Continuous time random walks and queues: explicit forms and approximations of the conditional law with respect to their local times *

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Abstract

A filtering problem is considered in the case when the state process is a continuous time random walk and the observation process is its local time. An explicit construction of the filter is given. This construction is then applied to a suitable approximation of a Brownian motion and to a rescaled M/M/1 queueing model. In both these cases a convergence result for the respective filters is given. The case of a queueing model when the observation is the idle time is also considered.

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Contents

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1 Introduction

The kind of problems we are interested in arises from the following situation. Suppose that in a queue we can observe, up to time t, whether the queue is busy or idle, but we cannot observe the size of the queue, so that the observation process is the total time the queue has spent in 0, i.e. the so called *idle time* (see Prabhu [10]). Then the problem is to evaluate the size of the queue at time t, given this information, i.e. to compute the conditional law (or the filter) of the queue given the observation process up to time t. In the setup of heavy traffic limit, the rescaled queue converges to a reflected Brownian motion and the observation process converges to its local time. Then the limit model can be constructed as $(W_t + \Lambda_t, \Lambda_t)$, where W_t is a Brownian motion and Λ_t is the local time in the sense of Skorohod definition, that is

$$\Lambda_t = \ell_t(W),$$

where

$$\ell : D_{\mathbb{R}}[0,\infty) \to D_{\mathbb{R}}[0,\infty),\ x \to \ell(x), \quad \text{such that} \quad \ell_t(x) = -\inf_{s\le t} x(s) \wedge 0. \tag{1}$$

In the limit model the corresponding filtering problem is the computation of the conditional law of a reflected Brownian motion $W_t + \Lambda_t$ when the observation process is its local time Λ_t, i.e. the computation of the filter $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big]$, for g in a sufficiently large class of functions. A first problem is to find the exact expression for the filters, both for the limit model and for the rescaled queue model. A second problem concerns the convergence of the latter to $E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big]$.

The filter of the limit Brownian motion model is derived in G. Nappo, B. Torti [9] (Sections 4 and 6), where it is obtained by means of a suitable sequence of processes Λ^n approximating the observation process Λ. Each process Λ^n is proportional to a counting process, and therefore the nonlinear filtering techniques for counting processes are used. The filter can also be derived by means of the Azéma martingale, and this derivation is shortly discussed in [9]. For sake of completeness we recall its explicit expression.

Theorem 1.1. *Let W_t be a Brownian motion with diffusion coefficient a^2 and drift $c \in \mathbb{R}$ and let Λ_t be its local time. Let g be a bounded measurable function.*
Denote by

$$\Pi(s,l;g) = \int_0^\infty g\left(-l + y\sqrt{s}\right) y \exp\left(-\frac{y^2}{2}\right) dy \quad s \ge 0, \tag{2}$$

$$\Pi_{a^2,c}(s,l;g) = \frac{\Pi\left(a^2 s, l; g(\cdot)\exp(\frac{c}{a^2}\cdot)\right)}{\Pi\left(a^2 s, l; \exp(\frac{c}{a^2}\cdot)\right)} \tag{3}$$

$$\hat{\Pi}_{a^2,c}(s;g) = \Pi_{a^2,c}(s,0;g) \tag{4}$$

Then

$$\pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] = \Pi_{a^2,c}(\zeta_t, \Lambda_t; g), \tag{5}$$

and

$$\hat{\pi}_t(g) = E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big] = \hat{\Pi}_{a^2,c}(\zeta_t; g) \tag{6}$$

where $\mathcal{F}_t^\Lambda$ is the history generated by Λ_u up to time t, ζ_t is the elapsed time from the last visit to 0 for the process $W_t + \Lambda_t$, i.e.

$$\zeta_t = \gamma_t^0(W + \Lambda) = \gamma_t(\Lambda), \tag{7}$$

with

$$\gamma_t^0(x) = t - \sup\{s < t : x_s = 0\}. \tag{8}$$

$$\gamma_t(x) = t - \sup\{s < t : x_s < x_t\}. \tag{9}$$

Note that $\Pi_{1,0}(s,l;g) = \Pi(s,l;g) = E\Big[g(-l+W_s^*)/\Lambda_s^* = 0\Big]$, where W^* is any standard Brownian motion and Λ^* is its local time.

In this paper we start by considering a somehow simplified version of the motivating problem: we consider a continuous time random walk Y_t and its conditional law w.r.t. its local time $L_u = \ell_u(Y)$ up to time t. Indeed in several cases a queueing model can be represented as the reflection of a continuous time random walk. It turns out that the filter of Y_t w.r.t. L_u up to time t can be expressed as a probability measure depending deterministically on L_t and $\gamma_t(L)$, where γ_t is defined in (9). We derive also a more explicit expression for the filter under the assumption that the process Y_t can be decomposed as $Y_t = V_{Z_t}$, where Z_t is a renewal process, V_k is a discrete time random walk, with Z_t and V_k mutually independent. Similar results hold also for rescaled random walks.

We are interested in the situation when a sequence X_t^n of rescaled random walks converges to a Brownian motion W_t in $D_{\mathbb{R}}[0,+\infty)$. Then a continuous map argument applies to show that $(X_t^n, L_t^n) = (X_t^n, \ell_t(X^n))$ converges to $(W_t, \Lambda_t) = (W_t, \ell_t(W))$, and analogously the systems with the reflected random walk $(X_t^n + L_t^n, L_t^n)$ converge to the system with the reflected Brownian motion $(W_t + \Lambda_t, \Lambda_t)$ (see Section 5 for the details). In analogy with the second equality of (7) we denote

$$\xi_t^n = \gamma_t(L^n) = t - \sup\{u \leq t \quad \text{such that} \quad L_u^n < L_t^n\}. \tag{10}$$

the elapsed time from last jump time of L^n. Then for random walk systems the corresponding filter is

$$\pi_t^n(g) = E[g(X_t^n)/\mathcal{F}_t^{L^n}] = \Sigma^n(\xi_t^n, L_t^n; g) \tag{11}$$

with $\Sigma^n(s,l)$ defined in (23), while for the reflected systems is

$$\hat{\pi}_t^n(g) = E[g(X_t^n + L_t^n)/\mathcal{F}_t^{L^n}] = \hat{\Sigma}^n(\xi_t^n; g) \tag{12}$$

with $\hat{\Sigma}^n(s) = \Sigma^n(s,0)$.

The problem whether $\pi_t^n = \Sigma^n(\gamma_t(L^n), L_t^n)$ converge weakly to the filter $\pi_t = \Pi_{a^2,c}(\zeta_t, \Lambda_t)$ of the limit is strictly related to the convergence of the filters $\hat{\pi}_t^n = \hat{\Sigma}^n(\gamma_t(L^n))$ for the reflected random walks to $\hat{\pi}_t = \hat{\Pi}_{a^2,c}(\zeta_t)$. Depending on the model we privilege one or the other problem. On the other hand, from a computational point of view, it is quite difficult to use the exact expression of $\hat{\Sigma}^n(s)$ to compute the filter $\hat{\pi}_t^n$. Then it is also interesting to find a good approximation of the filter of the discrete system, depending on the actually observed process L^n, so that it can be used in applications. For the reflected random walk system, a natural choice in order to give a manageable approximation of the filter is to use the limit functional $\hat{\Pi}_{a^2,c}(s)$, evaluated at $s = \xi_t^n$.

These problems have been studied in more general situations by many authors, among which we recall in particular Bhatt et al. in [1] and Goggin ([6], [7]). Most of these results concern diffusive models and do not apply to our case. Moreover, usually the applications concern the problem to approximate a given signal/observation process with a suitably chosen sequence of signal/observation processes so that the corresponding sequence of filters converges to the the filter of the original process. We start from a different point of view: the sequence of processes is given and the problem is to show the convergence of the filters to the filter of the state/observation limit, in the sense specified above.

The problem of weak convergence of the sequence of filters is not a trivial problem as even strong convergence of random variables does not imply convergence of the conditional laws. This is clearly explained by the following simple and illuminating example (see E. Goggin [6]).
Let ξ be a real random variable, and $(\xi_n, \eta_n) = (\xi, \xi/n)$. Then (ξ_n, η_n) converges strongly to (ξ, η), with $\eta = 0$. Nevertheless, for any measurable function g, $E(g(\xi_n)/\eta_n) = g(\xi)$, so that the conditional law of ξ_n given η_n is the measure concentrated in $\xi(\omega)$, while $E(g(\xi)/\eta) = E(g(\xi))$, so that the conditional law of ξ given η coincides with the (deterministic) law $P \circ \xi^{-1}$ of ξ.
In this example, although the sequence of the conditional laws of ξ_n given η_n does not converge to the conditional law of the limit, it is a constant sequence and therefore is a converging sequence. This is not surprising indeed in the light of the next general result, which is a slight generalization of a result of E. Goggin [6] (proof of Theorem 2.1, Step 1): one has only to replace the sequence of $\sigma-$algebras used in [6] with a general sequence.

Lemma 1.2. *Let R_n be a sequence of random variables with values in a Polish space, let $\mathcal{H}^n$ be a sequence of $\sigma-$algebras, let α^n be a regular version of the conditional distribution of R_n given $\mathcal{H}^n$. If $\{R_n, n \in \mathbb{N}\}$ is tight, then $\{\alpha^n, n \in \mathbb{N}\}$ is tight.*

Then, as far as weak convergence is concerned, for the systems converging to a Brownian motion, the main problem is to check whether the limit points of the sequence of filters are all equal to the filter of a standard Brownian motion w.r.t. its local time.

The first model we consider is a non-Markovian queueing model arising when a Brownian motion W is approximated by a sequence of continuous time random walks W^n, obtained with a suitable interpolation procedure. The approximation scheme we propose for W follows some of the ideas used in [7] to study a filter approximation problem in diffusive models, and is related with the approximation scheme used in [9]. In this particular case we get a strong convergence result for the approximating filter.
The second model we consider is the case when the renewal process of the random walk is a Poisson process, so that the reflected random walk is an M/M/1 queue. The main results are weak convergence of the corresponding filters (see Theorem 5.1) and approximation in $L^p(\Omega \times [0,T])$-norm (see Theorem 5.2) and are based on the weak convergence of (ξ_t^n, L_t^n) to (ζ_t, Λ_t) (Proposition 5.3) and on the convergence of $\hat{\Sigma}^n(\cdot)$ to $\hat{\Pi}_{a^2,c}(\cdot)$, in the sense that

$$\lim_{n\to\infty} \hat{\Sigma}^n(s_n; g) = \hat{\Pi}_{2\lambda,c}(s; g), \tag{13}$$

for any g in a convergence determining class, whenever s_n converges to s, with $s > 0$. We first consider the case when the random walk is symmetric and so the drift of the limit Brownian motion is zero, and, under suitable conditions, we prove the above key convergence result (13) by using the reflection principle (see Section 5.1). Then we consider the general case in Section 5.2, where we prove (13) by reformulating the problem in terms of the symmetric random walk by using a suitable change of measure.
It is important to note that when we deal with a queue $Q_t^n = X_t^n + L_t^n$, i.e. with a reflected random walk, the previous results concern the filter w.r.t. the filtration

$$\mathcal{G}_t^n = \mathcal{F}_t^{L^n}$$

generated by the local time of the random walk, while the motivating problem concerns the filter of the queue w.r.t. the filtration

$$\mathcal{H}_t^n = \mathcal{F}_{t^+}^{C^n}$$

generated by the idle time C_t^n, i.e. the total time spent in 0 up to t. These two problems are strictly related, for instance (Q_t^n, C_t^n), as well as (Q_t^n, L_t^n), converges weakly to the reflected system $(W_t+\Lambda_t, \Lambda_t)$. This property, among others, allows to extend the previous convergence and approximation results to this situation (see Theorem 6.4 and Theorem 6.6 in the last section).

2 The model

Fix a probability space $(\Omega, \mathcal{F}, P)$ and consider on it a sequence $\{(T_j, U_j), j \geq 1\}$, satisfying the assumption

H0 The $\mathbb{R}^+ \times \{+1, -1\}$−valued random variables (T_j, U_j), for $j \geq 1$, are identically distributed and mutually independent.

Put

$$\tau_0 = 0,\ \ \tau_k = \sum_{j=1}^{k} T_j \ \text{ for } k \geq 1,$$

consider the renewal process

$$Z_t = \sum_{j=1}^{\infty} \mathbb{I}\left(\tau_j \leq t\right)$$

and the random walk $\{V_j, j \geq 0\}$ defined by

$$V_0 = 0,\ \ V_j = V_{j-1} + U_j, j \geq 1.$$

Finally, consider the continuous time random walk

$$Y_t = V_{Z_t} = \sum_{j=1}^{\infty} U_j \mathbb{I}\left(\tau_j \leq t\right) = \sum_{j \leq Z_t} U_j. \tag{14}$$

The solution of the Skorohod problem [1] for the process Y_t is given by the pair $(Y_t + L_t, L_t)$ where L_t is the *local time at level 0* for the process Y_t, that is

$$L_t = \ell_t(Y),$$

where ℓ is defined in (1).
It is easy to see that, for our model, L_t admits the representation

$$L_t = \sum_{j=1}^{\infty} \mathbb{I}\left(\sigma_j \leq t\right), \tag{15}$$

and that the sequence of its jump times $\{\sigma_j, j \geq 0\}$ is the subsequence of $\{\tau_j, j \geq 0\}$ defined by $\sigma_0 = 0$ and

$$\sigma_j = \inf\left\{\tau_k \quad \text{s.t.} \quad Y_{\tau_k} \leq -j\right\} = \inf\left\{t > 0 \quad \text{s.t.} \quad Y_t \leq -j\right\} \quad \text{for } j \geq 1.$$

[1] For any $x \in D_{\mathbb{R}}[0, \infty)$, with $x(0) \geq 0$ the pair $(z, v) = (x + \ell(x), \ell(x))$ is the unique solution of the Skorohod problem, i.e. is the unique pair of functions satisfying $z(t) = x(t) + v(t)$, and such that $z(t) \geq 0$, for all $t \geq 0$, $v(0) = 0$, v is nondecreasing and increases only when $z(t) = 0$.

Set

$$\mathcal{G}_t = \mathcal{F}_t^L = \sigma\left\{L_s, s \le t\right\},$$

obviously $\{\sigma_j, j \ge 0\}$ are stopping times w.r.t. both the histories $\mathcal{G}_t$ and $\mathcal{F}_t^Y$.

A first representation for the filter of Y_t given $\mathcal{G}_t$ is stated in the following Proposition.

Proposition 2.1. *Assume* **H0***. Then the conditional law of Y_t given $\mathcal{G}_t$ admits the following representation P-a.s.*

$$E\big[g(Y_t)/\mathcal{G}_t\big] = \sum_{j=0}^{\infty} \frac{E\Big[g(-j+Y_{s+\sigma_j}-Y_{\sigma_j})\mathbb{I}\left(S_{j+1}>s\right)\Big]_{s=t-\sigma_j}}{E\Big[\mathbb{I}(S_{j+1}>s)\Big]_{s=t-\sigma_j}}\mathbb{I}\big\{\sigma_j \le t < \sigma_{j+1}\big\}, \tag{16}$$

where $S_{j+1} = \sigma_{j+1} - \sigma_j$, $j \ge 0$.

Proof. The thesis can be proved by a slight modification of the argument in [9], Proposition 3.1. □

Remark 2.2. *The main ingredient in the proof of the previous Proposition is that condition* **H0** *implies that the process $Y_{s+\sigma_j} - Y_{\sigma_j}$ is independent of $\mathcal{G}_{\sigma_j}$ and is equal in law to the process Y_s. In its turn the last property implies that (16) admits the representation*

$$E\big[g(Y_t)/\mathcal{G}_t\big] = \sum_{j=0}^{\infty} \frac{E\Big[g(-j+Y_s^*)\mathbb{I}\left(\sigma_1^*>s\right)\Big]_{s=t-\sigma_j}}{E\Big[\mathbb{I}(\sigma_1^*>s)\Big]_{s=t-\sigma_j}}\mathbb{I}\big\{\sigma_j \le t < \sigma_{j+1}\big\}, \tag{17}$$

where Y_s^* is another birth and death process, with the same law as Y_t, defined by the rule (14) starting from a sequence $\{(T_j^*, U_j^*), \ge 1\}$ satisfying condition **H0***, and σ_1^* is its first visit time to the state -1.*
Then the problem reduces to the computation of the probability measure defined by

$$\frac{E\Big[g(-j+Y_s^*)\mathbb{I}\left(\sigma_1^*>s\right)\Big]}{E\Big[\mathbb{I}(\sigma_1^*>s)\Big]} = \frac{E\Big[g(-j+Y_s)\mathbb{I}\left(\sigma_1>s\right)\Big]}{E\Big[\mathbb{I}(\sigma_1>s)\Big]},$$

which has to be evaluated at $j = L_t$ and $s = \gamma_t(L)$.

The previous Remark implies also that the process L_t is a renewal process, with inter-arrival times $\{S_h, h \ge 1\}$. It is possible to represent them in terms of the sequences $\{T_i, i \ge 1\}$ and $\{M_i, i \ge 0\}$, where the sequence $\{M_i, i \ge 0\}$ is recursively defined by the rule

$$\begin{cases} M_0 = 0 \\ M_i = \inf\big\{k \ge 0 : V_{M_0+\ldots+M_{i-1}+k} - V_{M_0+\ldots+M_{i-1}} = -1\big\}. \end{cases} \tag{18}$$

Then

$$\sigma_0 = 0, \quad \sigma_h = \sum_{i=1}^{M_1+\ldots+M_h} T_i = \tau_{M_1+\ldots+M_h}, \qquad h \ge 1,$$

and

$$S_h = \sigma_h - \sigma_{h-1} = \sum_{i=M_1+\ldots+M_{h-1}+1}^{M_1+\ldots+M_h} T_i, \qquad h \geq 1.$$

Under Condition **H0** the sequence $\{M_i, i \geq 1\}$ is a sequence of i.i.d. random variables, moreover, for a suitable distribution function F and a suitable value of $p \in (0,1)$, the following conditions hold:

H1 the random variables $\{T_j, j \geq 1\}$ are non-negative, mutually independent, with common distribution function F,

H2 $\{U_j, j \geq 1\}$ is a sequence of i.i.d. random variables such that

$$P(U_j = 1) = p \quad P(U_j = -1) = 1 - p = q, \quad p \in (0,1).$$

Under the further condition

H3 the sequences $\{T_j, j \geq 1\}$ and $\{U_j, j \geq 1\}$ are mutually independent;

clearly also $\{T_i, i \geq 1\}$ and $\{M_i, i \geq 1\}$ are mutually independent. Next result provides a more explicit expression for the terms of the sum in (16).

Proposition 2.3. *Assume conditions* **H1**, **H2**, **H3**. *Then the filter* $E\big[g(Y_t)/\mathcal{G}_t\big]$ *admits the representation*

$$E\big[g(Y_t)/\mathcal{G}_t\big] = \\ = \sum_{j=0}^{\infty} \frac{\sum_{k=1}^{\infty} E\Big[\mathbb{I}\,(M_1 \geq k)\, g(-j + V_{k-1})\Big]\,(F_{k-1}(t-\sigma_j) - F_k(t-\sigma_j))}{\sum_{m=1}^{\infty} P\big(M_1 = m\big)\,(1 - F_m(t-\sigma_j))} \mathbb{I}\big\{\sigma_j \leq t < \sigma_{j+1}\big\} \qquad (19)$$

where F_k *is the distribution function of* τ_k *i e.* $F_k = F^{*k}$, *the* k*-fold convolution of* F.

Proof. As explained in Remark 2.2 we need only to compute $E\Big[\mathbb{I}(\sigma_1 > s)\Big]$, and $E\Big[g(-j+Y_s)\mathbb{I}\,(\sigma_1 > s)\Big]$, for all $j \geq 0$.
Taking into account the equality $\sigma_1 = \tau_{M_1}$, and the independence of the sequences $\{T_i, i \geq 1\}$ and $\{M_i, i \geq 0\}$

$$E\big[\mathbb{I}\,(\sigma_1 > s)\big] = E\big[\mathbb{I}\,(\tau_{M_1} > s)\big] = E\left[\sum_{m=1}^{\infty} \mathbb{I}\,(M_1 = m)\,\mathbb{I}\,(\tau_m > s)\right] = \sum_{m=j}^{\infty} P\big(M_1 = m\big)\big(1 - F_m(s)\big).$$

Finally, taking into account also (14),

$$\begin{aligned}
E\Big[g(-j+Y_s)\mathbb{I}\,(\sigma_1 > s)\Big] =& E\left[g\left(-j + V_{Z_s}\right)\mathbb{I}\,(\tau_{M_1} > s)\right] = E\left[\sum_{m=1}^{\infty} \mathbb{I}\,(M_1 = m)\, g\left(-j + V_{Z_s}\right)\mathbb{I}\,(\tau_m > s)\right] = \\
=& E\left[\sum_{m=1}^{\infty}\sum_{i=1}^{m} \mathbb{I}\,(M_1 = m)\, g\left(-j + V_{i-1}\right)\mathbb{I}\,(\tau_{i-1} \leq s < \tau_i)\right] = \\
=& \sum_{m=1}^{\infty}\sum_{i=1}^{m} E\Big[\mathbb{I}\,(M_1 = m)\, g(-j + V_{i-1})\Big] P\left(\tau_{i-1} \leq s < \tau_i\right) = \\
=& \sum_{i=1}^{\infty}\sum_{m=i}^{\infty} E\Big[\mathbb{I}\,(M_1 = m)\, g(-j + V_{i-1})\Big]\Big(F_{i-1}(s) - F_i(s)\Big) = \\
=& \sum_{i=1}^{\infty} E\Big[\mathbb{I}\,(M_1 \geq i)\, g(-j + V_{i-1})\Big]\Big(F_{i-1}(s) - F_i(s)\Big).
\end{aligned}$$

□

The state space of the process Y_t being discrete, the filter $E\big[g(Y_t)/\mathcal{G}_t\big]$ is determined by its discrete density $\nu_t(x),\ x \in \mathbb{Z}$, i.e. $\nu_t(x) = E\big[g(Y_t)/\mathcal{G}_t\big]$ with $g(z) = \mathbb{I}_{\{x\}}(z),\ z \in \mathbb{Z}$. Then the expression of $\nu_t(x)$ is

$$\nu_t(x) = \sum_{j=0}^{\infty} \frac{\sum_{k=1}^{\infty} P\left(V_{k-1} = x + j, M_1 \geq k\right)\left(F_{k-1}(t-\sigma_j) - F_k(t-\sigma_j)\right)}{\sum_{m=1}^{\infty} P\big(M_1 = m\big)\big(1 - F_m(t-\sigma_j)\big)} \mathbb{I}\{\sigma_j \leq t < \sigma_{j+1}\},$$

where

$$P\left(V_{k-1} = a, M_1 \geq k\right) =$$

$$= \begin{cases} \dfrac{a+1}{k}\dbinom{k}{\frac{k+a+1}{2}} p^{\frac{k+a-1}{2}} q^{\frac{k-a-1}{2}} & \text{if} \quad k+a-1 \quad \text{is even and} \quad |a| \leq k-1 \\ \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

3 Scaling

Let $\{\tilde{X}^n,\ n \in \mathbb{N}\}$ be a sequence of continuous time random walks defined in $(\Omega^n, \mathcal{F}^n, P^n)$. We assume that $\tilde{X}^n_t = \tilde{V}^n_{\tilde{Z}^n_t}$, where $\tilde{V}^n_k$ and $\tilde{Z}^n_t$ are defined as in Section 2 starting from a sequence $\{(\tilde{T}^n_j, \tilde{U}^n_j); j \geq 1\}$. For each n, we assume that the sequence $\{(\tilde{T}^n_j, \tilde{U}^n_j); j \geq 1\}$ satisfies the assumption **H0** stated in Section 2.

Consider the deterministic linear time-space scaling

$$X^n_t = b_n \tilde{X}^n_{a_n t},$$

where $\{a_n,\ n \in \mathbb{N}\}$ and $\{b_n,\ n \in \mathbb{N}\}$ are suitable sequences of real numbers. The local time $L^n_t = \ell_t(X^n)$ of the rescaled process X^n_t can be obtained just applying the same scaling to the process $\tilde{L}^n_t = \ell_t(\tilde{X}^n)$, i.e.

$$L^n_t = b_n \tilde{L}^n_{a_n t}.$$

We are interested in the conditional law of X^n_t w.r.t. $\mathcal{F}^{L^n}_t = \mathcal{F}^{\tilde{L}^n}_{a_n t}$, i.e. the filter

$$\pi^n_t(g) = E^{P^n}\left[g(X^n_t)/\mathcal{F}^{L^n}_t\right] = E^{P^n}\left[g(b_n \tilde{X}^n_{a_n t})/\mathcal{F}^{\tilde{L}^n}_{a_n t}\right],$$

where E^{P^n} denotes the expectation w.r.t. P^n. For the sake of notational convenience we will denote $\mathcal{F}^{L^n}_t$ as $\mathcal{G}^n_t$ and, when unnecessary, drop the symbol P^n in the expectation, so that the filter becomes

$$\pi^n_t(g) = E\big[g(X^n_t)/\mathcal{G}^n_t\big].$$

Following the same lines as in the previous section we get the representation

$$\pi^n_t(g) = \sum_{j=0}^{\infty} \frac{E\Big[g(-b_n j + X^{n*}_s)\mathbb{I}\left(\sigma^{n*}_1 > s\right)\Big]_{s=t-\sigma^n_j}}{E\Big[\mathbb{I}(\sigma^{n*}_1 > s)\Big]_{s=t-\sigma^n_j}} \mathbb{I}\left(\sigma^n_j \leq t < \sigma^n_{j+1}\right), \tag{21}$$

where $\{\sigma_j^n, j \geq 0\}$ is defined by

$$\sigma_j^n = \inf\{t > 0 \ \text{ s. t. } \ X_t^n \leq -b_n j\}$$

and represents a renewal process that coincides with the local time L_t^n up to a space scaling, and X^{n*} is a process with the same law as X^n and σ_1^{n*} is its first exit time from the set $(-b_n, \infty)$. Note that, when $t \in [\sigma_j^n, \ \sigma_{j+1}^n)$, the stopping time σ_j^n can be written as

$$\sigma_j^n = \sup\{u \leq t \quad \text{s. t.} \quad L_u^n < L_t^n\},$$

and therefore (21) can be rewritten as

$$\pi_t^n(g) = \frac{E\Big[g(-l + X_s^n)\mathbb{I}\,(\sigma_1^n > s)\Big]}{E\Big[\mathbb{I}(\sigma_1^n > s)\Big]}\Bigg|_{l=L_t^n,\ s=\xi_t^n}$$

where, ξ_t^n is defined in (10), i.e.

$$\xi_t^n = \gamma_t(L^n) = t - \sup\{u \leq t \quad \text{s. t.} \quad L_u^n < L_t^n\}.$$

Equivalently we can rewrite

$$\pi_t^n(g) = E\Big[g(-l + X_s^n)/\{L_s^n < b_n\}\Big]\Big|_{l=L_t^n,\ s=\xi_t^n}. \tag{22}$$

Then, denoting by $\Sigma^n(s,l)$ the probability measure such that

$$\Sigma^n(s,l;g) = \frac{E\Big[g(-l + X_s^n)\mathbb{I}\,(\sigma_1^n > s)\Big]}{E\Big[\mathbb{I}(\sigma_1^n > s)\Big]} \tag{23}$$

(21) can be shortly written as

$$\pi_t^n(g) = E\big[g(X_t^n)/\mathcal{G}_t^n\big] = \Sigma^n(\xi_t^n, L_t^n; g). \tag{24}$$

Remark 3.1. *Note that $\Sigma^n(s,l)$ depends also on the probability measure P^n, i.e. $\Sigma^n(s,l) = \Sigma^n_{P^n}(s,l)$. This dependence will be emphasized when necessary.*

By taking into account that

$$\hat{\pi}_t^n(g) = E\big[g(X_t^n + L_t^n)/\mathcal{G}_t^n\big] = E\big[g(X_t^n + m)/\mathcal{G}_t^n\big]\big|_{m=L_t^n}$$

the filter can be written as

$$\hat{\pi}_t^n(g) = E\big[g(X_t^n + L_t^n)/\mathcal{G}_t^n\big] = \hat{\Sigma}^n(\xi_t^n; g), \tag{25}$$

where

$$\hat{\Sigma}^n(s;g) = \Sigma^n(s,0;g) = \frac{E\Big[g(X_s^n)\mathbb{I}\,(\sigma_1^n > s)\Big]}{E\Big[\mathbb{I}(\sigma_1^n > s)\Big]}. \tag{26}$$

We end this Section by noting that, when the process $\tilde{X}^n = \tilde{V}^n_{\tilde{Z}^n_t}$ satisfies the assumptions **H1**, **H2** and **H3** stated in Section 2, then Proposition 2.3 easily provides the explicit expressions of $\Sigma^n(s,l)$ and $\hat{\Sigma}^n(s)$

$$\Sigma^n(s,l;g) = \frac{\sum_{k=1}^{\infty} E\Big[\mathbb{I}\,(M_1^n \geq k)\, g(b_n \tilde{V}^n_{k-1} - l)\Big]\big(F^n_{k-1}(s) - F^n_k(s)\big)}{\sum_{m=1}^{\infty} P\big(M_1^n = m\big)\big(1 - F^n_m(s)\big)} \tag{27}$$

$$\hat{\Sigma}^n(s;g) = \frac{\sum_{k=1}^{\infty} E\Big[\mathbb{I}\,(M_1^n \geq k)\, g(b_n \tilde{V}_{k-1}^n)\Big]\big(F_{k-1}^n(s) - F_k^n(s)\big)}{\sum_{m=1}^{\infty} P\big(M_1^n = m\big)\big(1 - F_m^n(s)\big)}, \tag{28}$$

where $M_1^n,\ F_k^n$ have a similar meaning as $M_1,\ F_k$ in (19). In particular it is interesting to note that if $\tilde{F}_1^n$ denotes the distribution function of

$$\tilde{\sigma}_1^n = \inf\Big\{t > 0 \ \text{ s. t. } \ \tilde{X}_t^n \leq -1\Big\},$$

then $\sigma_1^n = \tilde{\sigma}_1^n / a_n$, and therefore $F_1^n(s) = \tilde{F}_1^n(a_n s)$, and $F_k^n(s) = \tilde{F}_k^n(a_n s)$, where $\tilde{F}_k^n$ is the $k-$fold convolution of $\tilde{F}_1^n$.

The next three sections are devoted to the problem of finding the limit of the filter (24) for the random walk (or of the filter (25) for the reflected random walk), or some good approximation for it, in the case when the rescaled sequence X_t^n converges to a Brownian motion. For the above models we investigate whether the limit of the filter is the corresponding filter of the limits.

4 The interpolating Brownian motion model

In this Section we study the case of a continuous time random walk arising when a Brownian motion W is approximated with a sequence of processes W^n. The processes W^n are defined on the probability space of W, and are obtained pathwise by an interpolation procedure. For this model we are able to get a strong convergence result for the filter. We start by introducing the approximating model, then we show the convergence result. Successively we discuss how the approximating models W^n fall into the frame of the previous sections. In particular, when the process W is a standard Brownian motion the processes W^n corresponds to the case examined at the end of the previous section, with scaling parameters $a_n = 2^{2n}$ and $b_n = 1/2^n$, $p_n = q_n = 1/2$, $\tilde{F}_1^n = \tilde{F}_1$, where

$$\tilde{F}_1(t) = 4\sum_{j=0}^{+\infty}(-1)^j \frac{1}{\sqrt{2\pi}} \int_{\frac{(2j+1)}{\sqrt{t}}}^{+\infty} \exp\left(-\frac{x^2}{2}\right) dx. \tag{29}$$

The basic idea is to approximate the state W by the stepwise interpolation of the random points where W hits a uniform grid and consider as approximating observation the local time of the approximating state. This procedure is a deterministic one and therefore we describe it in the deterministic case.

Let $z \in D_{\mathbb{R}}[0,+\infty)$ and let $h \in \mathbb{R}^+$ be a fixed threshold. Consider the sequence $\{\widehat{\tau}_k^h(z), k \geq 0\}$

$$\begin{cases} \widehat{\tau}_0^h(z) = 0 \\ \widehat{\tau}_k^h(z) = \inf\big\{t > \widehat{\tau}_{k-1}^h(z) : |z(t) - z(\widehat{\tau}_{k-1}^h(z))| > h\big\} \quad , \quad k \geq 1, \end{cases} \tag{30}$$

and the function $z^h \in D_{\mathbb{R}}[0,+\infty)$

$$z^h(t) = \sum_{k=0}^{\infty} \mathbb{I}_{\big[\widehat{\tau}_k^h(z), \widehat{\tau}_{k+1}^h(z)\big)}(t) z(\widehat{\tau}_k^h(z)). \tag{31}$$

We need the following result.

Lemma 4.1. *Let $z \in D_{\mathbb{R}}[0,+\infty)$. Then, for each $t \in \mathbb{R}^+$*

$$|z^h(t) - z(t)| \leq h, \tag{32}$$

and

$$z^h \underset{h\to 0}{\longrightarrow} z \quad \textit{and} \quad \ell(z^h) \underset{h\to 0}{\longrightarrow} \ell(z)$$

w.r.t. the topology of the uniform convergence, where the functional ℓ is defined by (1).

Proof. Observing that $|z^h(t) - z(t)| = |z(\widehat{\tau}_k^h(z)) - z(t)|$, for $t \in \big[\widehat{\tau}_k^h(z), \widehat{\tau}_{k+1}^h(z)\big)$, and that $\widehat{\tau}_k^h(z) \uparrow \infty$, the bound (32) and the uniform convergence of z^h to z follow. The continuity of the functional ℓ implies the other limit. □

Remark 4.2. *When z is a continuous function, the process $\ell(z^h)$ admits the representation*

$$\ell_t(z^h) = 0 \vee (-z_0) - \sum_{j=0}^{\infty} z(\sigma_j^h(z)) \mathbb{I}_{[\sigma_j^h(z), \sigma_{j+1}^h(z))}(t),$$

where

$$\sigma_j^h(z) = \inf\big\{t \quad s.t. \quad z(t) - z_0 \le -jh\big\} = \inf\big\{\widehat{\tau}_k^h(z) \quad s.t. \quad z(\widehat{\tau}_k^h(z)) - z_0 \le -jh\big\}. \tag{33}$$

When furthermore $z_0 = 0$, then $z(\sigma_j^h(z)) = -jh$ and

$$\ell_t(z^h) = \sum_{j=0}^{\infty} jh \mathbb{I}(\sigma_j^h(z) \le t < \sigma_{j+1}^h(z)) = \sum_{j=0}^{\infty} h \mathbb{I}(\sigma_j^h(z) \le t). \tag{34}$$

We now apply this approximating procedure to the Brownian motion W_t.

The local time Λ_t of the process W_t is $\Lambda_t = \ell_t(W)$, where ℓ is the deterministic functional defined by (1). Observe that $W_0 = 0$ implies

$$\Lambda_t = -\inf_{0\le s\le t} \big(W_s\big). \tag{35}$$

Fix now the sequence of thresholds $h_n = \frac{1}{2^n}$, and consider the stopping times $\tau_k^n := \widehat{\tau}_k^h(W)$, when using $h = h_n = \frac{1}{2^n}$ in (30). Then the approximating signal/observation process (W^n, Λ^n) is a $D_{\mathbb{R}^2}[0, +\infty)$-valued process, where

$$W_t^n = \sum_{k=0}^{\infty} W(\tau_k^n)\, \mathbb{I}_{\left[\tau_k^n, \tau_{k+1}^n\right)}(t), \quad \text{and} \quad \Lambda_t^n = \ell_t(W^n). \tag{36}$$

Note that Lemma 4.1 provides the following convergence result.

Lemma 4.3. *Let (W^n, Λ^n) be defined as in (36). Then, for each $t \in \mathbb{R}^+$*

$$|W_t^n - W_t| \le \frac{1}{2^n}, \tag{37}$$

and

$$(W^n, \Lambda^n) = (W^n, \ell(W^n)) \underset{n\to\infty}{\longrightarrow} (W, \Lambda) = (W, \ell(W)) \quad a.s.,$$

w.r.t. the topology of the uniform convergence.

By (34)

$$\Lambda_t^n = \sum_{j=0}^{\infty} \frac{1}{2^n} \mathbb{I}(\sigma_j^n \le t), \tag{38}$$

where

$$\sigma_j^n = \inf\big\{t \quad \text{s.t.} \quad W_t \le -\frac{j}{2^n}\big\} = \inf\big\{t \quad \text{s.t.} \quad \Lambda_t \ge \frac{j}{2^n}\big\}. \tag{39}$$

Moreover, with the above choice of the threshold, the n-th grid is generated by considering the dyadic intervals of rank n. Then in the passage from the n-th grid to the $(n+1)$-th grid each threshold is split into two parts, and therefore $\sigma_{2j}^{n+1} = \sigma_j^n$. This property is decisive since it guarantees that, for any t, $\{\mathcal{G}_t^n = \mathcal{F}_t^{\Lambda^n}, n \in \mathbb{N}\}$ is an increasing family of σ-algebras, with $\mathcal{G}_t^n \uparrow \mathcal{F}_t^\Lambda$ (see Lemma 2.3 of [9], where the process Λ_t^n is defined as in (38)). The last fact allows us to show the claimed strong convergence result, which is a slight generalization of Theorem 2.4 of [9].

Theorem 4.4. *Let $g \in C_b(\mathbb{R})$ uniformly continuous. Then*

$$\pi_t^n(g) = E\big[g(W_t^n)/\mathcal{G}_t^n\big] \to \pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big], \quad \textit{a.s. and in } L^1, \tag{40}$$

where $E\big[g(W_t)/\mathcal{F}_t^\Lambda\big]$ is given in (5).

Proof. Observe that

$$\Big|E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] - E\big[g(W_t^n)/\mathcal{G}_t^n\big]\Big| \le \Big|E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] - E\big[g(W_t)/\mathcal{G}_t^n\big]\Big| + \\ + \Big|E\big[g(W_t)/\mathcal{G}_t^n\big] - E\big[g(W_t^n)/\mathcal{G}_t^n\big]\Big|.$$

As in Theorem 2.4 of [9], we apply Doob's convergence Theorem to the discrete time martingale $E\big[g(W_t)/\mathcal{G}_t^n\big]$ and get

$$E\big[g(W_t)/\mathcal{G}_t^n\big] \to E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] \qquad \text{a.s. and in } L^1, \tag{41}$$

and therefore the first term converges to zero. As far as the second term is concerned, we have

$$\Big|E\big[g(W_t)/\mathcal{G}_t^n\big] - E\big[g(W_t^n)/\mathcal{G}_t^n\big]\Big| \le \Big|E\big[\big|g(W_t) - g(W_t^n)\big|/\mathcal{G}_t^n\big]\Big| \le \omega_g\left(1/2^n\right) \to 0,$$

where $\omega_g(\delta) = \sup_{|x-y|\le\delta} \big|g(x) - g(y)\big|$ is the modulus of continuity of g. □

If we consider the space of probability measures on $\mathbb{R}$ endowed with the topology of convergence in distribution, then (40) states that the conditional laws π_t^n converge a.s. to the conditional law π_t defined by (5).
Indeed the problem is that apparently the set of probability zero depends on g. Nevertheless in $\mathbb{R}$ it is possible to find a countable convergence determining class. For example the piecewise linear functions $\{f_{s,r}(y), s \le r \in \mathbb{Q}\}$, with $f_{s,r}(y) = 0$ for $y \le s$, $f_{s,r}(y) = 1$ for $y \ge r$. This can be seen directly or by using Theorem 4.5 in [4].
One can get the analogous convergence results for the corresponding queueing model generated by reflecting W^n. In particular the conditional laws defined by $E\big[g(W_t^n + \Lambda_t^n)/\mathcal{G}_t^n\big]$ converge a.s. to the conditional law $\hat{\pi}_t$ defined by (6).

Now we show that the approximating model falls into the frame of the previous sections. The sequence $\{(\tilde{T}_k^n, \tilde{U}_k^n),\, k \ge 1\}$ is defined as

$$\tilde{T}_k^n = (\tau_k^n - \tau_{k-1}^n)/2^{2n}, \qquad \tilde{U}_k^n = 2^n\big(W_{\tau_k^n} - W_{\tau_{k-1}^n}\big),$$

which clearly satisfies condition **H0**. Then

$$W_t^n = \frac{1}{2^n} \tilde{V}^n_{\tilde{Z}^n_{2^{2n}t}},$$

where $\tilde{Z}^n_t$ is the renewal process defined by the sequence of i.i.d. interarrival times $\{\tilde{T}^n_k,\ k \in \mathbb{N}\}$, and $\tilde{V}^n_k = 2^n W_{\tau^n_k}$. Therefore, recalling (24), $E\big[g(W^n_t)/\mathcal{G}^n_t\big] = \Sigma^n(\gamma_t(W^n), \ell_t(W^n); g)$, with $\Sigma^n(s, l; g)$ as in (26), and γ_t as in (9). Similar result holds also for $E\big[g(W^n_t + \ell_t(W^n))/\mathcal{G}^n_t\big]$.

In addition the strong convergence of Theorem 4.4 implies the weak convergence for the filters of any rescaled model X^n sharing the same law as W^n. As an example we can take $\tilde{X}^n = W^0$ for all n, and $X^n_t = \frac{1}{2^n} W^0_{2^{2n}t}$.

When the drift coefficient of W is zero the random variables $\tilde{T}^n_k$ have common law (see for example [5] page 342)

$$F^n(t) = P\big(\tilde{T}^n_k \le t\big) = \tilde{F}_1(2^{2n} a^2 t) \tag{42}$$

where $\tilde{F}_1$ is defined in (29), and $\tilde{U}^n_k$ is symmetric and independent of $\tilde{T}^n_k$, for each k. Therefore $\tilde{V}^n_k = 2^n W_{\tau^n_k}$ is a symmetric random walk, independent of the renewal process $\tilde{Z}^n_t$. Then Conditions **H1**, **H2**, **H3** hold and one can use (27) to define $\Sigma^n(s, l)$.
When the drift coefficient is $c \neq 0$ the processes $\tilde{Z}^n_t$ and $\tilde{V}^n_k$, defined as above, are not mutually independent, then one cannot use (27). Nevertheless (27), with F^n as in (42) above, could be used to get the approximate expression

$$\tilde{\pi}^n_t(g) = \frac{E^{P_0}\big[g(W^n_t)\exp(\frac{c}{a^2}W^n_t)/\mathcal{G}^n_t\big]}{E^{P_0}\big[\exp(\frac{c}{a^2}W^n_t)/\mathcal{G}^n_t\big]} = \left.\frac{\Sigma^n\left(s, l; g(\cdot)\exp(\frac{c}{a^2}\cdot)\right)}{\Sigma^n\left(s, l; \exp(\frac{c}{a^2}\cdot)\right)}\right|_{s=\gamma_t(W^n), l=\ell_t(W^n)}.$$

Indeed by Kallianpur Striebel formula and Girsanov Theorem

$$E\big[g(W^n_t)/\mathcal{G}^n_t\big] = \frac{E^{P_0}\big[g(W^n_t)\exp(\frac{c}{a^2}W_t)/\mathcal{G}^n_t\big]}{E^{P_0}\big[\exp(\frac{c}{a^2}W_t)/\mathcal{G}^n_t\big]}$$

where P_0 is equivalent to P, and under P_0 the process W has drift coefficient zero. Then

$$E\big[g(W^n_t)/\mathcal{G}^n_t\big] = \frac{E^{P_0}\big[g(W^n_t)\exp(\frac{c}{a^2}W^n_t)\exp(\frac{c}{a^2}(W_t - W^n_t))/\mathcal{G}^n_t\big]}{E^{P_0}\big[\exp(\frac{c}{a^2}W^n_t)\exp(\frac{c}{a^2}(W_t - W^n_t))/\mathcal{G}^n_t\big]},$$

Moreover, taking into account that $|W_t - W^n_t| \le \frac{1}{2^n}$, one can get that

$$\big|E\big[g(W^n_t)/\mathcal{G}^n_t\big] - \tilde{\pi}^n_t(g)\big| \le 4\exp\big(2\frac{|c|}{a^2}\frac{1}{2^n}\big)\frac{|c|}{a^2}\frac{1}{2^n}\|g\|_\infty. \tag{43}$$

5 The M/M/1 queueing model

In this Section we consider a random walk with exponential interarrival times, and the M/M/1 queue generated by reflecting the random walk. We use the techniques introduced in Section 2 to derive the filter of the M/M/1 queue (and therefore of the random walk) with respect to the local time of the random walk. Moreover, under a suitable set of conditions, which are related to the heavy traffic conditions, we get also the weak limit of the filter of the rescaled system (Theorem 5.1) and an approximation for the filter (Theorem 5.2). In order to do so, we need the key result (13) , which is proven in Proposition 5.5 of Subsection 5.1 in the case when the arrival intensity λ_n and the service potential μ_n of the M/M/1 queue are equal and constant in n (*the* *symmetric case*), and in Proposition 5.10 of Subsection 5.2 for the sequences of M/M/1 queues when $\{(\lambda_n, \mu_n), n \in \mathbb{N}\}$ satisfy conditions **C1**, **C2**, **C3** below (*the general case*). The approach of Subsection 5.2 is quite similar to the approach of Subsection 5.1 and does not require the knowledge of the previous results, but it is technically more complicated. This is the reason why, even if the results in the first Subsection can be obtained from the general case, we choose to discuss them separately.

The sequence of random walks we consider is defined by means of the same rule as in (14), namely for each $n \in \mathbb{N}$

$$\tilde{X}_t^n = \tilde{V}_{\tilde{Z}_t^n}^n = \sum_{j=1}^{\tilde{Z}_t^n} \tilde{U}_j^n,$$

where

A1 $\tilde{Z}_t^n$ is a Poisson process with intensity $(\lambda_n + \mu_n)$

A2 $\tilde{V}_j^n$ is defined by $\tilde{V}_j^n = \tilde{V}_{j-1}^n + \tilde{U}_j^n$, where $\{\tilde{U}_j^n, j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables with $P^n(\tilde{U}_k^n = +1) = \frac{\lambda_n}{\lambda_n+\mu_n}$ and $P^n(\tilde{U}_k^n = -1) = \frac{\mu_n}{\lambda_n+\mu_n}$.

A3 $\{\tilde{U}_k^n, k \in \mathbb{N}\}$ and $\tilde{Z}_t^n$ are mutually independent.

In this case the interarrival times $\tilde{T}_k^n$ of the renewal process $\tilde{Z}_t^n$ are exponential random variables with expectation $1/(\lambda_n + \mu_n)$. Therefore we are in the situation discussed at the end of Section 3, with $p_n = \lambda_n/(\lambda_n + \mu_n)$, $\tilde{F}_1^n$ the distribution function of an exponential random variable of parameter $\lambda_n + \mu_n$, moreover the scaling parameters are $a_n = n$ and $b_n = \sqrt{n}$, and then F_k^n is the distribution function Gamma of parameter $(k, n(\lambda_n + \mu_n))$.

The conditions **C1**, **C2**, and **C3** are defined as follows.

C1 $\lambda_n, \mu_n > 0$

C2 $(\lambda_n, \mu_n) \underset{n\to+\infty}{\to} (\lambda, \lambda)$

C3 $\sqrt{n}(\lambda_n - \lambda) \underset{n\to+\infty}{\to} c(1) \qquad \sqrt{n}(\mu_n - \lambda) \underset{n\to+\infty}{\to} c(2)$

Condition **C1** avoids to consider pure birth or pure death processes, and **C3** clearly implies condition **C2** and condition

C3* $\sqrt{n}(\lambda_n - \mu_n) \underset{n\to+\infty}{\to} c = c(1) - c(2)$.

We recall that, when $\lambda_n < \mu_n$, the set of conditions **C1**, **C2**, **C3*** are known in literature as the *heavy traffic conditions*, and in this case $c \leq 0$. These conditions guarantee the existence of the diffusive limit of the rescaled system, more precisely, the sequence of processes $X_t^n = \tilde{X}_{nt}^n/\sqrt{n}$ converges weakly in $D_{\mathbb{R}}[0, +\infty)$ to a Brownian motion W_t, with diffusion coefficient 2λ and drift coefficient c. The reason why we require the stronger condition **C3** will be clarified in Remark 5.8. Here we only point out that the above conditions are equivalent to require also the weak convergence of the processes $Z_t^n = (\tilde{Z}_{nt}^n - 2\lambda nt)/\sqrt{n}$.

The solution of the Skorohod problem for the process $\tilde{X}_t^n$ is the pair $(\tilde{Q}_t^n, \tilde{L}_t^n)$, where

$$\tilde{Q}_t^n = \tilde{X}_t^n + \tilde{L}_t^n$$

is a M/M/1 queue and $\tilde{L}_t^n$ is the local time for the process $\tilde{X}_t^n$ (see, for instance, [2]). Then, thanks to the weak convergence of X^n, a continuous map argument applies to show that

$$(X_t^n, Q_t^n, L_t^n) = \left(\frac{\tilde{X}_{nt}^n}{\sqrt{n}}, \frac{\tilde{Q}_{nt}^n}{\sqrt{n}}, \frac{\tilde{L}_{nt}^n}{\sqrt{n}} \right) \Rightarrow (W_t, W_t + \Lambda_t, \Lambda_t), \tag{44}$$

where $\Lambda_t = \ell_t(W)$ is the local time of W_t. Indeed the functional ℓ_t is continuous with respect to the topology of uniform convergence on bounded intervals of time, and X_t^n converges to W_t with respect to this topology, since W_t has continuous trajectories.

The main results are stated in the following theorems which are proven at the end of this section. We recall (see (11) and (12)) that π_t^n and $\hat{\pi}_t^n$ denote the filters of X_t^n and of Q_t^n given the filtration $\mathcal{G}_t^n$, respectively, where, as in Section 3, $\mathcal{G}_t^n$ denotes the filtration generated by L_t^n.

Theorem 5.1. *Assume conditions* **C1**, **C2**, **C3** *and* **A1**, **A2**, **A3**, *then, for any $t \geq 0$, π_t^n converge weakly to π_t, and $\hat{\pi}_t^n$ converge weakly to $\hat{\pi}_t$, as random variables with values in the space of probability measures endowed with the topology of weak convergence.*
In particular, for any $t \geq 0$, and for any bounded continuous function g

$$\pi_t^n(g) = E\big[g(X_t^n)/\mathcal{G}_t^n\big] \Rightarrow \pi_t(g) = E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] \tag{45}$$

and

$$\hat{\pi}_t^n(g) = E\big[g(Q_t^n)/\mathcal{G}_t^n\big] \Rightarrow \hat{\pi}_t(g) = E\big[g(W_t + \Lambda_t)/\mathcal{F}_t^\Lambda\big]. \tag{46}$$

As explained in the introduction it is interesting to find a good approximation for the filter $\hat{\pi}_t^n = \hat{\Sigma}^n(\xi_t^n)$ which is, at the same time, simpler to handle, and depends on the actually observed trajectory.
In the light of the key result (13), and in particular of Proposition 5.10, a natural candidate is

$$\hat{\Pi}_{2\lambda,c}(\xi_t^n), \tag{47}$$

where $\hat{\Pi}_{2\lambda,c}(s)$ is defined in (4).

We can show that (47) approximates the filter $E^{P^n}\big[g(Q_t^n)/\mathcal{G}_t^n\big]$ in the $L_p\left(\Omega \times [0,T]\right)$-norm, for each $T > 0$, $p > 0$.

Theorem 5.2. *Under the same assumption of Theorem 5.1, for all g bounded and continuous and for each $T > 0$, $p > 0$*

$$\int_0^T E\left|\hat{\Sigma}^n(\xi_t^n; g) - \hat{\Pi}_{2\lambda,c}(\xi_t^n; g)\right|^p dt \underset{n\to\infty}{\longrightarrow} 0.$$

The proof of Theorem 5.1, as well as the proof of Theorem 5.2, is based on the representations (24) and (25) for π_t^n and $\hat{\pi}_t^n$, respectively, and therefore we need a preliminary result concerning the weak convergence of $\xi_t^n = \gamma_t(L^n)$ to $\zeta_t = \gamma_t^0(W + \Lambda) = \gamma_t(\Lambda)$, with γ_t^0 and γ_t defined in (8) and (9) respectively.
We show a slightly stronger result concerning the weak convergence of $\gamma_t^0(X^n + L^n) = \gamma_t^0(Q^n)$ to ζ_t. This result is used later in Section 6.

Proposition 5.3. *Assume conditions* **C1**, **C2**, **C3***. *For each* $t > 0$

$$(\gamma_t^0(Q^n), \gamma_t(L^n), L_t^n) \Rightarrow (\zeta_t, \zeta_t, \Lambda_t).$$

Proof. Define

$$\eta_t^n = \sup\{s < t : L_s^n < L_t^n\}, \quad \eta_t = \sup\{s < t : \Lambda_s < \Lambda_t\},$$
$$\beta_t^n = \sup\{s < t : \quad Q_s^n = 0\}, \quad \beta_t = \sup\{s < t : W_s + \Lambda_s = 0\},$$

with $\eta_t^n = t$, $\eta_t = t$, $\beta_t^n = t$ and $\beta_t = t$ if the corresponding set is empty.
Note that

$$\eta_t^n = \sup\{s < t : X_t^n - X_s^n < Q_t^n - Q_s^n\},$$
$$\eta_t \ = \sup\{s < t : W_t - W_s < W_t + \Lambda_t - W_s - \Lambda_s\},$$

Applying the Skorohod representation theorem, we can assume that all the processes live on the same probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, and that

$$\sup_{s\leq t}(|X_s^n - W_s| + |Q_s^n - W_s - \Lambda_s|) \to 0 \quad \bar{P}\text{-a.s.} \tag{48}$$

This implies that $L^n \to \Lambda$ uniformly in $[0, t]$, $\bar{P}$-a.s., and

$$\liminf_{n\to\infty} \ \eta_t^n \geq \eta_t.$$

Then, since

$$\gamma_t^0(Q^n) = t - \beta_t^n, \qquad\qquad \gamma_t(L^n) = t - \eta_t^n,$$

the result is achieved once we prove that the sequence (β_t^n, η_t^n) converges $\bar{P}$-a.s. to (η_t, η_t) and $\zeta_t = t - \eta_t$.
Let $\beta_t^\infty = \limsup_{n\to\infty} \beta_t^n$ and note that $Q^n(\beta_t^n)$ assumes only the values 0 or $\frac{1}{\sqrt{n}}$.
Then, by (48), $W_{\beta_t^\infty} + \Lambda_{\beta_t^\infty} = 0$. It follows that

$$\limsup_{n\to\infty} \beta_t^n \leq \beta_t.$$

Moreover, if $\eta_t^n < t$, then $Q^n(\eta_t^n) = 0$, if $\eta_t^n = t$, then $\beta_t^n = t$, and it follows that

$$\eta_t^n \leq \beta_t^n \qquad\qquad \text{for all } t, \ \bar{P}\text{-a.s.}$$

and then

$$\eta_t \leq \liminf_{n\to\infty} \ \eta_t^n \leq \limsup_{n\to\infty} \beta_t^n \leq \beta_t, \qquad\qquad \text{for all } t, \ \bar{P}\text{-a.s.}$$

The proof is achieved since

$$\bar{P}(\eta_t = \beta_t) = 1$$

and then $\zeta_t = \gamma_t^0(W + \Lambda) = t - \beta_t = t - \eta_t = t - \gamma_t(\Lambda)$. For sake of completeness we prove the previous statement. Note that $W_s + \Lambda_s = 0$ if and only if $W_s = \inf_{r<s} W_r \leq 0$. If $\eta_t < \beta_t$, then for $\eta_t \leq s \leq \beta_t$,

$$W_t - W_s = W_t + \Lambda_t - W_s - \Lambda_s \leq W_t + \Lambda_t,$$

and for any rational r satisfying $\eta_t < r < \beta_t$, we must have

$$\inf_{s\leq r} W_s = \inf_{r\leq s\leq t} W_s.$$

But $\bar{P}\{\inf_{s\leq r} W_s = \inf_{r\leq s\leq t} W_s\} = 0$, and hence

$$\bar{P}\{\eta_t < \beta_t\} \leq \sum_{r\in\mathbb{Q}} \bar{P}\{\inf_{s\leq r} W_s = \inf_{r\leq s\leq t} W_s\} = 0.$$

□

Remark 5.4. *In the Skorohod space $(\bar\Omega, \bar{\mathcal{F}}, \bar P)$ used in the proof of Proposition 5.3, choose a jointly measurable version of $\xi^n_t = \gamma_t(L^n) = t - \eta^n_t$. Then $M = \big\{(\omega, t) \in \Omega \times [0,T] \;\; s.t.\; \xi^n_t(\omega) \not\to \zeta_t(\omega)\big\}$ is a zero $d\bar P \times dt$-measure set. Moreover a similar result holds for $\gamma^0_t(Q^n) = t - \beta^n_t$, namely $M_0 = \big\{(\omega, t) \in \Omega \times [0,T] \;\; s.t.\; \gamma^0_t(Q^n)(\omega) \not\to \zeta_t(\omega)\big\}$ is a zero $d\bar P \times dt$-measure set.*

Lemma 1.2 combined with (44), guarantees the sequences π^n_t and $\hat\pi^n_t$ to be tight. In order to obtain their weak limits, it is crucial to represent $E\big[g(X^n_t)/\mathcal{G}^n_t\big]$ and $E\big[g(Q^n_t)/\mathcal{G}^n_t\big]$ as in (24) and (25), where they are written as the deterministic functionals $\Sigma^n(s, l : g)$ and $\hat\Sigma^n(s; g)$ evaluated at $s = \xi^n_t$ and $l = L^n_t$.
Similarly, also the filters $\pi_t(g) = E\big[g(W_t)/\mathcal{F}^\Lambda_t\big]$ and $\hat\pi_t(g) = E\big[g(W_t + \Lambda_t)/\mathcal{F}^\Lambda_t\big]$ for the limit model are given as the compositions of the deterministic functionals $\Pi_{2\lambda,c}(s, l; g)$ and $\hat\Pi_{2\lambda,c}(s; g)$ evaluated at $(s, l) = (\zeta_t, \Lambda_t)$ and $s = \zeta_t$, respectively (see Theorem 1.1), and moreover Proposition 5.3 proves that $(\xi^n_t, L^n_t) \Rightarrow (\zeta_t, \Lambda_t)$.
These observations induce to investigate whether the sequences of probability measures $\hat\Sigma^n(s)$ and $\Sigma^n(s, l)$ (see (26) and (23)) converge to $\hat\Pi_{2\lambda,c}(s)$ and $\Pi_{2\lambda,c}(s, l)$ respectively. In Proposition 5.5 and Proposition 5.10 we show the convergence of $\hat\Sigma^n(s; g)$ to $\hat\Pi_{2\lambda,c}(s; g)$ for the symmetric case and for the general case, respectively. The convergence is uniform on bounded intervals contained in $(0,\infty)$, indeed we prove that, for every g in a convergence determining class, and whenever s_n converges to s, with $s > 0$,

$$\lim_{n\to\infty} \hat\Sigma^n(s_n; g) = \hat\Pi_{2\lambda,c}(s; g), \tag{49}$$

which is exactly the key result (13). Then the previous limit is valid for any bounded and continuous function g.
We now have all the ingredients for the proof of the main results of this section.

Proof of Theorem 5.1
The weak convergence for filters of the reflected random walk follows since we can use the Skorohod representation probability space as in Proposition 5.3, and in this space, for each $t > 0$, $\bar P(\xi^n_t = \gamma_t(L^n) \to \zeta_t) = 1$, and, on the other hand $\bar P\{\omega : \;\zeta_t(\omega) > 0\} = 1$, as observed in Remark 5.4. As a consequence, taking into account (49),

$$\bar P\big(\hat\Sigma^n(\xi^n_t; g) \underset{n\to\infty}{\longrightarrow} \hat\Pi_{2\lambda,c}(\zeta_t; g), \quad \text{for every } g : \mathbb{R}^+ \to \mathbb{R} \text{ bounded and continuous}\big) = 1,$$

and the above property is equivalent to show that $\hat\pi^n_t$ converges weakly to $\hat\pi_t$.
The proof of the weak convergence for the filter of the random walk is similar, since the convergence of $\Sigma^n(s_n, l_n; g)$ to $\Pi_{2\lambda,c}(s, l; g)$ whenever (s_n, l_n) converges to (s, l), with $s > 0$ is just a slight extension of the previous result (49), that is for any $g : \mathbb{R}^+ \to \mathbb{R}$ bounded and uniformly continuous,

$$\Sigma^n(s_n, l_n; g) \underset{n\to\infty}{\longrightarrow} \Pi_{2\lambda,c}(s, l; g). \tag{50}$$

Indeed $|\Sigma^n(s_n, l_n; g) - \Sigma^n(s_n, l; g)| \le \omega_g(|l_n - l|)$, where $\omega_g(\delta) = \sup_{|x-y|\le\delta} |g(x) - g(y)|$ is the modulus of continuity of g, and by (49) $\Sigma^n(s_n, l; g) = \hat\Sigma^n(s_n; g_l)$ converge to $\Pi(s, l; g) = \hat\Pi(s; g_l)$, where $g_l(x) = g(-l+x)$. The set of bounded and uniformly continuous functions is a convergence determining class and then (50) follows for bounded continuous functions g. Then, using again the Skorohod representation space, we get that $\bar P((\xi^n_t, L^n_t) \to (\zeta_t, \Lambda_t), \zeta_t > 0) = 1$, and

$$\bar P\big(\Sigma^n(\xi^n_t, L^n_t; g) \underset{n\to\infty}{\longrightarrow} \Pi_{2\lambda,c}(\zeta_t, \Lambda_t; g), \quad \text{for every } g : \mathbb{R}^+ \to \mathbb{R} \text{ bounded and continuous}\big) = 1.$$

Therefore π^n_t converge weakly to π_t, and Theorem 5.1 is completely achieved.

□

Proof of Theorem 5.2

The limit we are looking for depends only on the distribution of ξ_t^n, therefore using the Skorohod representation space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ as in the proof of Proposition 5.3, the thesis is equivalent to

$$\int_0^T E^{\bar{P}} \left| \hat{\Sigma}^n(\xi_t^n; g) - \hat{\Pi}_{2\lambda,c}(\xi_t^n; g) \right|^p dt \underset{n\to\infty}{\longrightarrow} 0.$$

As observed in Remark 5.4, we can assume that

$$\xi_t^n(\omega) \to \zeta_t(\omega) \quad d\bar{P} \times dt - \text{a.e.}$$

Proposition 5.10 implies that

$$\hat{\Sigma}^n(s_n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(s; g), \tag{51}$$

whenever $s_n \to s$, $s > 0$, moreover it is clear that

$$\hat{\Pi}_{2\lambda,c}(s_n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(s; g),$$

whenever $s_n \to s$, $s > 0$.
Combining these results we get

$$\hat{\Sigma}^n(\xi_t^n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(\zeta_t; g) \qquad \text{and} \qquad \hat{\Pi}_{2\lambda,c}(\xi_t^n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(\zeta_t; g) \tag{52}$$

for each (ω, t) such that $\zeta_t(\omega) > 0$.

The observation that $\{(\omega, t) \in \Omega \times [0, T]$ such that $\zeta_t(\omega) = 0\}$ is a zero measure set with respect to $d\bar{P} \times dt$, and an easy application of the dominated convergence theorem imply that

$$\int_0^T E^{\bar{P}} \left[\left| \hat{\Sigma}^n(\xi_t^n; g) - \hat{\Pi}_{2\lambda,c}(\zeta_t; g) \right|^p \right] \to 0, \qquad \text{for any } p > 0$$

and

$$\int_0^T E^{\bar{P}} \left[\left| \hat{\Sigma}^n(\xi_t^n; g) - \hat{\Pi}_{2\lambda,c}(\xi_t^n; g) \right|^p \right] \to 0, \qquad \text{for any } p > 0. \tag{53}$$

□

5.1 The symmetric case

In this subsection we assume that

$$\lambda_n = \mu_n = \lambda. \tag{54}$$

Therefore, without loss of generality, we can assume that $\tilde{Z}_t^n = \tilde{Z}_t$ and $\tilde{V}_k^n = \tilde{V}_k$, where $\tilde{Z}_t$ is a Poisson process with intensity $2\,\lambda$ and $\tilde{V}_k$ is a symmetric random walk, defined on a suitable space $(\Omega, \mathcal{F}, P)$. Then

$$\tilde{X}_t^n = \tilde{X}_t = \tilde{V}_{\tilde{Z}_t},$$

and, with this position, the process $X_t^n = \tilde{X}_{nt}/\sqrt{n}$ depends on the index n just by the time-space scaling, moreover conditions **C1**, **C2** and **C3** are satisfied with $c(1) = c(2) = 0$, and consequently $c = 0$.
In this case the process W_t defined by (44) is a driftless Brownian motion with diffusion coefficient 2λ and Λ_t is its local time.
Without loss of generality we can assume $\lambda = \frac{1}{2}$. Otherwise we can use the deterministic change of time $t/(2\lambda)$ instead of t and consider the sequence of processes $X^n_{t/(2\lambda)}$, which converges to a standard Brownian motion.

For the sake of notational convenience we denote by $\hat{\Pi}(s)$ the probability measure defined by

$$\hat{\Pi}(s;g) = \hat{\Pi}_{1,0}(s;g) = \Pi(s,0;g) = \int_0^\infty g(y\sqrt{s})y\exp\left(-\frac{y^2}{2}\right)dy \tag{55}$$

for each $g : \mathbb{R}^+ \to \mathbb{R}$ bounded and continuous, where $\Pi(s,l)$ and $\hat{\Pi}_{1,0}(s)$ are defined in Theorem 1.1.

Proposition 5.5. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a bounded continuous function. Under the assumption (54), for each s,*

$$\hat{\Sigma}^n(s;g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}(s;g). \tag{56}$$

Moreover for any sequence s_n converging to $s > 0$

$$\hat{\Sigma}^n(s_n;g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}(s;g). \tag{57}$$

In order to prove Proposition 5.5 we need a preliminary result.

Lemma 5.6. *For each $s > 0$ and $n \in \mathbb{N}$, let F_s^n be the distribution function of the random variable X_s^n. Then*

$$F_s^n(x) - \Phi\left(\frac{x}{\sqrt{s}}\right) = o\left(\frac{1}{\sqrt{ns}}\right), \quad \textit{uniformly in } x \in \Gamma_n, \tag{58}$$

where

$$\Gamma_n = \{\frac{1}{\sqrt{n}}\left(z+\frac{1}{2}\right), z \in \mathbb{Z}\},$$

and $\Phi(x)$ is the distribution function of a standard normal random variable.

Proof. The random variable X_s^n can be written as follows

$$X_s^n = \frac{1}{\sqrt{n}}\sum_{k=0}^{n}\Delta_s^k,$$

where $\{\Delta_s^k = \tilde{X}_{ks} - \tilde{X}_{(k-1)s},\ k \in \mathbb{N}\}$. Under the assumption stated at the beginning of this Subsection, the sequence $\{\Delta_s^k,\ k \in \mathbb{N}\}$ is a sequence of i.i.d. symmetric random variables, with characteristic function $\exp\{s(\cos(u)-1)\}$, with first and third moments equal to zero and $Var\left(\Delta_s^1\right) = s$. Moreover their common distribution function F_s is concentrated on the lattice Γ_n, as well as the distribution function F_s^n. Then, since the third moments of Δ_s^k are zero, by Theorem 2, in [5], Chapter XVI.4 we get

$$\hat{F}_s^n(x) - \Phi(x) = o\left(\frac{1}{\sqrt{n}}\right), \quad \text{uniformly in } x \text{ s.t.} \frac{x}{\sqrt{s}} \in \Gamma_n. \tag{59}$$

where $\hat{F}_s^n(x) = F_s^n(x\sqrt{s})$ is the distribution function of $\frac{X_s^n}{\sqrt{s}}$.
The expansion (59) does not imply immediately the thesis, nevertheless (58) can be proved by repeating and adapting the proof to the case when the limit distribution function is $\Phi\left(\frac{x}{\sqrt{s}}\right)$ instead of $\Phi(x)$. □

Remark 5.7. *Let $s_n \to s$ and $s > 0$, so that there exists $\bar{n}$ such that $s_n \geq \frac{1}{2}s$, for any $n > \bar{n}$. Then, under the assumptions of Lemma 5.6*

$$F_{s_n}^n(x) - \Phi\left(\frac{x}{\sqrt{s_n}}\right) = o\left(\frac{1}{\sqrt{n}}\right), \qquad \text{uniformly in } x \in \Gamma_n. \tag{60}$$

Proof of Proposition 5.5. We prove in detail only (56). The proof of (57) for the case of a sequence s_n converging to $s > 0$, is treated in a similar way, by using Lemma 5.6 and Remark 5.7.

When $s = 0$ (56) is trivial, since clearly $\hat{\Pi}(0; g) = \hat{\Sigma}^n(0; g) = g(0)$. When $s > 0$, taking into account (22), we note that $\hat{\Sigma}^n(s)$ is also the conditional law of X_s^n given the event $\{L_s^n < \frac{1}{\sqrt{n}}\}$. Then

$$\hat{\Sigma}^n(s; g) = \frac{\Theta^n(s; g)}{\Theta^n(s; 1)},$$

where 1 denotes the constant function $1(x) = 1$, and where $\Theta^n(s)$ is the measure defined by

$$\Theta^n(s; g) = E\left[g(X_s^n)\mathbb{I}\{L_s^n < \frac{1}{\sqrt{n}}\}\right].$$

We can restrict to the functions $g \in C^1(\mathbb{R})$ with $g'(0) = 0$ since this class is convergence determining. Without loss of generality we can also assume that $g(0) = 0$. In fact

$$\hat{\Sigma}^n(s; g) = \hat{\Sigma}^n(s; g - g(0)) + g(0).$$

By an explicit computation we get

$$\Theta^n(s; g) = \sum_{k=0}^{\infty} g\left(\frac{k}{\sqrt{n}}\right) P\left(X_s^n = \frac{k}{\sqrt{n}}, \min_{0 \leq u \leq s} X_s^n > \frac{-1}{\sqrt{n}}\right).$$

Moreover, the reflection principle yields

$$\begin{aligned} P\left(X_s^n = \frac{k}{\sqrt{n}}, \min_{0 \leq u \leq s} X_s^n > \frac{-1}{\sqrt{n}}\right) =& P\left(X_s^n = \frac{k}{\sqrt{n}}\right) - P\left(X_s^n = -\frac{k+2}{\sqrt{n}}\right) = \\ =& P\left(X_s^n = \frac{k}{\sqrt{n}}\right) - P\left(X_s^n = \frac{k+2}{\sqrt{n}}\right), \end{aligned}$$

and so

$$\begin{aligned} \Theta^n(s; g) =& \sum_{k=0}^{\infty} g\left(\frac{k}{\sqrt{n}}\right) \left[P\left(X_s^n = \frac{k}{\sqrt{n}}\right) - P\left(X_s^n = \frac{k+2}{\sqrt{n}}\right)\right] = \\ =& E\left[g(X_s^n)\mathbb{I}\{X_s^n \geq 0\} - g(X_s^n - \frac{2}{\sqrt{n}})\mathbb{I}\{X_s^n - \frac{2}{\sqrt{n}} \geq 0\}\right]. \end{aligned} \tag{61}$$

Set $\tilde{g} : \mathbb{R} \to \mathbb{R}$, $x \to \tilde{g}(x) = g(x)\mathbb{I}(x \geq 0)$. Then

$$\Theta^n(s;g) = E\left[\tilde{g}(X^n_s) - \tilde{g}(X^n_s - \frac{2}{\sqrt{n}})\right] = \frac{2}{\sqrt{n}} E\left[\tilde{g}'(X^n_s - \theta\frac{2}{\sqrt{n}})\right],$$

where θ is a $(0,1)$-valued random variable.
Then the sequence of measures $\hat{\Sigma}^n(s)$ admits the following representation

$$\hat{\Sigma}^n(s;g) = E\left[\tilde{g}'(X^n_s - \theta\frac{2}{\sqrt{n}})\right] \frac{2}{\sqrt{n}} \frac{1}{\Theta^n(1)(s)}, \tag{62}$$

for each $g \in C^1(\mathbb{R})$ with $g'(0) = 0$.
On the other hand

$$\lim_{n\to\infty} E\left[\tilde{g}'(X^n_s - \theta\frac{2}{\sqrt{n}})\right] = E\left[\tilde{g}'(W_s)\right], \tag{63}$$

and an explicit computation yields

$$E\left[\tilde{g}'(W_s)\right] = \frac{1}{\sqrt{2\pi s}} \int_0^\infty g'(y) \exp\left(-\frac{y^2}{2s}\right) dy = \frac{1}{\sqrt{2\pi s}} \left(-g(0) + \int_0^\infty g(y)\frac{y}{s} \exp\left(-\frac{y^2}{2s}\right) dy\right).$$

By the change of variable $x = \frac{y}{\sqrt{s}}$ in the integral, and recalling that $g(0) = 0$ we get

$$E\left[\tilde{g}'(W_s)\right] = \frac{1}{\sqrt{2\pi s}} \left(\int_0^\infty g(\sqrt{s}x)x \exp\left(-\frac{x^2}{2}\right) dx\right) = \frac{1}{\sqrt{2\pi s}} \hat{\Pi}(s;g).$$

Then, by (62), the proof of (56) is completely achieved by showing that

$$\lim_{n\to\infty} \frac{\sqrt{n}\Theta^n(s;1)}{2} = \frac{1}{\sqrt{2\pi s}}. \tag{64}$$

By (61) we get

$$\Theta^n(s;1) = P\left(X^n_s \in \left[0, \frac{1}{\sqrt{n}}\right]\right) = F^n_s\left(\frac{1}{\sqrt{n}}\right) - F^n_s\left(-\frac{1}{\sqrt{n}}\right), \tag{65}$$

and, by using the expedient to write the r.h.s. of the above formula as

$$F^n_s\left(\frac{3}{2\sqrt{n}}\right) - F^n_s\left(-\frac{1}{2\sqrt{n}}\right),$$

we can use the expansion (58), so that

$$\Theta^n(s;1) = \Phi\left(\frac{3}{2}\frac{1}{\sqrt{ns}}\right) - \Phi\left(-\frac{1}{2}\frac{1}{\sqrt{ns}}\right) + o\left(\frac{1}{\sqrt{ns}}\right) \simeq \frac{2}{\sqrt{ns}}\Phi'(\gamma^s_n) + o\left(\frac{1}{\sqrt{ns}}\right),$$

where $\gamma^s_n \in \left(-\frac{1}{2}\frac{1}{\sqrt{ns}}, \frac{3}{2}\frac{1}{\sqrt{ns}}\right)$. Then

$$\lim_{n\to\infty} \frac{\sqrt{n}\Theta^n(s;1)}{2} = \lim_{n\to\infty} \frac{1}{\sqrt{s}}\Phi'(\gamma^s_n) + o(1) = \frac{1}{\sqrt{2\pi s}}.$$

□

5.2 The general case

In this subsection our aim is to prove Theorem 5.1 in the general case defined by conditions **C1**, **C2**, **C3** and **A1**, **A2**, **A3**, with the same space-time scaling. Observe that the processes $\tilde{X}_t^n$ and $\tilde{Z}_t^n$ can be represented as

$$\tilde{X}_t^n = \tilde{A}_t^n - \tilde{N}_t^n, \quad \tilde{Z}_t^n = \tilde{A}_t^n + \tilde{N}_t^n$$

where, if $\tilde{\tau}_k^n$ are the jump times of $\tilde{Z}^n$,

$$\tilde{A}_t^n = \sum_{k=0}^{\infty} \mathbb{I}(\tilde{U}_k^n = 1)\mathbb{I}(\tilde{\tau}_k^n \leq t) \tag{66}$$

$$\tilde{N}_t^n = \sum_{k=0}^{\infty} \mathbb{I}(\tilde{U}_k^n = -1)\mathbb{I}(\tilde{\tau}_k^n \leq t). \tag{67}$$

The processes $\tilde{A}^n$ and $\tilde{N}^n$ are mutually independent Poisson processes with intensities λ_n and μ_n respectively. Indeed $\tilde{A}^n$ and $\tilde{N}^n$ do not have common jumps and they are Poisson processes with parameters λ_n, μ_n respectively, and the independence follows by Watanabe's Theorem (see, for instance, [2]).
Then, without loss of generality, we can assume that all the above processes are defined on the same measurable space, but with different probability measures P^n. We can use the measurable space $(\Omega, \mathcal{F})$ used in Subsection 5.1 for the symmetric case, and take again $\tilde{X}_t^n = \tilde{X}_t = \tilde{V}_{\tilde{Z}_t}$, where, under the measure P, the process $\tilde{Z}_t$ is a Poisson process $\tilde{Z}_t$ of intensity 2λ and $\tilde{V}_k = \sum_{j=1}^k \tilde{U}_j$ is a symmetric random walk. Starting from the processes $\tilde{X}_t$ and $\tilde{Z}_t$, and in analogy with (66) and (67), we can define the process $\tilde{A}_t$ as the process counting the positive jumps of $\tilde{X}_t$, and the process $\tilde{N}_t$ as the process counting the negative jumps of $\tilde{X}_t$.

On $(\Omega, \mathcal{F})$ we consider the filtration $\{\mathcal{F}_t^n, \, t \in [0, T]\}$ generated by the time-rescaled processes $(\hat{A}_t^n, \hat{N}_t^n) = (\tilde{A}_{nt}, \tilde{N}_{nt})$, and the probability measure P^n, absolutely continuous with respect to P, such that

$$\frac{dP^n}{dP}\Big|_{\mathcal{F}_t^n} = \mathcal{L}_t^n = \left(\frac{\lambda_n}{\lambda}\right)^{\hat{A}_t^n} \exp\left\{-n\left(\lambda_n - \lambda\right)t\right\} \left(\frac{\lambda_n}{\lambda}\right)^{\hat{N}_t^n} \exp\left\{-n\left(\mu_n - \mu\right)t\right\} \tag{68}$$

Under the measure P, the processes $\hat{A}_t^n$, $\hat{N}_t^n$ are mutually independent Poisson processes with intensities $n\lambda$, $n\lambda$ while (see [2], Chapter VIII) under P^n the processes $\hat{A}_t^n$, $\hat{N}_t^n$ are mutually independent Poisson processes with intensities $n\lambda_n$, $n\mu_n$. Moreover, in the probability space $(\Omega, \mathcal{F}, P^n)$, the conditions **A1**, **A2**, **A3** are satisfied with $\tilde{Z}_t^n = \tilde{Z}_t = \tilde{A}_t + \tilde{N}_t$, $\tilde{U}_j^n = \tilde{U}_j$.

From now to the end of this Section, we denote by E^P and E^{P^n} the expectations with respect to the probability measures P and P^n, respectively.

By Kallianpur Striebel formula, we get that

$$E^{P^n}\left[g(X_t^n)/\mathcal{G}_t^n\right] = \frac{E^P\left[g(X_t^n)\mathcal{L}_t^n/\mathcal{G}_t^n\right]}{E^P\left[\mathcal{L}_t^n/\mathcal{G}_t^n\right]}. \tag{69}$$

Moreover, setting

$$\begin{cases} \hat{Z}_t^n = \hat{A}_t^n + \hat{N}_t^n \\ \hat{X}_t^n = \hat{A}_t^n - \hat{N}_t^n, \end{cases} \tag{70}$$

the rescaled processes are $X_t^n = \frac{\hat{X}_t^n}{\sqrt{n}}$ and $Z_t^n = \frac{\hat{Z}_t^n - 2n\lambda t}{\sqrt{n}}$, and under the measure P the sequence of processes (X_t^n, Z_t^n) converge weakly in $D_{\mathbb{R}^2}([0,\infty))$ to two independent Brownian motions $(W_t, B_t) := (W_t^A - W_t^N, W_t^A + W_t^N)$, indeed

$$X_t^n = \frac{\hat{X}_t^n}{\sqrt{n}} = \frac{\tilde{A}_{nt} - \tilde{N}_{nt}}{\sqrt{n}} = \frac{\tilde{A}_{nt} - n\lambda t}{\sqrt{n}} - \frac{\tilde{N}_{nt} - n\lambda t}{\sqrt{n}} \Rightarrow W_t^A - W_t^N,$$

$$Z_t^n = \frac{\hat{Z}_t^n - 2n\lambda t}{\sqrt{n}} = \frac{\tilde{A}_{nt} - n\lambda t}{\sqrt{n}} + \frac{\tilde{N}_{nt} - n\lambda t}{\sqrt{n}} \Rightarrow W_t^A + W_t^N,$$

where W_t^A and W_t^N are clearly independent Brownian motion. The last property implies the independence of W and B.

Therefore it is natural to get an alternative expression of $\mathcal{L}_t^n$ in terms of the processes X_t^n and Z_t^n. Taking into account that, by (70)

$$\begin{cases} \hat{A}_t^n = (\hat{Z}_t^n + \hat{X}_t^n)/2 \\ \hat{N}_t^n = (\hat{Z}_t^n - \hat{X}_t^n)/2 \end{cases}$$

and that

$$log(\mathcal{L}_t^n) = \log\left(\frac{\lambda_n}{\lambda}\right)\hat{A}_t^n - n\left(\lambda_n - \lambda\right)t + \log\left(\frac{\mu_n}{\lambda}\right)\hat{N}_t^n - n\left(\mu_n - \lambda\right)t,$$

we get immediately that [2]

$$log(\mathcal{L}_t^n) = c_n X_t^n + d_n Z_t^n + e_n t, \tag{71}$$

where

$$c_n = \frac{\sqrt{n}}{2}\left[\log\left(\frac{\lambda_n}{\lambda}\right) - \log\left(\frac{\mu_n}{\lambda}\right)\right] \tag{72}$$

$$d_n = \frac{\sqrt{n}}{2}\left[\log\left(\frac{\lambda_n}{\lambda}\right) + \log\left(\frac{\mu_n}{\lambda}\right)\right] \tag{73}$$

$$e_n = n\left[\log\left(\frac{\lambda_n}{\lambda}\right) + \log\left(\frac{\mu_n}{\lambda}\right)\right]\lambda - n\left(\lambda_n + \mu_n - 2\lambda\right). \tag{74}$$

Therefore (69) can be rewritten as

$$E^{P^n}\left[g(X_t^n)/\mathcal{G}_t^n\right] = \frac{E^P\left[g(X_t^n)\exp(c_n X_t^n)\exp(d_n Z_t^n)/\mathcal{G}_t^n\right]}{E^P\left[\exp(c_n X_t^n)\exp(d_n Z_t^n)/\mathcal{G}_t^n\right]}. \tag{75}$$

[2] Indeed

$$log(\mathcal{L}_t^n) = \log\left(\frac{\lambda_n}{\lambda}\right)(\hat{Z}_t^n + \hat{X}_t^n)/2 - n\left(\lambda_n - \lambda\right)t + \log\left(\frac{\mu_n}{\lambda}\right)(\hat{Z}_t^n - \hat{X}_t^n)/2 - n\left(\mu_n - \lambda\right)t,$$

$$= \frac{1}{2}\left[\log\left(\frac{\lambda_n}{\lambda}\right) + \log\left(\frac{\mu_n}{\lambda}\right)\right]\hat{Z}_t^n + \frac{1}{2}\left[\log\left(\frac{\lambda_n}{\lambda}\right) - \log\left(\frac{\mu_n}{\lambda}\right)\right]\hat{X}_t^n - n\left(\lambda_n + \mu_n - 2\lambda\right)t$$

$$= \frac{\sqrt{n}}{2}\left[\log\left(\frac{\lambda_n}{\lambda}\right) + \log\left(\frac{\mu_n}{\lambda}\right)\right]\frac{\hat{Z}_t^n - 2n\lambda t}{\sqrt{n}} + \frac{\sqrt{n}}{2}\left[\log\left(\frac{\lambda_n}{\lambda}\right) - \log\left(\frac{\mu_n}{\lambda}\right)\right]\frac{\hat{X}_t^n}{\sqrt{n}} + \left[\log\left(\frac{\lambda_n}{\lambda}\right) + \log\left(\frac{\mu_n}{\lambda}\right)\right]n\lambda t - n\left(\lambda_n + \mu_n - 2\lambda\right)t$$

Under conditions **C1**, **C2** and **C3**, the sequence (c_n, d_n) converges to $(\bar{c}, \bar{d})$, where $\bar{c} = c/(2\lambda)$ (see Lemma 5.12 at the end of the section). If we substitute in the right hand side of the above expression the formal limits we get

$$\frac{E\big[g(W_t)\exp(\bar{c}W_t)\exp(\bar{d}B_t)/\mathcal{F}_t^{\Lambda}\big]}{E\big[\exp(\bar{c}W_t)\exp(\bar{d}B_t)/\mathcal{F}_t^{\Lambda}\big]} = \frac{E\big[g(W_t)\exp(\bar{c}W_t)/\mathcal{F}_t^{\Lambda}\big]}{E\big[\exp(\bar{c}W_t)/\mathcal{F}_t^{\Lambda}\big]} = \hat{\Pi}_{2\lambda,c}(\zeta_t; g), \tag{76}$$

where the first equality holds since $\mathcal{F}_t^{\Lambda} \subset \mathcal{F}_t^{W}$ and the processes W and B are independent, while the second equality follows by the fact that W_t has drift zero, variance 2λ, the value of the limit $\bar{c}$ and by using Kallianpur-Striebel formula again.

Remark 5.8. *We also observe that under condition* **C3** *the sequence* e_n *converges to* $\bar{e} = -(c_1^2 + c_2^2)/2\lambda$ *(see Lemma 5.12 again). Though this fact is irrelevant for our purposes, it is interesting to note that the convergence of* (c_n, d_n, e_n) *to* $(\bar{c}, \bar{d}, \bar{e})$ *could be used to prove that the laws under* P^n *of the pair of processes* (X_t^n, Z_t^n) *converge weakly in* $D_{\mathbb{R}^2}([0,\infty))$ *to a pair of independent Brownian motions* (W_t, B_t) *with drift* $c = c(1) - c(2)$ *and* $d = c(1) + c(2)$ *respectively, and both with variance* 2λ*. This can be seen also directly by observing that*

$$X_t^n = \frac{\hat{X}_t^n}{\sqrt{n}} = \frac{\tilde{A}_{nt} - \tilde{N}_{nt}}{\sqrt{n}} = \frac{\tilde{A}_{nt} - n\lambda_n t}{\sqrt{n}} - \frac{\tilde{N}_{nt} - n\mu_n t}{\sqrt{n}} + \sqrt{n}(\lambda_n - \mu_n)t,$$

$$Z_t^n = \frac{\hat{Z}_t^n - 2n\lambda t}{\sqrt{n}} = \frac{\tilde{A}_{nt} - n\lambda_n t}{\sqrt{n}} + \frac{\tilde{N}_{nt} - n\mu_n t}{\sqrt{n}} + \sqrt{n}(\lambda_n + \mu_n - 2\lambda)t,$$

that under P^n

$$\Big(\frac{\tilde{A}_{nt} - n\lambda_n t}{\sqrt{n}}, \frac{\tilde{N}_{nt} - n\mu_n t}{\sqrt{n}}\Big) \Rightarrow \big(W_t^{A,0}, W_t^{N,0}\big),$$

and to get the weak convergence of X^n *it is necessary condition* **C3***, *i.e. that* $\sqrt{n}(\lambda_n - \mu_n)$ *converges to* c*, while to get the weak convergence of* Z^n *it is necessary that* $\sqrt{n}(\lambda_n + \mu_n - 2\lambda)$ *converges to a constant* d*, which is implied by condition* **C3** *with* $d = c(2) + c(1)$*. Then*

$$W_t = W_t^{A,0} - W_t^{N,0} + ct \ \text{and}\ B_t = W_t^{A,0} + W_t^{N,0} + dt.$$

Finally we observe that

$$\sqrt{n}(\lambda_n - \mu_n) \to c, \qquad \sqrt{n}(\lambda_n + \mu_n - 2\lambda) \to d$$

is equivalent to

$$2\sqrt{n}(\lambda_n - \lambda) \to c + d, \qquad 2\sqrt{n}(\mu_n - \lambda) \to d - c,$$

which is exactly condition **C3***, with* $c(1) = (c+d)/2$ *and* $c(2) = (d-c)/2$*.*

The above considerations leads to a heuristic proof of our main result, nevertheless we do not formalize the above heuristic reasoning to get the proof. We instead use the following expression for the filter of the queue

$$E^{P^n}\big[g(Q_t^n)/\mathcal{G}_t^n\big] = \frac{E^P\big[\ g(X_s^n)\ \exp(c_n X_s^n)\ \exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]}{E^P\big[\exp(c_n X_s^n)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]}\Bigg|_{s=\xi_t^n}, \tag{77}$$

which holds since (25), i.e. $E^{P^n}\Big[g(Q_t^n)/\mathcal{G}_t^n\Big] = \hat{\Sigma}_{P^n}^n(\xi_t^n; g)$, where, by (26) and the definition of P^n

$$\hat{\Sigma}_{P^n}^n(s;g) = \frac{E^{P^n}\Big[g(X_s^n)\mathbb{I}(\sigma_1^n > s)\Big]}{E^{P^n}\Big[\mathbb{I}(\sigma_1^n > s)\Big]} = \frac{E^P\Big[\mathcal{L}_s^n g(X_s^n)\mathbb{I}(\sigma_1^n > s)\Big]}{E^P\Big[\mathcal{L}_s^n\mathbb{I}(\sigma_1^n > s)\Big]}$$

and finally by using (71) we get (77).

As we did for the symmetric case, in order to get the limit of previous filter the idea is to show the convergence of the function $\hat{\Sigma}_{P^n}^n(s;g)$

$$\hat{\Sigma}_{P^n}^n(s;g) = \frac{E^P\big[\, g(X_s^n)\exp(c_n X_s^n)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]}{E^P\big[\exp(c_n X_s^n)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]}. \tag{78}$$

Then, a first essential step consists in evaluating

$$E^P\big[\, f(X_s^n)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]$$

either for $f(x) = g_n(x) = g(x)\exp(c_n x)$ or for $f(x) = \exp(c_n x)$, and this can be found in the following lemma.

Lemma 5.9. *Let f be a function with continuous derivative f'. Then*

$$\begin{aligned}E^P\big[\, f(X_s^n)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big] =& E^P\Big[\frac{2}{\sqrt{n}}\mathbb{I}(X_s^n \geq 2/\sqrt{n}) f'(X_s^n - 2\theta_f^n/\sqrt{n})\exp(d_n Z_s^n)\Big]\\ &+ E^P\Big[\mathbb{I}(0 \leq X_s^n < 2/\sqrt{n}) f(X_s^n)\exp(d_n Z_s^n)\Big],\end{aligned}$$

where θ_f^n is a random variable with values in $(0,1)$.

Proof. Observe that

$$\begin{aligned}E^P\big[\, f(X_s^n)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big] &= E^P\big[\, f(X_s^n)\mathbb{I}(X_s^n \geq 0)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n > s)\big]\\
&= E^P\big[\, \tilde{f}(X_s^n)\exp(d_n Z_s^n)\big] - E^P\big[\, \tilde{f}(X_s^n)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n \leq s)\big]\\
&= E^P\big[\, \{\tilde{f}(X_s^n) - \tilde{f}(X_s^n - \frac{2}{\sqrt{n}})\}\exp(d_n Z_s^n)\big],\end{aligned} \tag{79}$$

where $\tilde{f}(x) = f(x)\mathbb{I}(x \geq 0)$, and where we apply the reflection principle in order to get the last equality. Indeed, if $\bar{X}_s^n$ is the process[3] obtained by reflecting X_s^n at time σ_1^n, then

$$\begin{aligned}f(X_s^n)\mathbb{I}(X_s^n \geq 0)\exp(d_n Z_s^n)\mathbb{I}(\sigma_1^n \leq s) &= f(-\bar{X}_s^n - 2/\sqrt{n})\mathbb{I}(\bar{X}_s^n \leq -2/\sqrt{n})\exp(d_n Z_s^n)\,\mathbb{I}(\bar{\sigma}_1^n \leq s)\\ &= f(-\bar{X}_s^n - 2/\sqrt{n})\mathbb{I}(\bar{X}_s^n \leq -2/\sqrt{n})\exp(d_n Z_s^n),\end{aligned} \tag{80}$$

since
(i) $\sigma_1^n \leq s$ if and only if $\bar{\sigma}_1^n \leq s$, where $\bar{\sigma}_1^n = \inf\{u$ such that $\bar{X}_u^n \leq -1/\sqrt{n}\}$,
(ii) if $\bar{\sigma}_1^n \leq s$ then $X_s^n + 1/\sqrt{n} = -1/\sqrt{n} - \bar{X}_s^n$,

[3] If $(\hat{A}_s^n, \hat{N}_s^n)$ are the independent Poisson processes used in constructing (X_s^n, Z_s^n), then the reflected process $\bar{X}_s^n$ can be obtained as $\bar{X}_s^n = (\bar{A}_s^n - \bar{N}_s^n)/\sqrt{n}$, where $(\bar{A}_s^n, \bar{N}_s^n)$ as $(\hat{A}_s^n, \hat{N}_s^n)$ for $s < \sigma_1^n$, and as $(\hat{A}_{\sigma_1^n}^n + (\hat{N}_s^n - \hat{N}_{\sigma_1^n}^n), \hat{N}_{\sigma_1^n}^n + (\hat{A}_s^n - \hat{A}_{\sigma_1^n}^n))$ for $s \geq \sigma_1^n$.

and therefore, when $\bar{\sigma}_1^n \leq s$,
(iii) $X_s^n \geq 0$ if and only if $-2/\sqrt{n} \geq \bar{X}_s^n$, which implies $\bar{\sigma}_1^n \leq s$,
and (79) follows by (80), taking into account that $(-\bar{X}_s^n, Z_s^n)$ has the same law as (X_s^n, Z_s^n) under P.

Moreover, by using the identity $f(x) - f(x - 2/\sqrt{n}) = f'(x - 2v_f^n(x)/\sqrt{n})\frac{2}{\sqrt{n}}$, for $v_f^n(x) \in (0,1)$, we observe that

$$\begin{aligned}\tilde{f}(x) - \tilde{f}(x - 2/\sqrt{n}) =& \mathbb{I}(x \geq 0) f(x) - \mathbb{I}(x \geq 2/\sqrt{n}) f(x - 2/\sqrt{n}) = \\ =& \mathbb{I}(x \geq 2/\sqrt{n})\big(f(x) - f(x - 2/\sqrt{n})\big) + \mathbb{I}(0 \leq x < 2/\sqrt{n}) f(x) = \\ =& (2/\sqrt{n}\,)\,\mathbb{I}(x \geq 2/\sqrt{n}) f'(x - 2v_f^n(x)/\sqrt{n}) + \mathbb{I}(0 \leq x < 2/\sqrt{n}) f(x).\end{aligned}$$

If we substitute x with X_s^n, we define $\theta_f^n = v_f^n(X_s^n)$, and finally we multiply by $\exp(d_n Z_s^n)$, then we get the result by taking the expectation with respect to P and by using (79).

□

We are now ready to prove the main result of this subsection.

Proposition 5.10. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a bounded continuous function. Under the assumptions* **A1**, **A2** *and* **A3** *and conditions* **C1**, **C2** *and* **C3** *for each s,*

$$\hat{\Sigma}_{P^n}^n(s; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(s; g). \tag{81}$$

Moreover whenever $s_n \to s$, with $s > 0$

$$\hat{\Sigma}_{P^n}^n(s_n; g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(s; g). \tag{82}$$

Proof. The case $s = 0$ is trivial. When $s > 0$, in order to show (81) we multiply the numerator and the denominator of (78) by $\sqrt{n}$. Without loss of generality, we can assume that g has continuous bounded derivative. Then, by Lemma 5.9, we need to evaluate the limit of

$$\begin{aligned}&\sqrt{n} E^P\Big[\frac{2}{\sqrt{n}}\,\mathbb{I}(X_s^n \geq 2/\sqrt{n}) g_n'(X_s^n - 2\theta_g/\sqrt{n}) \exp(d_n Z_s^n)\Big] \\ &+ \sqrt{n} E^P\Big[\mathbb{I}(0 \leq X_s^n < 2/\sqrt{n}) g_n(X_s^n)\, \exp(d_n Z_s^n)\Big],\end{aligned}$$

when $g_n(x) = g(x)\exp(c_n x)$, for the numerator, and then the limit of the denominator follows taking $g(x) = 1$. Recalling that (under P) (X_t^n, Z_t^n) converge weakly in $D_{\mathbb{R}^2}([0,\infty))$ to two independent Brownian motions (W_t, B_t), the limit of the first addend is

$$\begin{aligned}&2E^P\Big[\mathbb{I}(0 < W_s < \infty) f'(W_s))\;\exp\big(\bar{d} B_s\big)\Big] \\ &= 2E^P\Big[\mathbb{I}(0 < W_s < \infty) f'(W_s)\Big]\; E^P\Big[\exp\big(\bar{d} B_s\big)\Big],\end{aligned}$$

with $f(x) = g(x)\exp(\bar{c}x)$.
Moreover, as far as the second addend is concerned, in Lemma 5.11 below we prove that

$$\sqrt{n} E^P\Big[\mathbb{I}(0 \leq X_s^n < 2/\sqrt{n}) g_n(X_s^n) \exp(d_n Z_s^n)\Big] \to \frac{2}{\sqrt{2\pi}\sqrt{2\lambda s}} g(0) E^P\Big[\exp\big(\bar{d} B_s\big)\Big].$$

Taking into account that (under P) W has drift zero and diffusion coefficient 2λ, and that $\bar{c} = \frac{c}{2\lambda}$, $E^P\Big[\mathbb{I}(0 < W_s < \infty) f'(W_s)\Big]$ can be written as

$$
\begin{aligned}
&2\int_0^\infty f'(x)\ \frac{1}{\sqrt{2\pi}\sqrt{\ 2\lambda s}}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx\\
&=\frac{2}{\sqrt{2\pi}\sqrt{\ 2\lambda s}}\left(\Big[f(x)\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}\Big]_0^\infty+\int_0^\infty f(x)\frac{x}{2\lambda s}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx\right)\\
&=\frac{2}{\sqrt{2\pi}\sqrt{\ 2\lambda s}}\left(-f(0)+\int_0^\infty f(x)\frac{x}{2\lambda s}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx\right)\\
&=\frac{2}{\sqrt{2\pi}\sqrt{\ 2\lambda s}}\left(-g(0)+\int_0^\infty g(x)\exp\left(\frac{c}{2\lambda}x\right)\frac{x}{2\lambda s}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx\right).
\end{aligned}
$$

Then, summarizing, we obtain

$$
\begin{aligned}
&\sqrt{n}E^P\big[\ g_n(X_s^n)\ \exp(d_nZ_s^n)\mathbb{I}(\sigma_1^n>s)\big]\to\\
&E^P\Big[\exp\left(\bar{d}B_s\right)\Big]\ \frac{2}{\sqrt{2\pi}\sqrt{\ 2\lambda s}}\int_0^\infty g(x)\exp\left(\frac{c}{2\lambda}x\right)\frac{x}{2\lambda s}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx,
\end{aligned}
$$

that, together with (78), yields

$$
\lim_{n\to\infty}\Sigma^n_{P^n}(s:g)=\lim_{n\to\infty}\frac{\sqrt{n}E^P\big[\ g(X_s^n)\exp(c_nX_s^n)\ \exp(d_nZ_s^n)\mathbb{I}(\sigma_1^n>s)\big]}{\sqrt{n}E^P\big[\exp(c_nX_s^n)\exp(d_nZ_s^n)\mathbb{I}(\sigma_1^n>s)\big]}
$$

$$
=\frac{\int_0^\infty g(x)\exp(\bar{c}x)\frac{x}{2\lambda s}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx}{\int_0^\infty \exp(\bar{c}x)\frac{x}{2\lambda s}\exp\{-\frac{1}{2}\frac{x^2}{2\lambda s}\}dx}
$$

$$
=\frac{\Pi\big(2\lambda s,0;g(\cdot)\exp(\frac{c}{2\lambda}\cdot)\big)}{\Pi\big(2\lambda s,0;\exp(\frac{c}{2\lambda}\cdot)\big)}=\hat{\Pi}_{2\lambda,c}(s;g).
$$

The proof of (82) is similar to the previous one and can be achieved by using the weak convergence of (X_s^n, Z_s^n) to (W_s, B_s) in the uniform norm on bounded intervals, and Lemma 5.11. □

Lemma 5.11. *Under the same conditions of Proposition 5.10, let*

$$
q_n(s):=E^P\Big[\mathbb{I}(0\le X_s^n<2/\sqrt{n})\,g(X_s^n)\ \exp(c_nX_s^n)\exp(d_nZ_s^n)\Big],
$$

then

$$
\lim_{n\to\infty}\sqrt{n}q_n(s_n)=\frac{2}{\sqrt{2\pi}\sqrt{\ 2\lambda s}}\ g(0)\ E^P\Big[\exp\left(\bar{d}B_s\right)\Big], \tag{83}
$$

whenever $s_n\to s$, with $s>0$.

Proof. First of all we observe that

$$
\begin{aligned}
q_n(s)=&\sum_{h=0}^\infty P(\tilde{A}_{ns}=h)P(\tilde{N}_{ns}=h)g(0)\mathrm{e}^{c_n0}\mathrm{e}^{d_n\frac{2h-2n\lambda s}{\sqrt{n}}}\\
&+\sum_{h=0}^\infty P(\tilde{A}_{ns}=h+1)P(\tilde{N}_{ns}=h)g(1/\sqrt{n})\mathrm{e}^{c_n\frac{1}{\sqrt{n}}}\mathrm{e}^{d_n\frac{2h+1-2n\lambda s}{\sqrt{n}}}
\end{aligned}
$$

In order to evaluate (83) we observe that $\sqrt{n}\ q_n(s_n)$ has the same behaviour as $g(0)\sqrt{n}\bar{q}_n(s_n)$, where

$$\bar{q}_n(s) := \sum_{h=0}^{\infty} P(\tilde{A}_{ns}=h)P(\tilde{N}_{ns}=h)\mathrm{e}^{d_n \frac{2h-2n\lambda s}{\sqrt{n}}} + \sum_{h=0}^{\infty} P(\tilde{A}_{ns}=h+1)P(\tilde{N}_{ns}=h)\mathrm{e}^{d_n \frac{2h+1-2n\lambda s}{\sqrt{n}}},$$

as can be immediately seen[4].
Taking into account that $\tilde{Z}_{ns}=\tilde{A}_{ns}+\tilde{N}_{ns}$ is a Poisson random variable of parameter $2\lambda ns$, one can see that[5]

$$P(\tilde{A}_{ns}=h)P(\tilde{N}_{ns}=h) = P(\tilde{Z}_{ns}=2h)\frac{(2h)!}{h!\ h!}\frac{1}{2^{2h}}$$

and that

$$P(\tilde{A}_{ns}=h+1)P(\tilde{N}_{ns}=h) = P(\tilde{Z}_{ns}=2h+1)\frac{(2h+1)!}{(h+1)!\ h!}\frac{1}{2^{2h+1}}.$$

Then, setting

$$r(k) = \frac{k!}{(k-[k/2])!\ [k/2]!}\frac{1}{2^k},$$

we can rewrite

$$\bar{q}_n(s) = \sum_{k=0}^{\infty} P(\tilde{Z}_{ns}=k)\mathrm{e}^{d_n \frac{k-2n\lambda s}{\sqrt{n}}} r(k) = E^P\Big[r(\tilde{Z}_{ns})\ \exp(d_n \frac{\tilde{Z}_{ns}-2n\lambda s}{\sqrt{n}})\Big] = E^P\Big[r(\tilde{Z}_{ns})\ \exp(d_n Z^n_s)\Big]$$

Now by Stirling formula

$$\frac{r(m)}{\frac{1}{\sqrt{m}}} \to \frac{2}{\sqrt{2\pi}}, \qquad \text{as } m\to\infty$$

and therefore, we rewrite

$$\sqrt{n}\ \bar{q}_n(s) = \sqrt{n}\ P(\tilde{Z}_{ns}=0) + E^P\left[\mathbb{I}(\tilde{Z}_{ns}>0)\ \frac{r(\tilde{Z}_{ns})}{\frac{1}{\sqrt{\tilde{Z}_{ns}}}}\frac{1}{\sqrt{\frac{\tilde{Z}_{ns}}{n}}}\ \exp\big(d_n Z^n_s\big)\right].$$

---

[4]Indeed

$$\min\big(g(0), g(1/\sqrt{n})\big)\sqrt{n}\bar{q}_n(s) \le \sqrt{n}q_n(s) \le \mathrm{e}^{c_n \frac{1}{\sqrt{n}}} \max\big(g(0), g(1/\sqrt{n})\big)\sqrt{n}\bar{q}_n(s).$$

[5]By the following simple calculations:

$$P(\tilde{A}_{ns}=h)P(\tilde{N}_{ns}=h) = \frac{(\lambda ns)^h}{h!}\mathrm{e}^{-\lambda ns}\frac{(\lambda ns)^h}{h!}\mathrm{e}^{-\lambda ns} = \frac{(\lambda ns)^{2h}}{h!\ h!}\mathrm{e}^{-2\lambda ns} = \frac{(2\lambda ns)^{2h}}{(2h)!}\mathrm{e}^{-2\lambda ns}\frac{(2h)!}{h!\ h!}\frac{1}{2^{2h}} = P(\tilde{Z}_{ns}=2h)\frac{(2h)!}{h!\ h!}\frac{1}{2^{2h}}$$

$$P(\tilde{A}_{ns}=h+1)P(\tilde{N}_{ns}=h) = \frac{(\lambda ns)^{h+1}}{(h+1)!}\mathrm{e}^{-\lambda ns}\frac{(\lambda ns)^h}{h!}\mathrm{e}^{-\lambda ns} = \frac{(\lambda ns)^{2h+1}}{(h+1)!\ h!}\mathrm{e}^{-2\lambda ns} = \frac{(2\lambda ns)^{2h+1}}{(2h)!}\mathrm{e}^{-2\lambda ns}\frac{(2h+1)!}{(h+1)!\ h!}\frac{1}{2^{2h+1}} = P(\tilde{Z}_{ns}=2h+1)\frac{(2h)!}{(h+1)!\ h!}\frac{1}{2^{2h+1}}$$

We are interested in the asymptotic behaviour of $\sqrt{n}\ \bar{q}_n(s_n)$, and first of all we note that the sequence $nP(Z^n_{s_n}=0)$ converges to zero, as can be seen by direct calculations. Then we observe that for each $T>0$, by Kolmogorov inequality

$$P\big(\sup_{s\le T}\big|\frac{\tilde{Z}_{ns}}{n}-2\lambda s\big|\ge\varepsilon\big)\le\frac{Var(\tilde{Z}_{nT})}{n^2\varepsilon^2}=\frac{2n\lambda T}{n^2\varepsilon^2}.$$

Furthermore Z^n_s converge weakly to B_s in $D_R([0,\infty))$, w.r.t. the topology of uniform convergence on bounded intervals, the limit process having continuous paths. Therefore the pair $(\tilde{Z}_{ns}/n, Z^n_s)$ converges in $D_R([0,\infty))\times D_R([0,\infty))$, each component endowed with the topology of uniform convergence, and then

$$\left(\frac{\tilde{Z}_{ns}}{n}, Z^n_s\right)\Rightarrow(2\lambda s, B_s)\,,\qquad \text{in}\ \ D_{R^2}([0,\infty))$$

with the topology of uniform convergence on bounded intervals.

By Skorohod theorem we can assume w.l.o.g. that the above pair converges $P-$a.s., and uniformly on bounded intervals. Then,

$$\frac{r(\tilde{Z}_{ns_n})}{\frac{1}{\sqrt{\tilde{Z}_{ns_n}}}}\,\mathbb{I}(\tilde{Z}_{ns_n}>0)\ \frac{1}{\sqrt{\frac{\tilde{Z}_{ns_n}}{n}}}\ \exp\left(d_n Z^n_{s_n}\right)\ \to\ \frac{2}{\sqrt{2\pi}}\frac{1}{\sqrt{2\lambda s}}\exp\left(\bar{d}B_s\right)\qquad P-a.s. \tag{84}$$

whenever $s_n\to s$, with $s>0$. The above convergence is equivalent to the uniform convergence on bounded and compact intervals of $(0,\infty)$, and its proof is straightforward. We only observe that, if we set $h_n(s)=\sqrt{\tilde{Z}_{ns}/n}$, then, for any $T>0$, $h_n(s)$ converge to $h(s)=\sqrt{2\lambda s}$ uniformly in $[0,T]$, and therefore, for any $\underline{h}>0$, $1/h_n(s)$ converge to $1/h(s)$ uniformly in the set $\{s, \text{such that } h(s)\ge\underline{h}\}$.
Moreover the sequence at the r.h.s. of (84) is uniformly integrable since

$$\sup_n E^P\left[\left(\sqrt{n}\ r(\tilde{Z}_{ns_n})\ \exp\left(d_n Z^n_{s_n}\right)\right)^2\right]\le L<\infty.$$

Indeed[6]

$$\sup_n E^P\left[\mathbb{I}(\tilde{Z}_{ns_n}>0)\ \frac{n}{\tilde{Z}_{ns_n}}\ \exp\left(2\ d_n Z^n_{s_n}\right)\right]\le L'<\infty,$$

$$\sup_m\frac{r(m)}{\frac{1}{\sqrt{m}}}\le L''<\infty.$$

Then the thesis is achieved since

$$\lim_{n\to\infty}\sqrt{n}\ \bar{q}_n(s_n)=\frac{2}{\sqrt{2\pi}}\frac{1}{\sqrt{2\lambda s}}E^P\left[\exp\left(\bar{d}B_s\right)\right].$$

□

[6] Note that

$$E^P\left[\mathbb{I}(\tilde{Z}_{ns_n}>0)\ \frac{n}{\tilde{Z}_{ns_n}}\ \exp\left(2\ d_n Z^n_{s_n}\right)\right]=\sum_{k=1}^{\infty}\frac{n}{k}\exp\left(2d_n\frac{k-2\lambda ns}{\sqrt{n}}\right)\frac{(2\lambda ns_n)^k}{k!}\mathrm{e}^{-2\lambda ns_n}$$

$$=\exp(-2d_n/\sqrt{n})\frac{1}{2\lambda s_n}\sum_{k=1}^{\infty}\frac{k+1}{k}\exp\left(2d_n\frac{k+1-2\lambda ns}{\sqrt{n}}\right)\frac{(2\lambda ns_n)^{k+1}}{(k+1)!}\mathrm{e}^{-2\lambda ns_n}$$

$$\le\exp(-2d_n/\sqrt{n})\frac{1}{\lambda s_n}E^P[\exp(2d_n Z^n_{s_n})]$$

which is bounded and converges to $\frac{1}{\lambda s}E[\exp\left(2\bar{d}B_s\right)]$

We end this section with the following elementary technical lemma.

Lemma 5.12. *If condition* **C3** *holds, then*

$$\lim_{n\to\infty}(c_n, d_n, e_n) = (\bar{c}, \bar{d}, \bar{e}), \tag{85}$$

where c_n, d_n and e_n are defined in (72),(73) and (74), and where

$$\bar{c} = \frac{c}{2\lambda} = \frac{c(1)-c(2)}{2\lambda}, \quad \bar{d} = \frac{d}{2\lambda} = \frac{c(1)+c(2)}{2\lambda}, \quad \bar{e} = -(c_1^2+c_2^2)/2\lambda.$$

Proof. Set $\alpha_n = \lambda_n - \lambda$ and $\beta_n = \mu_n - \lambda$, and $\varepsilon_n = \beta_n - \alpha_n = \mu_n - \lambda_n$. Then,

$$c_n = \frac{\sqrt{n}}{2}\left[\log\left(\frac{\lambda_n}{\lambda}\right) - \log\left(\frac{\mu_n}{\lambda}\right)\right] = \frac{\sqrt{n}}{2}\log\left(1 - \frac{\varepsilon_n}{\mu_n}\right)$$

and

$$d_n = \frac{\sqrt{n}}{2}\left[\log\left(\frac{\lambda_n}{\lambda}\right) + \log\left(\frac{\mu_n}{\lambda}\right)\right] = \frac{\sqrt{n}}{2}\left[\log\left(1+\frac{\alpha_n}{\lambda}\right) + \log\left(1+\frac{\beta_n}{\lambda}\right)\right]$$

By expanding $\log(1+x) = x + o(x)$ we get

$$c_n = \frac{\sqrt{n}}{2}\left[-\frac{\varepsilon_n}{\mu_n} + o\left(\frac{\varepsilon_n}{\mu_n}\right)\right], \qquad d_n = \frac{\sqrt{n}}{2}\left[\frac{\alpha_n+\beta_n}{\lambda} + o(\alpha_n) + o(\beta_n)\right],$$

and then

$$c_n = \frac{\sqrt{n}}{2}\left[\frac{1}{\sqrt{n}}\frac{c}{\mu_n} + o\left(\frac{1}{\sqrt{n}}\right)\right] \qquad d_n = \frac{\sqrt{n}}{2}\left[\frac{1}{\sqrt{n}}\frac{d}{\lambda} + o(\frac{1}{\sqrt{n}})\right],$$

since **C3** implies that $\alpha_n = \frac{c(1)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$, $\beta_n = \frac{c(2)}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$, $\varepsilon_n = -\frac{c}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)$.

The limit (85) for the first two components follows from the convergence of μ_n to λ. Using the expansion $\log(1+x) = x - x^2/2 + o(x^2)$ one can prove that

$$\begin{aligned}
e_n &= n\left[\log\left(\frac{\lambda_n}{\lambda}\right) + \log\left(\frac{\mu_n}{\lambda}\right)\right]\lambda - n\left(\lambda_n + \mu_n - 2\lambda\right) \\
&= n\left\{\left[\frac{\alpha_n}{\lambda} - \frac{1}{2}\left(\frac{\alpha_n}{\lambda}\right)^2 + o\left(\alpha_n^2\right) + \frac{\beta_n}{\lambda} - \frac{1}{2}\left(\frac{\beta_n}{\lambda}\right)^2 + o\left(\beta_n^2\right)\right]\lambda - (\alpha_n+\beta_n)\right\} \\
&= -\frac{1}{2\lambda}\left(\sqrt{n}(\lambda_n-\lambda)\right)^2 - \frac{1}{2\lambda}\left(\sqrt{n}(\mu_n-\lambda)\right)^2 + n\,o\left(\frac{1}{n}\right) \to -\frac{1}{2}\frac{c_1^2+c_2^2}{\lambda}
\end{aligned}$$

□

Remark 5.13. *From the proof it is easy to see that for the convergence of c_n to $\bar{c}$ we only need conditions* **C2** *and* **C3***, *which imply that the sequence $\sqrt{n}\epsilon_n/\mu_n$ converges to c/λ. On the other hand, condition* **C3*** *does not assure that d_n is converging, and d_n can be either bounded or unbounded. Indeed one can take for instance $\mu_n = \lambda + c(2)/\sqrt{n} + (-1)^n b/\sqrt{n}$ and $\lambda_n = \lambda + c(1)/\sqrt{n} + (-1)^n b/\sqrt{n}$, so that $d_n = (c(2)+c(1)+2(-1)^n b)/(2\lambda) + o(1)$ is a bounded (but not convergent) sequence, or $\mu_n = \lambda + c(2)/\sqrt{n} + b/n^\gamma$ and $\lambda_n = \lambda + c(1)/\sqrt{n} + b/n^\gamma$, for $0 < \gamma < 1/2$, so that d_n is unbounded, though in either cases conditions* **C2** *and* **C3*** *are satisfied.*

6 The M/M/1 queueing model: observing the idle time process

In this section we are interested in the conditional law of the M/M/1 queue $\tilde{Q}^n_s$, when the observation process is the *idle time* process, i.e.

$$\tilde{C}^n_t = \int_0^t \mathbb{I}\left(\tilde{Q}^n_s = 0\right) ds,$$

the cumulative time the queue has spent in 0, up to t.

Equivalently one can consider as observation process the bivariate point process $(\tilde{I}^n_t, \tilde{B}^n_t)$, where $\tilde{I}^n_t$ is the process that counts the times when the system starts an idle period and $\tilde{B}^n_t$ is the process that counts the times when the system starts a busy period, that is

$$\tilde{I}^n_t = \int_0^t \mathbb{I}(\tilde{Q}^n_{s-} = 1) d\tilde{N}^n_s \tag{86}$$

$$\tilde{B}^n_t = \int_0^t \mathbb{I}(\tilde{Q}^n_{s-} = 0) d\tilde{A}^n_s. \tag{87}$$

Indeed the filtration generated by the idle time process $\tilde{C}^n_t$ and the filtration generated by the observation process $(\tilde{I}^n_t, \tilde{B}^n_t)$ coincide, or more precisely $\mathcal{F}^{\tilde{C}^n}_{t^+} = \mathcal{F}^{\tilde{I}^n, \tilde{B}^n}_t$.

Our first aim is to study the conditional law

$$E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big], \tag{88}$$

where for the notational convenience we denote

$$\tilde{\mathcal{H}}^n_t = \mathcal{F}^{\tilde{I}^n, \tilde{B}^n}_t = \mathcal{F}^{\tilde{C}^n}_{t^+},$$

and the explicit expression for the filter (88) in terms of $\gamma^0_t(\tilde{Q}^n)$, is given in (95).
Then we consider the rescaled processes

$$Q^n_t := \frac{\tilde{Q}^n_{nt}}{\sqrt{n}}, \qquad I^n_t := \frac{\tilde{I}^n_{nt}}{\sqrt{n}}, \qquad B^n_t := \frac{\tilde{B}^n_{nt}}{\sqrt{n}}, \qquad C^n_t := \sqrt{n}\mu_n \tilde{C}^n_{nt},$$

and the conditional law of the rescaled queue

$$E\big[g(Q^n_t)/\mathcal{H}^n_t\big], \tag{89}$$

where $\mathcal{H}^n_t$ is the filtration generated by the rescaled observation

$$\mathcal{H}^n_t = \tilde{\mathcal{H}}^n_{nt} = \mathcal{F}^{I^n, B^n}_t = \mathcal{F}^{C^n}_{t^+}.$$

We are interested in the limit behaviour of the filter (89) under the same assumption **A1**, **A2**, **A3** and conditions **C1**, **C2**, **C3** of Section 5. Under these assumptions we already know that Q^n_t converge weakly to a Brownian motion W_t with diffusion coefficient 2λ and drift coefficient c.
If one defines $\bar{X}^n_t := Q^n_t - C^n_t$, then clearly $Q^n_t = \bar{X}^n_t + C^n_t$, and therefore, since by definition C^n_t increases only when $Q^n_t = 0$, the pair (Q^n_t, C^n_t) is the solution of the Skorohod problem corresponding to $\bar{X}^n_t$. Moreover

$$\left(\bar{X}^n_t, Q^n_t, C^n_t\right) \Rightarrow \left(W_t, W_t + \Lambda_t, \Lambda_t\right),$$

where as usual Λ_t is the local time of W_t (for a deeper investigation of these results, we refer to Kurtz [8]). It is therefore natural to expect that $E\big[g(Q_t^n)/\mathcal{H}_t^n\big]$ converges weakly to $E\big[g(W_t+\Lambda_t)/\mathcal{F}_t^{\Lambda}\big]= \hat{\Pi}_{2\lambda,c}(\zeta_t;g)$. This result is proven in Theorem 6.4. Moreover $\hat{\Pi}_{2\lambda,c}(\gamma_t^0(Q^n);g)$ is a good approximation of the filter for the rescaled model (see Theorem 6.6).

Let $\{\sigma_k^{Bn},\ k\in\mathbb{N}\}$ and $\{\sigma_k^{In},\ k\in\mathbb{N}\}$ be the jump times of the process $\tilde{I}_t^n$ and the process $\tilde{B}_t^n$, respectively. Under the assumption $\tilde{Q}_0^n=0$, it easy to verify that

$$\sigma_k^{Bn}<\sigma_k^{In}<\sigma_{k+1}^{Bn}<\sigma_{k+1}^{In}, \qquad \text{for each } k\geq 1,$$

and that $Q_t^n=0$ when $\sigma_k^{In}\leq t<\sigma_{k+1}^{Bn}$, for $k\geq 0$, while $Q_t^n>0$ otherwise.
We start by observing some regenerative properties of the above jump times, which are fundamental in the sequel.

Lemma 6.1. *For each $k\in\mathbb{N}$ the processes $\tilde{Q}_{k,t}^{In}=\tilde{Q}_{t+\sigma_k^{In}}^n-\tilde{Q}_{\sigma_k^{In}}^n$ and $\tilde{Q}_{k,t}^{Bn}=\tilde{Q}_{t+\sigma_k^{Bn}}^n-\tilde{Q}_{\sigma_k^{Bn}}^n$ are independent of $\mathcal{F}_{\sigma_k^{In}}^{\tilde{Q}^n}$ and $\mathcal{F}_{\sigma_k^{Bn}}^{\tilde{Q}^n}$ respectively. Moreover, the process $\tilde{Q}_{k,t}^{In}$ has the same law as the process $\tilde{Q}_t^n$.*

Proof. Clearly $\tilde{Q}_{k,0}^{In}=\tilde{Q}_0^n=0$, moreover it is easy to see that $\tilde{Q}_{k,t}^{In}$ solves the same martingale problem as $\tilde{Q}_t^n$, hence these processes share the same law. The independence property follows by the strong Markov property of $\tilde{Q}_t^n$, since $\tilde{Q}_{\sigma_k^{In}}^n=0$:

$$\begin{aligned}
&E\left[\exp\left(iu\left(\tilde{Q}_{t+\sigma_k^{In}}^n-\tilde{Q}_{\sigma_k^{In}}^n\right)\right)/\mathcal{F}_{\sigma_k^{In}}^{\tilde{Q}^n}\right]=E\left[\exp\left(iu\left(\tilde{Q}_{t+\sigma_k^{In}}^n-\tilde{Q}_{\sigma_k^{In}}^n\right)\right)/\tilde{Q}_{\sigma_k^{In}}^n\right]=\\
=&E\left[\exp\left(iu\left(\tilde{Q}_{t+\sigma_k^{In}}^n-\tilde{Q}_{\sigma_k^{In}}^n\right)\right)\right]=E\left[\exp\left(iu\left(\tilde{Q}_{t+\sigma_k^{In}}^n-\tilde{Q}_{\sigma_k^{In}}^n\right)\right)\right]\\
=&E\left[\exp\left(iu\left(\tilde{Q}_t^n\right)\right)\right].
\end{aligned}$$

Similar arguments apply to show that the process $\tilde{Q}_{k,t}^{Bn}$ is independent of $\mathcal{F}_{\sigma_k^{Bn}}^{\tilde{Q}^n}$. ☐

The process $\tilde{I}_t^n$ is a renewal process, and $\tilde{B}_t^n$ is a delayed renewal process, i.e. the random variables $\sigma_{k+1}^{Bn}-\sigma_k^{Bn}$ are mutually independent for $k\geq 0$ and identically distributed for $k\geq 1$. Also $\{\sigma_k^{In}-\sigma_k^{Bn}\}_{k\geq 1}$ is a sequence of mutually independent random variables.
In the setting of this section, the above considerations and Lemma 6.1 guarantee that the filter of $\tilde{Q}_t^n$ given $\tilde{\mathcal{H}}_t^n$ admits a representation similar to that given in Proposition 2.1. More precisely

Proposition 6.2. *The conditional law of $\tilde{Q}_t^n$ given $\tilde{\mathcal{H}}_t^n$ admits the following representation*

$$\begin{aligned}
&E\big[g(\tilde{Q}_t^n)/\tilde{\mathcal{H}}_t^n\big]=\mathbb{I}\left(\tilde{Q}_t^n=0\right)g(0)+ \qquad (90)\\
&+\mathbb{I}\left(\tilde{Q}_t^n>0\right)\sum_{j=1}^{\infty}\frac{E\Big[g(\tilde{Q}_{s+\sigma_j^{Bn}}^n-\tilde{Q}_{\sigma_j^{Bn}}^n+1)\mathbb{I}\left(\sigma_j^{In}-\sigma_j^{Bn}>s\right)\Big]_{s=t-\sigma_j^{Bn}}}{E\Big[\mathbb{I}(\sigma_j^{In}-\sigma_j^{Bn}>s)\Big]_{s=t-\sigma_j^{Bn}}}\mathbb{I}\{\sigma_j^{Bn}\leq t<\sigma_j^{In}\}.
\end{aligned}$$

Even if this Proposition is quite similar to Proposition 2.1, we sketch the proof because the assumptions on the processes involved are a little different.
Moreover we point out that since the processes involved are all Markovian, the proof of (90) could be given by the techniques used in [3].

Proof. We begin by noting that $\mathbb{I}\left(\tilde{Q}_t^n > 0\right)$ is $\tilde{\mathcal{H}}_t^n$-adapted. Moreover we can state (see [2] Chapter. III T5)

$$\tilde{\mathcal{H}}_t^n \cap \{\sigma_j^{Bn} \le t < \sigma_j^{In}\} \cap \{\tilde{Q}_t^n > 0\} = \tilde{\mathcal{H}}_{\sigma_j^{Bn}}^n \cap \{\sigma_j^{Bn} \le t < \sigma_j^{In}\} \cap \{\tilde{Q}_t^n > 0\}.$$

By using, as in Proposition 2.1 the arguments of Proposition 3.1 in [9], we obtain

$$\mathbb{I}\left(\tilde{Q}_t^n > 0\right) E\big[g(\tilde{Q}_t^n)/\tilde{\mathcal{H}}_t^n\big]\mathbb{I}\{\sigma_j^{Bn} \le t < \sigma_j^{In}\} =$$
$$\mathbb{I}\left(\tilde{Q}_t^n > 0\right) \frac{E\Big[g(\tilde{Q}_t^n)\mathbb{I}\left(\sigma_j^{In} - \sigma_j^{Bn} > t - \sigma_j^{Bn}\right)/\tilde{\mathcal{H}}_{\sigma_j^{Bn}}^n\Big]}{P\Big(\sigma_j^{In} - \sigma_j^{Bn} > t - \sigma_j^{Bn}/\tilde{\mathcal{H}}_{\sigma_j^{Bn}}^n\Big)}\mathbb{I}\{\sigma_j^{Bn} \le t < \sigma_j^{In}\}.$$

Note that

$$\{\sigma_j^{In} - \sigma_j^{Bn} > s\} = \left\{\inf_{0 \le u \le s} \tilde{Q}_{j,u}^{Bn} > 0\right\}$$

and is independent of $\mathcal{F}_{\sigma_j^{Bn}}^{\tilde{Q}^n}$ by Lemma 6.1.
Then, denoting by $s = t - \sigma_j^{Bn}$, so that

$$\tilde{Q}_t^n = \tilde{Q}_{\sigma_j^{Bn}+s}^n = \tilde{Q}_{\sigma_j^{Bn}+s}^n - \tilde{Q}_{\sigma_j^{Bn}}^n + 1 = \tilde{Q}_{j,s}^{Bn} + 1$$

$$\frac{E\Big[g(\tilde{Q}_{\sigma_j^{Bn}+s}^n)\mathbb{I}\left(\sigma_j^{In} - \sigma_j^{Bn} > s\right)\Big|_{s=t-\sigma_j^{Bn}}/\tilde{\mathcal{H}}_{\sigma_j^{Bn}}^n\Big]}{P\Big(\left(\sigma_j^{In} - \sigma_j^{Bn} > s\right)\Big|_{s=t-\sigma_j^{Bn}}/\tilde{\mathcal{H}}_{\sigma_j^{Bn}}^n\Big)}$$

$$= \frac{E\Big[g\left(\tilde{Q}_{j,s}^{Bn} + 1\right)\mathbb{I}\left(\sigma_j^{In} - \sigma_j^{Bn} > s\right)\Big]_{s=t-\sigma_j^{Bn}}}{E\Big[\mathbb{I}\left(\sigma_j^{In} - \sigma_j^{Bn} > s\right)\Big]_{s=t-\sigma_j^{Bn}}}, \tag{91}$$

Then the thesis follows by writing $g(\tilde{Q}_t^n)$ as

$$g(\tilde{Q}_t^n) = \mathbb{I}\left(\tilde{Q}_t^n = 0\right) g(0) + \mathbb{I}\left(\tilde{Q}_t^n > 0\right) \sum_{j=1}^{\infty} g(\tilde{Q}_t^n)\mathbb{I}\{\sigma_j^{Bn} \le t < \sigma_j^{In}\},$$

and, therefore, $E\big[g(\tilde{Q}_t^n)/\tilde{\mathcal{H}}_t^n\big]$ as

$$E\big[g(\tilde{Q}_t^n)/\tilde{\mathcal{H}}_t^n\big] = \mathbb{I}\left(\tilde{Q}_t^n = 0\right) E\big[g(0)/\tilde{\mathcal{H}}_t^n\big] + \mathbb{I}\left(\tilde{Q}_t^n > 0\right) \sum_{j=1}^{\infty} E\big[g(\tilde{Q}_t^n)/\tilde{\mathcal{H}}_t^n\big]\mathbb{I}\{\sigma_j^{Bn} \le t < \sigma_j^{In}\}.$$

□

It is important to note that

$$\sigma_j^{In} - \sigma_j^{Bn} = \inf\left\{u \ge 0 : \ \tilde{Q}_{j,u}^{Bn} + 1 = 0\right\}, \tag{92}$$

and that the process $\tilde{Q}^{Bn}_{j,s}+1$ for $s<\sigma^{In}_j-\sigma^{Bn}_j$ behaves like the continuous time random walk $\tilde{X}^n_s+1$ for $s<\tilde{\sigma}^n_1=\inf\Big\{u\geq 0:\ \tilde{X}^n_u=-1\Big\}$, and hence

$$\frac{E\Big[g\left(\tilde{Q}^{Bn}_{j,s}+1\right)\mathbb{I}\left(\sigma^{In}_j-\sigma^{Bn}_j>s\right)\Big]}{E\Big[\mathbb{I}\left(\sigma^{In}_j-\sigma^{Bn}_j>s\right)\Big]}=\frac{E\Big[g\left(\tilde{X}^n_s+1\right)\mathbb{I}\left(\tilde{\sigma}^n_1>s\right)\Big]}{E\Big[\mathbb{I}\left(\tilde{\sigma}^n_1>s\right)\Big]}. \tag{93}$$

As a consequence

$$E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big]=\mathbb{I}\left(\tilde{Q}^n_t=0\right)g(0)+ \tag{94}$$
$$+\mathbb{I}\left(\tilde{Q}^n_t>0\right)\sum_{j=1}^{\infty}\frac{E\Big[g\left(\tilde{X}^n_s+1\right)\mathbb{I}\left(\tilde{\sigma}^n_1>s\right)\Big]_{s=t-\sigma^{Bn}_j}}{E\Big[\mathbb{I}\left(\tilde{\sigma}^n_1>s\right)\Big]_{s=t-\sigma^{Bn}_j}}\mathbb{I}\{\sigma^{Bn}_j\leq t<\sigma^{In}_j\}.$$

Observing that, by definition (8),

$$\gamma^0_t(\tilde{Q}^n)=t-\sup\{s<t\text{ such that }\tilde{Q}^n_s=0\}=\sum_{j=1}^{\infty}(t-\sigma^{Bn}_j)\mathbb{I}\{\sigma^{Bn}_j\leq t<\sigma^{In}_j\}$$

we can rewrite

$$E\big[g(\tilde{Q}^n_t)/\tilde{\mathcal{H}}^n_t\big]=\mathbb{I}\left(\tilde{Q}^n_t=0\right)g(0)+\mathbb{I}\left(\tilde{Q}^n_t>0\right)\left.\frac{E\Big[g\left(\tilde{X}^n_s+1\right)\mathbb{I}\left(\tilde{\sigma}^n_1>s\right)\Big]}{E\Big[\mathbb{I}\left(\tilde{\sigma}^n_1>s\right)\Big]}\right|_{s=\gamma^0_t(\tilde{Q}^n)}. \tag{95}$$

The above considerations leads us to state the following result

Theorem 6.3. *Consider the rescaled process $Q^n_t=\frac{\tilde{Q}^n_{nt}}{\sqrt{n}}$, the rescaled observation processes $I^n_t=\frac{\tilde{I}^n_{nt}}{\sqrt{n}}$ and $B^n_t=\frac{\tilde{B}^n_{nt}}{\sqrt{n}}$, and denote by $\mathcal{H}^n_t$ the history generated by (I^n_u,B^n_u) for $u\leq t$, i.e.*

$$\mathcal{H}^n_t=\mathcal{F}^{I^n,B^n}_t=\tilde{\mathcal{H}}^n_{nt}.$$

Then

$$E\big[g(Q^n_t)/\mathcal{H}^n_t\big]=\mathbb{I}\left(Q^n_t=0\right)g(0)+\mathbb{I}\left(Q^n_t>0\right)\bar{\Sigma}^n(\gamma^0_t(Q^n);g), \tag{96}$$

where

$$\bar{\Sigma}^n(s;g)=\hat{\Sigma}^n(s;\overline{g}_n),$$

with $\hat{\Sigma}^n(s)$ the probability measure defined in (26), and $\overline{g}_n(x)=g(x+\frac{1}{\sqrt{n}})$.

Proof. Equality (95) implies

$$E\big[g(Q^n_t)/\mathcal{H}^n_t\big]=\mathbb{I}\left(Q^n_t=0\right)g(0)+\mathbb{I}\left(Q^n_t>0\right)\left.\frac{E\Big[g\left(X^n_s+\frac{1}{\sqrt{n}}\right)\mathbb{I}\left(\sigma^n_1>s\right)\Big]}{E\Big[\mathbb{I}\left(\sigma^n_1>s\right)\Big]}\right|_{s=\gamma^0_t(Q^n)}$$

and clearly

$$\frac{E\Big[g\left(X^n_s+\frac{1}{\sqrt{n}}\right)\mathbb{I}\left(\sigma^n_1>s\right)\Big]}{E\Big[\mathbb{I}\left(\sigma^n_1>s\right)\Big]}=\hat{\Sigma}^n(s;\overline{g}_n).$$

□

As a consequence of the above Theorem, the conditional law of Q_t^n given $\mathcal{H}_t^n$ can be written as

$$\mathbb{I}\left(Q_t^n = 0\right)\left[\delta_{\{0\}} - \bar{\Sigma}^n(\gamma_t^0(Q^n))\right] + \bar{\Sigma}^n(\gamma_t^0(Q^n)).$$

Moreover, for any g uniformly continuous

$$\bar{\Sigma}^n(\gamma_t^0(Q^n); g) = \hat{\Sigma}^n(\gamma_t^0(Q^n); g) + \varepsilon(n, g) \tag{97}$$

with

$$|\varepsilon(n,g)| \leq \omega_g\left(\frac{1}{\sqrt{n}}\right) = \sup_{|x-y|\leq \frac{1}{\sqrt{n}}} \big|g(x) - g(y)\big|.$$

We can now prove the main result of this section.

Theorem 6.4. *Assume conditions* **C1**, **C2**, **C3** *and* **A1**, **A2**, **A3**. *Then, for any $t \geq 0$, the sequence of measure-valued random variables defined by (89) converge weakly to $\hat{\pi}_t$, on the space of probability measures endowed with the topology of weak convergence. In particular, for any bounded and continuous function g*

$$E\big[g(Q_t^n)/\mathcal{H}_t^n\big] \Rightarrow E\big[g(W_t)/\mathcal{F}_t^\Lambda\big] = \hat{\Pi}_{2\lambda,c}(\zeta_t; g), \qquad \text{for any } t \geq 0.$$

Proof. As in the proof of Theorem 5.1, using (97), it is possible to show that in the Skorohod space of Proposition 5.3

$$\bar{P}\big(\bar{\Sigma}^n(\gamma_t^0(Q^n); g) \underset{n\to\infty}{\longrightarrow} \hat{\Pi}_{2\lambda,c}(\zeta_t; g), \quad \text{for every } g: \mathbb{R}^+ \to \mathbb{R} \text{ bounded and continuous}\big) = 1,$$

since in that space $\gamma_t^0(Q^n)$ converges to ζ_t almost surely.
On the other hand, the total variation of the measure $\delta_{\{0\}} - \bar{\Sigma}^n(\gamma_t^0(Q^n))$ is at most 2, so that the result is achieved once we prove that

$$\mathbb{I}\left(Q_t^n = 0\right) \xrightarrow{Prob} 0, \qquad \text{for any } t \geq 0$$

Indeed, as recalled in (44), the sequence Q_t^n converges weakly to a reflected Brownian motion $W_t + \Lambda_t$. Then, the above convergence can be obtained by noting that

- the function $\mathbb{I}\left(x = 0\right)$ has a discontinuity point at $x = 0$,
- $P(W_t + \Lambda_t = 0) = 0$,
- the continuous mapping theorem implies that $\mathbb{I}\left(Q_s^n = 0\right) \Rightarrow 0$.

□

Remark 6.5. *As already observed at the beginning of this section,*

$$\big(\bar{X}_t^n, Q_t^n, C_t^n\big) \Rightarrow (W_t, W_t + \Lambda_t, \Lambda_t)$$

where $\bar{X}_t^n := Q_t^n - C_t^n$. Thanks to the continuity of the limit processes, the convergence can be considered in the space $D_{\mathbb{R}^3[0,\infty)}$ endowed with the topology of the uniform convergence on compact sets. Moreover it is interesting to note that $\gamma_t^0(Q^n) = \gamma_t(C^n) \Rightarrow \gamma_t(\Lambda) = \zeta_t$,
Then, similarly to Proposition 5.3, it is possible to prove that

$$\big(\gamma_t^0(Q^n), \gamma_t(C^n), C_t^n\big) \Rightarrow \big(\gamma_t^0(W_t + \Lambda_t), \gamma_t^0(\Lambda_t), \Lambda_t\big) = (\zeta_t, \zeta_t, \Lambda_t)\,,$$

and therefore an alternative proof of the previous Theorem can be achieved, by using these properties.

We end this section by noting that, even in this new situation, it is possible to give the same approximation for the filter as in Theorem 5.2, namely for $E\big[g(Q_t^n)/\mathcal{H}_t^n\big]$.
More precisely the following result holds

Theorem 6.6. *For all g bounded and continuous and for each $T > 0$, $p > 0$*

$$\int_0^T E\Big|E\big[g(Q_t^n)/\mathcal{H}_t^n\big] - \hat{\Pi}_{2\lambda,c}(\gamma_t^0(Q^n); g)\Big|^p dt \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. Note that, by (97),

$$\begin{aligned}
&\Big|E\big[g(Q_t^n)/\mathcal{H}_t^n\big] - \hat{\Pi}_{2\lambda,c}(\gamma_t^0(Q^n); g)\Big|^p = \\
&= \Big|\mathbb{I}\,(Q_t^n = 0)\, g(0) + \mathbb{I}\,(Q_t^n > 0)\, \hat{\Sigma}^n(\gamma_t^0(Q^n); g) + \varepsilon(n,g) - \hat{\Pi}_{2\lambda,c}(\gamma_t^0(Q^n); g)\Big|^p \\
&\quad \le C(p)\Big|\mathbb{I}\,(Q_t^n = 0)\, \Big(g(0) + \hat{\Sigma}^n(\gamma_t^0(Q^n); g)\Big) + \varepsilon(n,g)\Big|^p \\
&\quad + C(p)\Big|\hat{\Sigma}^n(\gamma_t^0(Q^n); g) - \hat{\Pi}_{2\lambda,c}(\gamma_t^0(Q^n); g)\Big|^p,
\end{aligned}$$

where $C(p)$ is a suitable constant. Then

$$\begin{aligned}
&E\left[\Big|E\big[g(Q_t^n)/\mathcal{H}_t^n\big] - \hat{\Pi}_{2\lambda,c}(\gamma_t^0(Q^n); g)\Big|^p\right] \le \\
&\le C(p)E\left[\Big|\mathbb{I}\,(Q_t^n = 0)\, 2 \parallel g \parallel_\infty + \varepsilon(n,g)\Big|^p\right] + C(p)E\left[\Big|\hat{\Sigma}^n(\gamma_t^0(Q^n); g) - \hat{\Pi}_{2\lambda,c}(\gamma_t^0(Q^n); g)\Big|^p\right].
\end{aligned}$$

The thesis follows since both the addends at the right hand side of the previous inequality converge to zero. The first addend converges to zero by the bounded convergence theorem. To prove that the the second addend converges to zero one has just to substitute ξ_t^n with $\gamma_t^0(Q^n)$ in the proof of Theorem 5.2. □

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