1. (Quasi) Coherent sheaves on affine varieties.

Let $\mathbb{K}$ be an algebraically closed field.

Let X be an affine variety over $\mathbb{K}$, and let $\mathbb{K}[X]$ be the $\mathbb{K}$-algebra of regular functions on X.

Let M be a $\mathbb{K}[X]$-module. We associate to M a sheaf of ab. grps. $\tilde{M}$ on M as follows.

If $p \in X$ let $m_p \subset \mathbb{K}[X]$ be the maximal ideal of (regular) functions vanishing at p. Let M_{m_p} be the localization of M at the multiplicative system

$$S := \mathbb{K}[X] \setminus m_p .$$

Recall that M_{m_p} is the set of equivalence classes of fractions $\frac{m}{f}$ where $m \in M$, $f \in S$, and

$$\frac{m_1}{f_1} \sim \frac{m_2}{f_2} \quad \text{if } \exists g \in S \text{ s.t. } g \cdot (f_2 \cdot m_1 - f_1 \cdot m_2) = 0.$$

Recall that M_{m_p} is a module over the local ring

$$\mathbb{K}[X]_{m_p} \xrightarrow{\sim} \mathcal{O}_{X,p}$$
$$\left[\frac{\alpha}{\beta}\right] \longmapsto \left[\left(X_\beta, \frac{\alpha}{\beta}\right)\right]$$

For $U \subset X$ open let

$$\tilde{M}(U) := \left\{ (s_p)_{p \in U} \;\middle|\; \begin{array}{l} \forall p \in U \;\; s_p \in M_{\mathfrak{m}_p} \text{ and there exist an open } \mathcal{U} \subset U, \\ m \in M \text{ and } f \in \mathbb{K}[X] \text{ with } f(x) \neq 0 \;\; \forall x \in \mathcal{U} \text{ such that} \\ s_x = \left[\frac{m}{f}\right] \text{ for all } x \in \mathcal{U} \end{array} \right\}$$

Then pointwise addition gives $\tilde{M}(U)$ the structure of an ab. grp. If $V \subset U$ are open subsets of X we have an obvious restriction map

$$\tilde{M}(U) \xrightarrow{\rho^{\tilde{M}}_{U,V}} \tilde{M}(V),$$

which is a homomorphism of groups. Moreover the compatibility conditions which define a presheaf are satisfied. Thus $\tilde{M}$ is a presheaf of ab. grps., actually a sheaf.

We define a structure of sheaf of $\mathcal{O}_X$-modules on $\tilde{M}$ as follows. Let $U \subset X$ be open. Let $f \in \mathcal{O}_X(U)$ and $s = (s_p)_{p \in U} \in \tilde{M}(U)$. Then

$$f \cdot s := (f \cdot s_p)_{p \in U}.$$

Expl. 2 We have an isomorphism of sheaves

$$\mathcal{O}_X \xrightarrow{\sim} \widetilde{\mathbb{K}[X]}$$

defined as follows. For $U \subset X$ open let

$$\begin{aligned} \mathcal{O}_X(U) &\longrightarrow \widetilde{\mathbb{K}[X]}(U) \\ f &\longmapsto (f_p)_{p \in U} \end{aligned}$$

Expl. 2 Let $Y \subset X$ be a closed subset, and let $I \subset \mathbb{K}[X]$ be the ideal of Y. For $U \subset X$ open we have the isomorphism of $\mathcal{O}_X(U)$-modules

$$\begin{aligned} \mathcal{I}_{Y/X}(U) &\longrightarrow \tilde{I}(U) \\ f &\longmapsto (f_p)_{p \in U} \end{aligned}$$

Hence $\tilde{I} \cong \mathcal{I}_{Y/X}$.

Let X be an affine variety (over $\mathbb{K}$).
Let M be a $\mathbb{K}[X]$-module, and let $\tilde{M}$ be the corresponding sheaf of $\mathcal{O}_X$-modules. Let $p \in X$, and let $m_p \subset \mathbb{K}[X]$ be the maximal ideal of p. Then we may define

$$\begin{array}{ccc} \tilde{M}_p & \xrightarrow{\phi} & M_{m_p} \\ [(U, s)] & \longmapsto & s_p \end{array} \qquad (*)$$

$$s = \{s_x\}_{x \in U}$$

$s_x \in M_{m_x}$ and locally

$s_x = \frac{a}{f}$ $a \in M$, $f \in \mathbb{K}[X]$

PROP 1 The homomorphism ϕ in $(*)$ is an isomorphism of $\mathcal{O}_{X,p}$-modules.

PROOF Let $\left[\frac{a}{f}\right] \in M_{m_p}$, where $a \in M$, $f \in \mathbb{K}[X]$ with $f(p) \neq 0$.
Let $U := X \setminus V(f)$. Note that $p \in U$. Let $s = \{s_x\}_{x \in U} \in \tilde{M}(U)$ be defined by

$$s_x = \left[\frac{a}{f}\right] \in M_{m_x}$$

Then ϕ maps $[(U, s)]$ to $\left[\frac{a}{f}\right]$. This proves that ϕ is surjective. We finish by proving that ϕ is injective.
Suppose that $[(U, s)] \in \tilde{M}_p$ and $\phi([U, s]) = 0$, i.e. $s_p = 0$.

By shrinking U we may assume that there exist $a \in M$ and $f \in \mathbb{K}[X]$ with $f(x) \neq 0$ $\forall x \in U$ such that

$$s_x = \left[\frac{a}{f}\right] \in M_{m_x} \qquad \forall x \in U.$$

Since $s_p = 0$ there exists $g \in \mathbb{K}[X]$ with $g(p) \neq 0$ such that

$$g \cdot a = 0. \tag{+}$$

Let $V = U \setminus V(g)$. Then $s_x = 0$ for all $x \in V$, and hence $s|_V = 0$. This proves that $[(U, s)] = 0$. $\square$

<u>DEF 1</u> Let X be an algebraic variety over $\mathbb{K}$. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is

(a) <u>quasi-coherent</u> if there exist an open affine cover $X = \bigcup_{i \in I} X_i$ and $\mathbb{K}[X_i]$-modules M_i for each $i \in I$ such that

$$\mathcal{F}|_{U_i} \cong \widetilde{M}_i \qquad \text{for all } i \in I$$

(b) <u>coherent</u> if there exist an open affine cover $X = \bigcup_{i \in I} X_i$ and <u>finitely generated</u> $\mathbb{K}[X_i]$-modules M_i for each $i \in I$ such that

$$\mathcal{F}|_{U_i} \cong \widetilde{M}_i \qquad \text{for all } i \in I$$

Expl 2 Let X be an affine variety (over $\mathbb{K}$.) Then $\mathcal{O}_X$ is a coherent sheaf.

Expl 3 Let X be an algebraic variety, and let

$$\begin{array}{c} E \\ \pi \downarrow \\ X \end{array}$$

be an algebraic vector bundle of rank r. Then $\mathcal{O}_X(E)$ is a coherent sheaf. In fact there exists an open cover $X = \bigcup_{i \in I} X_i$ such that we have a commutative diagram

$$\begin{tikzcd} X_i \times \mathbb{K}^r \arrow[rr, "\varphi_i", "\sim"'] \arrow[dr, "\text{projection}"'] & & \pi^{-1}(X_i) \arrow[dl, "\pi|_{\dots}"] \\ & X_i & \end{tikzcd} \qquad (*)$$

Since open affine subsets of X form a basis for the (Zariski) topology, we may assume that each X_i is affine.

By $(*)$ we have

$$\mathcal{O}_X(E)(X_i) \cong \underbrace{\mathcal{O}_{X_i} \oplus \cdots \oplus \mathcal{O}_{X_i}}_{r} \cong \underbrace{\mathbb{K}[X_i] \oplus \cdots \oplus \mathbb{K}[X_i]}_{r},$$

and hence $\mathcal{O}_X(E)$ is coherent.

2. How to describe $\tilde{M}$

Here X is an affine variety (over $\mathbb{K}$), and M is a fin. gen. $\mathbb{K}[X]$-module.

How do we describe the sheaf $\tilde{M}$?

The easiest case is $M \cong \mathbb{K}[X]^{\oplus d}$, i.e. a free module.

Then $\tilde{M} \cong \mathcal{O}_X^{\oplus d}$.

PROP 2 Suppose that X is irreducible. Then there exists an open dense $U \subset X$ such that

$$\tilde{M}|_U \cong \mathcal{O}_U^{\oplus d}$$

for some $d \in \mathbb{N}$ (if $d=0$, this means that $\tilde{M}|_U \cong 0$).

PROOF Since X is irreducible the field of fractions $\mathbb{K}(X)$ is defined. Then $M \underset{\mathbb{K}[X]}{\otimes} \mathbb{K}(X)$ is a fin. dim'l $\mathbb{K}(X)$-vector space.

Let $\{\alpha_1, .., \alpha_\ell\}$ be a basis of $M \otimes \mathbb{K}(X)$. Then $\forall i \in \{1, .., \ell\}$

$$\alpha_i = m_i \otimes \frac{1}{f_i} \qquad m_i \in M, \quad f_i \in \mathbb{K}[X]^*.$$

Multiplying α_i by $\frac{1}{f_i}$ we may assume that $\alpha_i = m_i \otimes 1$. Let

$$\mathcal{O}_X^{\ell} \xrightarrow{\phi} \tilde{M}$$

be the morphism of sheaves defined by setting (for $V \subset X$ open)

$$(\lambda_1, .., \lambda_\ell) \longmapsto \sum_{i=1}^{\ell} \lambda_i (m_i|_V) \qquad \lambda_1, .., \lambda_\ell \in \mathcal{O}_V (= \mathcal{O}_X|_V).$$

It suffices to prove that there exists an open dense $U \subset X$ such that $\phi|_U$ is an isomorphism.

Let $n_1, \dots, n_a \in M$ be generators of M. Then $\forall i \in \{1, \dots, a\}$

$$(\#) \qquad n_i \otimes 1 = \sum_{j=1}^{\ell} \frac{h_{ij}}{g_{ij}} m_j \otimes 1 \qquad h_{i,j}, g_{ij} \in \mathbb{K}[X] \quad g_{ij} \neq 0.$$

Let $U = X \setminus \bigcup_{i,j} V(g_{ij})$. Then U is a Z-dense open subset of X, and by $(\#)$ the restriction of ϕ to U is surjective. But $\phi|_U$ is also injective, as is easily checked.

DEF 2 Let $p \in X$. The <u>fiber of $\tilde{M}$ at p</u> is the quotient

$$\tilde{M}(p) := \tilde{M}_p / m_p \tilde{M}_p = M_{m_p} / m_p M_{m_p} = M / m_p M$$

RMK 1 $\tilde{M}(p)$ is a module over $\mathcal{O}_{X,p} / m_p \mathcal{O}_{X,p} = \mathbb{K}$, i.e. a $\mathbb{K}$-vector space.

EXPL 4 If $M = \mathbb{K}[X]^{\oplus d}$ then $\dim_{\mathbb{K}} \tilde{M}(p) = d \quad \forall p \in X$.

EXPL 5 Let $X = \mathbb{A}^n$, $I \subset \mathbb{K}[x_1, \dots, x_n]$ ideal of $p_0 = (0, 0, \dots, 0)$. Thus

$$\tilde{I} = \mathcal{I}_{p_0} \mathbb{A}^n.$$

Then

$$\tilde{I}|_{(\mathbb{A}^n \setminus \{p_0\})} \cong \mathcal{I}_{\emptyset, \mathbb{A}^n \setminus \{p_0\}} = \mathcal{O}_{\mathbb{A}^n \setminus \{p_0\}}, \qquad \text{free of rk}=1.$$

In particular

$$\tilde{I}(p) \cong \mathbb{K} \qquad p \neq p_0$$

What about $\tilde{I}(p_0)$? We have an isomorphism

$$\begin{aligned} \mathbb{K}^n &\xrightarrow{\sim} \tilde{I}(p_0) \\ (\lambda_1, \dots, \lambda_n) &\longmapsto \left[\sum_{i=1}^n \lambda_i x_i\right] \end{aligned}$$

In general, since M is fin. gen. and $\mathbb{K}[X]$ is Noetherian, there exists an ex. seq.

$$\mathbb{K}[X]^a \xrightarrow{f} \mathbb{K}[X]^b \xrightarrow{g} M \to 0$$

If $p \in X$, then we have an ex. seq.

$$\underbrace{\mathbb{K}[X]^a \otimes \mathbb{K}[X]/_{m_p}}_{\left(\mathbb{K}[X]/_{m_p}\right)^a} \xrightarrow{f(p)} \underbrace{\mathbb{K}[X]^b \otimes \mathbb{K}[X]/_{m_p}}_{\left(\mathbb{K}[X]/_{m_p}\right)^b} \xrightarrow{g(p)} \underbrace{M \otimes \mathbb{K}[X]/_{m_p}}_{\cong\, \tilde{M}(p)} \to 0$$

Hence

$$\dim_{\mathbb{K}} \tilde{M}(p) = b - \operatorname{rk} f(p).$$

In particular the function

$$\begin{array}{rcl} X & \xrightarrow{\delta} & \mathbb{N} \\ p & \longmapsto & \dim_{\mathbb{K}} \tilde{M}(p) \end{array} \qquad (*)$$

is upper semicontinuous.

RMK 2 We have proved that if X is irreducible then $\tilde{M}$ is loc. free of $rk = r$ on an open dense subset $U \subset X$. In particular $\dim_{\mathbb{K}} \tilde{M}(p) = r$ for all $p \in U$. One's guess is that, conversely, if $V \subset X$ is open and $\dim_{\mathbb{K}} \tilde{M}(p)$ $\forall p \in V$, then $\tilde{M}_{|V}$ is loc. free. This is true, but needs a proof.

3. General properties of the $\sim$ construction.

[Here X is an affine variety (over $\mathbb{K}$).]

Let M be a $\mathbb{K}[X]$-module (not necessarily fin. gen.). Then we have a homomorphism of $\mathbb{K}[X]$-modules

$$(\star) \qquad \begin{aligned} M &\longrightarrow \Gamma(X, \tilde{M}) \\ a &\longmapsto (a_p)_{p \in X} \end{aligned} \qquad a_p = \frac{a}{1} \in M_{m_p}$$

PROP 3 The homomorphism in $(\star)$ is an isomorphism.

Before sketching the proof of Prop. 3 we "recall" the following result.

PROP 4 Let R be a (commutative, with 1) ring, and let M be an R-module. Let $a \in M$ be non zero. Then there exists a maximal ideal $m \subset R$ such that

$$M_m \ni \frac{a}{1} \neq 0.$$

PROOF Let

$$\mathrm{Ann}(a) := \{ f \in R \mid f \cdot a = 0 \}.$$

Then $\mathrm{Ann}(a)$ is a proper ideal of R (proper because $a \neq 0$), hence there exists a maximal ideal $m \subset R$ containing $\mathrm{Ann}(a)$.

Then $\frac{a}{1} \in M_m$ is non-zero. In fact $\frac{a}{1}=0$ if and only if there exists $f \in (R \setminus m)$ such that $f \cdot a = 0$, and the existence of such an f contradicts $\mathrm{Ann}(a) \subset m$.

PROOF OF PROP. 3 Injectivity follows at once from Prop. 4.

We prove surjectivity under the additional hypothesis that X is irreducible and M is a torsion-free $\mathbb{K}[X]$-module.

Let $s \in \Gamma(X, \tilde{M})$. By hypothesis there exist an open cover $X = \bigcup_{i \in I} X_i$ and $\forall i \in I$ $a_i \in M$, $f_i \in \mathbb{K}[X]$ such that

$$s_{i,p} = \frac{a_i}{f_i} \in M_{m_p} \qquad \forall p \in X_i \ \left(\text{it is understood that } f_i(p) \neq 0 \ \forall p \in X_i\right)$$

Since principal open subsets $X_g \subset X$ are a basis of the topology we may assume that $X_i = X_{f_i}$ $\forall i \in I$.

Since the Z-topology is precompact we may assume that I is finite, say $I = \{1, \dots, d\}$. Since

$$V(f_1, \dots, f_d) = \emptyset$$

there exist $\lambda_1, \dots, \lambda_d \in \mathbb{K}[X]$ such that

$$\lambda_1 \cdot f_1 + \dots + \lambda_d \cdot f_d = 1.$$

Thus

$$s = \lambda_1 \cdot f_1 s + \lambda_2 \cdot f_2 s + \ldots + \lambda_d f_d s$$

We claim that for all $i \in \{1, .., d\}$

$(\times)$ $\qquad f_i \cdot s = a_i \qquad$ (meaning that the section $f_i \cdot s$ is equal to the section corresponding to a_i for the homomorphism in $\star$).

In fact if $i, j \in \{1, .., d\}$ the equality for $p \in X_i \cap X_j = X_{f_i} \cap X_{f_j} = X_{f_i \cdot f_j}$

$$M_{m_p} \ni \frac{a_i}{f_i} = \frac{a_j}{f_j} \in M_{m_p}$$

means that there exist $\varphi \in \mathbb{K}[X]$ with $\varphi(p) \neq 0$ s.t.

$$\varphi \cdot (f_j \cdot a_i - a_j \cdot f_i) = 0 .$$

Since $\varphi \neq 0$ and M is torsion-free we get that

(ℓ) $\qquad f_j \cdot a_i - a_j \cdot f_i = 0$

Now we prove $(\times)$, i.e. that for all $p \in X$

$(\boxtimes)$ $\qquad f_i \cdot s_p = a_{i,p} \qquad$ (in M_{m_p})

If $p \in X_j = X_{f_j}$ we have

by (ℓ)

$$f_i \cdot s_p = f_i \cdot \frac{a_j}{f_j} \overset{\downarrow}{=} f_j \cdot \frac{a_i}{f_j} = \frac{a_i}{1},$$

and we are done.

The proof in the general case is obtained by fudging a bit the proof above. Note the similarity with the proof that every regular function on a closed subset $X \subset \mathbb{A}^n$ is the restriction of a polynomial, i.e. the case $M = \mathbb{K}[x_1, \dots, x_n]/I(X)$ as module over itself. $\square$

Let M, N be $\mathbb{K}[X]$-modules. If

$$M \xrightarrow{\varphi} N$$

is a homomorphism of $\mathbb{K}[X]$-modules we get a homomorphism

$$M_{m_p} \xrightarrow{\varphi_p} N_{m_p}$$

for each $p \in X$, and for each open $U \subset X$ we get a homomorphism of $\mathcal{O}_X(U)$-modules

$$\tilde{M}(U) \xrightarrow{\tilde{\varphi}_U} \tilde{N}(U)$$

$$(s_p)_{p\in U} \longmapsto (\varphi_p(s_p))_{p\in U}$$

This defines a morphism of sheaves of $\mathcal{O}_X$-modules

$$\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{N}.$$

On the other hand, let

$$\tilde{M} \xrightarrow{\varphi} \tilde{N}$$

be a morphism of sheaves of $\mathcal{O}_X$-modules. Then we get a homomorphism of $\mathbb{K}[X]$-modules

$$\underset{\substack{\uparrow \\ \text{Prop. 3}}}{M =} \Gamma(X, \tilde{M}) \xrightarrow{\Gamma(\varphi)} \Gamma(X, \tilde{N}) \underset{\substack{\uparrow \\ \text{Prop. 3}}}{=} N$$

These two constructions are inverses of each other, hence we get an equivalence between

$\{$Category of $\mathbb{K}[X]$-modules$\}$ $\leftrightarrow$ $\{$Category of $\mathcal{O}_X$-modules given by $\tilde{M}$ for a $\mathbb{K}[X]$-module $M\}$

Let M, N be $\mathbb{K}[X]$-modules, and let

$$M \xrightarrow{\varphi} N$$

be a homomorphism of modules. Then for every $p \in X$ we get a homomorphism of $\mathbb{K}[X]_{m_p} = \mathcal{O}_{X,p}$-modules

$$\tilde{M}_p \xrightarrow{\varphi_p} \tilde{N}_p$$

which gives rise to an associated morphism of sheaves of $\mathcal{O}_X$-modules

$$\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{N}$$

mapping a local section $\frac{a}{f}$ to $\frac{\varphi(a)}{f}$.

PROP 5 Let

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} P$$

be an ex. seq. of $\mathbb{K}[X]$-modules. The associated sequence of morphism of sheaves

$$\tilde{M} \xrightarrow{\tilde{\alpha}} \tilde{N} \xrightarrow{\tilde{\beta}} \tilde{P} \qquad (*)$$

is also exact.

PROOF

The sequence $(*)$ is exact if and only if for all $p \in X$ the sequence of $\mathcal{O}_{X,p}$-modules

$$M_{m_p} \xrightarrow{\alpha_p} N_{m_p} \xrightarrow{\beta_p} P_{m_p}$$

is exact. For a proof of this fact see Prop. 3.3. in [Atiyah-MacDonald]

$\square$

COR 1 Let

$$M \xrightarrow{\varphi} N$$

be a homomorphism of modules, and let

$$\tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{N}$$

be the associated morphism of sheaves of $\mathcal{O}_X$-modules. Then

$$\ker \tilde{\varphi} \cong \widetilde{\ker \varphi} \qquad \operatorname{coker} \tilde{\varphi} \cong \widetilde{\operatorname{coker} \varphi}$$

PROOF Let $K = \ker \varphi$. This is a $\mathbb{K}[X]$-module, and we have the ex. seq.

$$K \xrightarrow{i} M \xrightarrow{\varphi} N,$$

where $i: K \to M$ is the inclusion homomorphism. By Prop. 5 the associated sequence of $\mathcal{O}_X$-modules

$$\tilde{K} \xrightarrow{\tilde{i}} \tilde{M} \xrightarrow{\tilde{\varphi}} \tilde{N}$$

is exact, i.e. $\tilde{K} = \ker \tilde{\varphi}$. This proves that $\ker \tilde{\varphi} \cong \widetilde{\ker \varphi}$. The proof of the second isomorphism is analogous. $\square$

PROP 6 Let M, N be $\mathbb{K}[X]$-modules. Then

$$\tilde{M} \oplus \tilde{N} \cong \widetilde{M \oplus N} \qquad \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} \cong \widetilde{M \otimes N}$$

PROOF The first isomorphism holds because for every $p \in X$ we have the isomorphism

$$\begin{aligned} M_{m_p} \oplus N_{m_p} &\xrightarrow{\sim} (M \oplus N)_{m_p} \\ \left(\frac{a}{f}, \frac{b}{g}\right) &\longmapsto \frac{ga + fb}{f \cdot g} \end{aligned}$$

The second isomorphism holds because for every $p \in X$ we have the isomorphism (see Prop. 3.7 in [Atiyah-MacDonald])

$$\begin{aligned} M_{m_p} \otimes N_{m_p} &\xrightarrow{\sim} (M \otimes N)_{m_p} \\ \frac{a}{f} \otimes \frac{b}{g} &\longmapsto \frac{a \otimes b}{f \cdot g} \end{aligned}$$

$\square$

Before stating the next result we recall that if R is a (commutative) ring (with 1), and $a \in R$, then

$$S := \{a^n \mid n \in \mathbb{N}\} \qquad (*).$$

is a multiplicative system. One lets

$$R_a := S^{-1} \cdot R$$

where S is as in $(*)$.

Similarly, if M is an R-module one lets

$$M_a := S^{-1} R.$$

EXPL 6 Let $f \in \mathbb{K}[X]$. Then we have an isomorphism of rings.

$$\mathbb{K}[X]_f \xrightarrow{\sim} \mathbb{K}[X_f]$$
$$\frac{g}{f^n} \longmapsto \left(p \longmapsto \frac{g(p)}{f(p)^n}\right)$$

PROP 7 Let M be a $\mathbb{K}[X]$-module, and let $f \in \mathbb{K}[X]$. Then we have an isomorphism of sheaves

$$\widetilde{M_f} \xrightarrow{\sim} \tilde{M}|_{X_f}$$
$$\left(\text{loc. section } \frac{a}{g \cdot f^n}\right) \longmapsto \left(\text{loc. section } \frac{a}{g \cdot f^n}\right)$$

PROP 8 Let $\mathcal{F}$ be a quasi-coherent sheaf on an affine variety X. Then the morphism of sheaves

$$\widetilde{\Gamma(X,\mathcal{F})} \longrightarrow \mathcal{F}$$

is an isomorphism. If $\mathcal{F}$ is coherent then the space of sections $\Gamma(X,\mathcal{F})$ is a finitely generated $\mathbb{K}[X]$-module.

Rmk 3 Let $\mathcal{F}$ be a quasi-coherent sheaf on an algebraic variety X. By definition X is covered by open affine sets U_i such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ where M_i is a $\mathbb{K}[U_i]$-module, with M_i fin. gen. if $\mathcal{F}$ is coherent. The proposition above states that if X is affine then $\mathcal{F} \cong \tilde{M}$ for a $\mathbb{K}[X]$-module M, with M fin. gen. if $\mathcal{F}$ is coherent. In fact the two statements are equivalent by Prop. 3 p. 22.

PROOF OF PROP. 8 (Following [Mumford-Oda AG II, Prop.-Def. 5.1])

By Noetherianity of the Zariski topology we may assume that there is a finite open cover $X = \bigcup_{i \in I} U_i$ (I finite) by affine subsets and a $\mathbb{K}[U_i]$-module M_i for each $i \in I$, with M_i fin. gen. if $\mathcal{F}$ is coherent, such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.

Since principal open subsets X_f form a basis of the Z-top. we may assume that each U_i is principal:

$$U_i = X_{f_i} \qquad f_i \in \mathbb{K}[X].$$

(You must use Prop. 7 ....) For $i, j \in I$ define sheaves of $\mathcal{O}_X$-modules $\mathcal{F}_i^*$ and $\mathcal{F}_{i,j}^*$ by setting (for $V \subset X$ open)

$$\mathcal{F}_i^*(V) := \Gamma(V \cap U_i, \mathcal{F}) \qquad \mathcal{F}_{i,j}^*(V) := \Gamma(V \cap \underbrace{U_i \cap U_j}_{\substack{\| \\ X_{f_i} \cap X_{f_j} = X_{f_i f_j}}}, \mathcal{F})$$

We have an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \to \bigoplus_{i \in I} \mathcal{F}_i^* \longrightarrow \bigoplus_{(i,j) \in I^2} \mathcal{F}_{i,j}^*$$

$$\mathcal{F}(V) \ni s \longmapsto (\dots, s|_{V \cap U_i}, \dots)$$

$$(\dots, \underset{\mathcal{F}_i^*(V)}{s_i \in} \dots) \longmapsto (\dots, \overbrace{s_i|_{V \cap U_i \cap U_j} - s_j|_{V \cap U_i \cap U_j}}^{\mathcal{F}_{i,j}^*(V)}, \dots)$$

Hence (by Cor. 2 p. 17) in order to prove the first part of the proposition it suffices to prove that there exist $\mathbb{K}[X]$-modules N_i and $P_{i,j}$ (for $i,j \in I$) such that $\mathcal{F}_i^* \cong \widetilde{N_i}$ and $\mathcal{F}_{i,j}^* \cong \widetilde{P_{i,j}}$.

Let M_i° be M_i viewed as $\mathbb{K}[X]$ module via the homomorphism of rings

$$\mathbb{K}[X] \longrightarrow \mathbb{K}[X]_{f_i} = \mathbb{K}[\underbrace{X_{f_i}}_{= U_i}]$$

Similarly let $M_{i,j}^\circ$ be the $\mathbb{K}[U_i \cap U_j]$-module

$$(M_i)_{f_j} \cong (M_j)_{f_i}$$

viewed as $\mathbb{K}[X]$-module. One shows (do it) that

$$\mathcal{F}_i^* \cong \widetilde{M_i^\circ} \qquad \mathcal{F}_{i,j}^* \cong \widetilde{M_{i,j}^\circ}.$$

It remains to prove that if $\mathcal{F}$ is coherent then M is finitely generated. Note that since $\mathcal{F}$ is coherent each M_i is a fin. gen. $\mathbb{K}[X]_{f_i}$-module, but this does not imply that M_i° is a fin. gen. $\mathbb{K}[X]$-module, in fact it is only in trivial cases. One may argue as follows. Let $s_i(1), \dots, s_i(m_i) \in M_i$ be generators of M_i as $\mathbb{K}[X]_{f_i}$-module. Then there exists $N \in \mathbb{N}$ s.t. $f_i^N \cdot s_i(1), \dots, f_i^N \cdot s_i(m_i)$ extend to a section of $\mathcal{F}$ on X, i.e.

$$f_i^N \cdot s_i(1), \dots, f_i^N \cdot s_i(m_i) \in \Gamma(X, \mathcal{F}). \qquad (+)$$

Let $P \subset \Gamma(X, \mathcal{F})$ be the $\mathbb{K}[X]$-submodule generated by the sections in $(+)$ for all $i \in I$. The homomorphism of sheaves

$$\tilde{P} \longrightarrow \tilde{\mathcal{F}}$$

is an isomorphism. Since P is fin. gen. over $\mathbb{K}[X]$ we are done (note: this shows that $P = \Gamma(X, \mathcal{F})$). $\square$