Advanced Topics in Geometry, a. y. 2024-25

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Logbook

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Week 1.

(26/09/2024) Presheaves of abelian groups: definition, morphisms, kernel and image presheaf, subpresheaf and quotient presheaf. Sheaves. The quotient of a sheaf modulo a subsheaf is, in general, not a sheaf (example: sheaf of germs of continuous complex functions modulo sheaf of locally constant integer valued functions on \mathbb{C}^*).

Week 2.

- (01/10/2024) Stalk of a presheaf in a point. A morphism of sheaves which is an isomorphism between stalks is an isomorphism. Definition of injective/surjective morphism of sheaves. Espace étalé of a presheaf and sheafification of a presheaf.
- (03/10/2024) Quotient of a sheaf modulo a subsheaf, exact sequences of sheaves. (Pre)Sheaves of rings and modules. The sheaves $\mathcal{O}_{\mathbb{P}^n}(d)$ and their local sections.

Week 3.

- (08/10/2024) Discussion of the homework that had been assigned. Direct sum and tensor product of sheaves of modules. Dual of a sheaf of modules. Kernel, image and cokernel of a morphism of sheaves of modules. Given a continuous map $\varphi \colon X \to Y$, a morphism of sheaves of rings $\mathcal{R}_Y \to \varphi_* \mathcal{R}_X$ where \mathcal{R}_Y is a sheaf of rings on Y, and a sheaf \mathcal{F} of \mathcal{R}_X -modules we get the push-forward sheaf of \mathcal{R}_Y -modules $\varphi_* \mathcal{F}$.
- (10/10/2024) The sheaf \widetilde{M} of \mathcal{O}_X -modules on an affine variety X (over an algebraically closed field \mathbb{K}) associated to a $\mathbb{K}[X]$ -module M. The stalk of \widetilde{M} at $p \in X$ is isomorphic to the localization $M_{\mathfrak{m}_p}$. Definition of (Quasi)Coherent sheaf on an algebraic variety X over \mathbb{K} . If M is finitely generated (and X is irreducible) there exist an open dense subset $U \subset X$ and $r \in \mathbb{N}$ such that $\widetilde{M}_{|U} \cong \mathcal{O}_X^{\oplus r}$.

Week 4 (week 14-20 October canceled).

- (22/10/2024) Review of homework. The module of global sections of \widetilde{M} is identified with M. The morphism of sheaves $\widetilde{\varphi} \colon \widetilde{M} \to \widetilde{N}$ associated to a homomorphism of modules $\varphi \colon M \to N$; every morphism of sheaves $\alpha \colon \widetilde{M} \to \widetilde{N}$ is of this kind. The isomorphisms ker $\widetilde{\varphi} \cong \widetilde{\ker \varphi}$ coker $\widetilde{\varphi} \cong \widetilde{\operatorname{coker} \varphi}, \ \widetilde{M} \oplus \widetilde{N} \cong \widetilde{M \oplus N}$ and $\widetilde{M} \otimes \widetilde{N} \cong \widetilde{M \otimes N}$.
- (24/10/2024) If \mathcal{F} is a quasi-coherent sheaf on an affine variety X then $\mathcal{F} \cong \Gamma(X, \mathcal{F})$, with $\Gamma(X, \mathcal{F})$ finitely generated if \mathcal{F} is coherent. Kernels and cokernels of morphism between (quasi)coherent sheaves are (quasi)coherent, direct sum and tensor products of (quasi)coherent sheaves are (quasi)coherent. The Čhech complex of a quasi-coherent sheaf \mathcal{F} associated to a finite open cover \mathcal{U} of an algebraic variety X (note: the index set of a cover is always totally ordered). The Čhech cohomology groups $H^p_{\mathcal{U}}(X, \mathcal{F})$. The identification $H^0_{\mathcal{U}}(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$.

Week 5.

(29/10/2024) Let X be an algebraic variety, let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be an exact sequence of quasi-coherent sheaves on X, and let \mathcal{U} be a finite open *affine* cover of X; then there is long exact sequence

$$\cdots H^{p-1}_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\partial} H^p_{\mathcal{U}}(X,\mathcal{F}) \longrightarrow H^p_{\mathcal{U}}(X,\mathcal{G}) \longrightarrow H^p_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\partial} \cdots$$

Statement of two key results. THM 1: If X is an affine variety with a finite open affine cover \mathcal{U} and \mathcal{F} is a quasi-coherent sheaf on X then $H^p_{\mathcal{U}}(X,\mathcal{F}) = 0$ for all p > 0. THM 2: If X is an algebraic variety with finite open affine covers \mathcal{U}, \mathcal{V} and \mathcal{F} is a quasicoherent sheaf on X, then there is well-defined isomorphism $H^p_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{\sim} H^p_{\mathcal{V}}(X,\mathcal{F})$ for all p. Definition: $H^p(X,\mathcal{F}) := H^p_{\mathcal{U}}(X,\mathcal{F})$ where \mathcal{U} is any finite open affine cover of X. Example: the vanishing $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) = \{0\}$ gives that for any curve $C \subset \mathbb{P}^2$ the (linear) map $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) \longrightarrow \Gamma(C, \mathcal{O}_C(m))$ is surjective.

(31/10/2024) Example: proof that $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)) = \{0\}.$

LEMMA 1: Let X be an algebraic variety and let \mathcal{F} be a quasi-coherent sheaf on X. Let $\mathcal{V} := {\mathcal{U}_j}_{j \in J}$ be an open affine cover of X. Let $k \in J$ be such that $\mathcal{U} := {\mathcal{U}_i}_{i \in I}$ is an open affine cover of X, where $I := J \setminus {k}$. Suppose that $H^p_{\overline{\mathcal{U}}}(U_k, \mathcal{F}) = 0$ where $\overline{\mathcal{U}} := {U_k \cap U_i}_{i \in I}$. Then the natural (linear) map $H^p_{\mathcal{V}}(U_k, \mathcal{F}) \to H^p_{\mathcal{U}}(U_k, \mathcal{F})$ is an isomorphism.

LEMMA 2: Let X be an affine variety and let \mathcal{F} be a quasi-coherent sheaf on X. If $\mathcal{U} := {\mathcal{U}_i}_{i \in I}$ is a cover of X by principal (open) subsets then $H^p_{\mathcal{U}}(X, \mathcal{F}) = 0$ for all p > 0.

LEMMA 1 and LEMMA 2 imply THM 1 and THM 2.

Week 6.

- (05/11/2024) Recap: divisors on locally factorial algebraic varieties (e.g. smooth varieties) and invertible sheaves, in particular $\mathcal{O}_{\mathbb{P}^n}(d)$ for $d \in \mathbb{Z}$. Cohomology of line bundles on \mathbb{P}^n (only the statement). The cohomology $H^p(X, \mathcal{F})$ of a quasi-coherent sheaf \mathcal{F} on a quasi-projective variety X vanishes for $p > \dim X$, and is a finite dimensional \mathbb{K} -vector space for any pif X is projective and \mathcal{F} is coherent. Vanishing of $H^p(X, \mathcal{F}(k))$ for $k \gg 0$ and p > 0 for $X \subset \mathbb{P}^n$ closed, $\mathcal{O}_X(k) := \mathcal{O}_{\mathbb{P}^n}(k)_{|X}$ and \mathcal{F} coherent.
- (07/11/2024) For $\mathcal{F} \in \operatorname{Coh}(X)$, where X is a projective variety (over K), we let $h^p(X, \mathcal{F}) := \dim_{\mathbb{K}} H^p(X, \mathcal{F})$ and $\chi(X, \mathcal{F}) := \sum_p (-1)^p h^p(X, \mathcal{F})$. Recap: degree of a line bundle on a smooth (irreducible) projective curve X via the result $\operatorname{deg}(\operatorname{div}(f)) = 0$ for $f \in \mathbb{K}(X)^*$. New approach to the definition of $\operatorname{deg} \mathcal{L}$ for an invertible sheaf on X: define $\operatorname{deg} \mathcal{L}$ to be $\chi(X, \mathcal{L}) - \chi(X, \mathcal{O}_X)$. Proof that the new definition agrees with the old one. Weak Riemann-Roch for a (smooth projective) curve X is now a definition, but thanks to the vanishing of $h^1(X, \mathcal{L}(n))$ for $n \gg 0$ it gives a non trivial result.

Week 7.

- (12/11/2024) Review of homework. Koszul homology and regular sequences. Proof that $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{d+n}{n}$, $h^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{-d-1}{n}$, and that $h^p(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ if $p \notin \{0, n\}$.
- (14/11/2024) Class cancelled.

Week 8.

- (19/11/2024) Double complexes, condition guaranteeing that the maps $H^n(K_I) \to H^n(K)$, $H^n(K_{II}) \to H^n(K)$ are isomorphisms. The double complex $C_{\mathcal{U},\mathcal{V}}(X,\mathcal{F})$ associated to a quasicoherent sheaf \mathcal{F} on an algebraic variety X and two open coverings \mathcal{U}, \mathcal{V} of X. Proof of Theorems 1 and 2 of October 29 2024 (the proof given on October 31 is incorrect).
- (21/11/2024) Module $\Omega_{R/k}$ of differentials of a k-algebra R, its universal property. Definition: The sheaf of differentials Ω_X of an affine variety is the sheafification of $\Omega_{\mathbb{K}[X]/\mathbb{K}}$. The sheaf of differentials Ω_X of an algebraic variety over \mathbb{K} either as obtained by patching together the sheaves Ω_{U_i} 's for $\{U_i\}_{i\in I}$ an open affine cover of X, or as $\mathcal{I}_\Delta/\mathcal{I}_\Delta^2$, where $X \cong \Delta \subset X \times X$ is the diagonal. Definition: The tangent sheaf of Θ_X of an algebraic variety over \mathbb{K} is $Hom_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$. Identication of Ω_X, Θ_X with the sheaf of germs of sections of the cotangent and tangent vector bundle respectively if X is smooth.