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RELATIONS between KENDALL DISTRIBUTIONS and FAMILIES of BIVARIATE VALUES at RISK in EXCHANGEABLE SURVIVAL MODELS

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Outline of talk

The family $\mathcal{D}$ of Bivariate VaR Curves

(upper-orthant) Kendall distribution $\widehat{K}$

The problem of ***compatibility*** between $\mathcal{D}$ and $\widehat{K}$

* Semicopulas & representation of $\mathcal{D}$

* The relation between $\mathcal{D}$ and the survival copula $\widehat{C}$

A transformation result by Genest & Rivest (2001)

The Archimedean case

Necessary conditions for the general case

A non Archimedean Example

***based on previous papers of Bruno Bassan and Fabio Spizzichino**

X, Y non-negative exchangeable random quantities

with $\overline{F}(x, y)$ *joint survival function*

$$\overline{F}(x, y) := P\{X > x, Y > y\}, \quad x > 0, y > 0.$$

with $\overline{G}(x)$ *marginal survival function*

$$\overline{G}(x) := \overline{F}(x, 0) = P\{X > x\} = P\{Y > x\}$$

Definition: (Bivariate (upper-orthant) Value at Risk at a probability level v, [Embrechts & Puccetti (2006)])

$$\boldsymbol{\overline{VaR_v}\left(\overline{F}\right) := \partial A_v, \qquad v \in [0, 1]}$$

where

$$\boldsymbol{A_v := \{(x, y) \in \mathbb{R}^2_+ : \overline{F}(x, y) \leq 1 - v\}.}$$

(for $A \subset \mathbb{R}^2_+$, ∂A boundary of A)

Assumptions: $\overline{G}$ continuous, strictly decreasing, strictly positive on $[0, \infty)$, $\overline{G}(0) = 1$

∂A_v regular curve, $\forall v \in [0, 1]$

Notation: $\boldsymbol{\mathcal{D}_{\overline{F}}}$:= family of ***bivariate upper-orthant Value at Risk curves***:

$$\boldsymbol{\mathcal{D}_{\overline{F}} := \{\partial A_v ; v \in [0, 1]\}}$$

(upper-orthant) ***Bi**variate **P**robability **I**ntegral **T**ransformation*:

$$\boldsymbol{Z := \overline{F}(X, Y)} \qquad \textbf{(BIPIT)}$$

(upper-orthant) ***Kendall distribution:***

$$\boldsymbol{\widehat{K}_{\overline{F}}(v) := \; P\{Z \le v\} \; = \; P\{\overline{F}(X, Y) \le v\}}$$
$$\boldsymbol{= 1 - P\{(X, Y) \in A_{1-v}\}, \;\; v \in [0, 1]}$$

(”upper-orthant” used in order to distinguish from the objects respectively considered in [**Genest, Rivest (2001)**], [**Nelsen et al. (2003)**] and ref. therein)

Facts:

* $\widehat{K}_{\overline{F}}\big(= \widehat{K}_{\widehat{C}_{\overline{F}}}\big)$ only determined by the ***survival copula*** $\widehat{C}_{\overline{F}}$, where

$$\widehat{C}_{\overline{F}}(u, v) = \overline{F}\left(\overline{G}^{\,-1}(u), \overline{G}^{\,-1}(v)\right),$$

* $\widehat{K}_{\overline{F}}$ is a Kendall distribution, i.e. $\boldsymbol{K(t) \ge t}$**,** $\forall \boldsymbol{t} \in \boldsymbol{[0, 1]}$

Kendall distributions are characterized as distribution functions on [0,1], with the above property (upper-orthant) ***Kendall distribution:***

$$\widehat{K}_{\overline{F}}(v) := P\{Z \leq v\} = P\{\overline{F}(X,Y) \leq v\}$$
$$= P\big((X,Y) \in \{(x,y) \ : \ \overline{F}(x,y) \leq 1-(1-v)\}\big)$$
$$= P\{(X,Y) \in A_{1-v}\}, \qquad v \in [0,1]$$

Let $\widehat{K}$ and $\mathcal{D}$ be given, with

$\widehat{K}$ a Kendall distribution ($\widehat{K}(t) \geq t$)

$\mathcal{D}$ a family of biv. (upper-orthant) Value at Risk curves

Compatibility Problem: Can we find $\overline{F}$ such that

$$\boxed{\widehat{\boldsymbol{K}}_{\overline{\boldsymbol{F}}} = \widehat{\boldsymbol{K}}, \quad \boldsymbol{\mathcal{D}}_{\overline{\boldsymbol{F}}} = \boldsymbol{\mathcal{D}} \ ?}$$

<u>Responses will be given under the further assumptions</u>

* $\overline{F}$ strictly one-decreasing

* $\overline{G}$ abs. continuous $\qquad \Rightarrow \widehat{\boldsymbol{K}}_{\overline{\boldsymbol{F}}}(\boldsymbol{t}^-) > \boldsymbol{t}, \ \boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{1})$

* $P\big(\overline{F}(X,Y) = t\big) = 0, \quad \forall t \in [0,1]$

<u>Methods</u>

- Describe $\mathcal{D}_{\overline{F}}$ by means of a ***semicopula*** $B_{\overline{F}}$

- Find the relation between $\mathcal{D}_{\overline{F}}$ and $\widehat{C}_{\overline{F}}$ (i.e. between $B_{\overline{F}}$ and $\widehat{C}_{\overline{F}}$)

- Exploit (and extend the use of) a transformation result by Genest & Rivest (2001)

- Exploit the properties of the class $\mathcal{C}_{\widehat{K}}$ of survival copulas $\widehat{C}$ such that $\widehat{K}_{\widehat{C}} = \widehat{K}$

For $x \geq 0, y \geq 0$, let $\boxed{h(x,y) := \overline{G}^{-1}\left[\overline{F}(x,y)\right]}$

Properties of h

* $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is one-increasing, $\quad h(x,0) = x, \ \forall x \geq 0$

* **h describes $\mathcal{D}_{\overline{F}}$:**

$$\boxed{\overline{VaR_v}\left(\overline{F}\right) = \{(x,y) \ : \ h(x,y) = \overline{G}^{-1}(1-v)\}}$$

* $\overline{F}$ is determined from the knowledge of h and $\overline{G}$: in fact, trivially,

$$\overline{F}(x,y) = \overline{G}\left[h(x,y)\right]$$

Remark: In the above representation of $\overline{F}$ the information provided by the marginal is separated from the information provided by $\mathcal{D}$

Question: In the representation of $\mathcal{D}$, can one replace h by a copula $B_{\overline{F}}$?

The idea is to transform h into a function B such that

$$B_{\overline{F}} : [0,1] \times [0,1] \to [0,1]$$

Define: $\boldsymbol{B_{\overline{F}}(u,v) = \exp\{-h(-\log u, -\log v)\}}$

$$\boldsymbol{B_{\overline{F}}(u,v) = \exp\{-\overline{G}^{-1}\left(\overline{F}(-\log u, -\log v)\right)\}}$$

whence

$$\boldsymbol{\overline{F}(x,y) = \overline{G}\left(-\log B_{\overline{F}}\left(e^{-x}, e^{-y}\right)\right)}$$

$B_{\overline{F}}$ is called ***aging function***, and it turns out that [Bassan, Spizzichino (2001), (2003), (2005)]:

* $\boldsymbol{B_{\overline{F}}}$ **can be used to describe** $\boldsymbol{\mathcal{D}_{\overline{F}}}$

* $\boldsymbol{B_{\overline{F}}}$ **is a *semicopula*** ($B_{\overline{F}}$ has all the properties of a copula but ... it may be not 2-increasing)

If $\overline{F}(x,y) = \left(e^{-\beta x} + e^{-\beta y} - 1\right)^+$, with $\beta > 0$, then $B(u,v) = \left[(u^\beta + v^\beta - 1)^+\right]^{\frac{1}{\beta}}$. When $\beta > 1$, $\exists$ a u_β such that $\frac{1}{2} < u_\beta \leq \left(\frac{1}{2}\right)^{1/\beta}$, and therefore $B(1,1) - B(u_\beta,1) - B(1,u_\beta) + B(u_\beta,u_\beta) =$ $= 1 - 2u_\beta + \left(\left(2\,(u_\beta)^\beta - 1\right)^+\right)^{1/\beta} = 1 - 2\,u_\beta < 0.$

Equivalent Compatibility Problem:

Given a Kendall distribution $\widehat{K}$ and an aging function B, can we find a bivariate survival function $\overline{F}$ such that

$$\boldsymbol{\widehat{K}_{\overline{F}} = \widehat{K}, \quad B_{\overline{F}} = B \ ?}$$

Example 1. ***(perfect dependence):*** *When*

$$\overline{F}(x,y) = \overline{G}\,(x \vee y)$$

$B_{\overline{F}}$ is the maximal copula

$$B_{\overline{F}}(u,v) = u \wedge v$$

Example 2. ***("Schur-constant" case):*** *The condition*

$$\overline{F}(x,y) = \overline{G}\,(x+y)$$

holds if and only if $B_{\overline{F}}$ is the product copula:

$$B_{\overline{F}}(u,v) = u \cdot v.$$

Example 3. ***(i.i.d. variables):***

$$\overline{F}\,(x,y) = \overline{G}(x) \cdot \overline{G}(y)$$

if and only if

$$B_{\overline{F}}(u,v) = Q^{-1}[Q(u) + Q(v)],$$

with $Q(u) := -\log \overline{G}\,(-\log u)$*,*

$$Q(u) := -\log \overline{G}\,(-\log u)\,.$$

The relation between $B_{\overline{F}}$ and $\widehat{C}_{\overline{F}}$ is

$$\boxed{\boldsymbol{B_{\overline{F}}(u,v) = \gamma\left(\widehat{C}_{\overline{F}}\left(\gamma^{-1}(u),\gamma^{-1}(v)\right)\right)},}$$

$$\boxed{\boldsymbol{\widehat{C}_{\overline{F}}(u,v) = \gamma^{-1}\left(B_{\overline{F}}(\gamma(u),\ \gamma(v))\right)}}$$

where $\gamma : [0,1] \longrightarrow [0,1]$ is the increasing function defined by

$$\boldsymbol{\gamma(w) = \exp\{-\overline{G}^{-1}(w)\}},$$

and such that, then,

$$\boldsymbol{\gamma^{-1}(w) = \overline{G}\,(-\log w)}.$$

Indeed, starting from

$$\begin{aligned} B_{\overline{F}}(u,v) &= \exp\{-h(-\log u, -\log v)\} \\ \overline{F}(x,y) &= \overline{G}\left(-\log B_{\overline{F}}\left(e^{-x}, e^{-y}\right)\right) \\ \widehat{C}_{\overline{F}}(u,v) &= \overline{F}\left(\overline{G}^{-1}(u), \overline{G}^{-1}(v)\right), \end{aligned}$$

one can easily obtain

$$\begin{aligned} B_{\overline{F}}(u,v) &= \exp\{-\overline{G}^{-1}\left(\widehat{C}_{\overline{F}}(-\log \overline{G}(u), -\log \overline{G}(v))\right)\}, \\ \widehat{C}_{\overline{F}}(u,v) &= \overline{G}\left(-\log B_{\overline{F}}\left(e^{-\overline{G}^{-1}(u)}, e^{-\overline{G}^{-1}(v)}\right)\right), \end{aligned}$$

i.e.

Solutions to the *compatibility* problem $\boxed{\widehat{\boldsymbol{K}_{\overline{\boldsymbol{F}}}} = \widehat{\boldsymbol{K}}, \quad \boldsymbol{B}_{\overline{\boldsymbol{F}}} = \boldsymbol{B}}$ have been inspired by the following results (see [Genest & Rivest (2001)] and [Nelsen et. al. (2003)])

✓ The *Kendall distribution* $\boxed{K_F(v) := P\{F(X,Y) \leq v\}}$ of a bivariate distribution F only depends on its copula $C = C_F$:

$$\boxed{\boldsymbol{K_F(t) = K_C(t) = t - \underbrace{\int_t^1 \frac{\partial}{\partial u} C(u, v_{u,t}) du}_{\lambda_C(t)}}}$$

where $\quad \boldsymbol{v_{u,t} = C_u^{-1}(t),} \quad$ with $\boldsymbol{C_u(v) := C(u,v)}$.

✓ If C and C^* are copulas that are related via relation

$$BC^*(u,v) = \gamma^{-1}(C\ \ (\gamma(u),\ \gamma(v)))$$

by a strictly increasing, differentiable bijection γ, then

$$\boldsymbol{\lambda_{C^*}(t) = \frac{\lambda_C(\gamma(v))}{\gamma'(v)}, \quad 0 < v < 1.}$$

✓ For any Kendall distribution K, there is a unique ***associative*** copula C such that $K_C = K$ (i.e. the Kendall distribution of C coincides with K).

C is *associative* $\Leftrightarrow C(C(u,v),w) = C(u, C(u,w))$ holds $\forall\ u$, v and w in $[0,1]$.

<u>Remark A</u> **If $\widehat{K}(t^-) > t$ (necessarily true under our conditions) the unique associative copula $\widehat{C}$ in the equivalence class of the (upper orthant) Kendall distribution $\widehat{K}$ is actually *ARCHIMEDEAN*, with generators (determined up to a constant)**

$$\widehat{\theta}\,\widehat{\phi}(t), \text{ where } \widehat{\phi}(t) = \widehat{\phi}_{\widehat{K}}(t) = \exp\left\{-\int_{t_0}^{t} \frac{1}{\widehat{\lambda}(s)}\,ds\right\},$$

and where $\widehat{\lambda}(s) = s - \widehat{K}(s)$, and $\widehat{\theta} > 0$.

A copula C is *Archimedean* $\Leftrightarrow C(u,v) = \phi^{-1}\big[\phi(u) + \phi(v)\big]\ \forall\ u,\ v$ in $[0,1]$.
The function ϕ (decreasing, convex, $\phi(1) = 0$) is called generator of C.

<u>Remark B</u> **The survival copula $\widehat{C}_{\overline{F}}$ is Archimedean iff the function $B_{\overline{F}}$ is such:**

$$\boxed{\widehat{C}_{\overline{F}}(u,v) = \widehat{\phi}^{-1}\big[\widehat{\phi}(u) + \widehat{\phi}(v)\big]}$$

$$\Updownarrow$$

$$B_{\overline{F}}(u,v) = \gamma\left(\widehat{C}_{\overline{F}}\left(\gamma^{-1}(u), \gamma^{-1}(v)\right)\right) = \gamma\left(\widehat{\phi}^{-1}\big[\widehat{\phi}(\gamma^{-1}(u)) + \widehat{\phi}(\gamma^{-1}(v))\big]\right)$$

$$\Updownarrow$$

$$\boxed{B_{\overline{F}}(u,v) = \varphi^{-1}\big[\varphi(u) + \varphi(v)\big]} \quad \text{with} \quad \boxed{\varphi(t) = \widehat{\phi}\big(\gamma^{-1}(t)\big)}$$

Replace $\widehat{\phi} \rightsquigarrow \widehat{\theta}\,\widehat{\phi}$, then, for $\theta = 1/\widehat{\theta}$

$$\boxed{\theta\,\varphi(t) = \widehat{\phi}\big(\gamma^{-1}(t)\big)}$$

$$\Updownarrow$$

$$\boxed{\overline{G}(x) = \overline{G}_\theta(x) := \widehat{\phi}^{-1}\left(\theta\,\varphi(e^{-}x)\right)}$$

since $\gamma^{-1}(t) = \overline{G}(-\log t)$, and $t \rightsquigarrow e^{-x}$, $\Leftrightarrow \gamma^{-1}(e^{-x}) = \overline{G}(x)$.

Therefore, if the aging function

$$\boldsymbol{B(u,v) = \varphi^{-1}\big[\varphi(u) + \varphi(v)\big]}, \textit{ with } \boldsymbol{\varphi = \varphi_B}$$

is an Archimedean semicopula, with $\boldsymbol{\varphi(0^+) = \infty}$, *and*

$$\boldsymbol{\widehat{K}(t^-) > t},$$

then the compatibility problem

$$\boldsymbol{\widehat{K}_{\overline{F}} = \widehat{K}}, \quad \boldsymbol{B_{\overline{F}} = B}$$

has infinitely many solutions:

Indeed, **necessarily $\boldsymbol{\widehat{C}}$ is Archimedean**

$$\boldsymbol{\widehat{C}(u,v) = \widehat{C}^{\widehat{\phi}}(u,v) := \widehat{\phi}^{-1}\big[\widehat{\phi}(u) + \widehat{\phi}(v)\big]}, \quad \text{with } \boldsymbol{\widehat{\phi} = \widehat{\phi}_{\widehat{K}}}.$$

the possible marginal distributions are

$$\boldsymbol{\overline{G}_\theta(x) := \widehat{\phi}^{-1}\left(\theta\, \varphi(e^{-}x)\right), \quad \theta > 0}.$$

and the solutions are $F = F_\theta$

$$\overline{F}_\theta(x,y) = \widehat{C}^{\widehat{\phi}}\left(\overline{G}_\theta(x), \overline{G}_\theta(y)\right),$$

or equivalently

$$\boldsymbol{\overline{F}_\theta(x,y) = \overline{G}_\theta\left(-\log B\left(e^{-x}, e^{-y}\right)\right)}.$$

(If $\varphi(0^+) < \infty$, then $\overline{G}_\theta$ are distribution functions only when $\widehat{\phi}(0^+) < \infty$ and $\theta \geq \widehat{\phi}(0^+)/\varphi(0^+)$; furthermore $\overline{G}_\theta(x) > 0 \;\forall\; x > 0 \Leftrightarrow \theta \geq \widehat{\phi}(0^+)/\varphi(0^+)$)

Let B be an aging function (not necessarily Archimedean). It is possible to define:

✓ **A *pseudo Kendall distribution* $\mathcal{K}B$**

$$\boxed{\mathcal{K}B(t) = t - \underbrace{\int_t^1 \frac{\partial}{\partial u} B(u, v_{u,t}^B) du}_{\mu_B(t)}}$$

where $\quad v_{u,t}^B = B_u^{-1}(t), \qquad$ with $B_u(v) := B(u, v)$.

✓ **A generator φ_B of a semicopula**

$$\varphi_B(t) = \exp\left\{ -\int_{t_0}^t \frac{1}{s - \mathcal{K}B(s)}(s)\, ds \right\},$$

Therefore, under the assumptions that $\varphi_B(0^+) = \infty$ and $\widehat{K}(t^-) > 0$, the possible solutions to *compatibility* problem $\boxed{\widehat{K}_{\overline{F}} = \widehat{K}, \quad B_{\overline{F}} = B}$ are given by

$$\boxed{\overline{F}_\theta(x, y) = \overline{G}_\theta\left(-\log B\left(e^{-x}, e^{-y}\right)\right).}$$

where, recalling that $\widehat{\phi}_{\widehat{K}}$ is a generator of the unique Archimedean copula, with Kendall distribution $\widehat{K}$,

$$\boxed{\overline{G}_\theta(x) := \widehat{\phi}_{\widehat{K}}^{-1}\left(\theta\, \varphi_B(e^- x)\right) \quad \theta > 0.}$$

(The proof is based on an extension of the transformation result by Genest & Rivest (2001))

A non Archimedean Example

$$\boldsymbol{B(u,v) = (uv)^{\alpha}\,(u \wedge v)^{1-\alpha}\,,}$$

with $\alpha \in (0,1)$, i.e. B is a Cuadras-Augé copula.

(Note that the Cuadras-Augé copula is not associative, and therefore is not Archimedean.)

$$\boldsymbol{\widehat{K}(t) = t - H\,t\log t,\ \text{with}\ H = \tfrac{2\,\alpha}{1+\alpha}.}$$

Then

$$\boldsymbol{\phi_B(t) = \widehat{\phi}_{\widehat{K}}(t) = const\,|\log t|^{1/H}},$$

and therefore necessarily the marginal distributions are exponential $Exp(\theta)$, and the compatible models are the ***exchangeable Marshall-Olkin models***:

Consider three independent, exponentially distributed, non-negative variables A, B, τ, where $A, B \sim Exp(\lambda), \tau \sim Exp(\mu)$ and set

$$X = \min(A, \tau), \quad Y = \min(B, \tau).$$

with $\alpha = \frac{\lambda}{\lambda+\mu}$ and $\theta = \lambda + \mu$.

OPEN PROBLEM:

WHAT HAPPENS IF WE DROP THE ASSUMPTION THAT $\widehat{K}(t^-) > t$?

CONJECTURE:

THE ASSOCIATIVE COPULAS TAKE THE ROLE OF ARCHIMEDEAN COPULAS (?)

IF WE DROP THE ASSUMPTION THAT $\boldsymbol{\varphi_B(0^+) = \infty}$, THEN THERE IS ONLY ONE MARGINAL DISTRIBUTION, WHICH IS STRICTLY POSITIVE ON $[0, \infty)$.

EXAMPLE??? (still Archimedean) Take $B(u, v)$ a Clayton semicopula, but not a copula:

$$B(u, v) = \left[(u^{\beta} + v^{\beta} - 1)^{+}\right]^{\frac{1}{\beta}} = \left[(u^{-b} + v^{-b} - 1)^{+}\right]^{-\frac{1}{b}}.$$

When $\beta > 1$, i.e. $b \leq -1$ it is not a copula, but it is a copula for $\beta \in (0, 1]$. The generator is $\varphi(u) = 1 - u^{\beta} = 1 - u^{-b}$