Filtering of a reflected Brownian motion with respect to its local time *

Giovanna Nappo † Barbara Torti ‡

Abstract

We consider a filtering problem when the state process is a reflected Brownian motion X_t and the observation process is its local time Λ_s, for $s \leq t$. For this model we derive an approximation scheme based on a suitable interpolation of the observation process Λ_t. The convergence of the approximating filter to the original one combined with an explicit construction of the approximating filter allows us to derive the explicit form of the original filter. The last result can be obtained also by means of the Azéma martingale.

MSC: 60J55; 60J65; 60F99; 60G35; 93E11; 60G55

Keywords: Brownian motion; Local time; Nonlinear filtering; Approximation; Counting processes; Reflection principle; Skorohod problem

1 Introduction

Let (X_t, Y_t) be a stochastic system. We assume that the *state process* X_t of the system cannot be directly observed, while the other component Y_t is completely observable and is referred to as the *observation process*. The aim of stochastic filtering is the computation of the conditional law of the state process at time t, given the observation process up to time t. Equivalently the aim is the computation of

$$E\big[g(X_t)/\mathcal{F}_t^Y\big],$$

for all functions g belonging to a determining class, i.e. the best estimate of $g(X_t)$ given the $\sigma-$algebra of the observations up to time t, $\mathcal{F}_t^Y = \sigma\{Y_s,\, s \leq t\}$.

The classical situation arises when the observation process Y_t is a noisy function of the state X_t, and when the noise can be represented via martingales. The typical example is

$$Y_t = \int_0^t h(X_s)ds + B_t,$$

with B_t a Brownian motion, and is a generalization of the Kalman filter. In this case the problem can be solved either by the “innovation” method or by the “reference probability” method to nonlinear filtering (see, for example, the classical books by Elliott [6] and by Kallianpur [12]). These methods are based on martingale representation properties and allow to characterize the conditional distribution of X_t given $\mathcal{F}_t^Y$ as the solution of a system of equations, usually called

*Partially supported by MURST project *Metodi stocastici in finanza matematica Anno 2004 prot. 2004014572_007*

†Dipartimento di Matematica - Università di Roma ”La Sapienza”. Italy

‡Dipartimento di Matematica - Università di Roma ”Tor Vergata”, Italy

the filter equation.

Another situation of interest is the so-called singular filtering. It arises when the observation is a deterministic functional of the state. A first example arises when

$$Y_t = \int_0^t h(X_s)ds,$$

or more generally when Y_t is a functional of the state X up to time t and is absolutely continuous in time. A second example arises when the observation Y_t is a function of the state X at a fixed time t, i.e. when

$$Y_t = h(X_t).$$

In these cases the classical methods cannot be applied because the necessary martingale representation properties are not satisfied. There is not yet a general theory to solve these kind of problems. On the other hand much work has been done in order to give good approximations both for the first kind of example (see Korezlioglu and Runggaldier [14] and references therein) and the second one (see the papers by Joannides and Le Gland [10] and [11]).

In this paper we deal with the problem of filtering a reflected Brownian motion when the observation process is its local time at level zero.

We start with the reflection in the sense of the Skorohod problem: let W be a Brownian motion, we assume that (X,Y) is the unique solution of the Skorohod problem (see, e.g., Karatzas and Shreve [13]) for W, so that

$$X_t = W_t + \Lambda_t, \quad Y_t = \Lambda_t, \tag{1}$$

where

$$\Lambda_t = \ell_t(W) \tag{2}$$

and ℓ is the deterministic functional

$$\ell : D_{\mathbb{R}}[0,+\infty) \to D_{\mathbb{R}}[0,+\infty); \qquad \ell_t(x) = -\inf_{0\le s\le t}\big(x(s)\wedge 0\big) = -\inf_{0\le s\le t}\big(x(s)\big)\wedge 0. \tag{3}$$

Recall that $W_t + \Lambda_t \geq 0$ for every t and Λ_t is increasing. Moreover, as W_t has continuous paths, Λ_t is continuous and the corresponding measure $d\Lambda_t$ is carried by the set $\{u \text{ s.t. } W_u + \Lambda_u = 0\}$. Finally for any stopping time σ such that $W_\sigma + \Lambda_\sigma = 0$ on $\{\sigma < \infty\}$, the law of the shifted process $(W_{\sigma+t} + \Lambda_{\sigma+t}, \Lambda_{\sigma+t} - \Lambda_\sigma)$ conditioned to $\{\sigma < \infty\}$ is the same as the law of the process $(W_t + \Lambda_t, \Lambda_t)$. The above properties imply that Λ_t is (up to a constant) the local time of the reflected Brownian motion (see Chapter IV of Bertoin [3] and in Proposition 5 therein).

The above model (1) can be viewed as the diffusive approximation, in heavy traffic conditions, of a sequence of rescaled queueing systems (see, e.g., the papers by Harrison and Williams [8] and [9], and the more recent paper by Bramson and Dai [4]). In this set up the state is the size of the queue, and the observation is the total time the queue has spent in zero, the so called *idle time* (see Prabhu [16]). In a companion paper [15] we study the sequence of filters for rescaled M/M/1 queues, given the idle time, and prove that the sequence converges to the filter of model (1), under conditions which are slightly stronger than the usual heavy traffic conditions.

Our aim is the computation of $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big]$. The process Λ_t being $\mathcal{F}_t^\Lambda$-measurable, it is equivalent to consider the state-observation system (W, Λ) and compute

$$\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big]. \tag{4}$$

This problem naturally comes into the frame of singular filtering since it is a deterministic functional of the whole history of the state W up to time t as in the first example, but with the non trivial difference that in our model Λ_t is singular w.r.t. the Lebesgue measure. Therefore, to our knowledge, the techniques in the frame of singular filtering do not apply.

In the case of standard Brownian motion, we compute explicitly the filter, by means of a simple approximation procedure, using the classical techniques of filtering with respect to counting processes (see Brémaud [5]) and the basic properties of Brownian motion, such as the continuity of the paths, the independence and homogeneity of the increments, and the reflection principle. We obtain

$$E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^{\Lambda}\big] = \int_0^\infty g\left(y\sqrt{\zeta_t}\right) y e^{\frac{-y^2}{2}} dy, \tag{5}$$

where ζ_t is the *elapsed time from the last visit to 0* for the process $X_t = W_t + \Lambda_t$, that is

$$\zeta_t = t - \sup\{u < t \text{ s.t. } W_u + \Lambda_u = 0\}, \tag{6}$$

which is $\mathcal{F}_t^{\Lambda}-$measurable, since

$$\sup\{u < t \text{ s.t. } W_u + \Lambda_u = 0\} = \sup\{u < t \text{ s.t. } \Lambda_u < \Lambda_t\} \quad a.s. \tag{7}$$

A result similar to (5) can be obtained when considering the reflection $X_t = |B_t|$ of a standard Brownian motion B_t, and $Y_t = L_t(|B|)$, the local time of $|B_t|$, defined, as usual, via the Tanaka formula. In this case, we obtain

$$E\big[f(|B_t|)/\mathcal{F}_t^{L(|B|)}\big] = \int_0^\infty f(y\,|\mu_t|) y e^{-\frac{y^2}{2}} dy, \tag{8}$$

where

$$\mu_t = \operatorname{sgn}(B_t)\sqrt{t - g_t(B)}, \quad \text{with} \quad g_t(B) = \sup\{s < t : B_s = 0\}, \tag{9}$$

is the process introduced by Azéma and Yor in [2] and coincides, up to a constant with the so called the Azéma martingale (see Protter [17]). The above result (8) can be derived by taking into account some properties of the Azéma martingale and of local times (see Section 5). Though these kind of properties are well known and can be found in classical books such as Protter [17], or Revuz and Yor [18], it seems to us that these properties are less elementary than those used to obtain (5), though, on the other hand, they do not involve at all filtering theory.

Our approach consists in constructing an L^1-approximation for $\pi_t(g)$ for a large class of functions g, by conditioning to $\mathcal{F}_t^{\Lambda^n}$, where Λ_t^n is a suitable approximation for the observation Λ_t. The approximating sequence Λ_t^n (defined in Section 2) has the following properties.
(i) The process Λ_t^n is proportional to a counting process and therefore we can use the techniques of nonlinear filtering with counting observations (see, for instance [5]).
(ii) The sequence of processes Λ_t^n converges to Λ_t for all ω in the space $D_{\mathbb{R}}[0,\infty)$ w.r.t. the topology of the uniform convergence.
(iii) The information $\mathcal{F}_t^{\Lambda^n}$ carried by Λ^n up to time t is increasing in n and $\bigvee_n \mathcal{F}_t^{\Lambda^n}$ coincides with $\mathcal{F}_t^{\Lambda}$, information carried by Λ up to time t.
As a consequence (see Theorem 2.4), the sequence of approximating filter

$$\pi_t^n(g) = E\big[g(W_t)/\mathcal{F}_t^{\Lambda^n}\big]. \tag{10}$$

converges to $\pi_t(g)$ P-a.s. and in $L^1(\Omega, dP)$, and therefore in $L^1([0,T]\times\Omega, dt\times dP)$. When W is a standard Brownian motion we provide an explicit expression for $\pi_t^n(g)$ (see Theorem 3.3)

$$\pi_t^n(g) = \frac{1}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}\int_{-\frac{1}{2^n}}^{+\infty} g(-\tfrac{j}{2^n}+x)\left[\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} - \frac{e^{-\frac{(\frac{2}{2^n}+x)^2}{2s}}}{\sqrt{2\pi s}}\right] dx\Big|_{s=t-\sigma_j^n},$$

when $\sigma_j^n < t < \sigma_{j+1}^n$, where $\{\sigma_j^n,\ j\in\mathbb{N}\}$ are the jump times of Λ^n, and where Φ is the distribution function of a standard Gaussian random variable.

Successively (see Section 4) the previous expressions for $\pi_t^n(g)$ allow us to identify the original filter $\pi_t(g)$. In Theorem 4.4 we get its explicit expression

$$E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] = \int_0^\infty g\left(-l+y\sqrt{s}\right) y e^{\frac{-y^2}{2}} dy\Big|_{l=\Lambda_t, s=\zeta_t},$$

and then (5) immediately follows.

The previous steps ***(i)***$-$***(iii)*** can be repeated for a large class of processes, such as diffusions or Lévy processes without negative jumps, and in particular for a general Brownian motion with drift coefficient c and diffusion coefficient a^2 (see Section 6). In the latter case the approximating filters and the exact filter are modifications of the corresponding filters for the standard Brownian motion. We obtain them (see Theorems 6.1 and 6.2) starting from the standard Brownian motion case, via a deterministic time change, Girsanov Theorem and Kallianpur Striebel formula.

2 Approximation results

Let $(\Omega,\mathcal{F},P)$ be a probability space and let W be a standard Brownian motion defined on it. Let $(X,Y)=(W+\Lambda,\Lambda)$ be the solution of the Skorohod problem (1), then, as already mentioned in the Introduction, Λ_t is the local time at level 0 of the process X_t.

Observe that $W_0=0$, so that

$$\Lambda_t = -\inf_{0\le s\le t}\big(W_s\big). \tag{11}$$

For each $n\in\mathbb{N}$ consider the sequence of stopping times $\sigma_j^n,\ j\ge 0$ defined as the first time the process Λ_t crosses the threshold $\frac{j}{2^n}$, that is

$$\sigma_0^n=0,\quad \sigma_j^n = \inf\big\{t\quad \text{s.t.}\quad \Lambda_t\ge \frac{j}{2^n}\big\}, \tag{12}$$

or, equivalently, as the first time the process W_t crosses the threshold $-\frac{j}{2^n}$, that is

$$\sigma_0^n=0,\quad \sigma_j^n = \inf\big\{t\quad \text{s.t.}\quad W_t\le -\frac{j}{2^n}\big\}. \tag{13}$$

The approximating observation process Λ^n is the $D_{\mathbb{R}}[0,+\infty)$-valued process

$$\Lambda_t^n = \sum_{j=0}^\infty \frac{j}{2^n}\mathbb{I}(\sigma_j^n\le t<\sigma_{j+1}^n) = \frac{1}{2^n}\sum_{j=0}^\infty \mathbb{I}(\sigma_j^n\le t), \tag{14}$$

where $\mathbb{I}(A)$ denotes the indicator function of A.

Remark 2.1. *It is easy to see that* $\Lambda^n_t = \ell(W^n_t)$*, where* $W^n_t = \sum_{k=0}^{\infty} W_{\tau^n_k}\, \mathbb{I}(\tau^n_k \le t < \tau^n_{k+1})$*, with* $\tau^n_0 = 0$ *and* $\tau^n_{k+1} = \inf\{t > \tau^n_k,\ s.t. |W_t - W_{\tau^n_k}| > \frac{1}{2^n}\}$ *(see e.g. Section 4 of [15]). Then clearly* Λ^n *converge to* Λ*, uniformly on compact sets, since* ℓ *is a continuous function w.r.t. the topology of the uniform convergence on compact sets. Nevertheless the convergence holds in a stronger sense, as explained in the following lemma.*

Lemma 2.2. *Let* Λ *and* Λ^n *be defined as in* (11) *and* (14) *respectively. Then for every* $t \ge 0$ *and* $\omega \in \Omega$

$$\Lambda^n_t(\omega) \le \Lambda_t(\omega) \le \Lambda^n_t(\omega) + \frac{1}{2^n}. \tag{15}$$

Proof. The process Λ has nondecreasing paths, which are also continuous, since the Brownian motion has continuous paths. The continuity property implies $\Lambda_{\sigma^n_{j+1}} = \frac{j+1}{2^n} = \Lambda_{\sigma^n_j} + \frac{1}{2^n}$ and therefore

$$\Lambda^n_t = \Lambda^n_{\sigma^n_j} = \Lambda_{\sigma^n_j}, \text{ for any } t \in [\sigma^n_j, \sigma^n_{j+1}).$$

The nondecreasing property implies

$$\Lambda_{\sigma^n_j} \le \Lambda_t \le \Lambda_{\sigma^n_{j+1}} = \Lambda_{\sigma^n_j} + \frac{1}{2^n}, \text{ for any } t \in [\sigma^n_j, \sigma^n_{j+1}),$$

and (15) follows immediately.

□

For notational convenience from now on we set

$$\mathcal{G}^n_t = \mathcal{F}^{\Lambda^n}_t = \sigma\big(\Lambda^n_s, s \le t\big) \quad \text{and} \quad \mathcal{G}^\infty_t = \bigvee_n \mathcal{G}^n_t.$$

The following result is the most important tool for showing the convergence result.

Lemma 2.3. *Let* $t \in \mathbb{R}^+$*. Then* $\{\mathcal{G}^n_t, n \in \mathbb{N}\}$ *is an increasing sequence of* σ*-algebras and* $\mathcal{G}^\infty_t = \mathcal{F}^\Lambda_t$*.*

Proof. The choice of the thresholds $\frac{j}{2^n}$ yields $\sigma^{n+1}_{2j} = \sigma^n_j$ (see (12) or (13)), so that, by (14), Λ^n_t is $\mathcal{G}^{n+1}_t$-measurable. Therefore $\mathcal{G}^n_t \subseteq \mathcal{G}^{n+1}_t$. Moreover, for each choice of $t_1 \le t_2 \le ... \le t_m \le t$, and for each $\omega \in \Omega$ it occurs $\big(\Lambda^n_{t_1}, ..., \Lambda^n_{t_m}\big) \to \big(\Lambda_{t_1}, ..., \Lambda_{t_m}\big)$ so that $\big(\Lambda_{t_1}, ..., \Lambda_{t_m}\big) \in \mathcal{G}^\infty_t$, and therefore $\mathcal{G}^\infty_t \supseteq \mathcal{F}^\Lambda_t$. Finally, by (12) and (14) Λ^n_t is $\mathcal{F}^\Lambda_t$-measurable, i.e. $\mathcal{G}^n_t \subseteq \mathcal{F}^\Lambda_t$ for each $n \in \mathbb{N}$.

□

Theorem 2.4. *Let* g *be a bounded measurable function. Then*

$$\pi^n_t(g) = E\big[g(W_t)/\mathcal{G}^n_t\big] \to \pi_t(g) = E\big[g(W_t)/\mathcal{F}^\Lambda_t\big] \quad a.s. \text{ and in } L^1(\Omega, dP) \tag{16}$$

Proof. Lemma 2.3 guarantees that $\pi_t(g) = E\big[g(W_t)/\mathcal{F}^\Lambda_t\big] = E\big[g(W_t)/\mathcal{G}^\infty_t\big]$. Moreover, for each $t \in \mathbb{R}^+$ and g fixed $\pi^n_t(g)$ coincides with $M_n = E\big[g(W_t)/\mathcal{G}^n_t\big]$. The sequence $\{M_n, n \ge 1\}$ is a bounded uniformly integrable $\mathcal{G}^n_t$-martingale in discrete time, so that M_n converge to $M_\infty = E\big[g(W_t)/\mathcal{G}^\infty_t\big]$, that is (16).

□

3 The approximating filter

In order to derive the explicit expression of $\pi_t^n(g) = E\big[g(W_t)/\mathcal{G}_t^n\big]$ we need the following preliminary result.

Proposition 3.1. *The conditional law of W_t given $\mathcal{G}_t^n$ admits the following representation P-a.s.*

$$E\big[g(W_t)/\mathcal{G}_t^n\big] = \sum_{j=0}^{\infty} \frac{E\Big[g\left(-\frac{j}{2^n} + W_{s+\sigma_j^n} - W_{\sigma_j^n}\right)\mathbb{I}\left(S_{j+1}^n > s\right)\Big]_{s=t-\sigma_j^n}}{E\Big[\mathbb{I}(S_{j+1}^n > s)\Big]_{s=t-\sigma_j^n}} \mathbb{I}\big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\}, \quad (17)$$

where $\{\sigma_j^n, j \ge 0\}$ are defined by (13)*, and $\{S_j^n = \sigma_j^n - \sigma_{j-1}^n, j \ge 1\}$.*

Moreover, let W_s^* be a standard Brownian motion, let Λ_s^* be the local time at level 0 *of the reflected process $W_s^* + \Lambda_s^*$, then*

$$E\big[g(W_t)/\mathcal{G}_t^n\big] = \sum_{j=0}^{\infty} E\Big[g(-\tfrac{j}{2^n} + W_s^*)\Big/\Lambda_s^* < \tfrac{1}{2^n}\Big]_{s=t-\sigma_j^n} \mathbb{I}\big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\}. \quad (18)$$

Proof. Observe that

$$E\big[g(W_t)/\mathcal{G}_t^n\big] = \sum_{j=0}^{\infty} E\Big[g(W_t)/\mathcal{G}_t^n\Big]\mathbb{I}\big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\},$$

since $\big\{\{\sigma_j^n \le t < \sigma_{j+1}^n\},\, j \in \mathbb{N}\big\}$ is a $\mathcal{G}_t^n$-measurable partition. Moreover (see [5] chapter III T5)

$$\mathcal{G}_t^n \cap \big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\} = \sigma\Big\{\sigma_0^n, ..., \sigma_j^n\Big\} \cap \big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\} = \mathcal{G}_{\sigma_j^n}^n \cap \big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\}$$

so that, for any $\mathcal{G}_t^n$-measurable random variable Y, there exists a $\mathcal{G}_{\sigma_j^n}^n$-measurable random variable Y' s.t.

$$Y\mathbb{I}\big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\} = Y'\mathbb{I}\big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\}.$$

In particular, taking $Y = E\Big[g(W_t)/\mathcal{G}_t^n\Big]$ and conditioning both sides on $\mathcal{G}_{\sigma_j^n}^n$, we get

$$E\Big[E\big[g(W_t)\mathbb{I}\{\sigma_{j+1}^n > t\}/\mathcal{G}_t^n\big]/\mathcal{G}_{\sigma_j^n}^n\Big]\mathbb{I}\big\{\sigma_j^n \le t\big\} = Y'\mathbb{I}\big\{\sigma_j^n \le t\big\}P\big(\sigma_{j+1}^n > t/\mathcal{G}_{\sigma_j^n}^n\big).$$

Then, recalling that $\sigma_{j+1}^n = \sigma_j^n + S_{j+1}^n$,

$$Y' = \frac{E\Big[g(W_t)\mathbb{I}\left(S_{j+1}^n > t - \sigma_j^n\right)/\mathcal{G}_{\sigma_j^n}^n\Big]}{P\Big(S_{j+1}^n > t - \sigma_j^n/\mathcal{G}_{\sigma_j^n}^n\Big)}\mathbb{I}\big\{\sigma_j^n \le t\big\}.$$

Moreover $W_{\sigma_j^n} = -\Lambda_{\sigma_j^n} = -\frac{j}{2^n}$ and, on $\big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\}$, $W_t = -\frac{j}{2^n} + W_t - W_{\sigma_j^n}$. So

$$\begin{aligned}
&\frac{E\Big[g(W_t)\mathbb{I}\left(S_{j+1}^n > t - \sigma_j^n\right)/\mathcal{G}_{\sigma_j^n}^n\Big]}{P\Big(S_{j+1}^n > t - \sigma_j^n/\mathcal{G}_{\sigma_j^n}^n\Big)}\mathbb{I}\big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\} = \\
&= \frac{E\Big[g\left(-\frac{j}{2^n} + W_{s+\sigma_j^n} - W_{\sigma_j^n}\right)\mathbb{I}\left(S_{j+1}^n > s\right)\big|_{s=t-\sigma_j^n}/\mathcal{G}_{\sigma_j^n}^n\Big]}{E\Big[\mathbb{I}(S_{j+1}^n > s)\big|_{s=t-\sigma_j^n}/\mathcal{G}_{\sigma_j^n}^n\Big]}\mathbb{I}\big\{\sigma_j^n \le t < \sigma_{j+1}^n\big\}. \quad (19)
\end{aligned}$$

The process $W_{s+\sigma_j^n} - W_{\sigma_j^n}$ is a standard Brownian motion independent of $\mathcal{G}^n_{\sigma_j^n}$, S^n_{j+1} is a random variable independent of $\mathcal{G}^n_{\sigma_j^n}$ and $t-\sigma_j^n$ is $\mathcal{G}^n_{\sigma_j^n}$-measurable. Then (17) follows since (19) is equal to

$$\frac{E\Big[g\left(-\frac{j}{2^n}+W_{s+\sigma_j^n}-W_{\sigma_j^n}\right)\mathbb{I}\left(S^n_{j+1}>s\right)\Big]_{s=t-\sigma_j^n}}{E\Big[\mathbb{I}(S^n_{j+1}>s)\Big]_{s=t-\sigma_j^n}}\mathbb{I}\{\sigma_j^n\le t<\sigma_{j+1}^n\}.$$

Finally (18) follows by noting that $\{\sigma_1^{*n}>s\}=\left\{\Lambda_s^*<\frac{1}{2^n}\right\}$ and that $(W_{s+\sigma_j^n}-W_{\sigma_j^n}, S^n_{j+1})$ has the same distribution as (W^*_s,σ_1^{*n}), where σ_1^{*n} is first exit time of W^*_t from the region $(-\frac{1}{2^n},\infty)$, so that

$$E\Big[g(-\tfrac{j}{2^n}+W_{s+\sigma_j^n}-W_{\sigma_j^n})\mathbb{I}\left(S^n_{j+1}>s\right)\Big]=E\Big[g(-\tfrac{j}{2^n}+W^*_s)\mathbb{I}\left(\sigma_1^{*n}>s\right)\Big]. \tag{20}$$

□

Our next step is to give a more explicit expression of (18). First of all note that (18) can be rewritten as

$$\begin{aligned}E\big[g(W_t)/\mathcal{G}^n_t\big] &= E\Big[g(-l+W^*_s)\Big/\Lambda^*_s<\tfrac{1}{2^n}\Big]\Big|_{l=\Lambda^n_t,\ s=\zeta^n_t}\\ &= E\Big[g(-l+W^*_s)\Big/\Lambda^{*n}_s=0\Big]\Big|_{l=\Lambda^n_t,\ s=\zeta^n_t},\end{aligned} \tag{21}$$

where Λ^n_t is defined in (14),

$$\zeta^n_t = t-\sup\{u\le t \quad \text{s.t.} \quad \Lambda^n_u<\Lambda^n_t\}, \tag{22}$$

and Λ^{*n} is obtained from W^* as Λ^n from W. Then, denoting

$$\Pi^n(s,l;g)=E\Big[g(-l+W^*_s)\Big/\Lambda^*_s<\tfrac{1}{2^n}\Big]=E\Big[g(-l+W^*_s)\Big/\Lambda^{*n}_s=0\Big], \tag{23}$$

the conditional expectation (18) can be shortly rewritten as

$$E\big[g(W_t)/\mathcal{G}^n_t\big]=\Pi^n(\zeta^n_t,\Lambda^n_t;g). \tag{24}$$

To obtain the explicit expression of the above conditional expectation we need to compute the conditional law of W^*_s given $\{\Lambda^*_s<k\}$, for $k>0$, where (W^*_s,Λ^*_s) are the processes of Proposition 3.1.

Lemma 3.2. *When $s>0$, the conditional law of W^*_s given $\{\Lambda^*_s<k\}$ has density*

$$f_{W^*_s}(x/\Lambda^*_s<k)=\frac{1}{2\Phi\left(\frac{k}{\sqrt{s}}\right)-1}\left[\frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}-\frac{e^{-\frac{(2k+x)^2}{2s}}}{\sqrt{2\pi s}}\right]\mathbb{I}(x>-k), \tag{25}$$

where Φ is the distribution function of a normal standard random variable. When $s=0$ the conditional law of W^*_0 given $\{\Lambda^*_0<k\}$ is the Dirac measure $\delta_{\{0\}}(dx)$ in 0.

Proof. The case $s = 0$ is trivial. When $s > 0$, set $M_s^* = \max_{u \le s} W_u^*$. By the reflection principle one can compute the joint law of (W_s^*, M_s^*) (see [7], for example)

$$P\left(W_s^* \le x, M_s^* \le k\right) = \begin{cases} \Phi\left(\frac{x}{\sqrt{s}}\right) - \Phi\left(\frac{x-2k}{\sqrt{s}}\right) & \text{if } x \le k \\ P\left(M_s^* \le k\right) = \Phi\left(\frac{k}{\sqrt{s}}\right) - \Phi\left(\frac{-k}{\sqrt{s}}\right) & \text{if } x > k. \end{cases}$$

Then, by the symmetry of W, the joint law of (W_s^*, Λ_s^*) is

$$P\left(W_s^* \le x, \Lambda_s^* \le k\right) = \begin{cases} 0 & \text{if } -x \ge k \text{ or } k < 0 \\ 2\Phi\left(\frac{k}{\sqrt{s}}\right) - \Phi\left(\frac{-x}{\sqrt{s}}\right) - \Phi\left(\frac{x+2k}{\sqrt{s}}\right) & \text{if } -x < k \text{ and } k \ge 0. \end{cases}$$

Then the thesis trivially follows by observing that

$$P\left(W_s^* \le x / \Lambda_s^* < k\right) = \frac{2\Phi\left(\frac{k}{\sqrt{s}}\right) - \Phi\left(\frac{-x}{\sqrt{s}}\right) - \Phi\left(\frac{x+2k}{\sqrt{s}}\right)}{2\Phi\left(\frac{k}{\sqrt{s}}\right) - 1} \mathbb{I}(-x < k).$$

□

Now we have all the tools to give a more explicit expression for $E\big[g(W_t)/\mathcal{G}_t^n\big]$: the following result is an immediate consequence of (21)$-$(25).

Theorem 3.3. *Let g be a measurable bounded function, then*

$$E\big[g(W_t)/\mathcal{G}_t^n\big] = \sum_{j=0}^{\infty} \left\{ \int_{-\frac{1}{2^n}}^{+\infty} g(-\tfrac{j}{2^n} + x) f_n(x,s) dx \right\}\Bigg|_{s=t-\sigma_j^n} \mathbb{I}\{\sigma_j^n \le t < \sigma_{j+1}^n\}, \tag{26}$$

where $f_n(x,s) = f_{W_s^*}(x/\Lambda_s^* < \frac{1}{2^n})$ for $s > 0$ (see (25)), with the convention that $f_n(x,0) = \delta_0(x)$, the Dirac function at the point 0.

Equivalently, the conditional law of W_t given $\mathcal{G}_t^n$ admits the representation (24) P-a.s., and

$$\Pi^n(s,l;g) = \begin{cases} \int_{-\infty}^{+\infty} g(-l+x) f_n(x,s) dx & \text{if} \quad s > 0 \\ g(-l) & \text{if} \quad s = 0. \end{cases} \tag{27}$$

4 The original filter as an L^1 limit

In this Section we compute explicitly the L^1-limit of the approximating filter $\pi_t^n(g) = E\big[g(W_t)/\mathcal{G}_t^n\big]$. Thanks to the convergence result stated in Theorem 2.4, this limit is a version of the original filter $\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big]$. By (24) and by Theorem 3.3 we know that $\pi_t^n(g) = \Pi^n(\zeta_t^n, \Lambda_t^n; g)$, where $\Pi^n(s,l;g)$ is defined in (23), its expression is given in (27), ζ_t^n is defined in (22), and Λ_t^n in (14). We have already shown in Lemma 2.2 the uniform convergence of Λ_t^n to Λ_t. The idea is to show that ζ_t^n converge to ζ_t, the elapsed time from last visit to 0 for the process $W_t + \Lambda_t$ (see (6)), and that $\Pi^n(s_n, l; g)$ converge to a limit $\Pi(s,l;g)$, to be computed explicitly, whenever $s_n \to s$. Taking into account (23), the natural candidate for $\Pi(s,l;g)$ is

$$E\left[g(-l + W_s^*)/\Lambda_s^* = 0\right] := \int_{-\infty}^{\infty} g(-l+x) f_{W_s^*|\Lambda_s^*}(x|0) dx$$

This is the case, as explained in Remark 4.2 below by using the following Lemma.

Lemma 4.1. *Let $\Pi(s,l)$ be the probability measure defined by*

$$\Pi(s,l;g)=\int_0^\infty g(-l+y\sqrt{s})ye^{-\frac{y^2}{2}}\,dy \quad s\geq 0. \tag{28}$$

Let g be a continuous function with compact support, then

$$\Pi^n(s,l;g)\underset{n\to\infty}{\longrightarrow}\Pi(s,l;g) \qquad \textit{for } s\geq 0. \tag{29}$$

Moreover if $s_n \underset{n\to\infty}{\longrightarrow} s$ and $s>0$, then

$$\Pi^n(s_n,l;g)\underset{n\to\infty}{\longrightarrow}\Pi(s,l;g). \tag{30}$$

Proof. The proof of (29) when $s=0$ is trivial. Let s be strictly positive, and let s_n converge to s. Taking into account that we may rewrite

$$\Pi(s,l;g)=\int_0^\infty g(-l+x)\,\frac{x}{s}\,e^{-\frac{x^2}{2s}}\,dx, \tag{31}$$

$\Pi^n(s_n,l;g)$ is given by the expression (27) and $g(-l+x)$ is integrable, the dominated convergence theorem yields (30), if we prove that $f_n(x,s_n)$ is uniformly bounded and

$$\lim_{n\to\infty} f_n(x,s_n)=\frac{x}{s}\,e^{-\frac{x^2}{2s}}\,\mathbb{I}\left(x\geq 0\right), \tag{32}$$

where f_n is defined in Theorem 3.3.

Taking into account that $2\Phi(z)-1=2\big(\Phi(z)-\Phi(0)\big)$, and that for suitable $\theta_n^2\in\Big(0,\frac{1}{2^n\sqrt{s_n}}\Big)$ and $y_n(x)\in\Big(\frac{1}{2s_n}x^2,\frac{1}{2s_n}(\frac{1}{2^{n-1}}+x)^2\Big)$,

$$\Phi\left(\tfrac{1}{2^n\sqrt{s_n}}\right)-\Phi\left(0\right)=\frac{1}{\sqrt{2\pi}}\,e^{-\frac{\theta_n^2}{2}}\frac{1}{2^n\sqrt{s_n}},$$

$$e^{-\frac{x^2}{2s_n}}-e^{-\frac{(\frac{2}{2^n}+x)^2}{2s_n}}=e^{-y_n(x)}\tfrac{2}{2^ns_n}\left[x+\tfrac{1}{2^n}\right],$$

we may rewrite

$$\begin{aligned}
f_n(x,s_n)&=\frac{1}{\sqrt{2\pi s_n}}\frac{1}{2\left(\Phi\left(\frac{1}{2^n\sqrt{s_n}}\right)-\Phi(0)\right)}\left[e^{-\frac{x^2}{2s_n}}-e^{-\frac{(\frac{2}{2^n}+x)^2}{2s_n}}\right]\mathbb{I}\left(x>-\tfrac{1}{2^n}\right)\\
&=e^{\frac{\theta_n^2}{2}}e^{-y_n(x)}\frac{1}{s_n}\left[x+\tfrac{1}{2^n}\right]\mathbb{I}\left(x>-\tfrac{1}{2^n}\right).
\end{aligned}$$

The sequence θ_n is infinitesimal and the sequence $y_n(x)$ converge to $\frac{x^2}{2s}$, and therefore boundedness of f_n and (32) immediately follow.

□

Remark 4.2. *The joint density of (W_s^*,Λ_s^*) is*

$$f_{W_s^*,\Lambda_s^*}(x,k)=\frac{2}{s\sqrt{2\pi s}}(x+2k)\exp\left(-\tfrac{(x+2k)^2}{2s}\right)\ \mathbb{I}(k\geq(-x)\vee 0), \qquad \textit{for } s>0,$$

and then

$$f_{W_s^*|\Lambda_s^*}(x|k)=\frac{1}{s}(x+2k)\exp\left(-\tfrac{(x+2k)^2}{2s}+\tfrac{k^2}{2s}\right)\ \mathbb{I}(k\geq(-x)\vee 0).$$

By definition $f_n(x,s) = f_{W_s^*}(x/\Lambda_s^* < \frac{1}{2^n})$*, and therefore* (32) *is equivalent to*

$$f_{W_{s_n}^*}(x/\Lambda_{s_n}^* < \tfrac{1}{2^n}) \underset{n\to\infty}{\longrightarrow} f_{W_s^*|\Lambda_s^*}(x|0) = \frac{x}{s} e^{-\frac{x^2}{2s}} \mathbb{I}\,(x \geq 0)\,.$$

Taking into account (31) *we get the announced interpretation*

$$\Pi(s,l;g) = \int_0^\infty g(-l+x) f_{W_s^*|\Lambda_s^*}(x|0)dx.$$

It is also interesting to note that when g is differentiable, then

$$\int_0^\infty g(-l+x) f_{W_s^*|\Lambda_s^*}(x|0)dx = g(-l) + \int_0^\infty g'(-l+x) e^{-\frac{x^2}{2s}} dx \tag{33}$$

since $-\frac{x}{s}\, e^{-\frac{x^2}{2s}} = \frac{d}{dx} e^{-\frac{x^2}{2s}}$.

The last observation is a key point in the alternative probabilistic proof of the result stated in Lemma 4.1. We postpone this alternative proof to the Appendix to this Section.

The next result concerns the limit of the processes ζ_t^n.

Lemma 4.3. *Let ζ_t^n be defined by* (22) *and let ζ_t be the elapsed time from last visit to* 0 *for the process $W_t + \Lambda_t$ (see* (6)*). Then, for*

$$\zeta_t^n(\omega) \underset{n\to\infty}{\searrow} \zeta_t(\omega) \quad a.s. \tag{34}$$

Proof. A fundamental tool is statement (7), which, for the moment, we give for granted. Then $\zeta_t = t - \eta_t$ and $\zeta_t^n = t - \eta_t^n$, where η_t and η_t^n are defined as

$$\eta_t = \sup\{u < t \ \text{ s.t. } \Lambda_u < \Lambda_t\}$$

and

$$\eta_t^n = \sup\{u < t \ \text{ s.t. } \Lambda_u^n < \Lambda_t^n\} \ = \ \sup\{u < t \text{ s.t. } \Lambda_u^n = \Lambda_t^n - \tfrac{1}{2^n}\},$$

with the convention that $\sup\{\emptyset\} = 0$. Observe that

$$\eta_t^n = \sigma_j^n \ \text{ if and only if } \quad -\tfrac{j+1}{2^n} < \inf_{s\leq t} W_s \leq -\tfrac{j}{2^n}$$

and

$$\eta_t^{n+1} = \begin{cases} \sigma_{2j}^{2n} & \text{if and only if} \quad -\frac{2j+1}{2^{n+1}} < \inf_{s\leq t} W_s \leq -\frac{2j}{2^{n+1}} = -\frac{j}{2^n} \\ \sigma_{2j+1}^{2n} & \text{if and only if} \ \ \frac{j+1}{2^n} = -\frac{2j+2}{2^{n+1}} < \inf_{s\leq t} W_s \leq -\frac{2j+1}{2^{n+1}}\ . \end{cases}$$

This observation implies η_t^n to be increasing in n. Moreover $\eta_t^n \leq t$, so that $\eta_t^n \uparrow \eta_t^\infty$ for a suitable process η_t^∞, and the proof is achieved if we show successively the inequalities $\eta_t^\infty \leq \eta_t$ and $\eta_t^\infty \geq \eta_t$. The first inequality is equivalent to $\eta_t^n \leq \eta_t$, for any $n \in \mathbb{N}$. In case $t < \sigma_1^n$ this inequality is trivial since $\eta_t^n = 0 \leq \eta_t$. When $t \geq \sigma_1^n$ observe that $\eta_t^n \in \{\sigma_j^n, j \in \mathbb{N}\}$ and $\Lambda_{\sigma_j^n}^n = \Lambda_{\sigma_j^n}$ for all $n, j \in \mathbb{N}$ and so $\Lambda_{\eta_t^n} = \Lambda_{\eta_t^n}^n = \Lambda_t^n - \frac{1}{2^n}$. Moreover, by (15), $\Lambda_t^n \leq \Lambda_t$, so that $\Lambda_{\eta_t^n} \leq \Lambda_t - \frac{1}{2^n}$. The last inequality implies $\eta_t^n \leq \eta_t$ for all $n \in \mathbb{N}$. On the other hand, if $\eta_t^\infty < s \leq t$ then $\eta_t^n \leq \eta_t^\infty < s \leq t$ for all $n \in \mathbb{N}$, and therefore $\Lambda_s^n = \Lambda_t^n$ for all n. Passing to the limit in the previous equality we get $\Lambda_s = \Lambda_t$ and so $s \geq \eta_t$. Therefore $\eta_t^\infty \geq \eta_t$.

For the sake of completeness and for the ease of the reader it only remains to prove statement (7). Define

$$\beta_t = \sup\{s < t : X_s = W_s + \Lambda_s = 0\},$$

so that (7) is equivalent to $P(\eta_t = \beta_t) = 1$. First note that $\eta_t \le \beta_t$: indeed, by the properties of the solution of the Skorohod problem, if $\eta_t = t$, then $\beta_t = t$, and if $\eta_t < t$, then $X_{\eta_t} = W_{\eta_t} + \Lambda_{\eta_t} = 0$, and it follows that $\eta_t \le \beta_t$.

Note that $W_s + \Lambda_s = 0$ if and only if $W_s = \inf_{r<s} W_r \le 0$. If $\eta_t < \beta_t$, then $\Lambda_s = \Lambda_t$ for every $s \in [\eta_t, \beta_t]$, so that

$$W_t - W_s = W_t + \Lambda_t - W_s - \Lambda_s \le W_t + \Lambda_t,$$

and therefore, for any rational r satisfying $\eta_t < r < \beta_t$, we must have $-\Lambda_r = \inf_{s\le r} W_s = \inf_{r\le s\le t} W_s = -\Lambda_t$. But $P\{\inf_{s\le r} W_s = \inf_{r\le s\le t} W_s\} = 0$, and hence

$$P\{\eta_t < \beta_t\} \le \sum_{r\in\mathbb{Q}} P\{\inf_{s\le r} W_s = \inf_{r\le s\le t} W_s\} = 0.$$

$\square$

Finally we can state our main result

Theorem 4.4. *Let ζ_t, Λ_t and $\Pi(s,l)$ be defined in (6), (11) and (28), respectively. Then, for all continuous functions g, with compact support,*

$$\int_0^T E\Big|\pi_t^n(g) - \Pi(\zeta_t, \Lambda_t; g)\Big| dt \underset{n\to\infty}{\longrightarrow} 0,$$

and the process $\Pi(\zeta_t, \Lambda_t; g)$ is a cadlag version of $\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big]$, for all bounded measurable functions g.

Proof. Taking into account (10) and (24), our first aim is to show that, for all continuous functions g with compact support,

$$\int_0^T E\big|\Pi^n(\zeta_t^n, \Lambda_t^n; g) - \Pi(\zeta_t, \Lambda_t; g)\big| dt \underset{n\to\infty}{\longrightarrow} 0.$$

By triangular inequality the integrand is bounded above by

$$\big|\Pi^n(\zeta_t^n, \Lambda_t^n; g) - \Pi^n(\zeta_t^n, \Lambda_t; g)\big| + \big|\Pi^n(\zeta_t^n, \Lambda_t; g) - \Pi(\zeta_t, \Lambda_t; g)\big|.$$

By (15), $\big|\Lambda_t^n - \Lambda_t\big| \le \frac{1}{2^n}$, then, (27) and a direct computation yield

$$\big|\Pi^n(\zeta_t^n, \Lambda_t^n; g) - \Pi^n(\zeta_t^n, \Lambda_t; g)\big| \le \omega_g\left(\tfrac{1}{2^n}\right),$$

where $\omega_g(\delta) = \sup_{|s-u|<\delta} |g(s) - g(u)|$.

Furthermore, (30) and (34) imply that for each (t,ω) satisfying the condition $\zeta_t(\omega) > 0$

$$\big|\Pi^n(\zeta_t^n, \Lambda_t; g) - \Pi(\zeta_t, \Lambda_t; g)\big| \underset{n\to\infty}{\longrightarrow} 0.$$

The observation that $\{(t,\omega) \in [0,T] \times \Omega$ such that $\zeta_t(\omega) = 0\}$ is a zero measure set with respect to $dt \times dP$, and an easy application of the dominated convergence theorem provide

$$E\big[g(W_t)/\mathcal{G}_t^n\big] \underset{L^1([0,T]\times\Omega)}{\longrightarrow} \Pi(\zeta_t,\Lambda_t;g).$$

This first result, combined with Theorem 2.4, yields that $\pi_t(g) = \Pi(\zeta_t,\Lambda_t;g)\ dt\times dP$-a.e., for continuous functions with compact support. Then the probability measures π_t and $\Pi(\zeta_t,\Lambda_t)$ coincide $dt\times dP$-a.e., and the second statement is achieved.

□

We end this Section by observing that the cadlag version of the filter $\pi_t(g)$ can be also written as follows (see Remark 4.2)

$$\Pi(\zeta_t,\Lambda_t;g) = g(-\Lambda_t)\mathbb{I}(\zeta_t=0) + \int_0^\infty g(-\Lambda_t+x)f_{W_s^*|\Lambda_s^*}(x|0)dx\Big|_{s=\zeta_t}\mathbb{I}(\zeta_t>0), \qquad (35)$$

and consequently we can write down the cadlag version for the filter of $W_t+\Lambda_t$ w.r.t. $\mathcal{F}_t^\Lambda$

$$E\big[g(W_t+\Lambda_t)/\mathcal{F}_t^\Lambda\big] = \int_0^\infty g(x)f_{W_s^*|\Lambda_s^*}(x|0)dx\Big|_{s=\zeta_t} = \int_0^\infty g(y\sqrt{s})ye^{-\frac{y^2}{2}}dy\Big|_{s=\zeta_t}. \qquad (36)$$

Appendix to Section 4: an alternative proof of Lemma 4.1

We now give a different derivation of the limit $\Pi(s,l;g)$, under the hypothesis that g has continuous derivative and is uniformly continuous. For the sake of simplicity we consider only the case $s_n = s > 0$, but the general case when s_n converges to $s>0$ can be managed in the same manner. By the definition of $f_n(x,s)$ (see Theorem 3.3), we have

$$\Pi^n(s,l;g) = \frac{E\Big[g(-l+W_s^*)\mathbb{I}(W_s^*>-\frac{1}{2^n})\Big] - E\Big[g(-l+W_s^*-2\frac{1}{2^n})\mathbb{I}(W_s^*-2\frac{1}{2^n}>-\frac{1}{2^n})\Big]}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}.$$

Note that the above expression could also be derived directly from (20) by means of the reflection principle. Observe that, g being differentiable,

$$\begin{aligned}&g(-l+x)\mathbb{I}(x>-\tfrac{1}{2^n}) - g(-l+x-2\,\tfrac{1}{2^n})\,\mathbb{I}(x-2\,\tfrac{1}{2^n}>-\tfrac{1}{2^n})\\ =\,&g(-l+x)\,\mathbb{I}(-\tfrac{1}{2^n}<x\le\tfrac{1}{2^n}) + g'(-l+x-\vartheta^n(x)2\,\tfrac{1}{2^n})\,2\,\tfrac{1}{2^n}\,\mathbb{I}(x>\tfrac{1}{2^n}),\end{aligned}$$

where $\vartheta^n(x)\in(0,1)$. Then

$$\begin{aligned}\Pi^n(s,l;g) \;=\;& \frac{E\Big[g(-l+W_s^*)\,\mathbb{I}(-\frac{1}{2^n}<W_s^*\le\frac{1}{2^n})\Big]}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}\\ &+\frac{E\Big[g'(-l+W_s^*-\vartheta^n(W_s^*)2\,\frac{1}{2^n})\,\mathbb{I}(W_s^*>\frac{1}{2^n})\Big]\,2\,\frac{1}{2^n}}{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}.\end{aligned}$$

The first addend converges to $g(-l)$ since $2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1 = P(-\frac{1}{2^n}<W_s^*\le\frac{1}{2^n}))$ and $|g(-l+x)-g(-l)| \le \omega_g(\frac{1}{2^n})$ for $x\in(-\frac{1}{2^n},\frac{1}{2^n}]$. For the second addend observe that the expectation in the numerator converges to $E\Big[g'(-l+W_s^*)\mathbb{I}(W_s^*>0)\Big]$, while

$$\lim_{n\to\infty} \frac{2\Phi\left(\frac{1}{2^n\sqrt{s}}\right)-1}{2\frac{1}{2^n}} = \frac{1}{\sqrt{2\pi s}}\,.$$

The expression (31) of $\Pi(s,l;g)$ is achieved again since

$$E\Big[g'(-l+W_s^*)\mathbb{I}(W_s^*>0)\Big] = \frac{1}{\sqrt{2\pi s}}\int_0^\infty g'(-l+x)e^{-\frac{x^2}{2s}}dx$$

and since the following integration by part formula, which can be thought as the counterpart of (33), yields

$$\begin{aligned}E\Big[g'(-l+W_s^*)\mathbb{I}(W_s^*>0)\Big] &= \frac{1}{\sqrt{2\pi s}}\left\{g(-l+x)e^{-\frac{x^2}{2s}}\Big|_0^\infty + \int_0^\infty g(-l+x)\frac{x}{s}e^{-\frac{x^2}{2s}}dx\right\}\\ &= \frac{1}{\sqrt{2\pi s}}\left\{-g(-l)+\int_0^\infty g(-l+x)\frac{x}{s}e^{-\frac{x^2}{2s}}dx\right\}.\end{aligned}$$

5 Connections with the Azéma's martingale

Let B_t be a standard Brownian motion, then the local time of B is defined by the Tanaka formula as

$$L_t(B) = |B_t| - \int_0^t \operatorname{sgn}(B_s)dB_s, \tag{38}$$

with $\operatorname{sgn}(x)=1$ for $x>0$, and -1 otherwise. By a repeated use of Tanaka formula, it turns out that the local time of $|B|$ is

$$L_t(|B|) = 2\,L_t(B),$$

(see e.g. the classical books [17] and [18]). Therefore $\mathcal{F}_t^{L(|B|)} = \mathcal{F}_t^{L(B)}$ and consequently to prove (8) is equivalent to prove

$$E\big[f(|B_t|)/\mathcal{F}_t^{L(B)}\big] = \int_0^\infty f(y\,|\mu_t|)ye^{-\frac{y^2}{2}}dy, \tag{39}$$

where μ_t is defined as in (9).

It is well-known that the processes $(W_t+\Lambda_t,\Lambda_t)$ and $(|B_t|,L_t(B))$ have the same law on $D_{\mathbb{R}^2}[0,\infty)$ endowed with the Skorohod topology (see e.g. [18]). This result implies that the filter of $W_t+\Lambda_t$ given the history generated by the process Λ_t and the filter of $|B_t|$ given the history generated by the process $L_t(B)$ have the same law and that these filters are obtained just applying the same functional to the processes Λ_t and $L_t(B)$ respectively.

In the sequel we show how the filter of $|B_t|$ given the history generated by the process $L_t(B)$ can be directly obtained just starting from some results related to the Azéma martingale.

In [2] Azéma and Yor show that μ_t is a martingale w.r.t. a filtration containing $\mathcal{F}_t^{\operatorname{sgn}(B)} = \sigma\{\operatorname{sgn}(B_s),\ s\le t\}$. Observe that μ_t is adapted to the filtration $\mathcal{F}_t^{\operatorname{sgn}(B)}$ and therefore is an $\mathcal{F}_t^{\operatorname{sgn}(B)}$-martingale. The process μ_t coincides, up to a constant, with the martingale

$$E\big[B_t/\mathcal{F}_t^{\operatorname{sgn}(B)}\big], \tag{40}$$

which is known in the literature as the *Azéma martingale* (see, e.g. [17]).

From Lemma 1 and Proposition 4 in [2], one can draw that

$$E\big[f(|B_t|)/\mathcal{F}_t^{\mathrm{sgn}(B)}\big] = \int_0^\infty f(y\sqrt{t-g_t(B)})ye^{-\frac{y^2}{2}}dy \ . \tag{41}$$

Observe now that $g_t(B)$ is measurable w.r.t. the filtration $\mathcal{F}_t^{L(B)}$ generated by $L(B)$ up to time t since

$$g_t(B) = \sup\{s < t \text{ s.t. } |B_s| = 0\} = \sup\{s < t \text{ s.t. } L_s(B) < L_t(B)\} \quad a.s. \tag{42}$$

Finally, recalling that $\mathcal{F}_t^{L(B)} \subseteq \mathcal{F}_t^{\mathrm{sgn}(B)}$ (see e.g. [17]), from (41)

$$E\big[f(|B_t|)/\mathcal{F}_t^{L(B)}\big] = \int_0^\infty f(y\sqrt{t-g_t(B)})ye^{-\frac{y^2}{2}}dy \ . \tag{43}$$

Then the connection with our filtering problem is evident by comparing the above equality with (36). Indeed one can obtain one from the other just by changing $(W_t + \Lambda_t, \Lambda_t)$ with $(|B_t|, L_t(B))$. Indeed, if we define

$$\gamma : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty);\ x \to \gamma(x),\ \text{s.t. } \gamma_t(x) = t - \sup\{s < t, \quad \text{s.t.} \quad x_s < x_t\}$$

then the process $t - g_t(B)$ in (42) and the elapsed time process ζ_t defined in Lemma 4.3 are respectively defined by $t - g_t(B) = \gamma_t\left(L(B)\right)$ and $\zeta_t = \gamma_t\left(\Lambda\right)$.

6 The case of a general Brownian motion

We start by observing that all the results obtained in Section 2 can be generalized to a larg class of processes: they are still valid when applied to a process W_t with continuous paths (e.g. to a diffusion process) or to a process with cadlag paths and no negative jumps, without any further assumption on its law. Indeed in either cases $\Lambda_t = \ell_t(W)$ has continuous paths and therefore (15) of Lemma 2.2 holds. When in particular W_t is a Lévy process, then $W^*_s = W_{s+\sigma^n_j} - W_{\sigma^n_j}$ is independent of $\mathcal{G}^n_{\sigma^n_j}$, and has the same law as the process W_s, and therefore the the proof of Proposition 3.1 can be repeated verbatim.

Summarizing the result of Proposition 3.1 is valid for a Lévy process W_t without negative jumps, and clearly representation (21) is valid as well (this extension has been observed by Delia Silvino in her Master Thesis in Mathematics, under the guidance of G. Nappo).

As an example we consider a general Brownian motion W_t with diffusion coefficient a^2 and drift $c \in \mathbb{R}$, so that

$$E_{(a,c)}\big[g(W_t)/\mathcal{G}^n_t\big] = \sum_{j=0}^{\infty} E_{(a,c)}\Big[g(-\tfrac{j}{2^n} + W^*_s)\Big/\Lambda^*_s < \tfrac{1}{2^n}\Big]_{s=t-\sigma^n_j} \mathbb{I}\{\sigma^n_j \le t < \sigma^n_{j+1}\} \tag{44}$$

$$= E_{(a,c)}\Big[g(-l + W^*_s)/\Lambda^{*n}_s = 0\Big]\Big|_{(s,l)=(\zeta^n_t,\Lambda^n_t)}, \tag{45}$$

where we use the symbol $E_{(a,c)}$ to recall the diffusion and the drift coefficient, and we use the same notations of Sections 2 and 3. Note that $E_{(1,0)}$ corresponds to the standard Brownian motion case, denoted with E in the previous Sections.

It is immediate to see that $c = 0$ corresponds to the case of a standard Brownian motion up to the deterministic time change $s \to a^2 s$. Indeed

$$E_{(a,0)}\Big[g(-l + W_s^*)/\Lambda_s^{*n} = 0\Big] = E_{(1,0)}\Big[g(-l + W_{a^2 s}^*)/\Lambda_{a^2 s}^{*n} = 0\Big] = \Pi^n(a^2 s, l; g), \tag{46}$$

where $\Pi^n(s, l; g)$ is defined in (23). Then by the results of Section 4

$$E_{(a,0)}\big[g(W_t)/\mathcal{G}_t^n\big] = \Pi^n(a^2 \zeta_t^n, \Lambda_t^n; g), \tag{47}$$

and

$$\Pi^n(a^2 \zeta_t^n, \Lambda_t^n; g) \underset{L^1([0,T]\times\Omega)}{\longrightarrow} \Pi(a^2 \zeta_t, \Lambda_t; g), \tag{48}$$

where ζ_t is defined as in (6) and $\Pi(s, l; g)$ is defined as in (28), and therefore

$$\Pi(a^2 s, l; g) = E\Big[g(-l + W_{a^2 s}^*)/\Lambda_{a^2 s}^* = 0\Big] = \int_0^\infty g\left(-l + ya\sqrt{s}\right) y \exp\left(-\tfrac{y^2}{2}\right) dy \quad s \ge 0. \tag{49}$$

Theorem 6.1. *Let W_t be a Brownian motion with diffusion coefficient a^2 and drift coefficient c. Then the conditional law of W_t given $\mathcal{G}_t^n = \mathcal{F}_t^{\Lambda^n}$ admits the following representation P-a.s.*

$$E_{(a,c)}\big[g(W_t)/\mathcal{G}_t^n\big] = \left.\frac{\Pi^n\big(a^2 s, l; g(\cdot)\exp(\frac{c}{a^2}\cdot)\big)}{\Pi^n\big(a^2 s, l; \exp(\frac{c}{a^2}\cdot)\big)}\right|_{(s,l)=(\zeta_t^n, \Lambda_t^n)} \tag{50}$$

Proof. The proof can be given in two ways. The first one is via Kallianpur-Striebel formula and the exponential martingale $Z_t = \exp\{\frac{c}{a^2} W_t - \frac{1}{2}\frac{c^2}{a^2} t\}$

$$\begin{aligned} E_{(a,c)}\big[g(W_t)/\mathcal{G}_t^n\big] &= \frac{E_{(a,0)}\big[g(W_t) Z_t/\mathcal{G}_t^n\big]}{E_{(a,0)}\big[Z_t/\mathcal{G}_t^n\big]} \\ &= \frac{E_{(a,0)}\big[g(W_t)\exp\{\frac{c}{a^2} W_t\}/\mathcal{G}_t^n\big]}{E_{(a,0)}\big[\exp\{\frac{c}{a^2} W_t\}/\mathcal{G}_t^n\big]}. \end{aligned} \tag{51}$$

Then we apply (47) to $g(x)\exp(\frac{c}{a^2} x)$ and to $\exp(\frac{c}{a^2} x)$.

On the other hand, an elementary proof is based on the following observations. When W_s is a Brownian motion with diffusion coefficient a^2 and drift c, the joint density $f^{(a,c)}_{W_s,\Lambda_s}(x,k)$ of (W_s, Λ_s) is equal to $f^{(a,c)}_{W_s}(x) \cdot f^{(a,c)}_{\Lambda_s|W_s}(k|x)$, and the conditional density $f^{(a,c)}_{\Lambda_s|W_s}(k|x)$ of Λ_s given W_s does not depend on c (see, for instance, [1]), i.e.

$$f^{(a,c)}_{\Lambda_s|W_s}(k|x) = f^{(a,0)}_{\Lambda_s|W_s}(k|x) = f^{(1,0)}_{\Lambda_{a^2 s}|W_{a^2 s}}(k|x).$$

□

Following the same lines as in Section 4, and observing that the convergence result for the filter as stated in Theorem 2.4 holds either for standard Brownian motion or for general Brownian motion we can finally state the main result of this Section.

Theorem 6.2. *For all g continuous, with compact support*

$$E_{(a,c)}\big[g(W_t)/\mathcal{G}_t^n\big] \underset{L^1([0,T]\times\Omega)}{\longrightarrow} \left.\frac{\Pi\big(a^2 s, l; g(\cdot)\exp(\frac{c}{a^2}\cdot)\big)}{\Pi\big(a^2 s, l; \exp(\frac{c}{a^2}\cdot)\big)}\right|_{(s,l)=(\zeta_t, \Lambda_t)}.$$

Moreover, by (49)*, for all g measurable and bounded, a cadlag version of* $E_{(a,c)}\left[g(W_t)/\mathcal{F}^{\Lambda}_t\right]$ *is given by*

$$E_{(a,c)}\left[g(W_t)/\mathcal{F}^{\Lambda}_t\right] = \left.\frac{\int_0^\infty g\left(-l+ya\sqrt{s}\right)\exp(\frac{c}{a^2}\left(-l+ya\sqrt{s}\right))y\exp\left(-\frac{y^2}{2}\right)dy}{\int_0^\infty \exp(\frac{c}{a^2}\left(-l+ya\sqrt{s}\right))y\,\exp\left(-\frac{y^2}{2}\right)dy}\right|_{(s,l)=(\zeta_t,\Lambda_t)} . \quad (52)$$

Before giving the proof, we note that the above expression (52) can be obtained directly from Theorem 4.4, via the Kallianpur-Striebel formula and a deterministic time change as in the case of the conditional law given $\mathcal{G}^n_t$.

Proof. Using the representation (50) instead of (24), the proof of Theorem 4.4 can be repeated verbatim for a general Brownian motion, since, by Girsanov theorem, Lemma 4.3 still holds true, i.e.

1. $\zeta^n_t(\omega) \searrow \zeta_t(\omega)$ $P_{(a,c)} - a.s.$;

2. the set $\{(t,\omega) \in [0,T] \times \Omega$ such that $\zeta_t(\omega) = 0\}$ is a zero measure set with respect to the measure $dt \times dP_{(a,c)}$.

□

Acknowledgements
We wish to thank Anna Gerardi, Thomas G. Kurtz and an anonymous referee for their stimulating and helpful comments. Moreover one of us, G. Nappo, wishes to thank Delia Silvino for her observations, and in particular for the extension of Proposition 3.1 to Lévy processes without negative jumps.

References

[1] Asmussen, S., Glynn, P., and Pitman, J. Discretization error in simulation of one-dimensional reflecting Brownian motion. *Ann. Appl. Probab. 5*, 4 (1995), 875–896.

[2] Azéma, J., and Yor, M. Étude d'une martingale remarquable. In *Séminaire de Probabilités, XXIII*, Azéma, Meyer, Yor, Ed., Lecture Notes in Mathematics n. 1372. Springer, Berlin, 1989, pp. 88–130.

[3] Bertoin, J. *Lévy Processes*, first ed., vol. 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[4] Bramson, M., and Dai, J. G. Heavy traffic limits for some queueing networks. *Ann. Appl. Probab. 11* (2001), 49–90.

[5] Brémaud, P. *Point processes and queues. Martingale dynamics.* Springer-Verlag, New York, 1981. Springer Series in Statistics.

[6] Elliott, R. J. *Stochastic calculus and Applications.* Springer-Verlag, 1982.

[7] Harrison, J. M. *Brownian motion and stochastic flow systems.* J. Wiley & Sons, 1985.

[8] Harrison, J. M., and Williams, R. J. On the quasireversibility of a multiclass brownian service station. *Ann. Probab. 18* (1990), 1249–1268.

[9] Harrison, J. M., and Williams, R. J. Brownian models of feedforward queueing networks: Quasireversibility and product form solutions. *Ann. Appl. Probab. 2* (1992), 263–293.

[10] Joannides, M., and Le Gland, F. Nonlinear filtering with perfect discrete time observations. In *Proocedings of the 34-th conference on Decision and Control* (December 1995), IEEE-CSS, pp. 4012–4017.

[11] Joannides, M., and Le Gland, F. Small noise asymptotics of the bayesian estimator in nonidentifiable models. *Statistical Inference for Stochastic Processes 1* (2002), 95–130.

[12] Kallianpur, G. *Stochastic Filtering Theory*. Springer-Verlag, 1980.

[13] Karatzas, I., and Shreve, S. E. *Brownian Motion and Stochastic Calculus*. Springer, New York, 1988.

[14] Korezlioglu, H., and Runggaldier, W. J. Filtering for nonlinear systems driven by nonwhite noises: an approximation scheme. *Stoch. and Stoch Rep. 44* (1993), 65–102.

[15] Nappo, G., and Torti, B. Continuous time random walks and queues: explicit forms and approximations of the conditional law with respect to local times. *Stoch. Proc. Appl.*, (this number).

[16] Prabhu, N. U. *Stochastic storage processes*, second ed., vol. 15 of *Applications of Mathematics*. Springer-Verlag, New York, 1998. Queues, insurance risk, dams, and data communication.

[17] Protter, P. *Stochastic integration and differential equations*, vol. 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1990. A new approach.

[18] Revuz, D., and Yor, M. *Continuous martingales and Brownian motion*, third ed., vol. 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.