. \end{cases}$$

By L^p-theory

$$\|u_2\|_{W^{2m,p}(B_{4R}(x_0))} \le C(p,B)\|f\|_{L^p(B_{4R}(x_0))}, \tag{4.22}$$

with $C(p,B)$ depending on p and the chosen ball B. Together with (4.21), we find

$$\|u_1\|_{L^1(B_{4R}(x_0))} \le C(p,B)(\|f\|_{L^p(B_{4R}(x_0))} + \|f\|_{L^1(\Omega)}).$$

By Proposition 2.4

$$\|u_1\|_{W^{2m,p}(B_R(x_0))} \le C(p,B)(\|f\|_{L^p(B_{4R}(x_0))} + \|f\|_{L^1(\Omega)}),$$

and (4.13) follows. □

4.3 The case of a closed manifold

To prove Theorem 4.2 we assume that $\sup_M u_k \to \infty$ and we blow up at I suitably chosen sequences of points $x_{i,k} \to x^{(i)}$ with $u_k(x_{i,k}) \to \infty$ as $k \to \infty$, $1 \le i \le I$. We call the $x^{(i)}$'s concentration points. Then we show the following:

(i) If $x^{(i)}$ is a concentration point, then $Q_0(x^{(i)}) > 0$.

(ii) The profile of the u_k's at any concentration point is the function η_0 defined in (5.4), hence it carries the fixed amount of energy Λ_1, see (5.7).

(iii) $u_k \to -\infty$ locally uniformly in $M \backslash \{x^{(i)} : 1 \le i \le I\}$.

(iv) The *neck energy* vanishes in the sense of (4.43) below, hence in the limit only the energy of the profiles at the concentration points appears.

Parts (i) and (ii) (Proposition 4.8) follow from Lemma 4.7 below and the classification results of [Mar1] (or [Xu]) and [Mar2]. For parts (iii) and (iv) we adapt a technique of [DR], see also [Mal], [Ndi] for a different approach.

The following lemma (compare [Mal, Lemma 2.3]) is important, because its failure in the non-compact case is responsible for the rich concentration-compactness behavior in Theorem 4.1. Its proof relies on the existence and on basic properties of the Green function for the Paneitz operator P_g^{2m}, as proven in [Ndi, Lemma 2.1] (here we need the hypothesis $\ker P_g^{2m} = \{constants\}$).

Lemma 4.7 *Let (u_k) be a sequence of functions on (M, g) satisfying (4.7) and (4.8). Then for $\ell = 1, \dots, 2m-1$, we have*

$$\int_{B_r(x)} |\nabla^\ell u_k|^p \, \mathrm{dvol}_g \le C(p) r^{2m-\ell p}, \quad 1 \le p < \frac{2m}{\ell},$$

for every $x \in M$, $0 < r < r_{\mathrm{inj}}$ and for every k, where r_{inj} is the injectivity radius of (M, g).

Proof. Set $f_k := Q_k e^{2mu_k} - Q_g^{2m}$, which is bounded in $L^1(M)$ thanks to (4.8). Let G_ξ be the Green's function for P_g^{2m} on (M, g) such that

$$u_k(\xi) = \fint_M u_k \, \mathrm{dvol}_g + \int_M G_\xi(y) f_k(y) \, \mathrm{dvol}_g(y). \tag{4.23}$$

For $x, \xi \in M$, $x \ne \xi$, we have

$$|\nabla_\xi^\ell G_\xi(x)| \le \frac{C}{\mathrm{dist}(x,\xi)^\ell}, \quad 1 \le \ell \le 2m-1. \tag{4.24}$$

Then, differentiating (4.23) and using (4.24) and Jensen's inequality, we get

$$\begin{aligned} |\nabla^\ell u_k(\xi)|^p &\le C\Big(\int_M \frac{1}{\mathrm{dist}(\xi,y)^\ell} |f_k(y)| \, \mathrm{dvol}_g(y)\Big)^p \\ &\le C \int_M \left(\frac{\|f_k\|_{L^1(M)}}{\mathrm{dist}(\xi,y)^\ell}\right)^p \frac{|f_k(y)|}{\|f_k\|_{L^1(M)}} \, \mathrm{dvol}_g(y). \end{aligned}$$

From Fubini's theorem we then conclude

$$\begin{aligned} \int_{B_r(x)} |\nabla^\ell u_k(\xi)|^p \, \mathrm{dvol}_g(\xi) &\le C\|f_k\|_{L^1(M)}^p \sup_{y \in M} \int_{B_r(x_0)} \frac{1}{\mathrm{dist}(\xi,y)^{\ell p}} \, \mathrm{dvol}_g(\xi) \\ &\le C r^{2m-\ell p}. \end{aligned}$$

□

Let $\exp_x : T_x M \cong \mathbb{R}^{2m} \to M$ denote the exponential map at x.

Proposition 4.8 *Let (u_k) be a sequence of solutions to (4.7), (4.8) with*

$$\max u_k \to \infty \quad \text{as } k \to \infty.$$

Choose points $x_k \to x_0 \in M$ (up to a subsequence) such that $u_k(x_k) = \max_M u_k$. Then $Q_0(x_0) > 0$ and, setting

$$\mu_k := 2\left(\frac{(2m-1)!}{Q_0(x_0)}\right)^{\frac{1}{2m}} e^{-u_k(x_k)} \tag{4.25}$$

we find that the functions $\eta_k : B_{\frac{r_{\mathrm{inj}}}{\mu_k}} \subset \mathbb{R}^{2m} \to \mathbb{R}$, given by

$$\eta_k(y) := u_k(\exp_{x_k}(\mu_k y)) + \log \mu_k - \frac{1}{2m} \log \frac{(2m-1)!}{Q_0(x_0)},$$

converge up to a subsequence to $\eta_0(y) = \ln \frac{2}{1+|y|^2}$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$. Moreover

$$\lim_{R\to+\infty} \lim_{k\to\infty} \int_{B_{R\mu_k}(x_k)} Q_k e^{2mu_k} \, \mathrm{dvol}_g = \Lambda_1. \tag{4.26}$$

Proof. Step 1. Set $\sigma_k = e^{-u_k(x_k)}$, and consider on $B_{\frac{r_{\mathrm{inj}}}{\sigma_k}} \subset \mathbb{R}^{2m}$ the functions

$$z_k(y) := u_k(\exp_{x_k}(\sigma_k y)) + \log(\sigma_k) \le 0, \tag{4.27}$$

and the metrics

$$\tilde{g}_k := (\exp_{x_k} \circ T_k)^* g,$$

where $T_k : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, $T_k y = \sigma_k y$. Then, setting $\hat{Q}_k(y) := Q_k(\exp_{x_k}(\sigma_k y))$, and pulling back (4.7) via $\exp_{x_k} \circ T_k$, we get

$$P^{2m}_{\tilde{g}_k} z_k + Q^{2m}_{\tilde{g}_k} = \sigma_k^{-2m} \hat{Q}_k e^{2mz_k}. \tag{4.28}$$

Setting now $\hat{g}_k := \sigma_k^{-2} \tilde{g}_k$, we have $P^{2m}_{\hat{g}_k} = \sigma_k^{2m} P^{2m}_{\tilde{g}_k}$, $Q^{2m}_{\hat{g}_k} = \sigma_k^{2m} Q^{2m}_{\tilde{g}_k}$, and from (4.28) we infer

$$P^{2m}_{\hat{g}_k} z_k + Q^{2m}_{\hat{g}_k} = \hat{Q}_k e^{2mz_k}. \tag{4.29}$$

Then, since the principal part of the Paneitz operator is $(-\Delta_g)^m$, we can write

$$P_{\hat{g}_k} = (-\Delta_{\hat{g}_k})^m + A_k,$$

where A_k is a linear differential operator of order at most $2m-1$; moreover the coefficients of A_k are going to 0 locally in all norms, since $\hat{g}_k \to g_{\mathbb{R}^{2m}}$ locally in all norms, and $P_{g_{\mathbb{R}^{2m}}} = (-\Delta)^m$. Then (4.29) can be written as

$$(-\Delta_{\hat{g}_k})^m z_k + A_k z_k + Q^{2m}_{\hat{g}_k} = \hat{Q}_k e^{2mz_k}. \tag{4.30}$$

Step 2. We now claim that $z_k \to z_0$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where

$$(-\Delta)^m z_0 = Q_0(x_0) e^{2mz_0}, \quad \int_{\mathbb{R}^{2m}} e^{2mz_0} dx < \infty. \tag{4.31}$$

We first assume $m > 1$. Fix $R > 0$ and write $z_k = h_k + w_k$ on B_R, where $\Delta^m_{\hat{g}_k} h_k = 0$ and

$$\begin{cases} (-\Delta_{\hat{g}_k})^m w_k = (-\Delta_{\hat{g}_k})^m z_k. & \text{in } B_R(x_0) \\ w_k = \Delta w_k = \ldots = \Delta^{m-1} w_k = 0 & \text{on } \partial B_R(x_0) \end{cases} \tag{4.32}$$

From $z_k \leq 0$ we infer $\|\hat{Q}_k e^{2mz_k}\|_{L^\infty(B_R)} \leq C$, and clearly $Q^{2m}_{\hat{g}_k} = \sigma_k^{2m} Q^{2m}_{\tilde{g}_k} \to 0$ in $L^\infty_{\text{loc}}(\mathbb{R}^{2m})$. Lemma 4.7 implies that $(A_k z_k)$ is bounded in $L^p(B_R)$, $1 \leq p < \frac{2m}{2m-1}$, hence from (4.32) and elliptic estimates we get uniform bounds for (w_k) in $W^{2m,p}(B_R)$, $1 \leq p < \frac{2m}{2m-1}$, hence in $C^0(B_R)$. Again using Lemma 4.7, we get

$$\|\Delta_{\hat{g}_k} h_k\|_{L^1(B_R)} \leq C(\|z_k\|_{W^{2,1}(B_R)} + \|w_k\|_{W^{2,1}(B_R)}) \leq C.$$

Since $\Delta^{m-1}_{\hat{g}_k}(\Delta_{\hat{g}_k} h_k) = 0$, elliptic estimates (compare Proposition 2.4) give

$$\|\Delta_{\hat{g}_k} h_k\|_{C^\ell(B_{R/2})} \leq C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{4.33}$$

This, together with $|h_k(0)| = |w_k(0)| \leq C$, and $h_k \leq -w_k \leq C$ and elliptic estimates (e.g. [GT, Thm. 8.18]), implies that $\|h_k\|_{L^1(B_{R/2})} \leq C$, hence, again using elliptic estimates,

$$\|h_k\|_{C^\ell(B_{R/4})} \leq C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{4.34}$$

Therefore (z_k) is bounded in $W^{2m,p}(B_{R/4})$, $1 \leq p < \frac{2m}{2m-1}$. We now go back to (4.32), replacing R with $R/4$ and redefining h_k and w_k accordingly on $B_{R/4}$. We now have that $(A_k z_k)$ is bounded in $L^p(B_{R/4})$ for $1 \leq p < \frac{2m}{2m-2}$ by Sobolev's embedding, and we infer as above that (w_k) is bounded in $W^{2m,p}(B_{R/4})$, $1 \leq p < \frac{2m}{2m-2}$, and h_k is bounded in $C^\ell(B_{R/16})$, $\ell \geq 0$. Iterating, we find that (z_k) is bounded in $W^{2m,p}(B_{R/4^m})$ for every $p \in [1, \infty[$. Hence, for every $\alpha \in [0,1[$ there is a function $w \in C^{2m-1,\alpha}(B_{R/4^m})$ such that up to a subsequence

$$w_k \to w \quad \text{in } C^{2m-1,\alpha}(B_{R/4^m}).$$

By (4.34) and Ascoli-Arzelà's Theorem (z_k) converges in $C^{2m-1,\alpha}(B_{R/4^m})$ up to a subsequence. Then (4.31) follows from Fatou's lemma, letting $R \to \infty$, and the claim is proven.

When $m = 1$, since $P^2_g = -\Delta_g$, (4.30) implies at once that $(\Delta_{\hat{g}_k} z_k)$ is locally bounded in L^∞. Then, since $z_k \leq 0$ and $z_k(0) = 0$, the claim follows from elliptic estimates (e.g. [GT, Thm. 8.18]).

Step 3. We shall now rule out the possibility that $Q_0(x_0) \leq 0$.

Case $Q_0(x_0) = 0$. By Theorem 3.3, if $m = 1$ there exists no solution z_0 to (4.31), contradiction. If $m \geq 2$, still by Theorem 3.3, then z_0 is a non-constant polynomial of degree at most $2m - 2$, and there are $1 \leq j \leq m - 1$ and $a < 0$ such that $\Delta^j z_0 \equiv a$. Following an argument of [RS], see also [Mal], we shall find a contradiction. Indeed we have

$$\lim_{k\to\infty} \int_{B_R} |\Delta^j z_k| dx = \int_{B_R} |\Delta^j z_0| dx = \frac{|a|\omega_{2m}}{2m} R^{2m} + o(R^{2m}), \quad \text{as } R \to +\infty.$$

Scaling back to u_k, we find

$$\lim_{k\to\infty} \left(\sigma_k^{2j-2m} \int_{B_{R\sigma_k}(x_k)} |\nabla^{2j} u_k| \,\text{dvol}_g \right) \geq C^{-1} R^{2m} + o(R^{2m}), \quad \text{as } R \to +\infty,$$

while, from Lemma 4.7,

$$\int_{B_{R\sigma_k}(x_k)} |\nabla^{2j} u_k| \,\mathrm{dvol}_g \leq C(R\sigma_k)^{2m-2j}. \tag{4.35}$$

This yields the desired contradiction as $k, R \to +\infty$.

Case $Q_0(x_0) < 0$. By Theorem 3.1 there exists no solution to (4.31) for $m = 1$, contradiction. If $m \geq 2$, from Theorem 3.2 we infer that there are a constant $a \neq 0$ and $1 \leq j \leq m-1$ such that

$$\lim_{\substack{|x|\to+\infty \\ x\in\mathcal{C}}} \Delta^j z_0(x) = a,$$

where $\mathcal{C} := \{t\xi \in \mathbb{R}^{2m} : t \geq 0,\ \xi \in K\}$ and $K \subset S^{2m-1}$ is a compact set with $\mathcal{H}^{2m-1}(K) > 0$. Then, as above,

$$\begin{aligned}\lim_{k\to\infty} \left(\sigma_k^{2j-2m} \int_{B_{R\sigma_k}(x_k)} |\nabla^{2j} u_k| \,\mathrm{dvol}_g \right) &\geq C^{-1} \int_{B_R \cap \mathcal{C}} |\Delta^j z_0| dx \\ &\geq C^{-1} R^{2m} + o(R^{2m}),\end{aligned}$$

again contradicting (4.35). Then we have shown that $Q_0(x_0) > 0$.

Step 4. Since $Q_k(x_0) > 0$, μ_k and η_k are well-defined. Repeating the procedure of Step 2, we find a function $\overline{\eta} \in C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$ such that $\eta_k \to \overline{\eta}$ in $C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where (compare (4.31))

$$(-\Delta)^m \overline{\eta} = (2m-1)! e^{2m\overline{\eta}}, \quad \int_{\mathbb{R}^{2m}} e^{2m\overline{\eta}} dx < +\infty.$$

By Theorem 2.2, either $\overline{\eta}$ is a standard solution, i.e. there are $x_0 \in \mathbb{R}^{2m}$, $\lambda > 0$ such that

$$\overline{\eta}(y) = \log \frac{2\lambda}{1 + \lambda^2 |y - y_0|^2}, \tag{4.36}$$

or $\Delta^j \overline{\eta}(x) \to a$ as $|x| \to \infty$ for some constant $a < 0$ and for some $1 \leq j \leq m-1$. In the latter case, as in Step 3, we reach a contradiction. Hence (4.36) is satisfied. Since $\max_M \eta_k = \eta_k(0) = \log 2$ for every k, we have $y_0 = 0$, $\lambda = 1$, i.e. $\overline{\eta} = \eta_0$. Since, by Fatou's lemma

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{R\mu_k(x_k)} Q_k e^{2mu_k} \,\mathrm{dvol}_g = (2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx,$$

(4.26) follows from (5.7). □

Proof of Theorem 4.2. Assume first that $u_k \leq C$. Then $P_g^{2m} u_k$ is bounded in $L^\infty(M)$ and by elliptic estimates $u_k - \overline{u}_k$ is bounded in $W^{2m,p}(M)$ for every $1 \leq p < \infty$, hence in $C^{2m-1,\alpha}(M)$ for every $\alpha \in [0,1[$, where $\overline{u}_k := \fint_M u_k \,\mathrm{dvol}_g$. Observe that by Jensen's inequality and (4.8), $\overline{u}_k \leq C$.

If $\overline{u}_k$ remains bounded (up to a subsequence), then by Ascoli-Arzelà's theorem, for every $\alpha \in [0,1[$, u_k is convergent (up to a subsequence) in $C^{2m-1,\alpha}(M)$, and we are in case (i) of Theorem 4.2.

If $\overline{u}_k \to -\infty$, we have that $u_k \to -\infty$ uniformly on M and we are in case (ii) of the theorem, with $S = \emptyset$.

From now on we shall assume that $\max_M u_k \to \infty$ as $k \to \infty$.

Step 1. There are $I > 0$ converging sequences $x_{i,k} \to x^{(i)} \in M$ with $u_k(x_{i,k}) \to \infty$ as $k \to \infty$, such that

(A_1) $Q_0(x^{(i)}) > 0$, $1 \le i \le I$.

(A_2) $\frac{\mathrm{dist}(x_{i,k}, x_{j,k})}{\mu_{i,k}} \to +\infty$ as $k \to +\infty$ for all $1 \le i, j \le I$, $i \neq j$, where

$$\mu_{i,k} := 2\left(\frac{(2m-1)!}{Q_0(x^{(i)})}\right)^{\frac{1}{2m}} e^{-u_k(x_{i,k})}.$$

(A_3) Set $\eta_{i,k}(y) := u_k(\exp_{x_{i,k}}(\mu_{i,k} y)) - u_k(x_{i,k})$. Then for $1 \le i \le I$

$$\eta_{i,k}(y) \to \eta_0(y) = \log \frac{2}{1 + |y|^2} \quad \text{in } C^{2m}_{\mathrm{loc}}(\mathbb{R}^{2m}) \quad (k \to \infty). \tag{4.37}$$

(A_4) For $1 \le i \le I$

$$\lim_{R \to +\infty} \lim_{k \to +\infty} \int_{B_{R\mu_{i,k}}(x_{i,k})} Q_k e^{2mu_k} dx \to \Lambda_1. \tag{4.38}$$

(A_5) There exists $C > 0$ such that for all k

$$\sup_{x \in M} \left[e^{u_k(x)} R_k(x)\right] \le C, \quad R_k(x) := \min_{1 \le i \le I} \mathrm{dist}(x, x_{i,k}).$$

Step 1 follows from Proposition 4.8 and induction as follows. Define $x_{1,k} = x_k$ as in Proposition 4.8. Then (A_1), (A_3) and (A_4) are satisfied with $i = 1$. If $\sup_{x \in M} \left[e^{u_k(x)} \mathrm{dist}(x_{i,k}, x)\right] \le C$, then $I = 1$ and also (A_5) is satisfied, so we are done. Otherwise we choose $x_{2,k}$ such that

$$R_{1,k}(x_{2,k}) e^{u_k(x_{2,k})} = \max_{x \in M} R_{1,k}(x) e^{u_k(x)} \to \infty, \quad R_{1,k}(x) := \mathrm{dist}(x, x_{1,k}). \tag{4.39}$$

Then (A_2) with $i = 2$, $j = 1$ follows at once from (4.39), while (A_2) with $i = 1$, $j = 2$ follows from (A_3), as in [DR]. A slight modification of Proposition 4.8 shows that $(x_{2,k}, \mu_{2,k})$ satisfies (A_1), (A_3) and (A_4), and we continue so, until also property (A_5) is satisfied. The procedure stops after finitely many steps, thanks to (A_2), (A_4) and (4.26).

Step 2. We now prove that

$$\sup_{x \in M} R_k(x)^\ell |\nabla^\ell u_k(x)| \le C, \quad \ell = 1, 2, \ldots, 2m - 1. \tag{4.40}$$

We differentiate the Green representation formula (4.23) ℓ times and we use (4.24) to estimate

$$|\nabla^\ell u_k(\xi)| \le C \int_M \frac{e^{2mu_k(y)}}{\mathrm{dist}(\xi, y)^\ell} \, \mathrm{dvol}_g(y) + C.$$

Set for $1 \le i \le I$

$$\Omega_{i,k} := \{y \in M : R_k(y) = \mathrm{dist}(x_{i,k}, y)\}$$

and further, assuming $\xi \ne x_{i,k}$ for $1 \le i \le I$ (otherwise (4.40) is trivial), set

$$\Omega^{(1)}_{i,k} := \Omega_{i,k} \cap B_{\mathrm{dist}(x_{i,k},\xi)/2}(x_{i,k}), \quad \Omega^{(2)}_{i,k} := \Omega_{i,k} \backslash B_{\mathrm{dist}(x_{i,k},\xi)/2}(x_{i,k}).$$

Observing that for $y \in \Omega^{(1)}_{i,k}$ we have $\frac{1}{\mathrm{dist}(\xi,y)} \le \frac{2}{\mathrm{dist}(\xi,x_{i,k})}$ and using (A_5) from Step 1, we infer

$$\begin{aligned}\int_{\Omega_{i,k}} \frac{e^{2mu_k}}{\mathrm{dist}(\xi,y)^\ell}\,\mathrm{dvol}_g(y) \quad &\le \quad \frac{C}{\mathrm{dist}(\xi,x_{i,k})^\ell}\int_{\Omega^{(1)}_{i,k}} e^{2mu_k(y)}\,\mathrm{dvol}_g(y)\\ &\quad + C\int_{\Omega^{(2)}_{i,k}} \frac{\mathrm{dvol}_g(y)}{\mathrm{dist}(\xi,y)^\ell\,\mathrm{dist}(y,x_{i,k})^{2m}}.\end{aligned}$$

The first integral on the right-hand side is bounded by $\frac{C}{\mathrm{dist}(\xi,x_{i,k})^\ell}$. As for the integral over $\Omega^{(2)}_{i,k}$, write $\Omega^{(2)}_{i,k} = \Omega^{(3)}_{i,k} \cup \Omega^{(4)}_{i,k}$, with

$$\Omega^{(3)}_{i,k} = \Omega^{(2)}_{i,k} \cap B_{2\,\mathrm{dist}(\xi,x_{i,k})}(\xi), \quad \Omega^{(3)}_{i,k} = \Omega^{(2)}_{i,k} \backslash B_{2\,\mathrm{dist}(\xi,x_{i,k})}(\xi).$$

We have

$$\begin{aligned}\int_{\Omega^{(3)}_{i,k}} \frac{\mathrm{dvol}_g(y)}{\mathrm{dist}(\xi,y)^\ell\,\mathrm{dist}(y,x_{i,k})^{2m}} \quad &\le \quad \frac{C}{\mathrm{dist}(\xi,x_{i,k})^{2m}}\int_{\Omega^{(3)}_{i,k}} \frac{\mathrm{dvol}_g(y)}{\mathrm{dist}(\xi,y)^\ell}\\ &\le \quad \frac{C}{\mathrm{dist}(\xi,x_{i,k})\ell}.\end{aligned}$$

Observing that

$$\frac{1}{C}\,\mathrm{dist}(y,x_{i,k}) \le \mathrm{dist}(\xi,y) \le C\,\mathrm{dist}(y,x_{i,k}) \quad \text{on } \Omega^{(4)}_{i,k},$$

we estimate

$$\begin{aligned}\int_{\Omega^{(4)}_{i,k}} \frac{\mathrm{dvol}_g(y)}{\mathrm{dist}(\xi,y)^\ell\,\mathrm{dist}(y,x_{i,k})^{2m}} \quad &\le \quad C\int_{\Omega^{(4)}_{i,k}} \frac{\mathrm{dvol}_g(y)}{\mathrm{dist}(x_{i,k},y)^{2m+\ell}}\\ &\le \quad C\int_{\mathbb{R}^{2m}\backslash B_{\mathrm{dist}(x_{i,k},\xi)}} \frac{dz}{|z|^{2m-\ell}}\\ &\le \quad \frac{C}{\mathrm{dist}(x_{i,k},\xi)^\ell}.\end{aligned}$$

Putting these last inequalities together yields

$$|\nabla^\ell u_k(\xi)| \le \frac{C}{\inf_{1\le i\le I}\mathrm{dist}(\xi,x_{i,k})^\ell} = \frac{C}{R_k(\xi)^\ell},$$

whence (4.40).

Step 3. $u_k \to -\infty$ locally uniformly in $M\backslash S$, $S := \{x^{(i)} : 1 \le i \le I\}$. This follows easily from (4.40) above and (4.42) below (which implies that $u_k \to -\infty$

locally uniformly in $B_{\delta_\nu}(x^{(i)})\backslash\{x^{(i)}\}$ for any $1 \le i \le I$, $\nu \in [1,2[$ and δ_ν as in Step 4), but we also sketch an instructive alternative proof, which does not make use of (4.42).

Our Theorem 4.1 can be reproduced on a closed manifold, with a similar proof and using Proposition 3.1 from [Mal] instead of Theorem 2.7 above. Then either

(a) u_k is bounded in $C^{2m-1}_{\mathrm{loc}}(M\backslash S)$, or

(b) $u_k \to -\infty$ locally uniformly in $M\backslash S$, or

(c) There exists a closed set $S_0 \subset M\backslash S$ of Hausdorff dimension at most $2m-1$ and numbers $\beta_k \to +\infty$ such that

$$\frac{u_k}{\beta_k} \to \varphi \text{ in } C^{2m-1}_{\mathrm{loc}}(M\backslash(S_0 \cup S)),$$

where

$$\Delta^m \varphi \equiv 0, \quad \varphi \le 0, \quad \varphi \not\equiv 0, \quad \varphi \equiv 0 \text{ on } S_0. \tag{4.41}$$

Case (a) can be ruled out using (4.8) as in (4.19) at the end of the proof of Theorem 4.1. Case (c) contradicts Lemma 4.7, as in the proof of Proposition 5.7 below (compare (5.30), (5.31)). Hence Case (b) occurs, as claimed.

Step 4. We claim that for every $1 \le \nu < 2$, there exist $\delta_\nu > 0$ and $C_\nu > 0$ such that for $1 \le i \le I$

$$\mathrm{dist}(x, x_{i,k})^{2m\nu} e^{2mu_k(x)} \le C_\nu \mu_{i,k}^{2m(\nu-1)}, \quad \text{for } x \in B_{\delta_\nu}(x_{i,k}). \tag{4.42}$$

Then on the *necks* $\Sigma_{i,k} := B_{\delta_\nu}(x_{i,k})\backslash B_{R\mu_{i,k}}(x_{i,k})$ we have

$$\begin{aligned}\int_{\Sigma_{i,k}} e^{2mu_k}\,\mathrm{dvol}_g &\le C_\nu \mu_{i,k}^{2m(\nu-1)} \int_{\Sigma_{i,k}} \mathrm{dist}(x, x_{i,k})^{-2m\nu}\,\mathrm{dvol}_g(x)\\ &\le C_\nu \mu_{i,k}^{2m(\nu-1)} \int_{R\mu_{i,k}}^{\delta_\nu} r^{2m-1-2m\nu} dr\\ &= C_\nu R^{2m(1-\nu)} - C_\nu \mu_{i,k}^{2m(\nu-1)} \delta_\nu^{2m(1-\nu)},\end{aligned}$$

whence

$$\lim_{R\to+\infty}\lim_{k\to+\infty}\int_{\Sigma_{i,k}} Q_k e^{2mu_k}\,\mathrm{dvol}_g = 0. \tag{4.43}$$

This, together with (4.26) and Step 3 implies (4.11), assuming that $x^{(i)} \ne x^{(j)}$ for $i \ne j$. This we be shown in Step 4c below. Then (4.12) follows at once from (4.2).

Let us prove (4.42). Fix $1 \le \nu < 2$ and set for $1 \le i \le I$

$$\tilde{R}_{i,k} := \min_{j\ne i} \mathrm{dist}(x_{i,k}, x_{j,k}).$$

Step 4a. Let $i \in \{1, \dots, I\}$ be such that for some $\theta > 0$ we have

$$\tilde{R}_{i,k} \le \theta \tilde{R}_{j,k} \quad \text{for } 1 \le j \le I,\ k \ge 1. \tag{4.44}$$

Set

$$\varphi_{i,k}(r) := r^{2m\nu} \exp\bigg(\fint_{\partial B_r(x_{i,k})} 2mu_k d\sigma_g \bigg), \tag{4.45}$$

for $0 < r < r_{\mathrm{inj}}$, where $d\sigma_g$ is the measure on $\partial B_r(x_{i,k})$ induced by g. Observe that

$$\varphi'_{i,k}(r\mu_{i,k}) < 0 \quad \text{if and only if} \quad r\mu_{i,k} < -\nu\bigg(\fint_{\partial B_{r\mu_{i,k}}(x_{i,k})} \frac{\partial u_k}{\partial n} d\sigma_g \bigg)^{-1}. \tag{4.46}$$

From (4.37) we infer

$$\mu_{i,k} \frac{\partial u_k}{\partial n}\bigg|_{\partial B_{\mu_{i,k}r}(x_{i,k})} \to \frac{\partial}{\partial r} \log \frac{2}{1+r^2} = \frac{-2r}{1+r^2},$$

hence

$$\mu_{i,k} \fint_{\partial B_{\mu_{i,k}r}(x_{i,k})} \frac{\partial u_k}{\partial n} d\sigma_g \to -\frac{2r}{1+r^2}, \quad \text{for } r > 0 \text{ as } k \to \infty,$$

and (4.46) implies that for any $R \geq 2R_\nu := 2\sqrt{\frac{\nu}{2-\nu}}$, there exists $k_0(R)$ such that

$$\varphi'_{i,k}(r\mu_{i,k}) < 0 \quad \text{for } k \geq k_0(R),\ r \in [2R_\nu, R]. \tag{4.47}$$

Define

$$r_{i,k} := \sup\Big\{ r \in [2R_\nu \mu_{i,k}, \tilde{R}_{i,k}/2] : \varphi'_{i,k}(\rho) < 0 \text{ for } \rho \in [2R_\nu \mu_{i,k}, r) \Big\}. \tag{4.48}$$

From (4.47) we infer that

$$\lim_{k\to+\infty} \frac{r_{i,k}}{\mu_{i,k}} = +\infty. \tag{4.49}$$

Let us assume that

$$\lim_{k\to\infty} r_{i,k} = 0. \tag{4.50}$$

Consider

$$v_{i,k}(y) := u_k(\exp_{x_{i,k}}(r_{i,k}y)) - C_{i,k}, \quad C_{i,k} := \fint_{\partial B_{r_{i,k}}(x_{i,k})} u_k d\sigma_g, \tag{4.51}$$

and let

$$\hat{g}_{i,k} := r_{i,k}^{-2} (\exp_{x_{i,k}} \circ T_{i,k})^* g, \quad \hat{Q}_{i,k}(y) := Q_k(\exp_{x_{i,k}}(r_{i,k}y)),$$

where

$$T_{i,k}(y) := r_{i,k} y \quad \text{for } y \in \mathbb{R}^{2m}.$$

Then

$$\begin{aligned} P^{2m}_{\hat{g}_{i,k}} v_{i,k} + r_{i,k}^{2m} Q_{\hat{g}_{i,k}} &= r_{i,k}^{2m} \hat{Q}_{i,k} e^{2m(v_{i,k}+C_{i,k})} \\ &= r_{i,k}^{2m(1-\nu)} \varphi_{i,k}(r_{i,k}) \hat{Q}_{i,k} e^{2mv_{i,k}}. \end{aligned} \tag{4.52}$$

We also set

$$\mathcal{J}_i = \{ j \neq i : \mathrm{dist}(x_{i,k}, x_{j,k}) = O(r_{i,k}) \text{ as } k \to \infty \}, \tag{4.53}$$

and

$$\tilde{x}_{j,k}^{(i)} := \frac{1}{r_{i,k}} \exp_{x_{i,k}}^{-1}(x_{j,k}), \quad \tilde{x}_j^{(i)} = \lim_{k\to\infty} \tilde{x}_{j,k}, \tag{4.54}$$

after passing to a subsequence, if necessary. Thanks to (4.44) and (4.48), we have that $|\tilde{x}_j^{(i)}| \geq 2$ for all $j \in \mathcal{J}_i$ and that

$$|\tilde{x}_j^{(i)} - \tilde{x}_\ell^{(i)}| \geq \frac{2}{\theta} \quad \text{for all } j, \ell \in \mathcal{J}_i,\ j \neq \ell.$$

By (4.40) and the choice of $C_{i,k}$ in (4.51), $v_{i,k}$ is uniformly bounded in

$$C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash \{0, \tilde{x}_j^{(i)} : j \in \mathcal{J}_i\}).$$

Thanks to (4.48) and (4.49), given $R > 2R_\nu$, there exists $k_0(R)$ such that $\varphi_{i,k}(r_{i,k}) < \varphi_{i,k}(R\mu_{i,k})$ for all $k \geq k_0$. From (4.37), we infer

$$\begin{aligned} \mu_{i,k}^{2m} \exp\left(\fint_{\partial B_{R\mu_{i,k}(x_{i,k})}} 2mu_k d\sigma \right) &= \exp\left(\fint_{\partial B_{R\mu_{i,k}}(x_{i,k})} 2m(u_k + \log \mu_{i,k}) d\sigma \right) \\ &= C(R) + o(1), \quad \text{as } k \to \infty, \end{aligned} \tag{4.55}$$

where

$$C(R) \to 0, \quad \text{as } R \to \infty. \tag{4.56}$$

Then, together with (4.49), letting $k \to +\infty$ we get

$$\begin{aligned} r_{i,k}^{2m(1-\nu)} \varphi_{i,k}(r_{i,k}) &\leq r_{i,k}^{2m(1-\nu)} \varphi_{i,k}(R\mu_{i,k}) \\ &= \mu_{i,k}^{2m} \exp\left(\fint_{\partial B_{R\mu_{i,k}(x_{i,k})}} 2mu_k d\sigma \right) R^{2m\nu} \left(\frac{\mu_{i,k}}{r_{i,k}} \right)^{2m(\nu-1)} \\ &\to 0. \end{aligned} \tag{4.57}$$

Therefore the right-hand side of (4.52) goes to 0 locally uniformly in

$$\mathbb{R}^{2m} \backslash \{0, \tilde{x}_j^{(i)} : j \in \mathcal{J}_i\};$$

moreover

$$\hat{g}_{i,k} \to g_{\mathbb{R}^{2m}} \text{ in } C^k_{\mathrm{loc}}(\mathbb{R}^{2m}) \text{ for every } k \geq 0, \quad r_{i,k}^{2m} \hat{Q}_{i,k} \to 0 \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}^{2m}). \tag{4.58}$$

It follows that, up to a subsequence,

$$v_{i,k} \to h_i \text{ in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m} \backslash \{0, \tilde{x}_j^{(i)} : j \in \mathcal{J}_i\}), \tag{4.59}$$

where, taking (4.40) into account,

$$\Delta^m h_i(x) = 0, \quad x \in \mathbb{R}^{2m} \backslash \{0, \tilde{x}_j^{(i)} : j \in \mathcal{J}_i\}$$

and

$$\tilde{R}(x)^\ell |\nabla^\ell h_i(x)| \le C_\ell, \quad \text{for } \ell = 1, \dots, 2m-1,\ x \in \mathbb{R}^{2m} \backslash \{0, \tilde{x}_j^{(i)} : j \in \mathcal{J}_i\},$$

with $\tilde{R}(x) := \min\{|x|, |x - \tilde{x}_j^{(i)}| : j \in \mathcal{J}_i\}$. Then Proposition 4.9 from the appendix implies that

$$h_i(x) = -\lambda \log|x| - \sum_{j \in \mathcal{J}_i} \lambda_j \log|x - \tilde{x}_j^{(i)}| + \beta, \tag{4.60}$$

for some $\lambda, \beta, \lambda_j \in \mathbb{R}$. We now recall that the Paneitz operator is in divergence form, hence we can write

$$P^{2m}_{\hat{g}_{i,k}} v_{i,k} = \operatorname{div}_{\hat{g}_{i,k}}(A_{\hat{g}_{i,k}} v_{i,k}) \tag{4.61}$$

for some differential operator $A_{\hat{g}_{i,k}}$ of order $2m-1$, with coefficients converging to the coefficient of $(-1)^m \nabla \Delta^{m-1}$ uniformly in B_1, thanks to (4.58). Then integrating (4.52), using (4.58), (4.59) and (4.61), we get

$$\begin{aligned}
\lim_{k\to\infty} \int_{B_{r_{i,k}}(x_{i,k})} Q_k e^{2mu_k}\, \mathrm{dvol}_g &= \lim_{k\to\infty} \varphi_{i,k}(r_{i,k}) r_{i,k}^{2m(1-\nu)} \int_{B_1} \hat{Q}_{i,k} e^{2mv_{i,k}}\, \mathrm{dvol}_{\hat{g}_{i,k}} \\
&= \lim_{k\to\infty} \int_{B_1} \left(\operatorname{div}_{\hat{g}_{i,k}}(A_{\hat{g}_{i,k}} v_{i,k}) + r_{i,k}^{2m} Q_{\hat{g}_{i,k}} \right) \mathrm{dvol}_{\hat{g}_{i,k}} \\
&= \lim_{k\to\infty} \int_{\partial B_1} n \cdot (A_{\hat{g}_{i,k}} v_{i,k}) d\sigma_{\hat{g}_{i,k}} \\
&= (-1)^m \int_{\partial B_1} \frac{\partial \Delta^{m-1} h_i}{\partial n} d\sigma = \lambda \frac{\Lambda_1}{2},
\end{aligned} \tag{4.62}$$

where here n denotes the exterior unit normal to ∂B_1 and the last identity can be inferred using (2.22) and the following:

$$\begin{aligned}
\int_{\partial B_1} \frac{\partial \Delta^{m-1} h_i}{\partial n} d\sigma &= \lambda \int_{\partial B_1} \frac{\partial \Delta^{m-1} \log \frac{1}{|x|}}{\partial n} d\sigma \\
&\quad + \sum_{j \in \mathcal{J}_i} \lambda_j \int_{B_1} \underbrace{\Delta^m \log \frac{1}{|x - \tilde{x}_j^{(i)}|}}_{\equiv 0 \text{ on } B_1} dx
\end{aligned}$$

From (4.40) with $\ell = 1$, we get

$$|u_k(\exp_{x_{i,k}}(r_{i,k} y_1)) - u_k(\exp_{x_{i,k}}(r_{i,k} y_2))| \le C r_{i,k} r \sup_{\partial B_{r_{i,k} r}(x_{i,k})} |\nabla u_k| \le C, \tag{4.63}$$

for $0 \le r \le \frac{3}{2}$, $|y_1| = |y_2| = r$. For $2R_\nu \mu_{i,k} \le R\mu_{i,k} \le r \le r_{i,k}$, we infer from (4.55)

$$\varphi_{i,k}(r) \le \varphi_{i,k}(R\mu_{i,k}) \le C(R) \mu_{i,k}^{2m(\nu-1)} + o(\mu_{i,k}^{2m(\nu-1)}).$$

This, (4.45), (4.55), (4.56) and (4.63) imply that for any $\eta > 0$ there exist $R_\eta \ge 2R_\nu$ and $k_\eta \in \mathbb{N}$ such that

$$\operatorname{dist}(x, x_{i,k})^{2m\nu} e^{2mu_k} \le \eta \mu_{i,k}^{2m(\nu-1)} \quad \text{for } x \in B_{r_{i,k}}(x_{i,k}) \backslash B_{R_\eta \mu_{i,k}}(x_{i,k}),\ k \ge k_\eta. \tag{4.64}$$

It now follows easily that

$$\lim_{R\to+\infty}\lim_{k\to\infty}\int_{B_{r_{i,k}}(x_{i,k})\setminus B_{R\mu_{i,k}}(x_{i,k})} Q_k e^{2mu_k}dx = 0,$$

and from (4.38)

$$\lim_{k\to+\infty}\int_{B_{r_{i,k}}(x_{i,k})} Q_k e^{2mu_k}dx = \Lambda_1.$$

That implies that $\lambda = 2$. With a similar computation, integrating on $B_\delta(\tilde{x}_j^{(i)})$ for δ small instead of $B_1(0)$, one proves that $\lambda_j \geq 2$ for all $j \in \mathcal{J}_i$. Now set

$$\overline{h}_i(r) := \fint_{\partial B_r(0)} h_i d\sigma.$$

Then

$$\frac{d}{dr}\big(r^{2m\nu}e^{2m\overline{h}_i(r)}\big) = 2m\bigg(\nu - 2 - \bigg(\sum_{j\in\mathcal{J}_i}\frac{\lambda_j}{2|\tilde{x}_j^{(i)}|^2}\bigg)r^2\bigg)r^{2m\nu-1}e^{2m\overline{h}_i(r)},$$

for $0 < r < \frac{3}{2}$. In particular

$$\frac{d}{dr}\big(r^{2m\nu}e^{2m\overline{h}_i(r)}\big)\big|_{r=1} < 0$$

hence, for k large enough, $\varphi'_{i,k}(r_{i,k}) < 0$. This implies that

$$r_{i,k} = \frac{\tilde{R}_{i,k}}{2} \quad \text{for } k \text{ large.} \tag{4.65}$$

This in turn implies $\lim_{k\to\infty}\tilde{R}_{i,k} = 0$, when i satisfies (4.44) and $\lim_{k\to\infty} r_{i,k} = 0$. For i satisfying (4.44) and $\limsup_{k\to\infty}\tilde{R}_{i,k} > 0$, we infer, instead, that $\limsup_{k\to\infty} r_{i,k} > 0$. In both cases (4.64) holds.

Step 4b. Now assume that

$$\limsup_{k\to\infty}\tilde{R}_{i,k} > 0, \quad \text{for every } 1 \leq i \leq I. \tag{4.66}$$

Then (4.44) is satisfied for every $1 \leq i \leq I$, hence $\limsup_{k\to\infty} r_{i,k} > 0$, $1 \leq i \leq I$. Up to selecting a subsequence, we can set

$$\delta_\nu := \inf_{1\leq i\leq I}\frac{1}{2}\lim_{k\to\infty} r_{i,k} > 0.$$

Take now $\eta = 1$ in (4.64), and let R_1 be the corresponding R_η. Then (4.42) is true for $x \in B_{\delta_\nu}(x_{i,k})\setminus B_{R_1\mu_{i,k}}(x_{i,k})$. On the other hand, thanks to (A_3), we have $u_k(x) \leq u_k(x_{i,k}) + C$ on $B_{R_1\mu_{i,k}}(x)$. Then, using (4.25), we get

$$\begin{aligned}\operatorname{dist}(x, x_{i,k})^{2m\nu}e^{2mu_k(x)} &\leq C(R_1\mu_{i,k})^{2m\nu}e^{2mu_k(x_{i,k})}\\ &\leq CR_1^{2m\nu}\mu_{i,k}^{2m(\nu-1)} \quad \text{for } x \in B_{R_1\mu_{i,k}}(x_{i,k}).\end{aligned}$$

This completes the proof of (4.42), under the assumption that (4.66) holds.

Step 4c. We now prove that in fact (4.66) holds true. Choose $1 \le i_0 \le I$ so that, up to a subsequence,

$$\tilde{R}_{i_0,k} = \min_{1\le i\le I} \tilde{R}_{i,k} \quad \text{for every } k \in \mathbb{N},$$

and assume by contradiction that $\lim_{k\to\infty} \tilde{R}_{i_0,k} = 0$. Clearly (4.44) holds for $i = i_0$, hence also (4.65) holds for $i = i_0$, by Step 4a. Then, setting $\mathcal{J}_{i_0}$ as is (4.53), we claim that, for any $i \in \mathcal{J}_{i_0}$, there exists $\theta(i) > 0$ such that

$$\tilde{R}_{i,k} \le \theta(i)\tilde{R}_{j,k} \quad \text{for } 1 \le j \le I.$$

Indeed

$$\tilde{R}_{i,k} = O(r_{i_0,k}) = O(\tilde{R}_{i_0,k}) \quad \text{as } k \to \infty.$$

It then follows that (4.44) holds for all $i \in \mathcal{J}_{i_0}$, and that Step 4a applies to them. Observing that $\mathcal{J}_{i_0} \neq \emptyset$ thanks to Step 4a (Identity (4.65) with i_0 instead of i), we can pick $i \in \mathcal{J}_{i_0}$ such that, up to a subsequence,

$$\operatorname{dist}(x_{i,k}, x_{i_0,k}) \ge \operatorname{dist}(x_{j,k}, x_{i_0,k}) \quad \text{for all } j \in \mathcal{J}_{i_0},\ k > 0.$$

Recalling the definition of $\tilde{x}_j^{(i)}$ for $j \in \mathcal{J}_i$, we get $|\tilde{x}_{i_0}^{(i)}| \ge |\tilde{x}_j^{(i)} - \tilde{x}_{i_0}^{(i)}|$ for all $j \in \mathcal{J}_i$. A consequence of this inequality is that the scalar product

$$\tilde{x}_{i_0}^{(i)} \cdot \tilde{x}_j^{(i)} > 0 \tag{4.67}$$

for all $j \in \mathcal{J}_i$. In other words all the $\tilde{x}_j^{(i)}$'s with $j \in \mathcal{J}_i$ lie in the same half space orthogonal to $\tilde{x}_{i_0}^{(i)}$ and whose boundary contains $0 = \tilde{x}_i^{(i)}$. Multiplying (4.52) by $\nabla v_{i,k}$ and integrating over $B_\delta = B_\delta(0)$ ($\delta > 0$ small), we get

$$\begin{aligned}\int_{B_\delta} P^{2m}_{\hat{g}_{i,k}} v_{i,k} \nabla v_{i,k}\, \mathrm{dvol}_{\hat{g}_{i,k}} &= -\int_{B_\delta} r_{i,k}^{2m} \hat{Q}_{i,k} \nabla v_{i,k}\, \mathrm{dvol}_{\hat{g}_{i,k}} \\ &\quad + \frac{r_{i,k}^{2m(1-\nu)}}{2m} \varphi_{i,k}(r_{i,k}) \int_{B_\delta(0)} \hat{Q}_{i,k} \nabla e^{2mv_{i,k}}\, \mathrm{dvol}_{\hat{g}_{i,k}} \\ &=: (I)_k + (II)_k. \end{aligned} \tag{4.68}$$

Recalling (4.58) and (4.59), we see at once that $\lim_{k\to\infty}(I)_k = 0$. Integrating by parts, we also see that

$$\begin{aligned}|(II)_k| &\le C \frac{r_{i,k}^{2m(1-\nu)}}{2m} \varphi_{i,k}(r_{i,k}) \int_{B_\delta(0)} \frac{\nabla \hat{Q}_{i,k}}{\hat{Q}_{i,k}} \hat{Q}_{i,k} e^{2mv_{i,k}} d\, \mathrm{vol}_{\hat{g}_{i,k}} \\ &\quad + \frac{r_{i,k}^{2m(1-\nu)}}{2m} \varphi_{i,k}(r_{i,k}) \int_{\partial B_\delta(0)} O(1) d\sigma_{\hat{g}_{i,k}} \\ &\to 0 \quad \text{as } k \to \infty, \end{aligned}$$

where the last term vanishes thanks to (4.57), and the first term on the right of $(II)_k$ vanishes thanks to (4.62) and the remark that

$$\frac{\nabla \hat{Q}_{i,k}}{\hat{Q}_{i,k}} \to 0 \quad \text{in } L^\infty(B_\delta). \tag{4.69}$$

Recalling (4.59), using (4.40) and (4.58), we arrive at

$$\int_{B_\delta} \nabla h_i (-\Delta)^m h_i dx = 0. \tag{4.70}$$

Let us assume m even. Then, integrating by parts, we get

$$\begin{aligned} 0 \quad = \quad & \frac{1}{2}\int_{\partial B_\delta} ((-\Delta)^{\frac{m}{2}} h_i)^2 n d\sigma \\ & - \sum_{j=0}^{\frac{m}{2}-1} \int_{\partial B_\delta} (\nabla(-\Delta)^j h_i) \frac{\partial (-\Delta)^{m-1-j} h_i}{\partial n} d\sigma \\ & + \sum_{j=0}^{\frac{m}{2}-1} \int_{\partial B_\delta} \nabla\Big(\frac{\partial(-\Delta)^j h_i}{\partial n}\Big) (-\Delta)^{m-1-j} h_i d\sigma. \end{aligned} \tag{4.71}$$

Then, taking the limit as $\delta \to 0$, and writing

$$h_i(x) = 2\log\frac{1}{|x|} + G_i(x)$$

we see that all terms in (4.71) vanish (G_i is regular in a neighborhood of 0 and the vector function $\nabla \log\frac{1}{|x|}$ is anti-symmetric), up to at most

$$\lim_{\delta\to 0} \int_{\partial B_\delta} (-\nabla G_i)\partial_\nu (-\Delta)^{m-1}\Big(2\log\frac{1}{|x|}\Big) d\sigma = 2\gamma_m \nabla G_i(0),$$

see (2.22). But then (4.71) gives

$$2\gamma_m \nabla G_i(0).$$

Also when m is odd, in a completely analogous way, we get $\nabla G_i(0) = 0$, a contradiction with (4.60) and (4.67). This ends the proof of Step 4.

Step 5. Finally, if case (ii) occurs and $S \neq \emptyset$, then (4.38) implies

$$\limsup_{k\to\infty} \mathrm{vol}(g_k) \geq Q_0(x^{(1)})^{-1}\Lambda_1 > 0.$$

This justifies the last claim of the theorem. □

The proposition below was used in the above proof.

Proposition 4.9 *Let $S = \{x_1, \dots, x_I\} \subset \mathbb{R}^{2m}$ be a finite set and let $h \in C^\infty(\mathbb{R}^{2m}\backslash S)$ satisfy $\Delta^m h = 0$ and*

$$\mathrm{dist}(x, S)|\nabla h(x)| \leq C, \quad \textit{for } x \in \mathbb{R}^{2m}\backslash S. \tag{4.72}$$

Then there are constants β and λ_i, $1 \leq i \leq I$, such that

$$h(x) = \sum_{i=1}^{I} \lambda_i \log\frac{1}{|x - x_i|} + \beta. \tag{4.73}$$

Proof. Thanks to (4.72), $h \in L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$, so that $\Delta^m h$ is well defined in the sense of distributions and it is supported in S. Therefore

$$\Delta^m h = \sum_{i=1}^{I} \beta_i \delta_{x_i},$$

for some constants β_i. Then, recalling (2.22), if we set

$$v(x) := h(x) - \sum_{i=1}^{I} \lambda_i \log \frac{1}{|x - x_i|}, \quad \lambda_i := (-1)^m \frac{\beta_i}{\gamma_m},$$

we get $\Delta^m v \equiv 0$ in $\mathbb{R}^{2m}$ in the sense of distributions (hence v is smooth) and

$$|\nabla v(x)||x| \leq C \quad \text{in } \mathbb{R}^{2m}. \tag{4.74}$$

Then $|v(x)| \leq C(\log(1 + |x|) + 1)$. By Theorem 2.5 v is a polynomial, which (4.74) forces to be constant, say $v \equiv -\beta$. Now (4.73) follows at once. □

4.4 The case of S^{2m}

In the case of the $2m$-dimensional round sphere, the concentration-compactness of Theorem 4.2 becomes quite explicit: only one concentration point can appear and, by composing with suitable Möbius transformations, we have a global understanding of the concentration behavior. This was already noticed in [Str4] and [MS], in dimension 2 and 4 under the assumption, which we now drop, that the Q-curvatures are positive.

Theorem 4.10 *Let (S^{2m}, g) be the $2m$-dimensional round sphere, and let $u_k : M \to \mathbb{R}$ be a sequence of solutions of*

$$P_g u_k + (2m-1)! = Q_k e^{2mu_k}, \tag{4.75}$$

where $Q_k \to Q_0$ in C^0 for a given continuous function Q_0. Assume also that

$$\mathrm{vol}(g_k) = \int_{S^{2m}} e^{2mu_k} \mathrm{dvol}_g = |S^{2m}|, \tag{4.76}$$

where $g_k := e^{2mu_k} g$. Then one of the following is true.

(i) For every $0 \leq \alpha < 1$, a subsequence converges in $C^{2m-1,\alpha}(S^{2m})$.

(ii) There is a point $x_0 \in S^{2m}$ such that up to a subsequence $u_k \to -\infty$ locally uniformly in $S^{2m} \backslash \{x_0\}$. Moreover $Q_0(x_0) > 0$,

$$Q_k e^{2mu_k} \, \mathrm{dvol}_g \rightharpoonup \Lambda_1 \delta_{x_0}$$

and there exist Möbius diffeomorphisms Φ_k such that the metrics $h_k := \Phi_k^* g_k$ satisfy

$$h_k \to g, \text{ in } C^{2m-1,\alpha}_{\mathrm{loc}}(S^{2m}), \quad Q_{h_k} \to (2m-1)! \text{ in } L^2(S^{2m}). \tag{4.77}$$

Proof. On the round sphere

$$P_g^{2m} = \prod_{i=0}^{m-1} (-\Delta_g + i(2m-i-1)). \tag{4.78}$$

Moreover $\ker \Delta_g = \{constants\}$ and the non-zero eigenvalues of $-\Delta_g$ are all positive. That easily implies that $\ker P_g^{2m} = \{constants\}$. From Theorem 4.2, and the Gauss-Bonnet-Chern theorem, we infer that in case (ii) we have

$$\Lambda_1 = \int_M Q_g \,\mathrm{dvol}_g = I\Lambda_1,$$

hence $I = 1$, and $Q_k e^{2mu_k} \,\mathrm{dvol}_g \rightharpoonup \Lambda_1 \delta_{x_0}$.

To prove the second part of the theorem, for every k we define Möbius transformations $\Phi_k : S^{2m} \to S^{2m}$ such that the *normalized metrics* $h_k := \Phi_k^* g_k$ satisfy

$$\int_{S^{2m}} x \,\mathrm{dvol}_{h_k} = 0, \tag{4.79}$$

and are normalized with respect to rotations, so that $\Phi_k \to \Phi_0 \equiv x_0$ locally uniformly on $S^{2m} \backslash \{p\}$ for some $p \in S^{2m}$. The metrics h_k can be expressed in the form

$$h_k = e^{2v_k} g_{S^{2m}}, \quad v_k = u_k \circ \Phi_k + \frac{1}{2m} \log\det(d\Phi_k).$$

Also notice that

$$Q_{h_k} = Q_k \circ \Phi_k \to Q_0(x_0) \tag{4.80}$$

locally uniformly on $S^{2m} \backslash \{p\}$. We now claim that

$$\int_{S^{2m}} \Phi_k \cdot P_g^{2m} \Phi_k \,\mathrm{dvol}_g = C. \tag{4.81}$$

Indeed, using the conformal invariance of the Paneitz operator, and writing $\Phi_k = \pi^{-1} \circ \delta_k \circ \pi$, where π is the stereographic projection from a point which we may take to be the South Pole, and δ_k is a dilation of $\mathbb{R}^{2m}$, we have

$$\begin{aligned} \int_{S^{2m}} \Phi_k \cdot P_g^{2m} \Phi_k \,\mathrm{dvol}_g &= \int_{\mathbb{R}^{2m}} (-\Delta_{\mathbb{R}^{2m}})^m (\pi^{-1} \circ \delta_k) \cdot \pi^{-1} \circ \delta_k dx \\ &= \int_{\mathbb{R}^{2m}} |\nabla^m \pi^{-1}|^2 dx < \infty, \end{aligned}$$

where ∇^m denotes $\Delta^{\frac{m}{2}}$ for m even and $\nabla \Delta^{\frac{m-1}{2}}$ for m odd. Using (4.78), we infer that

$$P_g^{2m} = \sum_{i=1}^{m} C_i (-\Delta_g)^i,$$

where $C_m = 1$ and $C_1 > 0$ for $1 \leq i \leq m$. Then (4.81) and integration by parts imply

$$\sum_{i=1}^{m} \int_{S^{2m}} |\nabla^i \Phi_k|^2 \,\mathrm{dvol}_g = C.$$

Since Φ_k is a bounded map, elliptic estimates give uniform bounds for Φ_k in $H^m(S^{2m}, g)$:

$$\|\Phi_k\|_{H^m(S^{2m},g)} \leq C.$$

Hence up to a subsequence (Φ_k) converges weakly in $H^m(S^{2m}, g)$ to $\Phi_0 \equiv x_0$. By (4.80), we infer

$$\|Q_0 \circ \Phi_k - Q_0(x_0)\|_{L^2(S^{2m},h_k)} = \|Q_0 - Q_0(x_0)\|_{L^2(S^{2m},g_k)} \to 0, \quad \text{as } k \to \infty,$$

hence

$$\begin{aligned} \|Q_{h_k} - Q_0(x_0)\|_{L^2(S^{2m},h_k)} &= \|Q_{h_k} - Q_0 \circ \Phi_k\|_{L^2(S^{2m},g_k)} + o(1) \\ &= \|Q_k - Q_0\|_{L^2(S^{2m},g_k)} + o(1) \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

We can now apply a slight modification of Theorem 4.2 to the sequence h_k (Q_{h_k} is uniformly bounded in L^∞ and the above L^2 convergence is enough to make the proof work), and obtain that one of the following is true:

(a) For every $0 \le \alpha < 1$, up to a subsequence

$$h_k \to h_0, \quad \text{in } C^{2m-1,\alpha}(S^{2m}),$$

for some metric h_0.

(b) There is a point $x_1 \in S^{2m}$ such that up to a subsequence $v_k \to -\infty$ locally uniformly in $S^{2m} \backslash \{x_1\}$.

$$Q_{h_k} e^{2mv_k} \, \mathrm{dvol}_g \rightharpoonup \Lambda_1 \delta_{x_1}. \tag{4.82}$$

Since (4.82) contradicts (4.79), we are in case (a) and $h_k \to h_0$ in $C^{2m-1,\alpha}(S^{2m})$, where $Q_{h_0}^{2m} \equiv Q_0(x_0) > 0$. By Theorem 2.2 (vi),

$$h_0 = \left(\frac{(2m-1)!}{Q(x_0)} \right)^{\frac{1}{2m}} g.$$

On the other hand, the volume constraint (4.76) implies that

$$\mathrm{vol}(h_0) = \mathrm{vol}(h_k) = \mathrm{vol}(g_k) = |S^{2m}|,$$

hence $Q_0(x_0) = (2m-1)!$, and the theorem is proved. □

Chapter 5

Concentration-Compactness for an equation with critical quadratic exponential non-linearity

We now switch to a different problem. Given a bounded domain $\Omega \subset \mathbb{R}^{2m}$ with smooth boundary, and a sequence $0 < \lambda_k \to 0$, consider a sequence (u_k) of smooth solutions to

$$\begin{cases} (-\Delta)^m u_k = \lambda_k u_k e^{m u_k^2} & \text{in } \Omega \\ u_k > 0 & \text{in } \Omega \\ u_k = \partial_\nu u_k = \ldots = \partial_\nu^{m-1} u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Assume also that

$$\int_\Omega u_k (-\Delta)^m u_k dx = \lambda_k \int_\Omega u_k^2 e^{m u_k^2} dx \to \Lambda \geq 0 \quad \text{as } k \to \infty. \tag{5.2}$$

We have another concentration-compactness result:

Theorem 5.1 *Let (u_k) be a sequence of solutions to* (2.1), (5.2). *Then either*

(i) $\Lambda = 0$ and $u_k \to 0$ in $C^{2m-1,\alpha}(\Omega)$,[1] *or*

(ii) We have $\sup_\Omega u_k \to \infty$ as $k \to \infty$. Moreover there exists $I \in \mathbb{N} \setminus \{0\}$ such that $\Lambda \geq I\Lambda_1$, where $\Lambda_1 := (2m-1)!\mathrm{vol}(S^{2m})$, and up to a subsequence there are I converging sequences of points $x_{i,k} \to x^{(i)}$ and of positive numbers $r_{i,k} \to 0$, the latter defined by

$$\lambda_k r_{i,k}^{2m} u_k^2(x_{i,k}) e^{m u_k^2(x_{i,k})} = 2^{2m}(2m-1)!, \tag{5.3}$$

such that the following is true:

1. *For every $1 \leq i \leq I$ we have $\lim_{k \to \infty} \frac{\mathrm{dist}(x_{i,k}, \partial\Omega)}{r_{i,k}} = +\infty$.*

[1] Here and in the following $\alpha \in [0, 1)$ is an arbitrary Hölder exponent.

2. *If we define*

$$\eta_{i,k}(x) := u_k(x_{i,k})(u_k(x_{i,k}+r_{i,k}x) - u_k(x_{i,k})) + \log 2$$

for $1 \le i \le I$*, then*

$$\eta_{i,k}(x) \to \eta_0(x) = \log \frac{2}{1+|x|^2} \quad \text{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}) \quad (k \to \infty). \tag{5.4}$$

3. *For every* $1 \le i \ne j \le I$ *we have* $\lim_{k\to\infty} \frac{|x_{i,k}-x_{j,k}|}{r_{i,k}} = \infty$.

4. *Set* $R_k(x) := \inf_{1\le i \le I} |x - x_{i,k}|$*. Then*

$$\lambda_k R_k^{2m}(x) u_k^2(x) e^{m u_k^2(x)} \le C, \tag{5.5}$$

where C *does not depend on* x *or* k*.*

Finally $u_k \rightharpoonup 0$ *in* $H^m(\Omega)$ *and* $u_k \to 0$ *in* $C^{2m-1,\alpha}_{\mathrm{loc}}(\overline{\Omega}\backslash\{x^{(1)},\dots,x^{(I)}\})$*.*

The function η_0 in (5.4) satisfies $(-\Delta)^m \eta_0 = (2m-1)! e^{2m\eta_0}$, which is (4.9) with $Q_k \equiv (2m-1)!$ and $a_k \equiv 1$. This shows a surprising relation between (4.9) and (5.1). In fact η_0 has a remarkable geometric interpretation: If $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection, then

$$e^{2\eta_0} g_{\mathbb{R}^{2m}} = (\pi^{-1})^* g_{S^{2m}}, \tag{5.6}$$

where $g_{S^{2m}}$ is the round metric on S^{2m}. Then (5.6) implies

$$(2m-1)! \int_{\mathbb{R}^{2m}} e^{2m\eta_0} dx = \int_{S^{2m}} Q_{S^{2m}} \mathrm{dvol}_{g_{S^{2m}}} = (2m-1)! |S^{2m}| = \Lambda_1. \tag{5.7}$$

This is the reason why $\Lambda \ge I\Lambda_1$ in case (ii) of Theorem 5.1 above, compare Proposition 5.7.

Solutions to (2.1) arise from the Adams-Moser-Trudinger inequality [Ada] (see also [Mos], [Tru] and [BW]):

$$\sup_{u \in H_0^m(\Omega),\, \|u\|^2_{H_0^m} \le \Lambda_1} \fint_\Omega e^{mu^2} dx = c_0(m) < +\infty, \tag{5.8}$$

where $c_0(m)$ is a dimensional constant, and $H_0^m(\Omega)$ is the Beppo-Levi defined as the completion of $C_c^\infty(\Omega)$ with respect to the norm[2]

$$\|u\|_{H_0^m} := \|\Delta^{\frac{m}{2}} u\|_{L^2} = \left(\int_\Omega |\Delta^{\frac{m}{2}} u|^2 dx \right)^{\frac{1}{2}}, \tag{5.9}$$

and we used the following notation:

$$\Delta^{\frac{m}{2}} u := \begin{cases} \Delta^n u \in \mathbb{R} & \text{if } m = 2n \text{ is even,} \\ \nabla \Delta^n u \in \mathbb{R}^{2m} & \text{if } m = 2n+1 \text{ is odd.} \end{cases} \tag{5.10}$$

[2]The norm in (5.9) is equivalent to the usual Sobolev norm $\|u\|_{H^m} := \left(\sum_{\ell=0}^m \|\nabla^\ell u\|_{L^2}\right)^{\frac{1}{2}}$, thanks to elliptic estimates.

In fact (2.1) is the Euler-Lagrange equation of the functional

$$F(u) := \frac{1}{2}\int_\Omega |\Delta^{\frac{m}{2}} u|^2 dx - \frac{\lambda}{2m}\int_\Omega e^{mu^2} dx$$

(where $\lambda = \lambda_k$ plays the role of a Lagrange multiplier), which is well defined and smooth thanks to (5.8), but does not satisfy the Palais-Smale condition. For a more detailed discussion, in the context of Orlicz spaces, we refer to [Str3].

Theorem 5.1 has been proven by Adimurthi and M. Struwe [AS] and Adimurthi and O. Druet [AD] in the case $m = 1$, and by F. Robert and M. Struwe [RS] for $m = 2$. The extraction of a blow-up profile from a concentrating sequence of solutions to a nonlinear PDE was pioneered by J. Sack and K. Uhlenbeck [SU] and Wente [Wen]. Their ideas were later expanded in various ways by M. Struwe [Str1], [Str2], H. Brezis and J. M. Coron [BC1], [BC2] who, in particular, first wrote down separation conditions like conditions 1 and 3 in part (ii) of Theorem 5.1 (see also the works of T. H. Parker [Par], E. Hebey and F. Robert [HR] and many others). For further motivations and references we refer to M. Struwe [Str6]. Here, instead, we want to point out the main ingredients of our approach. Crucial to the proof of Theorem 5.1 are the gradient estimates in Lemma 5.6 and the blow-up procedure of Proposition 5.7. For the latter, we rely on a concentration-compactness result from [Mar2] and a classification result from [Mar1], which imply, together with the gradient estimates, that at the finitely many concentration points $\{x^{(1)}, \dots, x^{(I)}\}$, the profile of u_k is η_0, hence an energy not less that Λ_1 accumulates, namely

$$\lim_{R\to 0}\limsup_{k\to\infty}\int_{B_R(x^{(i)})} \lambda_k u_k^2 e^{mu_k^2} dx \geq \Lambda_1, \quad \text{for every } 1 \leq i \leq I.$$

As for the gradient estimates, if one uses (2.1) and (5.2) to infer $\|\Delta^m u_k\|_{L^1(\Omega)} \leq C$, then elliptic regularity gives $\|\nabla^\ell u_k\|_{L^p(\Omega)} \leq C(p)$ for every $p \in [1, 2m/\ell)$. These bounds, though, turn out to be too weak for Lemma 5.6 (see also the remark after Lemma 5.5). One has, instead, to fully exploit the integrability of $\Delta^m u_k$ given by (5.2), namely $\|\Delta^m u_k\|_{L(\log L)^{1/2}(\Omega)} \leq C$, where $L(\log L)^{1/2} \subsetneq L^1$ is the Zygmund space. Then an interpolation result from [BS] gives uniform estimates for $\nabla^\ell u_k$ in the Lorentz space $L^{(2m/\ell,2)}(\Omega)$, $1 \leq \ell \leq 2m-1$, which are sharp for our purposes (see Lemma 5.5).

We remark that when $m = 1$, things simplify dramatically, as we can simply integrate by parts (5.2) and get

$$\|\nabla u_k\|_{L^{(2,2)}(\Omega)} = \|\nabla u_k\|_{L^2(\Omega)} \leq C.$$

In the case $m = 2$, F. Robert and M. Struwe [RS] proved a slightly weaker form of our Lemma 5.6 by using subtle estimates in the BMO space, whose generalization to arbitrary dimensions appears quite challenging. Our approach, on the other hand, is simpler and more transparent.

Recently O. Druet [Dru] for the case $m = 1$, and M. Struwe [Str5] for $m = 2$ improved the previous results by showing that in case (ii) of Theorem 5.1 we have $\Lambda = L\Lambda_1$ for some positive $L \in \mathbb{N}$. Whether the same holds true for $m > 2$ is still an open question. In is also unknown whether $L = I$ in case $m = 1, 2$.

If we assume that $\sup_\Omega u_k \leq C$, we have that $\Delta^m u_k \to 0$ uniformly, since $\lambda_k \to 0$. By elliptic estimates we infer $u_k \to 0$ in $W^{2m,p}(\Omega)$ for every $1 \leq p < \infty$, hence $u_k \to 0$ in $C^{2m-1,\alpha}(\Omega)$, $\Lambda = 0$ and we have proven that we are in case (i) of the Theorem. Therefore in the following we shall assume that, up to a subsequence, $\sup_\Omega u_k \to \infty$ and show that we are in case (ii) of the Theorem. In Section 5.1 we analyze the asymptotic profile at blow-up points. In Section 5.2 we sketch the inductive procedure which completes the proof.

5.1 Analysis of the first blow-up

Let $x_k = x_{1,k}$ be a point such that $u_k(x_k) = \max_\Omega u_k$, and let $r_k = r_{1,k}$ be as in (5.3). Throughout this section $(u_k)_{k\in\mathbb{N}}$ is a sequence of functions satisfying (5.1), (5.2). We use the following notation: when m is odd $\Delta^{\frac{m}{2}}u := \nabla\Delta^{\frac{m-1}{2}}u$. Integrating by parts in (5.2), we find $\|\Delta^{\frac{m}{2}}u_k\|_{L^2(\Omega)} \leq C$ which, together with the boundary condition and elliptic estimates, gives

$$\|u_k\|_{H^m(\Omega)} \leq C. \tag{5.11}$$

Lemma 5.2 *We have*

$$\lim_{k\to\infty}\frac{\operatorname{dist}(x_k,\partial\Omega)}{r_k} = +\infty.$$

Proof. Set

$$\overline{u}_k(x) := \frac{u_k(r_k x + x_k)}{u_k(x_k)} \quad \text{for } x \in \Omega_k := \{r_k^{-1}(x - x_k) : x \in \Omega\}.$$

Then $\overline{u}_k$ satisfies

$$\begin{cases} (-\Delta)^m\overline{u}_k = \dfrac{2^{2m}(2m-1)!}{u_k^2(x_k)}\overline{u}_k e^{mu_k^2(x_k)(\overline{u}_k^2-1)} & \text{in } \Omega_k \\ \overline{u}_k > 0 & \text{in } \Omega_k \\ \overline{u}_k = \partial_\nu\overline{u}_k = \ldots = \partial_\nu^{m-1}\overline{u}_k = 0 & \text{on } \partial\Omega_k. \end{cases}$$

Assume for the sake of contradiction that up to a subsequence we have

$$\lim_{k\to\infty}\frac{\operatorname{dist}(x_k,\partial\Omega)}{r_k} = R_0 < +\infty.$$

Then, passing to a further subsequence, $\Omega_k \to \mathcal{P}$, where $\mathcal{P}$ is a half-space, and $\overline{u}_k \to \overline{u}$ in $C^{2m}_{\mathrm{loc}}(\overline{\mathcal{P}})$, where $\overline{u}(0) = \overline{u}_k(0) = 1$ and

$$\begin{cases} (-\Delta)^m\overline{u} = 0 & \text{in } \mathcal{P} \\ \overline{u} > 0 & \text{in } \mathcal{P} \\ \overline{u} = \partial_\nu\overline{u} = \ldots = \partial_\nu^{m-1}\overline{u} = 0 & \text{on } \partial\mathcal{P}. \end{cases}$$

By (5.11) and the Sobolev imbedding $H^{m-1}(\Omega) \hookrightarrow L^{2m}(\Omega)$, we find

$$\int_{\Omega_k}|\nabla\overline{u}_k|^{2m}dx = \frac{1}{u_k(x_k)^{2m}}\int_\Omega|\nabla u_k|^{2m}dx \leq \frac{C}{u_k(x_k)^{2m}} \to 0.$$

Then $\nabla\overline{u} \equiv 0$, hence $\overline{u} \equiv const = 0$ thanks to the boundary condition. That contradicts $\overline{u}(0) = 1$. ∎

Lemma 5.3 *We have*

$$u_k(x_k + r_k x) - u_k(x_k) \to 0 \quad \textit{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}) \textit{ as } k \to \infty. \tag{5.12}$$

Proof. Set

$$v_k(x) := u_k(x_k + r_k x) - u_k(x_k), \quad x \in \Omega_k$$

Then v_k solves

$$(-\Delta)^m v_k = 2^{2m}(2m-1)! \frac{\overline{u}_k(x)}{u_k(x_k)} e^{m u_k^2(x_k)(\overline{u}_k^2 - 1)} \leq 2^{2m} \frac{(2m-1)!}{u_k(x_k)} \to 0. \tag{5.13}$$

Assume that $m > 1$. By (5.11) and the Sobolev embedding $H^{m-2}(\Omega) \hookrightarrow L^m(\Omega)$, we get

$$\|\nabla^2 v_k\|_{L^m(\Omega_k)} = \|\nabla^2 u_k\|_{L^m(\Omega)} \leq C. \tag{5.14}$$

Fix now $R > 0$ and write $v_k = h_k + w_k$ on $B_R = B_R(0)$, where $\Delta^m h_k = 0$ and w_k satisfies the Navier-boundary condition on B_R. Then, (5.13) gives

$$w_k \to 0 \quad \text{in } C^{2m-1,\alpha}(B_R). \tag{5.15}$$

This, together with (5.14) implies

$$\|\Delta h_k\|_{L^m(B_R)} \leq C. \tag{5.16}$$

Then, since $\Delta^{m-1}(\Delta h_k) = 0$, we get from Proposition 2.4

$$\|\Delta h_k\|_{C^\ell(B_{R/2})} \leq C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{5.17}$$

By Pizzetti's formula (2.7),

$$\fint_{B_R} h_k dx = h_k(0) + \sum_{i=1}^{m-1} c_i R^{2i} \Delta^i h_k(0),$$

and (5.17), together with $|h_k(0)| = |w_k(0)| \leq C$ and $h_k \leq -w_k \leq C$, we find

$$\fint_{B_R} |h_k| dx \leq C.$$

Again by Proposition 2.4 it follows that

$$\|h_k\|_{C^\ell(B_{R/2})} \leq C(\ell) \quad \text{for every } \ell \in \mathbb{N}. \tag{5.18}$$

By Ascoli-Arzelà's theorem, (5.15) and (5.18), we have that up to a subsequence

$$v_k \to v \quad \text{in } C^{2m-1,\alpha}(B_{R/2}),$$

where $\Delta^m v \equiv 0$ thanks to (5.13). We can now apply the above procedure with a sequence of radii $R_k \to \infty$, extract a diagonal subsequence $(v_{k'})$, and find a function $v \in C^\infty(\mathbb{R}^{2m})$ such that

$$v \leq 0, \quad \Delta^m v \equiv 0, \quad v_{k'} \to v \quad \text{in } C^{2m-1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{2m}). \tag{5.19}$$

By Fatou's Lemma

$$\|\nabla^2 v\|_{L^m(\mathbb{R}^{2m})} \leq \liminf_{k\to\infty} \|\nabla^2 v_{k'}\|_{L^m(\Omega_k)} \leq C. \tag{5.20}$$

By Theorem 2.5 and (5.19), v is a polynomial of degree at most $2m-2$. Then (5.20) implies that v is constant, hence $v \equiv v(0) = 0$. Therefore the limit does not depend on the chosen subsequence $(v_{k'})$, and the full sequence (v_k) converges to 0 in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, as claimed.

When $m=1$, Pizzetti's formula and (5.13) imply at once that for every $R>0$ $\|v_k\|_{L^1(B_R)} \to 0$, hence $v_k \to 0$ in $W^{2,p}(B_R)$ as $k\to\infty$, $1 \leq p < \infty$. □

Now set

$$\eta_k(x) := u_k(x_k)[u_k(r_kx + x_k) - u_k(x_k)] + \log 2. \tag{5.21}$$

An immediate consequence of Lemma 5.3 is the following

Corollary 5.4 *The function η_k satisfies*

$$(-\Delta)^m \eta_k = V_k e^{2ma_k\eta_k}, \tag{5.22}$$

where

$$V_k(x) = 2^{m(1-\overline{u}_k)}(2m-1)!\overline{u}_k(x) \to (2m-1)!, \quad a_k = \frac{1}{2}(\overline{u}_k + 1) \to 1$$

in $C^0_{\mathrm{loc}}(\mathbb{R}^{2m})$.

Lemma 5.5 *For every $1 \leq \ell \leq 2m-1$, $\nabla^\ell u_k$ belongs to the Lorentz space $L^{(2m/\ell,2)}(\Omega)$ and*

$$\|\nabla^\ell u_k\|_{(2m/\ell,2)} \leq C. \tag{5.23}$$

Proof. We first show that $f_k := (-\Delta)^m u_k$ is bounded in $L(\log L)^{\frac{1}{2}}(\Omega)$, where

$$L(\log L)^\alpha(\Omega) := \left\{ f \in L^1(\Omega) : \|f\|_{L(\log L)^\alpha} := \int_\Omega |f| \log^\alpha(2+|f|)dx < \infty \right\}.$$

Indeed, set $\log^+ t := \max\{0, \log t\}$ for $t>0$. Then, using the simple inequalities

$$\log(2+t) \leq 2 + \log^+ t, \quad \log^+(ts) \leq \log^+ t + \log^+ s, \quad t,s>0,$$

one gets

$$\log(2+\lambda_k u_k e^{mu_k^2}) \leq 2 + \log^+ \lambda_k + \log^+ u_k + mu_k^2 \leq C(1+u_k)^2.$$

Then, since $f_k \geq 0$, we have

$$\begin{aligned} \|f_k\|_{L(\log L)^{\frac{1}{2}}} &\leq \int_\Omega f_k \log^{\frac{1}{2}}(2+f_k)dx \\ &\leq C\int_{\{x\in\Omega: u_k(x)\geq 1\}} \lambda_k u_k^2 e^{mu_k} dx + C|\Omega| \leq C \end{aligned}$$

by (5.2), as claimed. Now (5.23) follows from Theorem 5.15. □

Remark. The inequality (5.23) is intermediate between the L^1 and the $L \log L$ estimates. Indeed, the bound of $f_k := (-\Delta)^m u_k$ in L^1 implies $\|\nabla^\ell u_k\|_{L^p} \leq C$ for every $1 \leq \ell \leq 2m-1$, $1 \leq p < \frac{2m}{\ell}$, compare Lemma 4.6, and actually $\|\nabla^\ell u_k\|_{(2m/\ell,\infty)} \leq C$ (compare [Hél, Thm. 3.3.6]), but that is not enough for our purposes (Lemma 5.6 below). On the other hand, was f_k bounded in $L(\log L)$, we would have $\|\nabla^\ell u_k\|_{(2m/\ell,1)} \leq C$, which implies $\|u_k\|_{L^\infty} \leq C$ (compare [Hél, Thm. 3.3.8]). But we know that this is not the case in general.

Actually, the cases $1 \leq \ell \leq m$ in (5.23) follow already from (5.11) and the improved Sobolev embeddings, see [O'N]. What really matters here are the cases $m < \ell < 2m$. In fact, when $m = 1$ Lemma 5.5 reduces to (5.11).

The following lemma, which is reminiscent of Lemma 4.7, replaces Proposition 2.3 in [RS].

Lemma 5.6 *For any $R > 0$, $1 \leq \ell \leq 2m-1$ there exists $k_0 = k_0(R)$ such that*

$$u_k(x_k) \int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \leq C(Rr_k)^{2m-\ell}, \quad \text{for all } k \geq k_0.$$

Proof. We first claim that

$$\|\Delta^m(u_k^2)\|_{L^1(\Omega)} \leq C. \tag{5.24}$$

To see that, observe that

$$|\Delta^m(u_k^2)| \leq 2u_k(-\Delta)^m u_k + C \sum_{\ell=1}^{2m-1} |\nabla^\ell u_k||\nabla^{2m-\ell} u_k|. \tag{5.25}$$

The term $2u_k(-\Delta)^m u_k$ is bounded in L^1 thanks to (5.2). The other terms on the right-hand side of (5.25) are bounded in L^1 thanks to Lemma 5.5 and Proposition 5.11 belows. Hence (5.24) is proven.

Now set $f_k := (-\Delta)^m(u_k^2)$, and for any $x \in \Omega$, let G_x be the Green's function for $(-\Delta)^m$ on Ω with Dirichlet boundary condition. Then

$$u_k^2(x) = \int_\Omega G_x(y) f_k(y) dy.$$

Thanks to the basic estimate on the Greens's function (see [DAS, Thm. 12]), $|\nabla^\ell G_x(y)| \leq C|x-y|^{-\ell}$, we infer

$$|\nabla^\ell(u_k^2)(x)| \leq \int_\Omega |\nabla_x^\ell G_x(y)||f_k(y)| dy \leq C \int_\Omega \frac{|f_k(y)|}{|x-y|^\ell} dy.$$

Let μ_k denote the probability measure $\frac{|f_k(y)|}{\|f_k\|_{L^1(\Omega)}} dy$. By Fubini's theorem

$$\begin{aligned}
\int_{B_{Rr_k}(x_k)} |\nabla^\ell(u_k^2)(x)| dx &\leq C\|f_k\|_{L^1(\Omega)} \int_{B_{Rr_k}(x_k)} \int_\Omega \frac{1}{|x-y|^\ell} d\mu_k(y) dx \\
&\leq C \int_\Omega \int_{B_{Rr_k}(x_k)} \frac{1}{|x-y|^\ell} dx d\mu_k(y) \\
&\leq C \sup_{y \in \Omega} \int_{B_{Rr_k}(x_k)} \frac{1}{|x-y|^\ell} dx \leq C(Rr_k)^{2m-\ell}.
\end{aligned}$$

To conclude the proof, observe that Lemma 5.3 implies that on $B_{Rr_k}(x_k)$, for $1 \le \ell \le 2m-1$, we have $r_k^\ell \nabla^\ell u_k \to 0$ uniformly, hence

$$\begin{aligned} u_k(x_k)|\nabla^\ell u_k| &\le Cu_k|\nabla^\ell u_k| \le C\bigg(|\nabla^\ell(u_k^2)| + \sum_{j=1}^{\ell-1}|\nabla^j u_k||\nabla^{\ell-j}u_k|\bigg)\\ &\le C|\nabla^\ell(u_k^2)| + o(r_k^{-\ell}), \quad \text{as } k\to\infty. \end{aligned}$$

Integrating over $B_{Rr_k}(x_k)$ and using the above estimates we conclude. □

Proposition 5.7 *Let η_k be as in* (5.21). *Then $\eta_k(x) \to \eta_0(x) = \log\frac{2}{1+|x|^2}$ in $C^{2m}_{\mathrm{loc}}(\mathbb{R}^{2m})$, and*

$$\lim_{R\to\infty}\int_{B_{Rr_k}(x_k)} \lambda_k u_k^2 e^{mu_k^2}dx = \lim_{R\to\infty}(2m-1)!\int_{B_R(0)} e^{2m\eta_0}dx = \Lambda_1. \qquad (5.26)$$

Proof. Let a_k be as in Corollary 5.4. Notice that, thanks to Lemma 5.3,

$$\begin{aligned} \int_{B_R(0)} V_k e^{2ma_k\eta_k}dx &= \int_{B_{Rr_k}(x_k)} u_k(x_k)u_k\lambda_k e^{mu_k^2}dx \qquad (5.27)\\ &\le (1+o(1))\int_{B_{Rr_k}(x_k)} u_k^2\lambda_k e^{mu_k^2}dx \le \Lambda + o(1), \end{aligned}$$

where V_k and a_k are as in Corollary 5.4, and $o(1)\to 0$ as $k\to\infty$.

Step 1. We claim that $\eta_k \to \overline{\eta}$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, where $\overline{\eta}$ satisfies

$$(-\Delta)^m\overline{\eta} = (2m-1)!e^{2m\overline{\eta}}, \qquad (5.28)$$

and letting $R\to\infty$ in (5.27), from Corollary 5.4 we infer $e^{2m\overline{\eta}} \in L^1(\mathbb{R}^{2m})$.

Let us prove the claim. Corollary 5.4, Theorem 4.1, and (5.27), together with $\eta_k \le \log 2$, imply that up to subsequences either

(i) $\eta_k \to \overline{\eta}$ in $C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$ for some function $\overline{\eta} \in C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m})$, or

(ii) $\eta_k \to -\infty$ locally uniformly, or

(iii) there exists a closed set $S_0 \neq \emptyset$ of Hausdorff dimension at most $2m-1$ and numbers $\beta_k \to +\infty$ such that

$$\frac{\eta_k}{\beta_k} \to \varphi \text{ in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}\backslash S_0),$$

where

$$\Delta^m\varphi \equiv 0, \quad \varphi \le 0, \quad \varphi \not\equiv 0 \quad \text{on } \mathbb{R}^{2m}, \quad \varphi \equiv 0 \text{ on } S_0. \qquad (5.29)$$

Since $\eta_k(0) = \log 2$, (ii) can be ruled out. Assume now that (iii) occurs. From Liouville's theorem and (5.29) we get $\Delta\varphi \not\equiv 0$, hence for some $R > 0$ we have $\int_{B_R}|\Delta\varphi|dx > 0$ and

$$\lim_{k\to\infty}\int_{B_R}|\Delta\eta_k|dx = \lim_{k\to\infty}\beta_k\int_{B_R}|\Delta\varphi|dx = +\infty. \qquad (5.30)$$

On the other hand, we infer from Lemma 5.6

$$\int_{B_R} |\nabla^\ell \eta_k| dx = u_k(x_k) r_k^{\ell-2m} \int_{B_{Rr_k}(x_k)} |\nabla^\ell u_k| dx \leq C R^{2m-\ell}, \tag{5.31}$$

contradicting (5.30) when $\ell = 2$ and therefore proving our claim.

Step 2. We now prove that $\overline{\eta}$ is a standard solution of (5.28), i.e. there are $\lambda > 0$, $x_0 \in \mathbb{R}^{2m}$ such that

$$\overline{\eta}(x) = \log \frac{2\lambda}{1 + \lambda^2 |x - x_0|^2}. \tag{5.32}$$

Was this not the case, according to [Mar1, Thm. 2], there would exist $j \in \mathbb{N}$ with $1 \leq j \leq m-1$ and $a < 0$ such that

$$\lim_{|x| \to \infty} (-\Delta)^j \overline{\eta}(x) = a.$$

This would imply

$$\lim_{k\to\infty} \int_{B_R(0)} |\Delta^j \eta_k| dx = |a| \cdot \mathrm{vol}(B_1(0)) R^{2m} + o(R^{2m}) \quad \text{as } R \to \infty,$$

contradicting (5.31) for $\ell = 2j$. Hence (5.32) is established. Since $\eta_k \leq \eta_k(0) = \log 2$, it follows immediately that $x_0 = 0$, $\lambda = 1$, i.e. $\overline{\eta} = \eta_0$, and (5.26) follows from (5.7), (5.27) and Fatou's lemma. □

5.2 Exhaustion of the blow-up points and proof of Theorem 5.1

For $\ell \in \mathbb{N}$ we say that (H_ℓ) holds if there are ℓ sequences of converging points $x_{i,k} \to x^{(i)}$, $1 \leq i \leq \ell$ such that

$$\sup_{x \in \Omega} \lambda_k R_{\ell,k}^{2m}(x) u_k^2(x) e^{m u_k^2(x)} \leq C, \tag{5.33}$$

where

$$R_{\ell,k}(x) := \inf_{1 \leq i \leq \ell} |x - x_{i,k}|.$$

We say that (E_ℓ) holds if there are ℓ sequences of converging points $x_{i,k} \to x^{(i)}$ such that, if we define $r_{i,k}$ as in (5.3), the following hold true:

(E_ℓ^1) For all $1 \leq i \neq j \leq \ell$

$$\lim_{k\to\infty} \frac{\mathrm{dist}(x_{i,k}, \partial\Omega)}{r_{i,k}} = \infty, \qquad \lim_{k\to\infty} \frac{|x_{i,k} - x_{j,k}|}{r_{i,k}} = \infty.$$

(E_ℓ^2) For $1 \leq i \leq \ell$ (5.4) holds true.

(E_ℓ^3) $\lim_{R\to\infty} \lim_{k\to\infty} \int_{\cup_{i=1}^\ell B_{Rr_{i,k}}(x_{i,k})} \lambda_k u_k^2 e^{m u_k^2} dx = \ell \Lambda_1$.

To prove Theorem 5.1 we show inductively that (H_I) and (E_I) hold for some positive $I \in \mathbb{N}$, following the approach of [AD] and [RS]. First observe that (E_1) holds thanks to Lemma 5.2 and Proposition 5.7. Assume now that for some $\ell \geq 1$ (E_ℓ) holds and (H_ℓ) does not. Choose $x_{\ell+1,k} \in \Omega$ such that

$$\lambda_k R^{2m}_{\ell,k}(x_{\ell+1,k}) u_k^2(x_{\ell+1,k}) e^{mu_k^2(x_{\ell+1,k})} = \lambda_k \max_{\Omega} R^{2m}_{\ell,k} u_k^2 e^{mu_k^2} \to \infty, \tag{5.34}$$

and define $r_{\ell+1,k}$ as in (5.3). It easily follows from (5.34) that

$$\lim_{k\to\infty} \frac{|x_{\ell+1,k} - x_{i,k}|}{r_{\ell+1,k}} = 0, \quad 1 \leq i \leq \ell. \tag{5.35}$$

We now need to replace Lemma 5.2 and Lemma 5.3 with the lemma below.

Lemma 5.8 *Under the above assumptions and notation, we have*

$$\lim_{k\to\infty} \frac{\operatorname{dist}(x_{\ell+1,k}, \partial\Omega)}{r_{\ell+1,k}} = \infty \tag{5.36}$$

and

$$u_k(x_{\ell+1,k} + r_{\ell+1,k}x) - u_k(x_{\ell+1,k}) \to 0 \quad \text{in } C^{2m-1}_{\mathrm{loc}}(\mathbb{R}^{2m}), \quad \text{as } k \to \infty. \tag{5.37}$$

Proof. To simplify the notation, let us write $y_k := x_{\ell+1,k}$ and $\rho_k := r_{\ell+1,k}$. Evaluating the right-hand side of (5.34) at the point $y_k - \rho_k x$ we get

$$\begin{aligned} &\Big(\inf_{1\leq i\leq \ell} |y_k - x_{i,k} - \rho_k x|^{2m}\Big) u_k^2(y_k + \rho_k x) e^{mu_k^2(y_k+\rho_k x)} \\ &\leq \Big(\inf_{1\leq i\leq \ell} |y_k - x_{i,k}|^{2m}\Big) u_k^2(y_k) e^{mu_k^2(y_k)}, \end{aligned}$$

that is

$$\overline{u}^2_{\ell+1,k}(x) e^{mu_k^2(y_k)(\overline{u}^2_{\ell+1,k}(x)-1)} \leq \frac{\inf_{1\leq i\leq \ell} |y_k - x_{i,k}|^{2m}}{\inf_{1\leq i\leq \ell} |y_k - x_{i,k} - \rho_k x|^{2m}} = 1+o(1), \tag{5.38}$$

where $o(1) \to 0$ as $k \to \infty$ locally uniformly in x, as (5.35) immediately implies. Then (5.36) follows as in the proof of Lemma 5.2, since (5.38) implies

$$(-\Delta)\overline{u}_{\ell+1,k} = \frac{2^{2m}(2m-1)!}{u_k^2(y_k)} \overline{u}_{\ell+1,k} e^{mu_k^2(y_k)(\overline{u}^2_{\ell+1,k}-1)} = o(1), \tag{5.39}$$

where $o(1) \to 0$ as $k \to \infty$ uniformly locally in x.

Similarly, if we define $v_k(x) := u_k(x_{\ell+1,k} + r_{\ell+1,k}x) - u_k(x_{\ell+1,k})$, we can use (5.39) to replace (5.13) in the proof of Lemma 5.3 and get

$$(-\Delta)^m v_k \to 0$$

locally uniformly in $\mathbb{R}^{2m}$. Then the rest of the proof of Lemma 5.3 applies without changes, and also (5.37) is proved. □

Still repeating the arguments of the preceding section with $x_{\ell+1,k}$ instead of x_k and $r_{\ell+1,k}$ instead of r_k, we define

$$\eta_{\ell+1,k}(x) := u_k(x_{\ell+1,k})[u_k(r_{\ell+1,k}x + x_{\ell+1,k}) - u_k(x_{\ell+1,k})],$$

and we have

Proposition 5.9 *We have* $\eta_{\ell+1,k}(x) \to \eta_0(x) = \log \frac{2}{1+|x|^2}$ *in* $C^{2m}_{\mathrm{loc}}(\mathbb{R}^{2m})$ *and*

$$\lim_{R\to\infty} \int_{B_{Rr_{\ell+1,k}}(x_{\ell+1,k})} \lambda_k u_k^2 e^{mu_k^2} dx = \lim_{R\to\infty} \int_{B_R(0)} e^{2m\eta_0} dx = \Lambda_1. \tag{5.40}$$

Summarizing, we have proved that $(E^1_{\ell+1})$, $(E^2_{\ell+1})$ and (5.40) hold. These also imply that $(E^3_{\ell+1})$ holds, hence we have $(E_{\ell+1})$. Because of (5.2) and (E^3_ℓ), the procedure stops in a finite number I of steps, and we have (H_I).

Finally, we claim that $\lambda_k \to 0$ implies $u_k \rightharpoonup 0$ in $H^m(\Omega)$. This, (5.5) and elliptic estimates then imply that

$$u_k \to 0 \quad \text{in} \quad C^{2m-1,\alpha}_{\mathrm{loc}}(\Omega \backslash \{x^{(1)}, \dots, x^{(I)}\}).$$

To prove the claim, we observe that for any $\alpha > 0$

$$\begin{aligned}
\int_\Omega (-\Delta)^m u_k dx &= \int_\Omega \lambda_k u_k e^{mu_k^2} dx \\
&\leq \frac{\lambda_k}{\alpha} \int_{\{x\in\Omega : u_k \geq \alpha\}} u_k^2 e^{mu_k^2} dx + \lambda_k \int_{\{x\in\Omega : u_k < \alpha\}} u_k e^{mu_k^2} dx \\
&\leq \frac{C}{\alpha} + \lambda_k C_\alpha,
\end{aligned}$$

where C_α depends only on α. Letting k and α go to infinity, we infer

$$\Delta^m u_k \to 0 \quad \text{in } L^1(\Omega). \tag{5.41}$$

Thanks to (5.11), we infer that up to a subsequence $u_k \rightharpoonup u_0$ in $H^m(\Omega)$. Then (5.41) and the boundary condition imply that $u_0 \equiv 0$, in particular the full sequence converges to 0 weakly in $H^m(\Omega)$. This completes the proof of the theorem.

5.3 A Lorentz-space estimate

In this section we want to give a proof of an elliptic estimate for functions u satisfying Navier or Dirichlet (or even more general) boundary conditions and with $\Delta^m u$ being slightly more than integrable, Theorem 5.15 below. We start by recalling the definition and some properties of the Lorentz spaces.

Given a measurable function $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, we define the *distribution function* of u as

$$\lambda_u(t) := \mu\{x \in \Omega : |u(x)| > t\},$$

where μ denotes the Lebesgue measure in $\mathbb{R}^n$. Then we define the *equimeasurable decreasing rearrangement* of u as

$$u^*(s) := \inf\{t : \lambda_u(t) \leq s\}.$$

Definition 5.10 (Lorentz spaces) *For* $u : \Omega \to \mathbb{R}$ *measurable, set*

$$u^{**}(t) := \frac{1}{t} \int_0^t u^*(s) ds.$$

Set, for $1 \le p \le \infty$

$$\|u\|_{(p,q)} := \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} u^{**}(t)\right) q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } 1 \le q < \infty \\ \\ \sup_{t>0} t^{\frac{1}{p}} u^{**}(t) & \text{if } q = \infty. \end{cases}$$

The Lorentz space $L^{(p,q)}(\Omega)$ *is defined as*

$$L^{(p,q)}(\Omega) := \Big\{ u : \Omega \to R \text{ measurable} : \|u\|_{(p,q)} < \infty \Big\},$$

endowed with the norm $\| \cdot \|_{(p,q)}$.

The Lorentz spaces generalize the Lebesgue spaces in that $L^{(p,p)}(\Omega) = L^p(\Omega)$ for $1 < p < \infty$. Moreover $L^{(p,q)}(\Omega) \subset L^{(p,r)}$ for $1 < p < \infty$ and $1 \le q \le r \le \infty$.

The following is a generalization of Hölder's inequality, and can be found in [O'N].

Proposition 5.11 *Let* $1 < p < \infty$, $1 \le q \le \infty$ *and* $\Omega \subset \mathbb{R}^n$. *Let* p' *and* q' *be the conjugate exponents, i.e.*

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Let $f \in L^{(p,q)}(\Omega)$, $g \in L^{(p',q')}(\Omega)$. *Then* $fg \in L^1(\Omega)$, *and*

$$\|fg\|_{L^1(\Omega)} \le \|f\|_{(p,q)} \|g\|_{(p',q')}.$$

We also have the following improved Sobolev imbedding.

Proposition 5.12 *Let* $f \in L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$, $1 \le p \le n$, *and assume that* $\nabla f \in L^{(p,q)}(\Omega)$. *Then* $f \in L^{(p^*,q)}(\Omega)$, *where* $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. *Moreover*

$$\|f\|_{(p^*,q)} \le C(\|\nabla f\|_{(p,q)} + \|f\|_{L^p}).$$

Finally, we also need an interpolation result for Lorentz spaces.

Theorem 5.13 ([SW], [Hun]) *Let* Ω *be an open subset of* $\mathbb{R}^n$. *Let* r_0, r_1, p_0, *be real numbers such that*

$$1 \le r_0 < r_1 \le \infty$$

and

$$1 \le p_0 \ne p_1 \le \infty.$$

Let T *be a linear operator whose domain* D *contains*

$$\bigcup_{r_0 \le r \le r_1} L^r(\Omega)$$

and which maps continuously $L^{r_0}(\Omega)$ *to* $L^{p_0}(\mathbb{R})$ *and* $L^{r_1}(\Omega)$ *to* $L^{p_1}(\mathbb{R})$, *with norms*

$$\begin{aligned} \|Tf\|_{L^{p_0}(\mathbb{R})} &\le C_0 \|f\|_{L^{r_0}(\Omega)} \quad \text{for every } f \in L^{r_0}(\Omega) \\ \|Tf\|_{L^{p_1}(\mathbb{R})} &\le C_1 \|f\|_{L^{r_1}(\Omega)} \quad \text{for every } f \in L^{r_1}(\Omega). \end{aligned}$$

Then, for each $1 \leq q \leq \infty$*, and for every pair* (p, r) *such that there exists* $\theta \in (0,1)$ *with*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1},$$

T *maps continuously* $L^{(r,q)}(\Omega)$ *to* $L^{(p,q)}(\mathbb{R})$*, and*

$$\|Tf\|_{L^{(p,q)}(\mathbb{R})} \leq C\|f\|_{L^{(r,q)}(\Omega)} \quad \text{for every } f \in L^{(r,q)}(\Omega),$$

where $C = C(C_0, C_1, r, p, q, r_0, r_1, p_0, p_1)$*.*

The following theorem can be found in [BS, Cor. 6.16] (see also the discussion in [BS, Page 254]).

Theorem 5.14 *Let* T_λ *be the fractional integral operator defined by*

$$T_\lambda f = I_\lambda * f, \quad I_\lambda(x) := \frac{1}{|x|^{n-\lambda}},$$

for $0 < \lambda < n$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *measurable. Then, for any* $0 \leq \alpha < 1$*,* T_λ *is bounded from* $L(\log L)^\alpha(\mathbb{R}^n)$ *to* $L^{\left(\frac{n}{n-\lambda}, \frac{1}{\alpha}\right)}(\mathbb{R}^n)$*.*

Theorem 5.15 *Let* u *solve* $\Delta^m u = f \in L(\log L)^\alpha$ *in* Ω *with Dirichlet or Navier boundary conditions,* $0 \leq \alpha \leq 1$*,* $\Omega \subset \mathbb{R}^n$ *bounded and with smooth boundary,* $n \geq 2m$*. Then* $\nabla^{2m-\ell} u \in L^{\left(\frac{n}{n-\ell}, \frac{1}{\alpha}\right)}(\Omega)$*,* $1 \leq \ell \leq 2m-1$ *and*

$$\|\nabla^{2m-\ell} u\|_{\left(\frac{n}{n-\ell}, \frac{1}{\alpha}\right)} \leq C\|f\|_{L(\log L)^\alpha}. \tag{5.42}$$

Proof. Define

$$\hat{f} := \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

and let $w := K * \hat{f}$, where K is the fundamental solution of Δ^m. Then

$$|\nabla^{2m-1} w| = |(\nabla^{2m-1} K) * \hat{f}| \leq C I_1 * |\hat{f}|,$$

where $I_1(x) = |x|^{1-n}$. According to theorem 5.14, $|\nabla^{2m-1} w| \in L^{\left(\frac{n}{n-1}, \frac{1}{\alpha}\right)}(\mathbb{R}^n)$ and

$$\|\nabla^{2m-1} w\|_{\left(\frac{n}{n-1}, \frac{1}{\alpha}\right)} \leq C\|\hat{f}\|_{L(\log L)^\alpha} = C\|f\|_{L(\log L)^\alpha}. \tag{5.43}$$

We now use (5.43) to prove (5.42), following a method that we learned from [Hél]. Given $g : \Omega \to \mathbb{R}^n$ measurable, let v_g be the solution to $\Delta^m v_g = \operatorname{div} g$ in Ω, with the same boundary condition as u, and set $P(g) := |\nabla^{2m-1} v_g|$. By L^p estimates (see e.g. [ADN]), P is bounded from $L^p(\Omega; \mathbb{R}^n)$ into $L^p(\Omega)$ for $1 < p < \infty$. Then, thanks to Theorem 5.13 above, P is bounded from $L^{(p,q)}(\Omega; \mathbb{R}^n)$ into $L^{(p,q)}(\Omega)$ for $1 < p < \infty$ and $1 \leq q \leq \infty$. Choosing now $g = \nabla \Delta^{m-1} w$, we get $v_g = u$, hence $|\nabla^{2m-1} u| = P(\nabla \Delta^{m-1} w)$, and from (5.43) we infer

$$\|\nabla^{2m-1} u\|_{\left(\frac{n}{n-1}, \frac{1}{\alpha}\right)} \leq C\|\nabla \Delta^{m-1} w\|_{\left(\frac{n}{n-1}, \frac{1}{\alpha}\right)} \leq C\|f\|_{L(\log L)^\alpha}.$$

For $1 < \ell \leq 2m-1$, (5.42) follows from Proposition 5.12. □

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