1. PRESHEAVES

X a topological space

DEF 1 A presheaf of abelian groups $\mathcal{F}$ on X consists of the assignment of an abelian group $\mathcal{F}(U)$ to each open $U \subset X$, and of a homomorphism of groups

$$\mathcal{F}(U) \xrightarrow{\rho^{\mathcal{F}}_{U,V}} \mathcal{F}(V) \qquad \text{(often we set } \rho_{U,V} = \rho^{\mathcal{F}}_{U,V}\text{)}$$

to each inclusion $V \subset U$ of open subsets of X subject to the following requirements:

I. $\mathcal{F}(\emptyset) \cong \{0\}$

II. $\rho_{U,U} = \mathrm{Id}_{\mathcal{F}(U)}$

III. If $W \subset V \subset U$ are open subsets of X then

$$\rho^{\mathcal{F}}_{V,W} \circ \rho^{\mathcal{F}}_{U,V} = \rho^{\mathcal{F}}_{U,W}.$$

Let $\mathcal{F}$ be a presheaf on X. If $U \subset X$ is open the elements of $\mathcal{F}(U)$ are the sections of $\mathcal{F}$ over U.

The terminology could be motivated by Expl 2 below. A better motivation is given by the espace étalé associated to a presheaf. The $\rho^{\mathcal{F}}_{U,V}$ are the restriction maps. Sometimes we let

$$s|_V := \rho^{\mathcal{F}}_{U,V}(s).$$

Expl 1 Let G be a topological abelian group, e.g. $G=(\mathbb{Z},+)$ with the discrete topology, $G=(\mathbb{R},+)$ or $G=(\mathbb{R}^{\times},\cdot)$ with the Euclidean topology. For $U\subset X$ open let

$$\underline{G}_X(U) := \{ U \xrightarrow{s} G \mid s \text{ continuous} \}.$$

Pointwise addition gives $\underline{G}_X(U)$ the structure of an abelian group. If $V\subset U\subset X$ are open let

$$\begin{aligned} G_X(U) &\xrightarrow{\rho_{U,V}} G_X(V) \\ s &\longmapsto s|_V \end{aligned}$$

Then $\rho_{U,V}$ is a homomorphism of groups.

These assignments define a sheaf of abelian groups G_X on X.

Expl 2. Let $\begin{array}{c} E \\ \pi\downarrow \\ X \end{array}$ be a real or complex vector bundle.

For $U\subset X$ open let

$$\mathcal{S}_E(U) := \{ U \xrightarrow{s} E \mid s \text{ continuous and } \pi\circ s = \mathrm{Id}_U \}$$

Fiberwise addition gives $\mathcal{S}_E(U)$ the structure of an abelian group. If $V\subset U\subset X$ are open let

$$\begin{aligned} \mathcal{S}_E(U) &\xrightarrow{\rho_{U,V}} \mathcal{S}_E(V) \\ s &\longmapsto s|_V \end{aligned}$$

Then $\rho_{U,V}$ is a homomorphism of groups. These assignments define a sheaf of abelian groups $\mathcal{S}_E$ on X. This is the <u>sheaf of germs of sections of E.</u>

DEF 2 Let $\mathcal{F}, \mathcal{G}$ be presheaves of ab. grps on X. A <u>morphism of presheaves</u>

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$$

consists of the assignment of a homomorphism of groups

$$\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)$$

to each open $U \subset X$, such that for each inclusion $V \subset U$ of open subsets of X the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ {\scriptstyle \rho^{\mathcal{F}}_{U,V}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathcal{G}}_{U,V}} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

is commutative.

Rmk 1 Presheaves of ab. grps. on X with morphisms defined above form a category, i.e. we have $\mathcal{F} \xrightarrow{\mathrm{Id}_{\mathcal{F}}} \mathcal{F}$ defined by the identity $\mathcal{F}(U) \to \mathcal{F}(U)$ for each open $U \subset X$, and the composition of morphisms is a morphism.

In particular we have the notion of <u>isomorphism of presheaves</u>.

<u>Expl 3.</u> Let $G \xrightarrow{\Phi} H$ be a continuous homomorphism of topological groups. Then we get a morphism of presheaves $G_X \xrightarrow{\varphi} H_X$ by setting

$$\begin{aligned} \mathcal{G}_X(U) &\xrightarrow{\varphi_U} H_X(U) \\ s &\longmapsto \Phi \circ s \end{aligned}$$

<u>Expl 4</u> Let

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\theta"] \\ & X & \end{tikzcd}$$

be a continuous homomorphism of real or complex vector bundles on X. Then we get a morphism of presheaves

$$\mathcal{J}_E \xrightarrow{\varphi} \mathcal{J}_F$$

by setting

$$\begin{aligned} \mathcal{J}_E(U) &\xrightarrow{\varphi_U} \mathcal{J}_F(U) \\ s &\longmapsto \Phi \circ s \end{aligned}$$

<u>Expl 5</u> Given a sheaf $\mathcal{F}$ on X we have unique morphisms

$$0_X \longrightarrow \mathcal{F} \qquad \mathcal{F} \longrightarrow 0_X$$

Let $\mathcal{F}, \mathcal{G}$ be presheaves of ab. grps on X, and let

$$\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$$

be a morphism of presheaves

DEF 3 Let $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ be a morphism of presheaves.
Then φ is injective/surjective if

$$\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)$$

is respectively injective/surjective for all open $U \subset X$.

DEF 4 The kernel presheaf $\ker \varphi$ is defined by setting

$$(\ker \varphi)(U) := \ker \left(\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U) \right)$$

for $U \subset X$ open, and

$$\begin{aligned} (\ker \varphi)(U) &\longrightarrow (\ker \varphi)(V) \\ s &\longmapsto \rho^{\mathcal{F}}_{U,V}(s) \end{aligned}$$

for $V \subset U \subset X$ open.

DEF 5 The image presheaf $\mathrm{Im}\varphi$ is defined by setting

$$(\mathrm{Im}\varphi)(U) := \mathrm{Im}\left(\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)\right)$$

for $U \subset X$ open, and

$$\begin{aligned} (\mathrm{Im}\varphi)(U) &\longrightarrow (\mathrm{Im}\varphi)(V) \\ s &\longmapsto \rho^{\mathcal{F}}_{U,V}(s) \end{aligned}$$

for $V \subset U \subset X$ open.

Rmk 2 A morphism of presheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is injective if and only if $\ker\varphi \cong 0_X$, and it is surjective if and only if $\mathrm{Im}\varphi = \mathcal{G}$.

Direct sum of groups and quotients of groups give corresponding operations on presheaves of ab. grps.

DEF 6 Let $\mathcal{F}, \mathcal{G}$ be presheaves of ab. grps. on X. For $U \subset X$ open let

$$(\mathcal{F} \oplus \mathcal{G})(U) := \mathcal{F}(U) \oplus \mathcal{G}(U)$$

and for $V \subset U$ open subsets of X let

$$(\mathcal{F} \oplus \mathcal{G})(U) \longrightarrow (\mathcal{F} \oplus \mathcal{G})(V)$$

be the homomorphism $\rho^{\mathcal{F}}_{U,V} \oplus \rho^{\mathcal{G}}_{U,V}$.

Then $\mathcal{F} \oplus \mathcal{G}$ is a presheaf of ab. grps. on X.

DEF 7 Let $\mathcal{F}$ be a presheaf of ab. grps. on X. A <u>subpresheaf of $\mathcal{F}$</u> consists of the assignment of a subgroup

$$\mathcal{G}(U) < \mathcal{F}(U)$$

for every open $U \subset X$, such that

$$\rho^{\mathcal{F}}_{U,V}(\mathcal{G}(U)) \subset \mathcal{G}(V)$$

for every open $V \subset U \subset X$. Restricting $\rho^{\mathcal{F}}_{U,V}$ to $\mathcal{G}(U)$ we get restriction maps, and $\mathcal{G}$ is itself a presheaf of ab. grps. on X.

DEF 8 Let $\mathcal{F}$ be a presheaf of ab. grps. on X, and let $\mathcal{G} \subset \mathcal{F}$ be a subpresheaf.

For $U \subset X$ open let

$$\mathcal{F}/\mathcal{G}(U) := \mathcal{F}(U)/\mathcal{G}(U)$$

If $V \subset U \subset X$ are open the homomorphism $\rho^{\mathcal{F}}_{U,V}$ and $\rho^{\mathcal{G}}_{U,V}$ define

$$\mathcal{F}/\mathcal{G}(U) \xrightarrow{\rho_{U,V}} \mathcal{F}/\mathcal{G}(V)$$

With these data $\mathcal{F}/\mathcal{G}$ is a presheaf.

2. SHEAVES

DEF 9 A presheaf of abelian groups $\mathcal{F}$ on X is a <u>sheaf</u> if for all open $U \subset X$ and $U = \bigcup_{i \in I} V_i$ where each V_i is open the following hold:

I. If $s \in \mathcal{F}(U)$ and $\rho_{U,V_i}(s) = 0$ $\forall i \in I$ then $s = 0$.

II. If $\forall i \in I$ we are given $s_i \in \mathcal{F}(V_i)$ such that

$$\rho_{V_i, V_i \cap V_j}(s_i) = \rho_{V_j, V_i \cap V_j}(s_j)$$

for all $i, j \in I$, then there ~~exist~~ exists $s \in \mathcal{F}(U)$ such that

$$\rho_{U, V_i}(s) = s_i \quad \forall i \in I.$$

Rmk 3 Let $\mathcal{F}$ be a sheaf, and let $s, t \in \mathcal{F}(U)$ be such that $s|_{V_i} = t|_{V_i}$ for all $i \in I$. Then $s = t$ by I above.

Expl 6 Let G be a topological group. Then G_X is a sheaf.

Expl 7 Let $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ be a real or complex vector bundle.

Then $\mathcal{S}_E$ is a sheaf.

Expl 8 Let $\mathcal{C}(X)$ be the sheaf of germs of continuous complex functions on X, i.e. $\mathcal{C}(X) = \mathcal{J}_E$ where $E = \mathbb{C} \times X$
$$\downarrow$$
$$X$$

or $\mathcal{C}(X) = \mathbb{C}_X$ when $\mathbb{C}$ has the Euclidean topology.

Let $\mathbb{Z}_X \subset \mathcal{C}(X)$ be the subsheaf of locally constant integer valued functions. The presheaf $\mathcal{F} := \mathcal{C}(X)/\mathbb{Z}_X$ is in general not a sheaf.

In fact consider $X = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Let

$$U_1 := \mathbb{C}^* \setminus \mathbb{R}_- \qquad U_2 := \mathbb{C}^* \setminus \mathbb{R}_+$$

$$U_1 \xrightarrow{f_1} \mathbb{C} \qquad U_2 \xrightarrow{f_2} \mathbb{C}$$

f_1 continuous $\qquad$ f_2 continuous

$$e^{2\pi i f_1(z)} = z \qquad e^{2\pi i f_2(z)} = z$$

Let $\bar{f}_k := [f_k] \in \mathcal{F}(U_k)$. Then

$$[f_1]|_{U_1 \cap U_2} = [f_2]|_{U_1 \cap U_2},$$

but there does not exist $[f] \in \mathcal{F}(\mathbb{C}^*)$ such that

$$[f]\big|_{U_k} = [f_k] \qquad k \in \{1,2\}.$$

In fact if such an f exists, then $\frac{1}{2\pi i} f : \mathbb{C}^* \to \mathbb{C}$ is a continuous logarithm of $\mathbb{C}^* \xrightarrow{\mathrm{Id}} \mathbb{C}^*$, and this is a contradiction.

"Moral" Even if we deal with sheaves, simple operations (such as passing to the quotient) might produce presheaves which are not sheaves.

DEF 10 A morphism of sheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is a morphism of the corresponding presheaves.

$\boxed{\text{stalks}}$

By definition a sheaf is "more local" than a presheaf. The notion of stalk of a presheaf clarifies this point.

Let $\mathcal{F}$ be a presheaf (of ab. grps) on X, and let $p \in X$.

On the set

$$\mathcal{N}_p := \{ (U, s) \mid p \in \underset{\text{open}}{U} \subset X , \quad s \in \mathcal{F}(U) \}$$

let $\sim$ be the relation

$(U, s) \sim (V, t)$ if there exists $(W, u) \in \mathcal{N}_p$

such that

$$\begin{cases} W \subset U \cap V \\ s|_W = u \\ t|_W = u \end{cases}$$

This is an equivalence relation.

<u>DEF 11</u> The quotient $\mathcal{F}_p := \mathcal{N}_p / \sim$ is the <u>stalk of $\mathcal{F}$ at p</u>. Elements of $\mathcal{F}_p$ are <u>germs of sections of $\mathcal{F}$</u>.

<u>NOTATION</u> For $s \in \mathcal{F}(U)$ and $p \in U$ we let $s_p := [(U, s)]$

The operation (on $\mathcal{F}_p$)

$$[(U,s)] + [(V,t)] := [(U\cap V,\ s|_{U\cap V} + t|_{U\cap V})]$$

is well defined and gives $\mathcal{F}_p$ the structure of an ab. grp.

Rmk 4 Let $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ be a morphism of presheaves (of ab. grps.) on X, and let $p \in X$. Then we get a homomorphism of ab. grps

$$\begin{aligned} \mathcal{F}_p &\xrightarrow{\varphi_p} \mathcal{G}_p \\ [(U,s)] &\longmapsto [(U, \varphi_U(s))] \end{aligned}$$

PROP 1 Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves (note: sheaves not presheaves). If $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for all $p \in X$, then φ is an isomorphism.

PROOF Let $U \subset X$ be open. We must prove that

$$\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)$$

is an isomorphism of ab. grps. Suppose that $\varphi_U(s) = 0$. Then

$$\varphi_p(s_p) = \varphi_U(s)_p = 0$$

and hence $s_p = 0$ because s_p is an isomorphism. This means that there exists an open $V_p \subset U$ containing p such that $s|_{V_p} = 0$. Since U is covered by the open subsets V_p, it follows that $s = 0$. This proves that φ_U is injective.

Now let $t \in \mathcal{G}(U)$. Let $p \in U$. By hypothesis there exists $s_p \in \mathcal{F}_p$ such that

$$\varphi_p(s_p) = t_p$$

This means that there exist an open $V_p \subset U$ containing p and $s_{V_p} \in \mathcal{F}(V_p)$ such that

$$\varphi_{V_p}(s_{V_p}) = t|_{V_p}. \qquad (*)$$

We claim that for each $p, q \in U$ we have

$$s_{V_p}|_{V_p \cap V_q} = s_{V_q}|_{V_p \cap V_q} \qquad (\star)$$

In fact $(*)$ gives that

$$\varphi_{V_p \cap V_q}\left(s_{V_p}|_{V_p \cap V_q}\right) = t|_{V_p \cap V_q} = \varphi_{V_p \cap V_q}\left(s_{V_q}|_{V_p \cap V_q}\right)$$

Hence for all $x \in V_p \cap V_q$ we have

$$\varphi_x([s_{V_p}, x]) = \varphi_x([s_{V_q}, x])$$

Since φ_x is an isomorphism it follow that $s_{V_p}|_{V_p \cap V_q}$ and $s_{V_q}|_{V_p \cap V_q}$ have the same stalk at each $x \in V_p \cap V_q$, This implies that they are equal (because $\mathcal{F}$ is a sheaf). We have proved that $(\star)$ holds, and hence (since $\mathcal{F}$ is a sheaf) there exists $s \in \mathcal{F}(U)$ such that

$$s|_{V_p} = s_{V_p}.$$

Clearly $\varphi(s) = t$. (again because $\mathcal{G}$ is a sheaf).

Expl 9 Let $\mathcal{F} := \mathcal{C}(X)/\mathbb{Z}_X$ be the quotient presheaf of Expl. 8.

Let $\mathcal{G} := \mathbb{C}^*_X$ where $\mathbb{C}^*$ has the Euclidean topology, i.e.

$$\mathcal{G}(U) = \{ U \xrightarrow{s} \mathbb{C}^* \mid s \text{ continuous} \}. \qquad (\bullet)$$

Note that $\mathcal{G}$ is a sheaf, while in general $\mathcal{F}$ is a presheaf but not a sheaf. Let $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ be the morphism of presheaves defined by

$$\begin{aligned} \mathcal{F}(U) &\xrightarrow{\varphi_U} \mathcal{G}(U) \\ [f] &\longmapsto e^{2\pi i f} \end{aligned}$$

Suppose that each point of X has a simply connected open neighborhood (e.g. X a manifold). Then

$$\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p$$

is an isomorphism for each $p \in X$. On the other hand if $H^1(X;\mathbb{Z}) \neq 0$ then $\mathcal{F}$ is not a sheaf, and hence φ is not an isomorphism. (It suffices that X contains an open set U s.t. $H^1(U;\mathbb{Z}) \neq 0$.)

Rmk 5 Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Suppose that $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all $p \in X$. Arguing as in the proof of Prop. 1 one shows that for all $U \subset X$ open

$$\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)$$

is injective, i.e. that φ is injective as morphism of presheaves. $\boxed{Z}$ On the other hand, suppose that $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all $p \in X$. It does not follow that φ is surjective as morphism of presheaves.

An example: let $\mathcal{G}$ be as in (c) p. 15, and let

$$\mathcal{C}(X) \xrightarrow{\varphi} \mathcal{G} \quad \text{be defined by}$$

$$\mathcal{C}(X)(U) \xrightarrow{\varphi_U} \mathcal{G}(U)$$
$$f \longmapsto e^{2\pi i f}$$

($U \subset X$ open). If each point of X has a simply connected open neighborhood then $\varphi_p : \mathcal{C}(X)_p \to \mathcal{G}_p$ is surjective for all $p \in X$, but if X contains an open set U s.t. $H^1(U;\mathbb{Z}) \neq 0$ then φ is not a surjection of presheaves.

DEF 12 A morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ of sheaves is

I. injective (as a morphism of sheaves) if $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all $p \in X$.

II. surjective (as a morphism of sheaves) if $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all $p \in X$.

Rmk 6 By Remark 5 an injective morphism of sheaves is injective as morphism of presheaves, but in general a surjective morphism of sheaves is not a surjective morphism of presheaves.

3. SHEAFIFICATION

Let $\mathcal{F}$ be a presheaf on a topolical space X. There is a construction of a sheaf $\mathcal{F}^+$ and a morphism (of presheaves)

$$\mathcal{F} \to \mathcal{F}^+ \qquad (*)$$

which is universal in the sense that every morphism $\mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf factors through the morphism in $(*)$.

Definition of $\mathcal{F}^+$ Let

$$\coprod_{p\in X} \mathcal{F}_p \xrightarrow{\pi} X$$
$$s_p \longmapsto p$$

We define a topology on $\coprod_{p\in X} \mathcal{F}_p$ as follows.

Given $\mathcal{U} \subset X$ open and $s \in \Gamma(\mathcal{U}, \mathcal{F})$ let $A(\mathcal{U}, s)$ be the subset of $\coprod_{p\in X} \mathcal{F}_p$ defined by

$$A(\mathcal{U}, s) := \{ s_p \mid p \in \mathcal{U} \}.$$

the family $\{A(\mathcal{U}, s)\}$ for all $\mathcal{U} \subset X$ open and $s \in \Gamma(\mathcal{U}, \mathcal{F})$ is a basis for a topology on $\coprod_{p \in X} \mathcal{F}_p$.

DEF 13 The "espace étalé" associated to $\mathcal{F}$ is $\coprod_{p \in X} \mathcal{F}_p$ with the above topology. We denote it by $Et(\mathcal{F})$.

The map

$$\begin{aligned} Et(\mathcal{F}) &\xrightarrow{\pi} X \\ s \in \mathcal{F}_p &\longmapsto p \end{aligned}$$

is continuous and it is a local homeomorphism.

Given $\mathcal{V} \subset X$ open let

$$\mathcal{F}^+(\mathcal{V}) := \{\mathcal{V} \xrightarrow{t} Et(\mathcal{F}) \mid \pi \circ t = Id_{\mathcal{V}} \quad t \text{ is continuous}\}.$$

Rmk 7 Let $t \in \mathcal{F}^+(\mathcal{V})$. Then there exist an open covering

$$\mathcal{V} = \bigcup_{i \in I} \mathcal{V}_i \qquad (\star)$$

and $s_i \in \mathcal{F}(\mathcal{V}_i)$ for all $i \in I$ such that

$$t_p = s_{i,p} \qquad \forall\, p \in \mathcal{V}_i.$$

Viceversa, if we are given the open covering in $(\star)$

and $s_i \in \mathcal{F}(\mathcal{V}_i)$ for all $i \in I$ such that

$$s_{i,p} = s_{j,p} \qquad \forall i,j \in I \text{ and } \forall p \in \mathcal{V}_i \cap \mathcal{V}_j$$

then the map

$$\begin{aligned} \mathcal{V} &\xrightarrow{t} \coprod_{p \in X} \mathcal{F}_p \\ p &\longmapsto s_{i,p} \qquad (\text{for } i \in I \text{ s.t. } p \in \mathcal{V}_i) \end{aligned}$$

is continuous and $\pi \circ t = \mathrm{Id}_{\mathcal{V}}$.

Group structure Given $s, t \in \mathcal{F}^+(\mathcal{V})$ let

$$\begin{aligned} \mathcal{V} &\xrightarrow{s+t} \coprod_{p \in X} \mathcal{F}_p \\ p &\longmapsto s(p) + t(p) \end{aligned}$$

Then $s+t \in \mathcal{F}^+(\mathcal{V})$ and with this operation $\mathcal{F}^+(\mathcal{V})$ is an abelian group.

Fact (easy to check) $\mathcal{F}^+$ is a sheaf.

We have a morphism of presheaves $\mathcal{F} \xrightarrow{\alpha^+} \mathcal{F}^+$ defined by setting

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathcal{F}^+(U) \\ s &\longmapsto \left(\begin{aligned} U &\longrightarrow \coprod_{p \in X} \mathcal{F}_p \\ p &\longmapsto s_p \end{aligned} \right) \end{aligned}$$

Fact (easy to check) Let $\mathcal{G}$ be a sheaf and let $\mathcal{F} \xrightarrow{\beta} \mathcal{G}$ be a morphism (of presheaves). Then there exists a morphism $\mathcal{F}^+ \xrightarrow{\beta^+} \mathcal{G}$ such that $\beta = \beta^+ \circ \alpha^+$, and such a β^+ is unique.

By definition $\mathcal{F}^+(U) = \{ U \xrightarrow{s} Et(\mathcal{F}) \mid \substack{s \text{ continuous} \\ \pi \circ s = Id_U} \}$

Viceversa, if $E \xrightarrow{\pi} X$ is a local homeomorphism, we get a sheaf $\mathcal{S}$ on X by letting

$$\mathcal{S}(U) = \{ U \xrightarrow{s} E \mid \substack{s \text{ continuous} \\ \pi \circ s = Id_U} \}$$

Hence we may view the theory of sheaves as an extension of the theory of topological coverings.

Expl 10 Let $\mathcal{F}$ be a sheaf on a topological space X and let $\mathcal{G} \subset \mathcal{F}$ be a subsheaf. The presheaf Q on X with space of sections on an open $U \subset X$ given by

$$Q(U) := \mathcal{F}(U) / \mathcal{G}(U)$$

and the obvious restriction maps is, in general, not a sheaf. We define the quotient sheaf

$$\mathcal{F}/\mathcal{G}$$

to be the sheafification Q^+ of Q.

Expl 11 Let $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ be a morphism of sheaves on a top. space X. The presheaf $\mathcal{I}$ on X defined by letting (for $U \subset X$ open)

$$\mathcal{I}(U) := \operatorname{Im}\left(\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)\right)$$

with the obvious restriction maps, is in general not a sheaf.

We <u>define</u> the <u>image sheaf $\operatorname{Im}(\varphi)$</u> as the sheafification $\mathcal{I}^+$ of $\mathcal{I}$.

DEF 14 A sequence of morphisms of sheaves on X

$$\dots \to \mathcal{F}_{i-1} \xrightarrow{\varphi_{i-1}} \mathcal{F}_i \xrightarrow{\varphi_i} \mathcal{F}_{i+1} \to \dots \qquad i \in \mathbb{Z}$$

is a <u>complex</u> if

$$\varphi_i \circ \varphi_{i-1} = 0 \qquad \forall i \in \mathbb{Z}$$

A complex of sheaves on X is <u>exact</u> if for all $p \in X$ the complex of stalks

$$\cdots \to \mathcal{F}_{i-1,p} \xrightarrow{\varphi_{i-1,p}} \mathcal{F}_{i,p} \xrightarrow{\varphi_{i,p}} \mathcal{F}_{i+1,p} \to \cdots$$

is exact, i.e. $\mathrm{Im}(\varphi_{i-1,p}) = \ker(\varphi_{i,p}) \quad \forall i$.

Notation Most times one considers finite complexes, i.e. such that $\mathcal{F}_i = 0_X \, (=: 0)$ for $|i| \gg 0$, and one denotes them by

$$0 \to \mathcal{F}_a \xrightarrow{\varphi_a} \mathcal{F}_{a+1} \xrightarrow{\varphi_{a+1}} \cdots \xrightarrow{\varphi_{b-1}} \mathcal{F}_b \to 0$$

Expl 1.2 A short exact sequence is an exact complex

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$$

We spell out what this means:

I. α is injective, i.e. $\mathcal{F}_p \xrightarrow{\alpha_p} \mathcal{G}_p$ is injective for all $p \in X$.

II. $\mathrm{Im}(\alpha) = \ker(\beta)$

III. $\mathcal{H} = \mathrm{Im}(\beta)$

Note also that since α is injective we may view $\mathcal{F}$ as a subsheaf of $\mathcal{G}$, and since $\beta\circ\alpha = 0$ the morphism β induces a morphism

$$\mathcal{G}/\mathcal{F} \xrightarrow{\bar\beta} \mathcal{H}$$

which is an isomorphism.