Kendall distributions and level sets in bivariate exchangeable survival models

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A R T I C L E   I N F O

Article history:
Received 13 June 2008
Received in revised form 5 February 2009
Accepted 13 February 2009

Keywords:
Survival copulas
Bivariate aging functions
Semi-copulas
Archimedean copulas
Associative copulas
Bivariate VaR-curves

A B S T R A C T

For a given bivariate survival function \( F \), we study the relations between the set of the level curves of \( F \) and the Kendall distribution. Then we characterize the survival models simultaneously admitting a specified Kendall distribution and a specified set of level curves. Attention will be restricted to exchangeable survival models.

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1. Introduction

Let \( X, Y \) be a pair of non-negative random variables with joint distribution function \( F(x, y) \), and let \( F(x, y) \) denote their joint survival function, namely, for \( x \geq 0, y \geq 0 \),

\[
F(x, y) = P(X > x, Y > y).
\]

We concentrate our attention on the case when

H1 \( X \) and \( Y \) are exchangeable.

H2 \( F \) is continuous, strictly positive and strictly 1-decreasing, i.e. \( F \) is strictly decreasing in each variable.

We denote by \( \overline{F} = \overline{F} \) the common univariate marginal survival function of \( X \) and \( Y \), i.e.

\[
\overline{F}(t) = P(X > t) = P(Y > t) = \overline{F}(0, t),
\]

and by \( \overline{F} \) the survival copula. Under the previous assumptions, \( \overline{F} \) is continuous, strictly positive, and strictly decreasing all over the half-line \([0, \infty)\), with \( \overline{F}(0) = 1 \), and the survival copula is uniquely characterized by

\[
F(x, y) = \overline{F}(\overline{F}(x), \overline{F}(y)).
\]
In analogy with the Kendall distribution function associated to $F$ (see e.g. [18])

$$K_F(v) := P\{F(X, Y) \leq v\},$$

we set

$$\tilde{K}_F(v) := P\{\tilde{F}(X, Y) \leq v\},$$

and refer to it as the the upper-orthant Kendall distribution associated to $\tilde{F}$.

It can be easily shown (see Section 2) that $K_F$ depends only on the survival copula $C_F$. Namely that it is the Kendall distribution associated to the distribution $C_F$, and therefore $K_F$ is a Kendall distribution. The Kendall distributions are characterized (see [18, 19]) as the distribution functions $K$ such that

$$K(0^+) = 0, \ K(u) \geq u, \ u \in [0, 1].$$

(4)

Considering the level sets $A_v := \{(x, y) \in \mathbb{R}^2 | \tilde{F}(x, y) \leq v\}$, we can write $\tilde{K}_F(v) = P\{(X, Y) \in A_v\}$, and, furthermore, we will assume that

$${\cal H}_3$$

for $v \in (0, 1)$, the boundary $\partial A_v$,

$$\partial A_v := \{(x, y) \in \mathbb{R}^2 | \tilde{F}(x, y) = v\},$$

is a continuous curve, that will be referred to as a level curve.

We mention that, for $0 < v < 1$, $\partial A_v$ can be interpreted as the bivariate upper-orthant Value at Risk curve (VaR-curve) at level $1 - v$, as in [9] (see also [21]).

In the rest of the paper conditions $${\cal H}_1-\cal H}_3$$ will be assumed as our standing hypotheses, unless differently stated.

We will be interested in survival models $\tilde{F}$ such that the following set is fixed

$$\Psi_F := \{A_v | v \in (0, 1)\}.$$ 

Let $\tilde{F}$ be a given joint survival function satisfying conditions $${\cal H}_1-\cal H}_3$$, we wonder if there exists a different bivariate survival function $\tilde{F}$ such that, simultaneously, $K_F = \tilde{K}_F$, $\Psi_F = \Psi_{\tilde{F}}$, and, in case, which type of relations exist between $\tilde{F}$ and $\tilde{F}$. More generally, in this paper we face the following compatibility problem: let $K$ be a Kendall distribution (i.e. satisfying (4)), we wonder if the system of functional equations $\tilde{K}_F = K$, $\Psi_F = \Psi_{\tilde{F}}$, admits some solutions $\tilde{F}$. If this is the case we shall say that the pair $(K, \Psi_F)$ is compatible.

Our main result concerns such a compatibility problem and will be stated as Theorem 2 in the next Section 2. In words this result shows, under simple technical conditions, which is the form of possible solutions. It is also shown that, if a solution does exist, then the problem may have at most $\infty^3$ solutions, more precisely the possible solutions are indexed by a non-negative parameter $\theta$. The problem is completely solved in the following "Archimedean" case: the survival copula $C_F$ is Archimedean (see e.g. [18]), and one can find a joint survival function $\tilde{F}$, with $C_F$ Archimedean, and such that $K = \tilde{K}_F$. In such a case the analysis is rather direct, and its solution has also a key role in the interpretation of the result for the general case (see in particular Corollary 16, Example 25 and Remark 26).

Essentially, our analysis will be based on three different types of arguments:

(i) For a bivariate survival model satisfying our assumptions, the set $\Psi_F$ can be represented in terms of the so-called bivariate aging function $B = B_{\theta}$, introduced by Bassan and Spizzichino in [2] (the definition of $B$ will be recalled in Section 3, see [18]; see also, e.g. [4]).

(ii) To any Kendall distribution $K$ it can be associated the equivalence class $\mathbb{E}_K$ of all bivariate copulas admitting $K$ as their Kendall distribution. In Section 2, we shall briefly recall the results obtained in [10], [11], [18], and [19], concerning Kendall distributions and their associated equivalence classes. We shall in particular use a basic result showing that, under the condition $K(r^+) > r$, the class $\mathbb{E}_K$ contains one and only one Archimedean copula. The generator of such a copula will have a fundamental role in the solution of the compatibility problem.

(iii) We analyze some specific aspects of the relations existing between $\Psi_F$ and $\tilde{K}_F$. To this purpose we use (a slight modification of) a transformation result proven by Genest and Rivest in [11].

Some applications of our method can be found in the field of risk and bivariate models of interacting defaults. In fact, as mentioned above, the level curves of $\tilde{F}$ can be interpreted as upper-orthant bivariate VaR-curves.

A different type of application is, instead, the characterization of special bivariate statistical models in terms of the upper-orthant Kendall distribution function and of the levels sets, or equivalently of the aging function. A significant example is provided by Marshall–Olkin models (see Example 27). As a consequence of Theorem 2, for any choice of the pair $(K, \Psi_{\tilde{F}})$, these models are one-parameter models.

More in detail, the outline of the paper is as follows. Our main result will be stated in Section 2, and proven in Section 5, under the condition that $C_F$ admits a density function. In Section 2, we will also recall some basic facts about the Kendall distributions. Sections 3 and 4 will be devoted to collect further developments and results that are preliminary to the proof of Theorem 2; in Section 3 we discuss in which sense the level sets of a bivariate, exchangeable, survival function can be
described in terms of the aging function. We also provide some examples and details useful for our purposes. In Section 4 we analyze some further aspects of $K_F$ and recall the transformation result in [11] for copulas and extend it to semi-copulas (see Theorem 19 and Proposition 20, respectively). Section 5 will be devoted to present the proof of Theorem 2; we shall actually see that the compatibility problem can be formulated in terms of the aging function, and that this alternative formulation is a key point in our proof. Section 5 ends with some comments and the analysis of special cases.

2. Strict Archimedean Kendall distributions and solution to the compatibility problem

In this section we recall some relevant facts about Kendall distributions and introduce some terminology and notation, needed for our main result. In particular we focus on the concept of strict Archimedean Kendall distribution.

Let $X$ and $Y$ be two random variables, not necessarily exchangeable, with joint distribution function $F(x, y)$, and marginal distributions functions $F_X(x)$ and $F_Y(y)$. The Kendall distribution function $K_F(u)$, associated to $F$, denoted by $K_F(u)$ and given in (2), is the probability distribution function of the random variable $F(X, Y)$ (also known as BIPIT, the Bivariate Probability Integral Transformation, see e.g. [11]). In view of Sklar’s Theorem (see e.g. [16]), it is easily seen that, when $F$ satisfies condition H2, then $K_F$ only depends on $C_F$, the unique connecting copula of $F$, i.e. the bivariate distribution function such that $C_F(F_X(x), F_Y(y)) = F(x, y)$. More precisely we can write $K_F = K_{C_F}$. A similar relation holds between the upper-orthant distribution function $K_F$ and the survival copula $C_F$, the latter being characterized by $F(x, y) = C_F(F_X(x), F_Y(y))$. Namely the relation reads $K_F = K_{C_F}$. A proof can be easily obtained by recalling that $(U, V) := (G(X), G(Y))$ has distribution function $C_F$, and that $C_F(u, v) = u + v - 1 + C_F(1 - u, 1 - v)$: indeed then $(U, V) := (G(X), G(Y))$ has distribution function $C_F$ and therefore

$$\hat{K}_F(t) = P(F(X, Y) \leq t) = P(C_F(U, V) \leq t).$$

It is important to point out that therefore $\hat{K}_F$ is a Kendall distribution, as we have already mentioned in Section 1.

Under H2, $C_F$ and $C_F$ belong to the family of bivariate copulas $C$ such that $C_F(\cdot, \cdot) := C(\cdot, \cdot)$ is a strictly increasing and continuous function for every $v$. On this family, one can define the operator $\mathcal{A}$ as follows: setting $\nu^t_{u} = \nu^t_{u}(t)$,

$$\mathcal{A}C(t) := t + \int_{t}^{1} \frac{C}{C_{0}}(u, v) |_{u=\nu^t_{v}} ? du.$$

The interest of the operator $\mathcal{A}$ is in that the Kendall distribution associated to $C$ is given by $\mathcal{A}C(t)$ (see e.g. [11]), and consequently $K_F(t) = \mathcal{A}C_F(t)$. The latter equation, combined with (5), shows that

$$\hat{K}_F(t) = \mathcal{A}C_F(t).$$

We remind that a bivariate copula is Archimedean if it is of the form

$$C^\phi(u, v) = \phi^{-1}[\phi(u) + \phi(v)]$$

with $\phi : [0, 1] \rightarrow [0, +\infty)$ a convex, continuous, decreasing function. The function $\phi$ is called an (additive) generator of $C^\phi$ and it is obviously determined up to a positive multiplicative constant. The generator $\phi$ is said to be strict when $\phi(0) := \phi(0^+) = +\infty$.

In the sequel, when an Archimedean copula is a survival copula we will use the symbol $\hat{\phi}$, in place of $\phi$, to denote its generator, i.e. we will write $C_F = C^\phi$. In the latter case one has then

$$\hat{K}_F(t) = t - \frac{\hat{\phi}(t)}{\hat{\phi}(t^+)},$$

as it can be easily checked, e.g. by using (7) and (6). The above relation is the survival “counterpart” of the well known fact that $K_F(t) = t - \phi(t)/\phi(t^+)$, in the Archimedean case (see e.g. [16, p. 102]).

As already mentioned, Kendall distributions can be characterized as the distribution functions $K$ satisfying the conditions (4). Generally, for such functions, the equivalence class $\mathcal{K}$ of bivariate copulas $C$ such that $K_C = K$ is an infinite class. There is then a (generally infinite) class of bivariate survival functions such that their upper-orthant Kendall distribution is $K$. When $K$ satisfies the following further condition

$$\forall t \in (0, 1), \quad K(t^+) > t,$$

then there exists a unique Archimedean copula $C^\phi$ belonging to $\mathcal{K}$ (see [10], [18], and [19]). Letting $t_0$ be an arbitrary constant in $(0, 1)$, any possible generator of $C^\phi$ is a decreasing function $\phi : [0, 1] \rightarrow [0, +\infty)$ of the form

$$\phi(t) = \theta \cdot \exp \left\{ \int_{t_0}^{t} \frac{1}{s - K(s)} \, ds \right\},$$

with $\theta > 0$. This result had been proven by Genest and Rivest in [10] and has been exploited by Nelsen et al. in [19], to prove that, in the case when (4) holds but without assuming the condition (10), there exists a unique associative copula belonging to $\mathcal{K}$. As is well known, a copula $C$ is said to be associative, when
\( C(C(u, v), w) = C(u, C(v, w)) \) 

holds for every \( u, v \) and \( w \) in \([0, 1]\). When (10) holds, since an Archimedean copula is associative, the associative copula in \( \mathcal{C}_K \) coincides with \( C^0 \) above. In order to state our main result, we introduce the following:

**Definition 1.** A Kendall distribution \( K \) is an Archimedean Kendall distribution when it satisfies the condition in (10). We say that \( K \) is a strict Archimedean Kendall distribution when the generators of the associated Archimedean copula, that are then given by (11), are strict.

For an Archimedean Kendall distribution \( K \), we will use the notation

\[
\tau_{t_0, K}(t) = \exp \left\{ \int_{t_0}^{t} \frac{1}{s - K(s)} \, ds \right\},
\]

\( t \) to denote the generator (unique up to a constant) of the unique Archimedean copula in \( \mathcal{C}_K \).

Our result is the following:

**Theorem 2.** Let \( J \) be a bivariate survival function satisfying conditions H1–H3, let \( K \) be a strict Archimedean Kendall distribution. Assume furthermore that \( \tau_{J} \) is differentiable and that also \( \tau_{J} \) is a strict Archimedean Kendall distribution. Set

\[
\phi_{J}(t) := \tau_{t_0, \tau_{J}} \left( \tau_{J}^{-1}(\log t) \right),
\]

\[
\phi_{K}(t) := \tau_{t_0, K}(t)
\]

\[
\mathcal{T}_{\alpha}(x) := \phi_{K}^{-1} \left( \theta \phi_{J}(e^{-x}) \right) = \tau_{t_0, K}^{-1} \left( \theta \tau_{t_0, \tau_{J}} \left( \tau_{J}(x) \right) \right).
\]

Then

(a) \( \mathcal{T}_{\alpha}(x) \) is a survival function for any \( \alpha > 0 \).

(b) Let \( F \) be a bivariate survival function satisfying conditions H1–H3. Then \( F \) solves the compatibility equations

\[
\mathcal{K}_F = K, \quad \mathcal{S}_F = \mathcal{S}_J
\]

if and only if there exists some \( \alpha > 0 \) such that \( F = F_{\alpha} \), where

\[
F_{\alpha}(x, y) = \mathcal{T}_{\alpha} \left( \mathcal{T}_{\alpha}^{-1}(J(x, y)) \right).
\]

(c) \( \mathcal{T}_{\alpha} \) is the marginal survival function of \( F_{\alpha} \).

The proof will be given in Section 5, where, furthermore, we will be in a position to state (a) in an alternative and equivalent form (see (b), see also Eq. (34)). To this end, here we notice only that the statements of the above Theorem remain valid if we multiply \( \phi_{J} \) by a constant, the result of this multiplication being only a different parametrization of the possible solutions.

In order to better focus our developments, a few remarks and comments are now in order.

First we notice that the bivariate function \( \mathcal{T}_{\alpha}(J(x, y)) \) is not necessarily a joint survival function. Some conditions, sufficient to guarantee that \( F_{\alpha}(x, y) \) is a joint survival function, will be discussed in Sections 3 and 5. As a consequence of our analysis we will also single out some conditions under which a pair \((K, \mathcal{S}_J)\) cannot be compatible.

**Remark 3.** The solution of the compatibility problem rather depends on the set \( \mathcal{S}_J \), and not on the specific survival function \( J \). In fact the solution depends on \( \mathcal{T}_{\alpha}(J(x, y)) \) and all the different survival functions \( J \) such that \( \mathcal{S}_J = \mathcal{S}_J \) give rise to the same set of possible solutions (see also the arguments in the next Section). Furthermore, it cannot come as a surprise that a solution \( F \) is of the form \( F(x, y) = \mathcal{T}_{\alpha} \left( \mathcal{T}_{\alpha}^{-1}(J(x, y)) \right) \) (in this respect see, more specifically, Lemma 8 below). The actual problem is rather to understand whether we can establish compatibility for given \((K, \mathcal{S}_J)\) and to identify the possible acceptable marginals \( \mathcal{T}_{\alpha} \).

In order to prove Theorem 2 we need some preliminary notation and results that will be provided in the next two Sections.

### 3. On the semi-copula representation of \( \mathcal{S}_F \)

Let \( F \) be an exchangeable bivariate survival function satisfying conditions H1–H3. To start this Section, we first consider the function \( h : [0, \infty) \times [0, \infty) \rightarrow [0, \infty) \) defined by

\[
h(x, y) = h_F(x, y) = \mathcal{T}_{\alpha}^{-1}(F(x, y)).
\]
In view of our assumptions on \( \bar{F}, h \) is continuous and strictly 1-increasing; furthermore, \( \partial A_v \) is the image of a function \( \gamma_v : [0, 1] \rightarrow [0, \infty) \times [0, \infty) \), connecting the Cartesian axes, i.e. such that \( \gamma_v(0) = (\xi_v, 0) \) and \( \gamma_v(1) = (0, \eta_v) \), for some strictly positive \( \xi_v \) and \( \eta_v \). Concerning \( \xi_v, \eta_v \), and the curve \( \gamma_v \), the following properties can be easily checked:

\[
\xi_v = \eta_v = \theta^{-1}(v), \quad \eta \gamma(x, y) = h(x, y).
\]

Moreover, the curve \( \gamma_v \) can be parameterized as a function \( y(v, x) \) of the variable \( x \), with \( x \in [0, \xi_v] \), such that \( y(v, 0) = \xi_v \), and \( y(v, \xi_v) = 0 \). Setting \( h_v = h(x, \cdot) \), we obtain \( y(v, x) = h_v^{-1}([C_1, 1 - v]) \). We can also write

\[
A_v = \left\{ (x, y) \in \mathbb{R}^2 \mid h(x, y) \geq \theta^{-1}(v) \right\}
\]

and

\[
\partial A_v = \left\{ (x, y) \in \mathbb{R}^2 \mid h(x, y) = \theta^{-1}(v) \right\}.
\]

The function \( h \) then determines \( \mathcal{L}_\nu \) or, equivalently, the set of the level curves \( \partial A_v \) of \( \bar{F} \), while the marginal \( \theta \) determines the parametrization of \( \mathcal{L}_\nu \).

It is also immediate to see that the following Lemma holds.

**Lemma 4.** Let \( F \) and \( J \) be two different survival functions satisfying conditions \( H1-H3 \), then \( \mathcal{L}_\nu = \mathcal{L}_j \) if and only if \( h_\nu = h_j \).

For our purposes we follow however the approach introduced in [2], [4], and replace the function \( h = h_\nu \) by the function \( B = B_\nu : [0, 1] \times [0, 1] \rightarrow [0, 1] \) defined by:

\[
B(u, v) = B_\nu(u, v) = \exp\{-h(-\log u, -\log v)\}, \quad u, v \in [0, 1],
\]

i.e.

\[
B(u, v) = \exp\left\{-\theta^{-1}(\bar{F}(-\log u, -\log v))\right\}.
\]

We can then write

\[
F(x, y) = \theta(-\log B(e^{-x}, e^{-y})).
\]

Notice that Eq. (19) provides a representation of \( F(x, y) \) in terms of the pair \( (B_\nu, \theta) \); this is in a sense analogous to (but different from) the representation (1) in terms of \( (C_\nu, \theta) \).

The function \( B \) turns out to be a convenient tool in the study of certain notions of multivariate aging (see in particular [2], [4], [8]) and has also been termed bivariate aging function.

In order to fix ideas it is useful to consider the functions \( B \), corresponding to three very special cases of interest.

**Example 5 (Perfect dependence).** Let \( P\{X = Y\} = 1 \). Then \( F(x, y) = \theta(x \lor y) \) and, for \( 0 \leq u \leq 1 \),

\[
A_v = \left\{ (x, y) \mid x \lor y \geq \theta^{-1}(v) \right\}.
\]

In this case, \( B(u, v) = u \land v \), i.e. \( B \) is the maximal copula.

**Example 6 ("Schur-constant" \( F \)).** Here we consider the case

\[
F(x, y) = \theta(x + y)
\]

where \( \theta \) is a univariate continuous, convex, strictly positive and strictly decreasing survival function on \([0, +\infty)\). It is immediate to check that \( \theta \) has also the role of univariate marginal and that

\[
A_v = \left\{ (x, y) \mid x + y \geq \theta^{-1}(v) \right\}.
\]

The condition (20) holds if and only if \( B \) is the product copula \( B(u, v) = u \cdot v \).

**Example 7 (i.i.d. random variables).** When \( F(x, y) = \theta(x) \cdot \theta(y) \), we can write

\[
A_v = \left\{ (x, y) \mid A(x) + A(y) \geq -\log v \right\},
\]

by setting \( A(x) := -\log \theta(x) \). In terms of the function \( B \), we can say that the above condition holds if and only if

\[
B(u, v) = Q^{-1}[Q(u) + Q(v)],
\]

where we set

\[
Q(u) := A(-\log u) = -\log \theta(-\log u).
\]

In view of what has been discussed so far, \( B \) can be seen as a tool for representing \( \mathcal{L}_\nu \). Taking into account the definition (18), we can in fact replace Lemma 4 above by the following Lemma, whose proof is immediate (see also [3] and [8]).
Lemma 8. Let $\mathcal{F}$ and $\mathcal{J}$ be two different survival functions satisfying conditions H1–H3. Then the following conditions are equivalent:

(a) $\mathcal{L}_B = \mathcal{L}_C$;
(b) $B = C$;
(c) there exists a continuous, strictly increasing, function $\psi : [0, 1] \to [0, 1]$, such that
\[
\mathcal{F}(x, y) = \psi(\mathcal{J}(x, y));
\]
(d) $\mathcal{F}(x, y) = \mathcal{C}_\mathcal{F}[\mathcal{C}_\mathcal{J}^{-1}(\mathcal{J}(x, y))].
\]

Notice that statement (b) of Theorem 2 provides more detailed information than Eq. (23); the latter in fact has been obtained by imposing the condition (a) (or (b)) above and we do not take into account here possible information about $K_B$ (see also Remark 3 above). However statement (b) of Theorem 2 does not guarantee $\mathcal{F}_G$ to be a joint survival function. In this respect two relevant remarks follow.

Remark 9. For $\mathcal{J}$ a given bivariate survival function satisfying H1–H3, we wonder under which conditions on $\psi$, the function $\psi(\mathcal{J}(x, y))$ is a survival function satisfying H1–H3, as well. The candidate survival copula of $\psi(\mathcal{J}(x, y))$ being the transformation $\psi(\mathcal{C}_\mathcal{J}^{-1}(u), \psi^{-1}(v))$ of $\mathcal{C}_\mathcal{J}$, our problem is strictly related to the question whether the previous transformation of $\mathcal{C}_\mathcal{J}$ is still a copula (the latter problem is studied in the recent paper [5], see also [8]). Obviously, $\psi$ strictly increasing is a necessary condition, but it turns out that it is not a sufficient condition. On the other hand, as can be easily seen, $\psi$ increasing and convex is a sufficient condition (see also [5], where the copula-transformation problem is studied by using Proposition 4.B.2 in [15]), but it is not a necessary condition. Indeed, simple counterexamples may be constructed by taking $\psi(t) = K_B(t)$, and $\mathcal{J}$ with sufficiently regular marginals and with Archimedean survival copula, i.e. $\mathcal{C}_\mathcal{J} = C^\beta$, for a strict (convex) generator $\phi$. In such a case the explicit expression $\psi(t) = t - \phi(t)/\phi'(t^+)$ is obtained by Eq. (9). The choice
\[
\hat{\phi}(t) = |\log t| + \alpha(1 - t) + (1 - t)^2, \quad \beta > 3,
\]
yields the desired counterexamples for $\alpha \in (0, 1)$. As a matter of fact, for all the considered values of $\alpha$ and $\beta$, the above function $\psi$ is strictly increasing but not convex. Furthermore, independently of the marginal $\mathcal{C}_\mathcal{J}$, the function $\psi(\mathcal{J})$ is a survival function for $\alpha = 0$ and not for $\alpha = 1$.

Remark 10. For $\mathcal{J}$ a given bivariate survival function satisfying H1–H3, we wonder now under which conditions on the univariate survival function $\mathcal{M}$, the function $\mathcal{M}(\mathcal{C}_\mathcal{J}^{-1}(\mathcal{J}(x, y)))$ is a survival function satisfying H1–H3, as well. In view of the above Remark 9 a sufficient, but not necessary, condition is $\mathcal{M}(\mathcal{C}_\mathcal{J}^{-1})$ being convex. Such a sufficient condition may be stated as $\mathcal{M}\leq \mathcal{C}_G$, i.e. $\mathcal{M}$ is dominated by $\mathcal{G}$ in the likelihood order (see [20]).

Let us now look at analytical properties of the function $B = B_\mathcal{F}$, as defined by (18). Since $X$ and $Y$ are exchangeable one has $B(u, v) = B(v, u)$.

It is immediate to check that $B : [0, 1] \times [0, 1] \to [0, 1]$ is strictly increasing in each variable and also shares with a copula the property to be grounded and with uniform marginals, i.e.
\[
\forall 0 \leq u \leq 1, \quad B(u, 0) = 0, \quad B(u, 1) = u.
\]

However it is not necessarily a copula: in some cases it can happen that is not two-increasing, as for example in (21) whenever $\mathcal{Q}$ is not convex. Actually $\mathcal{J}$ is a semi-copula. The term semi-copula has been introduced (see [4], [7]) to describe functions $\mathcal{S} : [0, 1] \times [0, 1] \to [0, 1]$ that are grounded, with uniform marginals and strictly increasing in each variable; see in particular [7] for the analysis of some technical aspects of the notion of semi-copula and also [8] for the relations with the concept of capacity. This notion can have however other interesting applications in problems of rather different type (see e.g. [6], [13]).

In order to describe semi-copulas of the form (21), the term Archimedean semi-copulas had been used in some of the cited past papers. However, in such a case, $\mathcal{S}$ is an Archimedean $t$-norm (see [14]), and this term will be currently used in the sequel. Generally, for an Archimedean $t$-norm
\[
\mathcal{S}^\varphi (u, v) = \varphi^{-1}[\varphi(u) + \varphi(v)],
\]
the generator $\varphi$ is a decreasing function, with $\varphi(1) = 0$. When $\varphi(0^+) = \infty$, then we say that $\varphi$ is a strict generator. It is clear that if $\varphi$ is a generator, then $\theta \varphi$ is also a generator, for any $\theta > 0$. As mentioned just above, when the generator is also a convex function $\phi$, then $\mathcal{S}^\phi$ is a copula and it coincides with the Archimedean copula $C^\phi$ (here we are using the notation (8)).

For our purposes it is necessary to analyze the relations existing between $B = B_\mathcal{F}$ and $\mathcal{C} = \mathcal{C}_\mathcal{F}$. The latter relations involve the marginal survival function $\mathcal{C} = \mathcal{C}_\mathcal{F}$; in fact we can write, from (19) and (1),
\[
\begin{align*}
B(u, v) &= \exp \left\{ -G^{-1}\left( \hat{C} \left( \log C(u), \log C(v) \right) \right) \right\}, \\
\hat{C}(u, v) &= G\left( -\log B\left( e^{-G^{-1}(u)}, e^{-G^{-1}(v)} \right) \right),
\end{align*}
\]

or

\[
B(u, v) = \gamma\left( \hat{C}(\gamma^{-1}(u), \gamma^{-1}(v)) \right), \quad \hat{C}(u, v) = \gamma^{-1}(B(\gamma(u), \gamma(v)))
\]  

(24)

where \( \gamma, \gamma^{-1}: [0, 1] \rightarrow [0, 1] \) are the increasing functions defined by

\[
\gamma(w) = \exp\{-G^{-1}(w)\}, \quad \gamma^{-1}(u) = G(-\log u).
\]  

(25)

**Example 11.** In the perfect dependence case \( \hat{C}(u, v) = B(u, v) = u \land v \), as can be trivially seen. Notice that the maximal copula \( u \land v \) is a fixed point of both the transformations in (24), regardless of the marginal \( C \).

As to the Schur-constant case, with given convex marginal \( C \) (as in Example 6), we observe that \( B(u, v) = u \cdot v \) holds if and only if \( \hat{C} = C^{-1} \) (see [2], [17]), i.e.

\[
\hat{C}(u, v) = G^{-1}(G(u) + G(v)),
\]

Finally, we consider the case of independence \( \hat{C}(u, v) = u \cdot v \), with given marginal \( C \). By (24) we reobtain condition (21) for \( B \) (the latter condition can be rewritten as \( B = S^d \), with \( Q \) given by (22)).

**Example 12 (Exponential marginals).** When \( \tau(x) = e^{-x} \), with \( \beta > 0 \), then \( \gamma \) and \( \gamma^{-1} \) in (25) are given respectively by \( \gamma(w) = w^\beta \) and \( \gamma^{-1}(u) = u^\beta \), and (24) becomes

\[
B(u, v) = \left( \hat{C}(u^\beta, v^\beta) \right)^{1/\beta}, \quad \text{and} \quad \hat{C}(u, v) = \left( B(u^\beta, v^\beta) \right)^{1/\beta}.
\]

Note that if we take the copula \( \hat{C}(u, v) = (u + v - 1)^+ \), i.e. the lower Frechét–Hoeffding bound, then \( B \) coincides with the Clayton semi-copula, given by

\[
B(u, v) = \left( (u^\beta + v^\beta - 1)^+ \right)^{1/\beta}.
\]

The condition \( \tau(x) = e^{-x} \) implies \( \gamma(w) = w \), and therefore

\[
B(u, v) = \hat{C}(u, v).
\]

Thus we see that if an aging function \( B \) is compatible with the standard exponential marginal distribution, i.e. if \( F \) has a standard exponential marginal then necessarily \( B \) is a copula.

Notice that the condition \( \tau(x) = e^{-x} \) is not necessary for \( B = \hat{C} \) as we see, for instance, by considering the perfect dependence case. Another instance will be met in Section 5, where we will consider the case of Marshall–Olkin models, whose marginal distributions are exponential, but non-necessarily standard exponential.

We conclude this Section with the following remarks and preliminary results, the role of which will emerge in the rest of the paper.

**Remark 13.** The property of associativity (12) can immediately be extended to semi-copulas. It can be derived from (24) that \( B \) is associative if and only if such is \( \hat{C} \). In the latter case the aging function is then a t-norm (see [14]).

**Remark 14.** In view of (24), we see that \( B \) is Archimedean, with an invertible generator, if and only if such is \( \hat{C} \); if \( \hat{C} \) is a convex generator of \( \hat{C} \) then

\[
\hat{\phi}(\gamma^{-1}(u)) = \hat{\phi}(\hat{C}(-\log u))
\]

is a generator of \( B \). Furthermore, since generators of Archimedean t-norm are determined up to a constant, we can conclude as follows: if \( \hat{\phi} \) is an invertible, convex, generator of \( \hat{C} \) and \( \phi \) is a generator of \( B \), then there exists a constant \( \theta > 0 \) such that

\[
\theta \varphi(u) = \hat{\phi}(\gamma^{-1}(u)) = \hat{\phi}(\hat{C}(-\log u)).
\]  

(26)

From the above relation we also see that, in our framework, the generators of \( B \) are continuous.

We now consider a univariate survival function and two generators, not necessarily deriving from a given bivariate survival function. Eq. (26) suggests the following useful result.
Lemma 15

(a) Assume that $0 > \theta > 0$, $\phi$ and $\mathcal{T}$ are given, with $\phi$ a strict and convex generator, and $\mathcal{T}$ a strictly positive survival function with $\mathcal{T}(0) = 1$. Let $\varphi$ be defined by $\varphi(u) := \hat{\phi}(\mathcal{T}(\log u))/\theta$, then $\varphi$ is a strict generator of a $t$-norm. Furthermore if $\hat{\phi}$ and $\mathcal{T}$ are strictly decreasing, then the same holds for $\varphi$.

(b) Assume that $0 > \theta > 0$, $\phi$, and $\varphi$ are given, with $\phi$ a strict and convex generator and $\varphi$ a strict generator. Assume furthermore that $\phi$ and $\varphi$ are strictly decreasing, and that $\mathcal{T}_0$ is defined by means of (26), i.e.

$$\mathcal{T}_0(x) := \hat{\phi}^{-1}(\theta \varphi(e^{-x})).$$

(27)

Then $\mathcal{T}_0$ is a strictly decreasing survival function, with $\mathcal{T}_0(0) = 1$.

(c) Assume that $\phi$ and $\varphi$ satisfy the same properties of item (b), except the strictness property, i.e. $\hat{\phi}(0) = \hat{\phi}(0^+)$ and $\varphi(0) = \varphi(0^+)$ are finite. Then (27) defines a survival function $\mathcal{T}_0$ if and only if $0 > \theta > 0$, where $0 := \hat{\phi}(0)/\varphi(0)$. Furthermore $\mathcal{T}_0$ is strictly decreasing on $[0, \infty)$ if and only if $\theta = 0$.

Proof. The proof of part (a) is a simple verification: since $0 > \theta > 0$, the functions $\hat{\phi}$, $\mathcal{T}$ and $-\log$ are (strictly) decreasing, then $\varphi$ is (strictly) decreasing. Since $\lim_{u \to 0^-} - \log(u) = +\infty$, $\lim_{u \to \infty} \mathcal{T}(x) = 0$ and $\lim_{u \to 0^+} \hat{\phi}(u) = +\infty$, then $\lim_{u \to 0^+} \varphi(u) = +\infty$.

Finally we see that $\varphi(1) = 0$, since $-\log(1) = 0$, $\mathcal{T}(0) = 1$, and $\hat{\phi}(1) = 0$.

The proof of part (b) is similar: since $\hat{\phi}$, $\varphi$ and $e^{-x}$ are strictly decreasing, then $\hat{\phi}^{-1}$ and therefore $\mathcal{T}_0$ are also strictly decreasing. Since $\varphi(1) = 0$, $\hat{\phi}^{-1}(0) = 1$, then $\mathcal{T}_0(0) = \hat{\phi}^{-1}(\theta \varphi(1)) = 1$. Finally, since $\hat{\phi}$ and $\varphi$ are strict generators, then

$$\lim_{x \to \infty} \mathcal{T}_0(x) = \lim_{y \to 0^+} \hat{\phi}^{-1}(\theta \varphi(y)) = 0.$$

The proof of part (c) is readily obtained by observing that $\hat{\phi}^{-1}(\theta x) = 0$ if and only if $\theta x \geq \hat{\phi}(0)$.

Note that in the proof of Lemma 15 we have not actually used the fact that $\hat{\phi}$ is convex, while the convexity property has a key role in the following Corollary; the latter is a simple consequence of the previous Lemma 15, and its proof is left to the reader.

Corollary 16. Assume the same hypotheses of Lemma 15 (part (b)). Assume furthermore that $\varphi$ is continuous and that $\mathcal{T}_0$ is defined by (27). Then, for every $\theta > 0$, the bivariate function

$$\hat{\phi}^{-1}(\theta (\varphi(e^{-x}) + \varphi(e^{-y})))$$

coincides with the bivariate survival function defined by

$$F_\theta(x,y) := \hat{\phi}^{-1}\left(\hat{\phi}\left(\mathcal{T}_0(x)\right) + \hat{\phi}\left(\mathcal{T}_0(y)\right)\right).$$

Furthermore $F_\theta(x,y)$ satisfies conditions H1–H3, its marginal survival function is $\mathcal{T}_0$ and has Archimedean survival function and aging function with generators $\hat{\phi}$ and $\varphi$, respectively, i.e.

$$\tilde{C}_\mathcal{T} = C^\theta, \quad B_\mathcal{T} = S^\theta.$$

Vice versa, if $\tilde{F}$ is a bivariate survival function with $\tilde{C}_\mathcal{T} = C^\theta$ and $B_\mathcal{T} = S^\theta$ then $\tilde{F} = F_\theta$ and $\tilde{C}_\mathcal{T} = \mathcal{T}_0$, for some $\theta > 0$.

This result is the starting point in the solution of the compatibility problem, at least in the cases when given $\tilde{K}$ and $B$ are both Archimedean. In the next Section we shall analyze this point more in detail and in a larger generality.

4. Survival copulas and upper-orthant Kendall distributions

In this Section we concentrate our attention on the upper-orthant Kendall distribution $\tilde{K}_\mathcal{T}$ defined in (3). The following Lemma points out some properties of $\tilde{K}_\mathcal{T}$, that are important in our frame.

Lemma 17

$$\tilde{K}_\mathcal{T}(t^+) = \tilde{K}_\mathcal{T}(t) \quad \text{if and only if} \quad P(\tilde{F}(X,Y) = t) = 0,$$

(28)

i.e. if and only if $P((X,Y) \in \partial A_\mathcal{T}) = 0$.

Under the assumptions H1–H3 on $\tilde{F}$, the inequality in (4) is strict for all $u \in (0,1)$, namely

$$\forall u \in (0,1), \quad \tilde{K}_\mathcal{T}(u) > u.$$

(29)

Proof. The proof of (28) is readily obtained by observing that $\tilde{K}_\mathcal{T}(t^+) = P(\tilde{F}(X,Y) < t)$. We now prove that the strict inequality (29) holds; actually we want to show that the condition $\tilde{K}_\mathcal{T}(t) = t$ for some $t \in (0,1)$ is incompatible with our assumptions. To this purpose we notice that, for all $u \in (0,1)$,
Let us maintain the assumption of the Proposition above and assume furthermore that \( \bar{K}_F = \mathcal{K} \bar{C}_F \) is Archimedean, then \( \mathcal{T}_{t_0, \mathcal{B}_F} \) is well defined, with

\[
\mathcal{T}_{t_0, \mathcal{B}_F}(t) = \exp \left\{ \int_{t_0}^t \frac{1}{s - \mathcal{B}_F(s)} \, ds \right\} = \exp \left\{ \int_{\gamma^{-1}(t_0)}^{\gamma^{-1}(t)} \frac{1}{s - \mathcal{B}_F(s)} \, ds \right\},
\]  

(31)
and there exists a positive constant $\theta$ such that the following relation holds
\[
\theta \int_{t_0}^{x} y \rho_y(t) = \int_{t_0}^{x} \left( \frac{\partial}{\partial t} P_{t_0}(t) \right) dt.
\]  
(32)

**Proof.** The relation (31) is a direct consequence of Eq. (30), while the relation (32) follows by observing that
\[
\int_{t_0}^{x} y \rho_y(t) = \int_{t_0}^{x} \left( \frac{\partial}{\partial t} P_{t_0}(t) \right) dt.
\]
where $\theta = \exp \left\{ \int_{t_0}^{x} \left( \frac{1}{s - \lambda_{t_0}(s)} \right) ds \right\}$. □

In the next Example 22, setting for simplicity’s sake $(B, \hat{C}, \hat{\omega}) = (B_t, \hat{C}_t, \hat{\omega}_t)$, we compute the upper-orthant Kendall distribution functions $\mathcal{K}$ and the functions $\hat{\lambda}$, together with $\mathcal{K}^*$ and $\hat{\lambda}_{t, x}$, for the basic cases considered in Examples 5–7, replacing the Schur–constant case with the more general Archimedean case.

**Example 22.** In the perfect dependence case, in agreement with Remark 18, it holds $\hat{K}_t(x, y) = \mathcal{K}(x \wedge y)$, and $\hat{U} = \mathcal{U}(x)$ is a uniform random variable on $(0, 1)$. Furthermore, $\mathcal{K}^* B(t) = t$ and $\hat{\lambda}_x(t) = 0$, since $B = \hat{C}$.

In the case of an Archimedean survival copula with convex generator $\hat{\phi}$, i.e. (using the notation (8))
\[
\hat{C}(u, v) = \hat{C}^*(u, v) = \hat{\phi}^\prime \left( \frac{\hat{\phi}(u) + \hat{\phi}(v)}{2} \right),
\]
we obtain $\hat{\phi}(t) = \hat{\phi}^\prime(t)$, which is Archimedean with generator $-\log t$, it holds $\mathcal{K}(t) = t - \log t$ (see e.g. [11, p. 319]) and $\mathcal{K}^* B(t) = t - \frac{\hat{\phi}^\prime(t)}{\hat{\phi}(t)}$, with $Q$ defined as in (22).

5. **Conditions for compatibility between Kendall distributions and level sets**

Basic aspects of the arguments developed in the Sections 3 and 4 will be collected and used in this Section to prove the proof of Theorem 2 and to present examples and related comments.

Before proceeding with the announced proof, we focus the reader’s attention on some remarks concerning the role of the aging function. In view of the equivalence between (a) and (b) in Lemma 8, we can in fact restate the compatibility problem as follows: the pair $(K, \mathcal{K})$ is compatible if and only if we can find some solution $\mathcal{F}$ to the system of equations
\[
\mathcal{F}(x, y) = \mathcal{G}(x \wedge y), \quad \mathcal{K}(t) = t - \log t.$
\]  
(33)

Taking into account the first condition in (33) and (19), any solution $\mathcal{F}$ must satisfy $\mathcal{F}(x, y) = \mathcal{G}(x \wedge y, e^{-x}, e^{-y})$. Then, in order to identify a solution, if it exists, we only need to determine the marginal survival function $\mathcal{G}_x$, and the proof of Theorem 2 basically reduces to checking that the possible solutions are of the form
\[
\mathcal{F}(x, y) = \mathcal{G}_x \left( -\log B_t(x^{-}, e^{-y}) \right) = \mathcal{G}_x \left( \theta \mathcal{G}_{\hat{F}} \left( B_t(x^{-}, e^{-y}) \right) \right),
\]  
(34)

where we have taken into account the definition (15) of $\mathcal{G}_x$. To this end we use the following argument. Substituting (33) into Eq. (32) of Corollary 21, we obtain, for some $\theta > 0$
\[
\theta \bar{\mathcal{F}}_{t_0, x, \mathcal{F}}(t) = \bar{\mathcal{F}}_{t_0, \mathcal{F}} \bar{\mathcal{G}}_{\hat{\varphi}} \left( -\log(t) \right),
\]
for any solution $\mathcal{F}$ of the compatibility problem. On the other hand Eq. (32) applied to $\hat{J}$ shows that, for a suitable $\theta > 0$,
\[
\theta \bar{\mathcal{F}}_{t_0, x, \mathcal{F}}(t) = \bar{\mathcal{F}}_{t_0, \mathcal{F}} \left( -\log(t) \right),
\]
where the right hand side coincides with the definition (13) of $\hat{\varphi}_{\mathcal{F}}$. The previous relations, together with the definition (14) of $\hat{\varphi}_{\mathcal{F}}$, show that the possible candidates for $\mathcal{F}$ are necessarily of the form (15), and therefore the possible solutions $\mathcal{F}$ are necessarily of the form (34).
Finally we note that, in definition (13), the dependence of \( \phi_J \) on \( J \) is expressed in terms of \( (G_0, K_J) \). We point out that actually \( \phi_J \) is proportional to \( T_{o,x,R} \), as we have seen above. This observation allows us to replace the definition (13) with \( \phi_J(t) := T_{o,x,R} \) in (34). As a consequence, part (b) of Theorem 2 can be replaced by the following equivalent (up to a re-parametrization) statement. (b') Let \( F \) be a bivariate survival function satisfying conditions H1–H3. Then \( F \) solves the compatibility equations (16) (or equivalently (33)) if and only if there exists some \( \theta > 0 \) such that \( F \) has the form

\[
\hat{\phi}_K^{-1}\left( \theta T_{o,x,R} \left( B_J(e^{-x}, e^{-y}) \right) \right).
\]

The latter expression shows explicitly the fact that the possible solutions depends on \( K \) only through the convex generator \( \hat{\phi}_K \), and on \( J \) only through the aging function \( B_J \).

We now proceed more formally with the proof of our result.

**Proof (of Theorem 2)**

(a) First of all note that, since \( \hat{K}_J \) is a strict Archimedean Kendall distribution, the function \( T_{o,x,R} \) is a strict convex generator. It also follows from the assumptions on \( J \) that \( G_0 \) is strictly positive and strictly decreasing. Then \( \phi_J \) is a strict generator and (a) follows by Lemma 15 (part (b)).

(b) In view of (a), \( G_0(x) \) is a bona-fide one-dimensional survival function for any \( \theta > 0 \). In the above discussion we have already proven that the possible solutions are of the form (34), which can be also rewritten as

\[
F_{u,v}(x,y) = G_0 \left( - \log B_J(e^{-x}, e^{-y}) \right) = \hat{\phi}_K^{-1} \left( \theta \partial \partial T_{o,x,R} \left( B_J(e^{-x}, e^{-y}) \right) \right).
\]

The first equality in (34) is immediately seen to be equivalent to (17), as shown by (19), applied to \( J \). Finally Eq. (36) also shows that, up to a re-parametrization, the solutions are of the form (35).

(c) The result is obvious since

\[
F_{u}(x,0) = G_0 \left( \hat{G}_J^{-1}(u,0) \right) = G_0 \left( \hat{G}_J^{-1}(G_0(x)) \right) = G_0(x) \quad \square
\]

Theorem 2 provides conditions on the form of \( F \) that are necessary for \( F \) to be a solution of the compatibility problem, but it does not ensure that the bivariate functions \( F_{u} \) are actual survival functions, for a given choice of \( K \) and \( B_J \).

The following two remarks, that point out some further aspects about the aging function, can be of help in the discussion about sufficient conditions that guarantee that the functions \( F_{u} \) are survival functions.

**Remark 23.** Let \( (G_0, B) = (G_0, B_J) \) for a given joint survival function \( F \). Assume that \( F \) has a jointly continuous and strictly positive density \( f \), so that \( G = G_f \) has a strictly positive, differentiable, density \( g = g_f \) and \( B = B_f \) has continuous second-order derivatives. Then in view of Eq. (19), for all \( u, v \in (0,1) \),

\[
1 + \frac{g'(x)}{g(x)} \left|_{x = - \log B(u,v)} \right. \leq \frac{B(u,v)^{\phi''_{B}(u,v)} \frac{\partial \partial B(u,v)}{\partial \partial B(u,v)}}{B(u,v)^{\phi''_{B}(u,v)} \frac{\partial \partial B(u,v)}{\partial \partial B(u,v)}}.
\]

Note that when \( G_0(x) = e^{-x} \), then the compatibility condition (37) may be satisfied if and only if \( \phi''_{B}(u,v) > 0 \), i.e. if and only if \( B \) is a copula. In fact (see Example 12) we know that \( B \) has to be a copula, in order to be compatible with a standard exponential marginal. Observe also that when \( G_0(x) = e^{-\alpha x} \) the l.h.s. of (37) coincides with \( 1 - \alpha \). Again, when \( \theta < 1, \) \( B \) has to be a copula in order to be compatible with \( G \), while this is not the case when \( \theta > 1 \). Finally observe that if \( B \) is a copula, then clearly it is compatible with exponential \( G \) for all \( \theta \), while if \( B \) is not a copula, then it is compatible with \( G \) if and only if \( \theta > 1 - \inf_{u,v \geq (0,1) \times (0,1)} \frac{B(u,v)^{\phi''_{B}(u,v)} \frac{\partial \partial B(u,v)}{\partial \partial B(u,v)}}{B(u,v)^{\phi''_{B}(u,v)} \frac{\partial \partial B(u,v)}{\partial \partial B(u,v)}} \).

**Remark 24.** Let \( M \) be an arbitrary continuous, strictly decreasing and strictly positive survival function over \( [0, +\infty) \), and \( B_J \) be the aging function of a bivariate survival function \( J \). The bivariate function defined by

\[
\hat{M} = \left( - \log B_J(e^{-x}, e^{-y}) \right)
\]

is a bivariate survival function (with aging function \( B_J \) and marginal survival function \( M \)) provided appropriate compatibility conditions hold. For instance, if we assume that \( M \) has a strictly positive density \( m \in C^\infty \) and that \( J \) admits a jointly continuous and strictly positive density (and therefore \( B_J \in C^\infty \)), then, by Remark 23, the necessary and sufficient compatibility conditions are given by (37) with \( g \) and \( B \) replaced by \( m \) and \( B_J \). As a consequence, in this case, a sufficient condition is given by \( M \leq B_J \), i.e.

\[
\frac{m'(x)}{m(x)} \leq \frac{g'_J(x)}{g_J(x)} \iff \frac{d}{dx} \log m(x) \leq \frac{d}{dx} \log g_J(x).
\]

Indeed the latter condition implies that

\[
1 + \frac{m'(x)}{m(x)} \bigg|_{x = - \log B_J(u,v)} \leq 1 + \frac{g'_J(x)}{g_J(x)} \bigg|_{x = - \log B_J(u,v)} \iff \frac{B_J(u,v)^{\phi''_{B_J}(u,v)} \frac{\partial \partial B_J(u,v)}{\partial \partial B_J(u,v)}}{B_J(u,v)^{\phi''_{B_J}(u,v)} \frac{\partial \partial B_J(u,v)}{\partial \partial B_J(u,v)}} \leq \frac{B_J(u,v)^{\phi''_{B_J}(u,v)} \frac{\partial \partial B_J(u,v)}{\partial \partial B_J(u,v)}}{B_J(u,v)^{\phi''_{B_J}(u,v)} \frac{\partial \partial B_J(u,v)}{\partial \partial B_J(u,v)}}.
\]
For instance, this happens when \( g_j(x) = k \exp\{-A_j(x)\} \), with \( A_j \) positive and increasing, and \( m(x) := k_0 \exp\{-\theta A_j(x)\} \), with \( \theta \geq 1 \).

Coming back to the setting of Theorem 2, when \( \tilde{J} \) admits continuous second-order derivatives and under suitable regularity assumptions on \( K \), we may assume that \( g_j = -C_j \) is a \( C^1 \) probability density, and then we may use Remark 24 to find conditions for \( F _0 \), given by (34), to be a survival function: (i) a necessary and sufficient condition is (37), with \( g \) replaced by \( g_j \) and \( B \) replaced by \( B_j \) (ii) a sufficient condition is given by \( C_0 \leq C_2 \). The latter sufficient condition was already obtained in Remark 10.

In a number of cases it can be checked directly that the bivariate functions \( F _0 \) in (34) (see also (35)) are bivariate survival functions and therefore they are solutions of our compatibility problem. A relevant special case of this kind arises when, besides the assumptions of Theorem 2, we add the condition that \( B \) is Archimedean. This case can be analyzed as application of Corollary 16, however, we prefer to present the following example to summarize some details about the Archimedean cases and to briefly discuss a few related aspects.

**Example 25.** Consider the special case of exchangeable survival models \( H(x,y) \) characterized by the condition that the survival copula \( C_{\tilde{H}} \) is Archimedean with a (convex) invertible generator \( \phi \), i.e.

\[
H(x,y) = \phi^{-1}\left(\phi(C_{\tilde{H}}(x)) + \phi(C_{\tilde{H}}(y))\right).
\]

In this case \( B_{\tilde{H}} \) is Archimedean, with a continuous and invertible generator \( \varphi \), as well, that is \( B_{\tilde{H}} = S^{\varphi} \), with

\[
\varphi(u) = \phi\left(C_{\tilde{H}}(-u)\right).
\]

as can be easily checked by (18) (or by (24) and (25)). Thus, in terms of \( \hat{\phi} \) and \( \varphi \), it must be

\[
H(x,y) = \phi^{-1}(\varphi(e^{-x}) + \varphi(e^{-y})).
\]

Vice versa, for any choice of the invertible generators \( \hat{\phi} \) and \( \varphi \), with \( \hat{\phi} \) convex, the latter formula defines a true two-dimensional survival function. The latter satisfies also conditions (H2 and H3), when the generators are strict.

Now, back to the compatibility problem, let \( K \) and \( J \) be given as in Theorem 2, with the further hypothesis that the survival copula of \( J \) is Archimedean, having an invertible generator (w.l.o.g. we can take it to be equal to \( Y - \lambda \)). From above we see that the compatibility problem has a solution \( F \) which is determined by the pair \( \hat{\phi}_K \) and \( \varphi_J \), where \( \varphi_J \) defined as in (38), is the generator of \( B_J \) (and then of \( B_P \)), i.e. \( B_J = S^{\varphi_J} \).

Of course, for \( \theta > 0, \theta \varphi_J \) is also a generator of \( B_J = B_P \) and, correspondingly, any bivariate survival function having the form

\[
\hat{\phi}_K^{-1}\left(\theta(\varphi_J(e^{-x}) + \varphi_J(e^{-y}))\right) = \hat{\phi}_K^{-1}\left(\theta \varphi_J S^{\varphi_J}(e^{-x}, e^{-y})\right),
\]

is still a solution of the same compatibility problem.

Note that the generator \( \varphi_J \) defined in Example 25 coincides exactly with the generator defined in (13) of Theorem 2. Actually this circumstance is a key point for the following remark.

**Remark 26 (Role of the Archimedean case).** The form (34) of the possible solutions to the compatibility problem for the general case can be seen as the natural generalization of Eq. (39) for the Archimedean case. However Eq. (39) can be derived directly, as shown in the Example 25; the derivation of Eq. (34) on the contrary stands on the appropriate extension to semi-copulas (see Proposition 20) of the result by Rivest and Genest (here recalled as Theorem 19).

Furthermore we notice that, for the general case, the possible marginals \( C_{\tilde{H}}(x) \) coincide with the marginals for the Archimedean model (39). These considerations also explain the role of the hypothesis that \( K \) and \( K_J \) are strict Archimedean Kendall distributions. When the condition of strictness is dropped, then there may be some values of the parameter \( \theta \) such that \( C_{\tilde{H}} \) is not a survival function, and therefore \( F _0 \) cannot be a solution of the compatibility problem: by (c) of Lemma 15, when \( K \) and \( K_J \) are both not strict then, \( C_{\tilde{H}} \) is a survival function only for \( \theta \) sufficiently large. If \( K \) is strict and \( K_J \) is not, then \( C_{\tilde{H}} \) is not a survival function for any value of \( \theta \); indeed in such a case \( \hat{\phi}_K \) is a strict (convex) generator and \( \varphi_J \) is a non-strict generator, so that \( \lim_{x \to -\infty} C_{\tilde{H}}(x) = \hat{\phi}_K^{-1}(\theta \varphi_J(0)) \neq 0 \).

The condition that the aging function is Archimedean with strict generator \( \varphi \) characterizes the bivariate models such that the set of the level curves is the same as in the case of stochastic independence; in fact (see also [2] and (21) in Example 7) a pair of independent, identically distributed, non-negative variables with marginal survival function \( e^{-\varphi(e^{-x})} \) admits \( S^\varphi \) as its aging function. We can then conclude by saying that any strict Archimedean Kendall distribution is compatible with the set of the level curves associated to any independent model, provided that the marginal of the latter is strictly positive, strictly decreasing, and continuous all over \([0, +\infty)\).

Theorem 2 can also be used in cases where \( B_J \) is not Archimedean, nor admits second derivatives. The following example is related to the celebrated *Marshall-Olkin models*, where the corresponding \( F \) is not absolutely continuous, as it also happens in the basic perfect dependence case.
Example 27 (Marshall–Olkin models and Cuadras-Augé copulas). Consider independent, exponentially distributed, non-negative variables $A, B, \tau$, where $A, B \sim \text{Exp}(\lambda)$, $\tau \sim \text{Exp}(\mu)$, and set $X = \min(A, \tau), Y = \min(B, \tau)$. The joint survival function of $X, Y$ is the exchangeable Marshall–Olkin model

$$
\Phi(x, y) = \exp \{ -\lambda(x + y) - \mu(x \vee y) \}.
$$

In this case one has $\mathcal{C}_\Phi(x) = e^{-(\lambda + \mu)x}$, $\mathcal{C}_\Phi^{-1}(u) = -\frac{\log u}{x + y}$. By (1), and then by Example 12, with $\beta = \lambda + \mu$, we find for $(C, B) = (C, B_2)$

$$
\hat{C}(u, v) = B(u, v) = (uv)^{1-x},
$$

with $\alpha = \frac{1}{x} < 1$, i.e. $\hat{C} = B$ is a Cuadras-Augé copula. Notice that this is not an Archimedean copula, since it is not even associative (note that $\hat{C}(\hat{C}(u, v), v) \neq \hat{C}(u, \hat{C}(v, v))$, for $v^{1-x} < u < v$. As was shown in [12] and can be easily computed, $K_C(t) = \hat{C}(t) = \hat{C}(B(t) = t - \tilde{\rho} t \log t$, with $\tilde{\rho} = \frac{1}{x}$. Let us now consider the compatibility problem with $J$ such that $B_2$ is the Cuadras-Augé copula (41), and $K(t) = \mathcal{K}_B(t) = t - \tilde{\rho} t \log t$. Notice that $K$ is a strict Archimedean Kendall distribution, in fact

$$
\phi_K(t) = \mathcal{K}_B(t) = \exp \left\{ \int_{u=0}^{t} \frac{1}{1 - \log u} d \log u \right\} = \text{const} |t|^{1/\tilde{\rho}}.
$$

and $\phi_J$ is proportional to $\phi_K$. Therefore the compatible marginal survival functions $\mathcal{C}_\alpha$, given by (15), have to be exponential.

These are exactly the marginal survival functions of the above exchangeable Marshall–Olkin models. Summarizing, we obtained a characterization of the exchangeable Marshall–Olkin models. Indeed if $B_2(u, v) = (uv)^{1-x}$, i.e. the Cuadras-Augé copula, and $K(t) = \mathcal{K}_{B_2}(t) = t - \tilde{\rho} t \log t$, then $K$ and $B_2$ are compatible for any $\alpha \in (0, 1)$, and the possible solutions provided by Theorem 2 are the models in (40), with a different parametrization.

We end this discussion by mentioning an open problem that, in our setting, arises as a natural one. As we have seen, our

compatibility problem can be formulated as a compatibility problem between a Kendall distribution $K$ and a semi-copula $B_2$ (which is the aging function of a given survival function $J$). One can also consider the compatibility between $K$ and a semi-copula $B$, without assuming that $B$ is an aging function (i.e. that $B$ can be obtained by applying a continuous transformation as in (24) on a bivariate copula $C$). The latter problem naturally leads us to the following one: to characterize the semi-copulas $B$ that are aging functions (some results in this direction have been given in [1]). For the Archimedean case, Corollary 16 gives at least a sufficient condition: any Archimedean semi-copula with continuous, strict, and strictly decreasing generator $\phi$ is an aging function. We expect that our arguments can be used for further analysis in the above characterization problem.

Acknowledgements

We like to thank the anonymous referees for their numerous comments and suggestions that helped us to revise and improve the paper.

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