1. Sheaves of rings and of modules

DEF 1 A <u>presheaf of rings</u> on a top. space X is a presheaf of ab. grps $\mathcal{R}$ such that:

1. For each $U \subset X$ open $\mathcal{R}(U)$ has a structure of ring in which addition is the addition on the group of sections of the sheaf $\mathcal{R}$ of abelian groups.

2. For $V \subset U$ open in X the restriction map
$$\mathcal{R}(U) \xrightarrow{\rho_{UV}} \mathcal{R}(V)$$
is a homomorphism of rings.

A presheaf of rings $\mathcal{R}$ is a <u>sheaf of rings</u> if it is a sheaf when viewed as a presheaf of ab. grps.

Expl 1 X a C^∞ manifold, and
$$\mathcal{R}(U) := \{ U \xrightarrow{f} \mathbb{R} \mid f \text{ is } C^\infty \}$$
with pointwise addition and multiplication, and the obvious restriction maps.
This is a sheaf of rings.

Expl 2 X an algebraic variety over an algebraically closed field $\mathbb{K}$, with the Zariski topology.

For $U \subset X$ open let

$$\mathcal{O}_X(U) := \{ U \xrightarrow{f} \mathbb{K} \mid f \text{ regular} \}$$

with pointwise addition and multiplication, and the obvious restriction maps. Then $\mathcal{O}_X$ is a sheaf of rings, called the structure sheaf of X.

DEF 2 Let X be a top. space with a presheaf of rings $\mathcal{R}$.

A presheaf of $\mathcal{R}$-modules on X is a presheaf of ab. grps $\mathcal{M}$ such that:

1. For each $U \subset X$ open, $\mathcal{M}(U)$ has a structure of $\mathcal{R}(U)$-module in which addition is the addition on the group of sections of the sheaf $\mathcal{M}$ of abelian groups.

2. For $V \subset U$ open in X the restriction map

$$\mathcal{M}(U) \xrightarrow{\rho^{\mathcal{M}}_{UV}} \mathcal{M}(V)$$

is compatible with the module structures, i.e. for $\lambda \in \mathcal{R}(U)$ and $s \in \mathcal{M}(U)$ we have

$$\rho^{\mathcal{M}}_{U,V}(t \cdot s) = \rho^{\mathcal{R}}_{U,V}(t) \cdot \rho^{\mathcal{M}}_{U,V}(s).$$

A presheaf of $\mathcal{R}$-modules is a <u>sheaf of $\mathcal{R}$-modules</u> if it is a sheaf when viewed as a presheaf of ab. grps.

<u>Expl 3</u> Let X be an algebraic variety over $\mathbb{K}$, and let

$$\begin{array}{c} E \\ \pi \downarrow \\ X \end{array}$$

be an algebraic vector bundle. For $U \subset X$ open let

$$\mathcal{O}(E)(U) := \left\{ s : U \to E \;\middle|\; \begin{array}{l} s \text{ regular and} \\ \pi \circ s = \mathrm{Id}_U \end{array} \right\}$$

Pointwise addition and multiplication gives $\mathcal{O}(E)(U)$ the structure of an $\mathcal{O}_U$-module. For $V \subset U$ open in X we have restriction maps

$$\mathcal{O}(E)(U) \xrightarrow{\rho_{U,V}} \mathcal{O}(E)(V)$$

$\mathcal{O}(E)$ is a <u>sheaf</u> of $\mathcal{O}_X$-modules

Let X be a top. space with a presheaf of rings $\mathcal{R}$, and let $\mathcal{F}, \mathcal{G}$ be presheaves of $\mathcal{R}$-modules.

DEF 3 A morphism of presheaves of $\mathcal{R}$-modules $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ consists of a morphism of presheaves of ab. grps. such that for each open $U \subset X$ the homomorphism of ab. grps.

$$\mathcal{F}(U) \xrightarrow{\varphi_U} \mathcal{G}(U)$$

is a homomorphism of $\mathcal{R}(U)$ modules.

Injectivity/surjectivity as presheaves (respectively sheaves) of $\mathcal{R}$-modules means injectivity/surjectivity as presheaves (respectively sheaves) of ab. grps.

Exp 4 Let X be an algebraic, and let

$$\begin{tikzcd} E \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\rho"] \\ & X & \end{tikzcd}$$

be a morphism of algebraic vector bundles on X. We get a morphism

$$\mathcal{O}(E) \xrightarrow{\varphi} \mathcal{O}(F)$$

by setting

$$\begin{aligned} \mathcal{O}(E)(U) &\xrightarrow{\varphi_U} \mathcal{O}(F)(U) \\ s &\longmapsto \Phi \circ s \end{aligned}$$

Expl 5 Recall that the tautological line bundle on $\mathbb{P}^n(\mathbb{K}) = \left\{ \ell \subset \mathbb{K}^{n+1} \;\middle|\; \begin{array}{l} \ell \text{ vector} \\ \text{subspace, } \dim \ell = 1 \end{array} \right\}$ ($\mathbb{K}$ an algebraically closed field) is defined as

$$\begin{array}{ccc} (\ell, v) \in & L := \{ (\ell, v) \in \mathbb{P}^n(\mathbb{K}) \times \mathbb{K}^{n+1} \mid v \in \ell \} \\ \downarrow & \downarrow \pi \\ \ell & \mathbb{P}^n(\mathbb{K}) \end{array}$$

One sets

$$\mathcal{O}_{\mathbb{P}^n}(-1) := \mathcal{O}(L)$$

$$\mathcal{O}_{\mathbb{P}^n}(1) := \mathcal{O}(L^\vee)$$

$$\mathcal{O}_{\mathbb{P}^n}(d) := \begin{cases} \mathcal{O}(\underbrace{L^\vee \otimes \cdots \otimes L^\vee}_{d}) =: L^{-d} & \text{if } d > 0 \\ \mathcal{O}_{\mathbb{P}^n} =: L^0 & \text{if } d = 0 \\ \mathcal{O}(\underbrace{L \otimes \cdots \otimes L}_{|d|}) =: L^{d} & \text{if } d < 0 \end{cases}$$

How does one describe $\mathcal{O}_{\mathbb{P}^n}(d)(U)$ for $U \subset \mathbb{P}^n(\mathbb{K})$ open ?

For $i \in \{0, \dots, n\}$ let $A_i \subset \mathbb{P}^n(\mathbb{K})$ be the standard open affine space

$$A_i := \mathbb{P}^n(\mathbb{K}) \setminus V(z_i) = \{ [z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n] \mid (z_0, \dots, z_n) \in \mathbb{A}^n(\mathbb{K}) \}$$

Then L is trivial on each A_i. In fact the regular section

$$\begin{tikzcd} \pi^{-1}(A_i) = L|_{A_i} \arrow[d, "\pi|\dots"] \\ A_i \arrow[u, bend left, "s_i"] \end{tikzcd}$$

defined by

$$s_i([z_0,.,z_{i-1},1,z_{i+1},..z_n]) := (\mathrm{Span}\{(z_0,.,z_{i-1},1,z_{i+1},..z_n)\}, (z_0,.,z_{i-1},1,z_{i+1},..z_n))$$

is nowhere zero, and hence defines a trivialization of $L|_{A_i}$.

It follows that L^d is also trivial for all $d \in \mathbb{Z}$.

A nowhere zero section of L^{-1} on A_i is given as follows. For $i \in \{0,.,n\}$ let

$$Z_i \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$$

(For a sheaf $\mathcal{F}$ on a top. space X, and an open $U \subset X$, one denotes $\mathcal{F}(U)$ also by $\Gamma(U, \mathcal{F})$. We will often use this notation.)

be the global regular section defined by

$$Z_i((\ell, v)) := Z_i(v)$$

(if $v = [a_0,..,a_n]$, $Z_i(v) = a_i$). Note that

$$Z_i(s_i) = 1, \tag{$*$}$$

hence $Z_i|_{A_i}$ is nowhere zero, and defines a trivialization of $L^{\vee} (= L^{-1})$ on A_i. If $d > 0$ then Z_i^d defines a trivialization of L^{-d} over A_i. By $(*)$ it makes sense to let $z_i^{-1} = s_i$, and hence z_i^d defines a trivialization of L^{-d} over A_i for all i. It follows that we have a well defined isomorphism of modules over $\Gamma(A_i, \mathcal{O}_{A_i})$, i.e. modules over $\mathbb{K}[\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}]$,

$$\begin{array}{ccc} \text{deg-}d \text{ hom. compn't of } \left(\mathbb{K}[z_0, \dots, z_n]_{z_i} \right) & \xrightarrow{\sim} & \Gamma(A_i, \mathcal{O}_{\mathbb{P}^n}(d)) \\ \psi & & \\ f\left(\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}\right) \cdot z_i^d & \longmapsto & f\left(\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}\right) \cdot \left(\text{section } z_i^d\right) \end{array}$$

The conclusion is: if $U \subset \mathbb{P}^n(\mathbb{K})$ is open then an element of $\Gamma(U, \mathcal{O}_{\mathbb{P}^n}(d))$ is described as follows. For each open $U \cap A_i$ we must give

$$f_i \cdot z_i^d \qquad f_i \in \mathcal{O}(U \cap A_i)$$

and we must have

$$f_i \cdot z_i^d = f_j \cdot z_j^d \qquad \text{on } U \cap A_i \cap A_j$$

<u>Exercise 1</u> Let $d \geq 0$. If $P \in \mathbb{K}[z_0, .., z_n]_d$ we associate to P the section

$$s_P \in \Gamma(\mathbb{P}^n(\mathbb{K}), \mathcal{O}_{\mathbb{P}^n}(d))$$

by letting

$$f_i := \frac{P}{z_i^d}$$

Prove that the map

$$\begin{aligned} \mathbb{K}[z_0, .., z_n]_d &\longrightarrow \Gamma(\mathbb{P}^n(\mathbb{K}), \mathcal{O}_{\mathbb{P}^n}(d)) \\ P &\longmapsto s_P \end{aligned}$$

is an isomorphism of $\mathbb{K}$ vector spaces (the right hand side is a module over $\Gamma(\mathbb{P}^n(\mathbb{K}), \mathcal{O}_{\mathbb{P}^n}) = \mathbb{K}$, hence a $\mathbb{K}$ vector space).

DEF 4 Let X be a top. space, and let $\mathcal{R}$ be a presheaf of rings on X. Let $p \in X$. The stalk $\mathcal{R}_p$ of $\mathcal{R}$ as a presheaf of ab. grps. can be given the structure of a ring by defining multiplication as

$$[(U,f)] \cdot [(V,g)] := [(U \cap V, (f|_{U\cap V})(g|_{U \cap V}))].$$

Now let $\mathcal{F}$ be a presheaf of $\mathcal{R}$-modules. If $p \in X$ the stalk

$\mathcal{F}_p$ of $\mathcal{F}$ as a presheaf of ab. grps. can be given the structure of an $\mathcal{R}_p$ module by defining multiplication as

$$\begin{aligned} \mathcal{R}_p \times \mathcal{F}_p &\longrightarrow \mathcal{F}_p \\ ([(U,f)], [(V,s)]) &\longmapsto [(U \cap V, (f|_{U\cap V})(s|_{U \cap V}))]. \end{aligned}$$

2. Operations on sheaves of modules

Throughout this section X is a top. sp. with a sheaf of rings $\mathcal{R}$.

2a. Direct sum, Tensor product, dual

Let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{R}$-modules.

The sheaf of ab. grps. $\mathcal{F} \oplus \mathcal{G}$ has a structure of $\mathcal{R}$-module.

In fact if $U \subset X$ is open then

$$\mathcal{F}(U) \oplus \mathcal{G}(U)$$

is an $\mathcal{R}(U)$-module and this structure is compatible with restriction to $\mathcal{F}(V) \oplus \mathcal{G}(V)$. Since the presheaf of ab. grps. $\mathcal{F} \oplus \mathcal{G}$ is already the sheaf of ab. grps. $\mathcal{F} \oplus \mathcal{G}$, we are done.

In order to define the sheaf $\mathcal{F} \otimes \mathcal{G}$ we proceed as follows.

Let $U \subset X$ be open. Then

$$\mathcal{H}(U) := \mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U)$$

is an $\mathcal{R}(U)$-module. If $V \subset U$ is open we have the homomorphism

$$\mathcal{H}(U) = \mathcal{F}(U) \otimes_{\mathcal{R}(U)} \mathcal{G}(U) \xrightarrow{\rho^{\mathcal{F}}_{U,V} \otimes \rho^{\mathcal{G}}_{U,V}} \mathcal{F}(V) \otimes_{\mathcal{R}(V)} \mathcal{G}(V) = \mathcal{H}(V)$$

We get a presheaf $\mathcal{H}$ of $\mathcal{R}$-modules. Call it $\mathcal{F} \overset{pre}{\otimes} \mathcal{G}$

In general $\mathcal{H}$ is not a sheaf. The tensor product $\mathcal{F} \otimes \mathcal{G}$ is the sheafification of $\mathcal{F} \overset{\text{pre}}{\otimes} \mathcal{G}$:

$$\mathcal{F} \otimes \mathcal{G} := \left(\mathcal{F} \overset{\text{pre}}{\otimes} \mathcal{G}\right)^+$$

Exercise 2 Show that $\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \cong \mathcal{O}_{\mathbb{P}^n}$ and that $\mathcal{O}_{\mathbb{P}^n}(1) \overset{\text{pre}}{\otimes} \mathcal{O}_{\mathbb{P}^n}(-1)$ is not a sheaf.

For $V \subset U$ open in X we have the restriction map

$$\mathcal{H}om_{\mathcal{R}(U)}(\mathcal{F}|_U, \mathcal{G}|_U) \to \mathcal{H}om_{\mathcal{R}(V)}(\mathcal{F}|_V, \mathcal{G}|_V)$$

(the restriction of a sheaf of ab. grps., rings, $\mathcal{R}$-modules to an open set is defined in the obvious way)

and we get a sheaf of $\mathcal{R}$-modules that we denote

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}).$$

We let

$$\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{R})$$

Expl 6 $\mathcal{O}_{\mathbb{P}^n}(d)^\vee \cong \mathcal{O}_{\mathbb{P}^n}(-d)$.

2b. Kernel, image, cokernel.

Let $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ be a morphisms of sheaves of R-modules.
Then the sheaves of ab. grps.

$$\ker(\varphi), \quad \mathrm{im}(\varphi)$$

can be given the structure of sheaves of R-modules as follows.
Let $U \subset X$ be open. Then

$$\ker(\varphi_U) \subset \mathcal{F}(U) \quad \text{and} \quad \mathrm{im}(\varphi_U) \subset \mathcal{G}(U)$$

are $R(U)$ submodules and the restriction maps (for $V \subset U$ with V open) to $\mathcal{F}(V)$ and $\mathcal{G}(V)$ map them into $\ker(\varphi_V)$ and $\mathrm{im}(\varphi_V)$ respectively (and the restriction maps are compatible with the restriction maps $R(U) \to R(V)$).
Since the presheaf $\ker_{\mathrm{pre}}(\varphi)$ is already a sheaf, and hence equal to $\ker(\varphi)$, this gives $\ker(\varphi)$ a structure of R-module. In general $\mathrm{im}_{\mathrm{pre}}(\varphi)$ is not a sheaf, and $\mathrm{im}(\varphi) = \mathrm{im}_{\mathrm{pre}}(\varphi)^{+}$ is the sheafification of $\mathrm{im}_{\mathrm{pre}}(\varphi)$.
Recall that for $U \subset X$ open $\Gamma(U, \mathrm{im}(\varphi))$ is the set of

continuous sections over U of

$$\begin{array}{ccc} \mathrm{Et}(\mathcal{F}) & \longrightarrow & X \\ \cap & & \\ \coprod_{p\in X} \mathcal{F}_p & & \end{array}$$

Hence in order to give $\Gamma(U, \mathrm{im}(\varphi))$ a structure of $R(U)$-module compatible with restrictions it suffices to give $\mathcal{F}_p$ a structure of R_p-module for every $p \in X$. Such a structure is given by the structure of $R(U)$-module on $\mathrm{im}(\varphi)(U)$.

If $\mathcal{F}$ is a sheaf of R-modules and $\mathcal{G} \subset \mathcal{F}$ is a subsheaf of R-modules, i.e. $\mathcal{G}(U) \subset \mathcal{F}(U)$ is a sub $R(U)$-module for all open $U \subset X$, then $\mathcal{F}/\mathcal{G}$ is also a sheaf of R-modules and the morphism

$$\mathcal{F} \to \mathcal{F}/\mathcal{G}$$

is a surjective homomorphism of R-modules with kernel $\mathcal{G}$. This is done as above.

Exercise 3 Let $d \in \mathbb{N}$. On $\mathbb{P}^1(\mathbb{K})$ we have sections

$$z_0^d, z_1^d \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$$

Since $\mathcal{O}_{\mathbb{P}^1}(d) = \mathcal{O}_{\mathbb{P}^1}(-d)$ each of z_0^d, z_1^d gives a morphism of sheaves

$$\mathcal{O}_{\mathbb{P}^1}(-d) \xrightarrow{z_i^d} \mathcal{O}_{\mathbb{P}^1} \qquad i \in \{0,1\}.$$

Hence we have a morphism

$$\mathcal{O}_{\mathbb{P}^1}(-d) \xrightarrow{\varphi} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^2}$$

I. Show that φ is injective, hence it defines an inclusion

$$\mathcal{O}_{\mathbb{P}^1}(-d) \subset \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^2}$$

II. Show that we have an isomorphism

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \big/ \mathcal{O}_{\mathbb{P}^1}(-d) \cong \mathcal{O}_{\mathbb{P}^1}(d)$$

III. By (I) - (II) we have an exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-d) \to \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(d)$$

Check that if $d \geq 2$ then the complex of $\mathbb{K}$ vector spaces

$$0 \to \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(-d)) \xrightarrow{\alpha} \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{\beta} \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(d)) \to 0$$

is <u>not</u> at $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(d))$.

2c. Push forward, pull-back

Let
$$X \xrightarrow{\varphi} Y$$

be a continuous map of top. spaces, and let $\mathcal{F}$ be a presheaf of ab. grps. on X. If $U \subset Y$ is open let
$$(\varphi_*\mathcal{F})(U) := \mathcal{F}(\varphi^{-1}U) \qquad (\bullet)$$

If $V \subset U$ are open in X then $\varphi^{-1}V \subset \varphi^{-1}U$ and hence we have
$$(\varphi_*\mathcal{F})(U) = \mathcal{F}(\varphi^{-1}U) \xrightarrow{\rho^{\mathcal{F}}_{\varphi^{-1}U,\varphi^{-1}V}} \mathcal{F}(\varphi^{-1}V) = (\varphi_*\mathcal{F})(V). \qquad (\bullet\bullet)$$

The compatibility conditions are satisfied, hence we get

DEF 5 The push forward of $\mathcal{F}$ is the presheaf $\varphi_*\mathcal{F}$ on Y defined by $(\bullet)$ and $(\bullet\bullet)$.

If $\mathcal{F}$ is a sheaf then $\varphi_*\mathcal{F}$ is a sheaf.

RMK 1 If $\mathcal{R}_X$ is a (pre)sheaf of rings on X then $\varphi_*\mathcal{R}_X$ is a (pre)sheaf of rings on Y, the ring structure on $\varphi_*\mathcal{R}_X(V) = \mathcal{R}_X(\varphi^{-1}V)$ being obvious.

ASSUMPTION 1 We are given (pre)sheaves of rings $\mathcal{R}_X, \mathcal{R}_Y$ on X and Y respectively and a morphism
$$\mathcal{R}_Y \xrightarrow{\alpha} \varphi_*\mathcal{R}_X$$

of (pre)sheaves of rings.

RMK 2 The morphism α consists of a homomorphism of rings

$$\mathcal{R}_Y(U) \xrightarrow{\alpha_U} \mathcal{R}_X(\varphi^{-1}U)$$

for each $U \subset Y$ open commuting with the restriction maps.

Expl 7 Let $X \xrightarrow{\varphi} Y$ be a regular map of algebraic varieties. Then pull-back defines a morphism of sheaves of rings

$$\mathcal{O}_Y \xrightarrow{\varphi^*} \varphi_*(\mathcal{O}_X)$$

More precisely if $U \subset Y$ is open then

$$\begin{aligned} \mathcal{O}_Y(U) &\xrightarrow{\varphi^*_U} \varphi_*(\mathcal{O}_X)(U) = \mathcal{O}_X(\varphi^{-1}U) \\ f &\longmapsto f \circ (\varphi|_{\varphi^{-1}U}) \end{aligned}$$

Suppose that $\mathcal{F}$ is a (pre)sheaf of $\mathcal{R}_X$-modules, and that Assumption 1 holds.

Then $\varphi_* \mathcal{F}$ is a (pre)sheaf of $\mathcal{R}_Y$-modules. In fact if $U \subset Y$ ~~is~~ open then

$$\varphi_* \mathcal{F}(U) = \mathcal{F}(\varphi^{-1} U)$$

is a module over $\mathcal{R}_X(\varphi^{-1} U)$, and since we are given the homomorphism of rings

$$\mathcal{R}_Y(U) \longrightarrow \varphi_* \mathcal{R}_X(U) = \mathcal{R}_X(\varphi^{-1} U)$$

it is also a module over $\mathcal{R}_Y(U)$. With this structure $\varphi_* \mathcal{F}$ is a (pre)sheaf of $\mathcal{R}_Y$-modules.

Expl 8 Let Y be an algebraic variety and let $X \subset Y$ be a subvariety (this means that X is a closed subset of Y). For $U \subset Y$ open let

$$\mathcal{I}_{X/Y}(U) := \{ f \in \mathcal{O}_Y(U) \mid f|_X = 0 \}$$

then $\mathcal{I}_{X/Y}$ is a subsheaf of the sheaf (of rings) $\mathcal{O}_Y$: we let $\mathcal{I}_{X/Y} \xrightarrow{a} \mathcal{O}_Y$ be the inclusion.

Let

$$X \xrightarrow{i} Y$$

be the inclusion map. The morphism of sheaves of rings

$$\mathcal{O}_Y \xrightarrow{\beta} i_*\mathcal{O}_X$$

is surjective and its kernel is equal to $\mathcal{I}_{X/Y}$.
In other words we have an exact sequence of sheaves of $\mathcal{O}_Y$-modules

$$0 \to \mathcal{I}_{Y/X} \xrightarrow{\alpha} \mathcal{O}_Y \xrightarrow{\beta} i_*\mathcal{O}_X \to 0.$$

Exercise 4 Give examples in which the map

$$\Gamma(Y, \mathcal{O}_Y) \longrightarrow \Gamma(Y, i_*\mathcal{O}_X)$$

is not surjective.

Let $X \xrightarrow{\varphi} Y$ be as above, and let

$$\begin{array}{c} E \\ \pi \downarrow \\ Y \end{array}$$

be an alg. v.b. on Y. Then we have defined an algebraic v.b.

$$\begin{array}{c} \varphi^* E \\ \rho \downarrow \\ X \end{array}$$

as the closed subset

$$\varphi^* E := \{ (x, e) \in X \times E \mid \varphi(x) = \pi(e) \}$$

with ρ the restriction of the projection $X \times E \to X$.

Passing to the sheaf of germs of sections, we have associated to the sheaf of $\mathcal{O}_Y$-modules $\mathcal{O}_Y(E)$ the sheaf of $\mathcal{O}_X$-modules $\mathcal{O}_X(\varphi^* E)$.

There is a more general procedure which associates to a sheaf of $\mathcal{O}_Y$-modules $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules $\varphi^* \mathcal{F}$, provided Assumption 2 holds.

First let $\mathcal{F}$ be a sheaf of ab. grps on Y.
(No need for Assumption 2 for the moment being.)
For $U \subset X$ open one defines $\varphi_{pre}^{-1}\mathcal{F}(U)$ as the set of couples (V, s) where $\varphi(U) \subset V \subset Y$, with V open, and $s \in \mathcal{F}(V)$, modulo the equivalence relation

$(V_1, s_1) \sim (V_2, s_2)$ if there exists $\varphi(U) \subset W \subset Y$, with W open, such that

$$\rho^{\mathcal{F}}_{V_1, W}(s_1) = \rho^{\mathcal{F}}_{V_2, W}(s_2).$$

The addition on $\mathcal{F}(V)$ defines an operation on $\varphi_{pre}^{-1}\mathcal{F}(U)$, and with this operation $\varphi_{pre}^{-1}\mathcal{F}(U)$ is an ab. grp. If $U' \subset U$ with U' open the restriction map of $\mathcal{F}$ defines a homomorphism of groups

$$\varphi_{pre}^{-1}\mathcal{F}(U) \longrightarrow \varphi_{pre}^{-1}\mathcal{F}(U'),$$

and $\varphi_{pre}^{-1}\mathcal{F}$ is a presheaf of ab. grps. on X.

<u>DEF 6</u> The sheaf of ab. grps. $\varphi^{-1}\mathcal{F}$ is the sheafification of $\varphi_{pre}^{-1}\mathcal{F}$.

Note: if R_Y is a sheaf of rings on X, then $\varphi^{-1}R_Y$ is naturally a sheaf of rings on Y.

Now suppose that Assumption 2 holds. This gives a homomorphism of sheaves of rings on X:

$$\varphi^{-1}R_Y \xrightarrow{\alpha} R_X$$

Now $\varphi^{-1}\mathcal{F}$ is naturally an $\varphi^{-1}R_Y$-module, and hence

$$\varphi^*\mathcal{F} := \varphi^{-1}\mathcal{F} \otimes_{\varphi^{-1}R_Y} R_X$$

makes sense, and via multiplication on the 2nd factor it is an R_X-module.

Expl 9 Let X be an alg. var, and let $X \xrightarrow{\varphi} \{pt\}=Y$ be the constant map. Then

$$\varphi^{-1}(\mathcal{O}_Y)(U) = \{ U \xrightarrow{f} \mathbb{K} \mid f \text{ is locally constant} \}$$

$$\varphi^*\mathcal{O}_Y \cong \mathcal{O}_X.$$