Advanced Topics in Geometry, a. y. 2024-25

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Logbook

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Week 1.

(26/09/2024) Presheaves of abelian groups: definition, morphisms, kernel and image presheaf, subpresheaf and quotient presheaf. Sheaves. The quotient of a sheaf modulo a subsheaf is, in general, not a sheaf (example: sheaf of germs of continuous complex functions modulo sheaf of locally constant integer valued functions on $\mathbb{C}^*$).

Week 2.

(01/10/2024) Stalk of a presheaf in a point. A morphism of sheaves which is an isomorphism between stalks is an isomorphism. Definition of injective/surjective morphism of sheaves. Espace étalé of a presheaf and sheafification of a presheaf.

(03/10/2024) Quotient of a sheaf modulo a subsheaf, exact sequences of sheaves. (Pre)Sheaves of rings and modules. The sheaves $\mathcal{O}_{\mathbb{P}^n}(d)$ and their local sections.

Week 3.

(08/10/2024) Discussion of the homework that had been assigned. Direct sum and tensor product of sheaves of modules. Dual of a sheaf of modules. Kernel, image and cokernel of a morphism of sheaves of modules. Given a continuous map $\varphi\colon X \to Y$, a morphism of sheaves of rings $\mathcal{R}_Y \to \varphi_*\mathcal{R}_X$ where $\mathcal{R}_Y$ is a sheaf of rings on Y, and a sheaf $\mathcal{F}$ of $\mathcal{R}_X$-modules we get the push-forward sheaf of $\mathcal{R}_Y$-modules $\varphi_*\mathcal{F}$.

(10/10/2024) The sheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules on an affine variety X (over an algebraically closed field $\mathbb{K}$) associated to a $\mathbb{K}[X]$-module M. The stalk of $\widetilde{M}$ at $p \in X$ is isomorphic to the localization $M_{\mathfrak{m}_p}$. Definition of (Quasi)Coherent sheaf on an algebraic variety X over $\mathbb{K}$. If M is finitely generated (and X is irreducible) there exist an open dense subset $U \subset X$ and $r \in \mathbb{N}$ such that $\widetilde{M}_{|U} \cong \mathcal{O}_X^{\oplus r}$.

Week 4 (week 14-20 October canceled).

(22/10/2024) Review of homework. The module of global sections of $\widetilde{M}$ is identified with M. The morphism of sheaves $\widetilde{\varphi}\colon \widetilde{M} \to \widetilde{N}$ associated to a homomorphism of modules $\varphi\colon M \to N$; every morphism of sheaves $\alpha\colon \widetilde{M} \to \widetilde{N}$ is of this kind. The isomorphisms $\ker\widetilde{\varphi} \cong \widetilde{\ker\varphi}$ $\operatorname{coker}\widetilde{\varphi} \cong \widetilde{\operatorname{coker}\varphi}$, $\widetilde{M} \oplus \widetilde{N} \cong \widetilde{M \oplus N}$ and $\widetilde{M} \otimes \widetilde{N} \cong \widetilde{M \otimes N}$.

(24/10/2024) If $\mathcal{F}$ is a quasi-coherent sheaf on an affine variety X then $\mathcal{F} \cong \widetilde{\Gamma(X, \mathcal{F})}$, with $\Gamma(X,\mathcal{F})$ finitely generated if $\mathcal{F}$ is coherent. Kernels and cokernels of morphism between (quasi)coherent sheaves are (quasi)coherent, direct sum and tensor products of (quasi)coherent sheaves are (quasi)coherent. The Čhech complex of a quasi-coherent sheaf $\mathcal{F}$ associated to a finite open cover $\mathcal{U}$ of an algebraic variety X (note: the index set of a cover is always totally ordered). The Čhech cohomology groups $H^p_{\mathcal{U}}(X,\mathcal{F})$. The identification $H^0_{\mathcal{U}}(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F})$.

Week 5.

(29/10/2024) Let X be an algebraic variety, let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of quasi-coherent sheaves on X, and let $\mathcal{U}$ be a finite open *affine* cover of X; then there is long exact sequence

$$\cdots H^{p-1}_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\partial} H^p_{\mathcal{U}}(X,\mathcal{F}) \longrightarrow H^p_{\mathcal{U}}(X,\mathcal{G}) \longrightarrow H^p_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\partial} \cdots$$

Statement of two key results. THM 1: If X is an affine variety with a finite open *affine* cover $\mathcal{U}$ and $\mathcal{F}$ is a quasi-coherent sheaf on X then $H^p_{\mathcal{U}}(X,\mathcal{F}) = 0$ for all $p > 0$. THM 2: If X is an algebraic variety with finite open *affine* covers $\mathcal{U},\mathcal{V}$ and $\mathcal{F}$ is a quasi-coherent sheaf on X, then there is well-defined isomorphism $H^p_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{\sim} H^p_{\mathcal{V}}(X,\mathcal{F})$ for all p. Definition: $H^p(X,\mathcal{F}) := H^p_{\mathcal{U}}(X,\mathcal{F})$ where $\mathcal{U}$ is any finite open *affine* cover of X. Example: the vanishing $H^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(m)) = \{0\}$ gives that for any curve $C \subset \mathbb{P}^2$ the (linear) map $\Gamma(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(m)) \longrightarrow \Gamma(C,\mathcal{O}_C(m))$ is surjective.

(31/10/2024) Example: proof that $H^1(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(m)) = \{0\}$.

LEMMA 1: Let X be an algebraic variety and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $\mathcal{V} := \{\mathcal{U}_j\}_{j\in J}$ be an open affine cover of X. Let $k \in J$ be such that $\mathcal{U} := \{\mathcal{U}_i\}_{i\in I}$ is an open affine cover of X, where $I := J \setminus \{k\}$. Suppose that $H^p_{\overline{\mathcal{U}}}(U_k,\mathcal{F}) = 0$ where $\overline{\mathcal{U}} := \{U_k \cap U_i\}_{i\in I}$. Then the natural (linear) map $H^p_{\mathcal{V}}(U_k,\mathcal{F}) \to H^p_{\mathcal{U}}(U_k,\mathcal{F})$ is an isomorphism.

LEMMA 2: Let X be an affine variety and let $\mathcal{F}$ be a quasi-coherent sheaf on X. If $\mathcal{U} := \{\mathcal{U}_i\}_{i\in I}$ is a cover of X by principal (open) subsets then $H^p_{\mathcal{U}}(X,\mathcal{F}) = 0$ for all $p > 0$.

LEMMA 1 and LEMMA 2 imply THM 1 and THM 2.

Week 6.

(05/11/2024) Recap: divisors on locally factorial algebraic varieties (e.g. smooth varieties) and invertible sheaves, in particular $\mathcal{O}_{\mathbb{P}^n}(d)$ for $d \in \mathbb{Z}$. Cohomology of line bundles on $\mathbb{P}^n$ (only the statement). The cohomology $H^p(X,\mathcal{F})$ of a quasi-coherent sheaf $\mathcal{F}$ on a quasi-projective variety X vanishes for $p > \dim X$, and is a finite dimensional $\mathbb{K}$-vector space for any p if X is projective and $\mathcal{F}$ is coherent. Vanishing of $H^p(X,\mathcal{F}(k))$ for $k \gg 0$ and $p > 0$ for $X \subset \mathbb{P}^n$ closed, $\mathcal{O}_X(k) := \mathcal{O}_{\mathbb{P}^n}(k)_{|X}$ and $\mathcal{F}$ coherent.

(07/11/2024) For $\mathcal{F} \in \mathrm{Coh}(X)$, where X is a projective variety (over $\mathbb{K}$), we let $h^p(X,\mathcal{F}) := \dim_{\mathbb{K}} H^p(X,\mathcal{F})$ and $\chi(X,\mathcal{F}) := \sum_p (-1)^p h^p(X,\mathcal{F})$. Recap: degree of a line bundle on a smooth (irreducible) projective curve X via the result $\deg(\mathrm{div}(f)) = 0$ for $f \in \mathbb{K}(X)^*$. New approach to the definition of $\deg \mathcal{L}$ for an invertible sheaf on X: define $\deg \mathcal{L}$ to be $\chi(X,\mathcal{L}) - \chi(X,\mathcal{O}_X)$. Proof that the new definition agrees with the old one. Weak Riemann-Roch for a (smooth projective) curve X is now a definition, but thanks to the vanishing of $h^1(X,\mathcal{L}(n))$ for $n \gg 0$ it gives a non trivial result.

Week 6.

(12/11/2024) Review of homework. Koszul homology and regular sequences. Proof that $h^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{d+n}{n}$, $h^n(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{-d-1}{n}$, and that $h^p(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d)) = 0$ if $p \notin \{0,n\}$.

(14/11/2024) Class cancelled.