Chapter 3

Rational maps, dimension and degree

3.1 Introduction

3.2 Rational maps

Let X, Y be algebraic varieties. We define a relation on the set of couples (U, φ) where $U \subset X$ is open dense and $\varphi \colon U \to Y$ is a regular map as follows: $(U, \varphi) \sim (V, \psi)$ if the restrictions of φ and ψ to $U \cap V$ are equal. Then $\sim$ is an equivalence relation. In fact reflexivity and symmetry are trivially true. To prove transitivity suppose that $(U, \varphi) \sim (V, \psi)$ and $(V, \psi) \sim (W, \mu)$. Then the restrictions of φ and μ to $U \cap V \cap W$ are equal. Since V is open dense in X, the intersection $U \cap V \cap W$ is (open) dense in $U \cap W$. Since X is separable it follows that the restrictions of φ and μ to $U \cap W$ are equal, i.e. $(U, \varphi) \sim (W, \mu)$.

Definition 3.2.1. A *rational map* $f \colon X \dashrightarrow Y$ is a $\sim$-equivalence class of couples (U, φ) where $U \subset X$ is open dense and $\varphi \colon U \to Y$ is a regular map.

1. The map f is *regular* at $x \in X$ (equivalently x is a *regular point* of f), if there exists (U, φ) in the equivalence class of f such that $x \in U$. We let $\mathrm{Reg}(f) \subset X$ be the set of regular points of f. By definition $\mathrm{Reg}(f)$ is an open subset of X.

2. The *indeterminancy set* of f is $\mathrm{Ind}(f) := X \backslash \mathrm{Reg}(f)$ (notice that $\mathrm{Ind}(f)$ is closed). A point $x \in X$ is a *point of indeterminancy* if it belongs to $\mathrm{Ind}(f)$.

Example 3.2.2. If $f \colon X \to Y$ is a regular map, we may consider f as a rational map represented by (X, f).

Example 3.2.3. Let X be an algebraic variety, and let $U \subset X$ be open. Let $\iota \colon U \hookrightarrow X$ be the inclusion map. Then (U, ι) represents a rational map $f \colon X \dashrightarrow U$ (note that f goes in the "wrong" direction). Clearly $\mathrm{Reg}(f) = U$.

Example 3.2.4. Let V be a finitely generated vector space and let $[v_0] \in \mathbb{P}(V)$. Let $U := (\mathbb{P}(V) \backslash \{[v_0]\})$. We assume that $\dim V \geqslant 2$, and hence U is open dense in $\mathbb{P}(V)$. The map

$$\begin{array}{ccc} U & \overset{\varphi}{\dashrightarrow} & \mathbb{P}(V/\langle v_0 \rangle) \\ [w] & \mapsto & [\overline{w}] \end{array}$$

where $\overline{w}$ is the equivalence class of w, is regular. Hence (U, φ) represents a rational map $f \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V/\langle v_0 \rangle)$, which is called the *projection from* $[v_0]$. If $\dim V = 2$ then φ is constant and hence φ is regular. If $\dim V > 2$ then the regular locus of φ is equal to U.

From now on we will consider only rational maps between *irreducible* algebraic varieties. Let $f \colon X \dashrightarrow Y$ and $g \colon Y \dashrightarrow W$ be rational maps between (irreducible) algebraic varieties. It might happen that for all $x \in \mathrm{Reg}(f)$ the image $f(x)$ does not belong to $\mathrm{Reg}(g)$, and hence the composition $g \circ f$ makes no sense. In order to deal with compositions of rational maps, we give the following definition.

Definition 3.2.5. A rational map $f\colon X \dashrightarrow Y$ between irreducible algebraic varieties is *dominant* if it is represented by a couple (U, φ) such that $\varphi(U)$ is dense in Y.

Remark 3.2.6. Let $f\colon X \dashrightarrow Y$ be a dominant rational map between irreducible algebraic varieties. If (V, ψ) is an arbitrary representative of f then $\psi(V)$ is dense in Y. In fact by definition f is represented by a couple (U, φ) such that $\varphi(U)$ is dense in Y. Replacing V by $V \cap U$ (which is open dense in X) we may assume that $V \subset U$, and hence $\psi = \varphi_{|V}$. Suppose that $\psi(V)$ is not dense in Y, i.e. there exists a proper closed $W \subsetneq Y$ containing $\psi(V)$. Since $\varphi^{-1}(W) \subset U$ is closed and it contains the dense subset $V \subset U$, it is equal to U. Thus $\varphi(U) \subset W$, and this is a contradiction.

Let X, Y, W be irreducible algebraic varieties. Let

$$X \overset{g}{\dashrightarrow} Y \overset{f}{\dashrightarrow} W \tag{3.2.1}$$

be dominant rational maps, represented by (U, φ) and (V, ψ) respectively. Since $\varphi(U)$ is dense in Y, $\varphi(U) \cap V$ is non empty and hence $\varphi^{-1}(V)$ is non empty. Since $\varphi^{-1}(V)$ is open and X is irreducible, it follows that $\varphi^{-1}(V)$ is dense in X.

Definition 3.2.7. Keeping notation as above, the *composition* $f \circ g$ is the rational map $X \dashrightarrow W$ represented by $(\varphi^{-1}(V), \psi \circ \varphi)$. (The equivalence class of $(\varphi^{-1}(V), \psi \circ \varphi)$ is independent of the representatives (U, φ) and (V, ψ).)

Definition 3.2.8. A dominant rational map $f\colon X \dashrightarrow Y$ between irreducible algebraic varieties is *birational* if there exists a dominant rational map $g\colon Y \dashrightarrow X$ such that $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. An irreducible algebraic variety X is *rational* if it is birational to $\mathbb{P}^n$ for some n, it is *unirational* if there exists a dominant rational map $f\colon \mathbb{P}^n \dashrightarrow X$.

Example 3.2.9. Of course isomorphic irreducible quasi projective varieties are birational. Example 3.2.3 is a slightly less trivial instance of birational map. The inclusion map $\iota\colon U \hookrightarrow X$ has rational inverse the map $f\colon X \dashrightarrow U$ of Example 3.2.3.

Example 3.2.10. Let V be a $\mathbb{K}$ vector space of dimension $n+1$. Suppose that $P\colon V \to \mathbb{K}$ is a quadratic form of rank at least 3, i.e. $\ker P$ has codimension at least 3 (recall that $\ker P \subset V$ is the subspace of vectors u such that $P(u+v) = P(v)$ for all $v \in V$). Then P is not the product of linear functions and hence $Q := V(P) \subset \mathbb{P}(V)$ is an irreducible quadric. Let $[v_0] \in (Q \backslash \mathbb{P}(\ker P))$. The restriction of the projection from $[v_0]$ (see Example 3.2.4) is a rational map

$$Q \overset{f}{\dashrightarrow} \mathbb{P}(V/\langle v_0\rangle). \tag{3.2.2}$$

We claim that f is birational, and hence Q is rational. The reason is the following. First note that by associating to a line $\mathbb{P}(W) \subset \mathbb{P}(V)$ containing $[v_0]$ the element $W/\langle v_0\rangle$ of $\mathbb{P}(V/\langle v_0\rangle)$ we get a bijection between the set of lines containing $[v_0]$ and $\mathbb{P}(V/\langle v_0\rangle)$. Thus we view the latter as parametrizing lines through $[v_0]$. An open dense subset of lines through $[v_0]$ intersect Q in $[v_0]$ and another point (because P has degree 2). Thus for an open dense $U \subset \mathbb{P}(V/\langle v_0\rangle)$ we may define a map $U \to Q$ by associating to the line $\Lambda \in U$ the unique point in $\Lambda \cap Q$ other than $[v_0]$. This is a regular map $U \to Q$ defining a rational map $g\colon \mathbb{P}(V/\langle v_0\rangle) \dashrightarrow Q$ which is the rational inverse of f. More explicitly: in suitable coordinates $Z_0, \ldots, Z_n$ we have $v_0 = (0, 0, \ldots, 0, 1)$ and $F = Z_0 Z_n - G$, where $0 \neq G \in \mathbb{K}[Z_0, \ldots, Z_{n-1}]_2$. Then

$$\begin{array}{ccc} Q & \overset{f}{\dashrightarrow} & \mathbb{P}^{n-1} \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0, \ldots, Z_{n-1}] \end{array}$$

and

$$\begin{array}{ccc} \mathbb{P}^{n-1} & \overset{g}{\dashrightarrow} & Q^{n-1} \\ [T_0, \ldots, T_{n-1}] & \mapsto & [T_0^2, T_0T_1, \ldots, T_0T_{n-1}, G(T_0, \ldots, T_{n-1})] \end{array}$$

Notice that if $n = 2$, then f and g are regular (see Example 1.5.9). If $n \geqslant 3$ then neither f nor g is regular. Moreover the quadric Q is not isomorphic to $\mathbb{P}^{n-1}$. We cannot prove this now in general. For $\mathbb{K} = \mathbb{C}$ and $n = 3$ you may show that $Q \subset \mathbb{P}^3(\mathbb{C})$ with the Euclidean topology is not homeomorphic to $\mathbb{P}^2(\mathbb{C})$ with the Euclidean topology, and hence they are not isomorphic as algebraic varieties.

Proposition 3.2.11. *Irreducible algebraic varieties X, Y are birational if and only if there exist open dense subsets $U \subset X$ and $V \subset Y$ that are isomorphic.*

Proof. An isomorphism $\varphi\colon U \xrightarrow{\sim} V$ clearly defines a birational map $f\colon X \dashrightarrow Y$. To prove the converse, let

$$X \overset{g}{\dashrightarrow} Y \overset{f}{\dashrightarrow} X \tag{3.2.3}$$

be birational inverse maps. Let (U, φ) represent g and (V, ψ) represent f. Then $\varphi^{-1}(V)$ and $\psi^{-1}(U)$ are open dense subsets of U and V respectively. By hypothesis the composition

$$\psi \circ \left(\varphi_{|\varphi^{-1}(V)}\right) : \varphi^{-1}(V) \to U$$

is equal to the identity on an open non-empty subset of $\varphi^{-1}(V)$. By separability of X we get that $\psi \circ \left(\varphi_{|\varphi^{-1}(V)}\right) = \mathrm{Id}_{\varphi^{-1}(V)}$. In particular $\psi \circ \varphi\left(\varphi^{-1}(V)\right) \subset U$, i.e. $\varphi\left(\varphi^{-1}(V)\right) \subset \psi^{-1}(U)$. Similarly

$$\varphi \circ \left(\psi_{|\psi^{-1}(U)}\right) = \mathrm{Id}_{\psi^{-1}(U)}, \quad \psi\left(\psi^{-1}(U)\right) \subset \varphi^{-1}(V).$$

Thus the restrictions of φ and ψ define regular maps $\varphi^{-1}(V) \xrightarrow{\sim} \psi^{-1}(U)$ and $\psi^{-1}(U) \xrightarrow{\sim} \varphi^{-1}(V)$ which are inverse of each other. ☐

Example 3.2.12. Let f, g be the birational maps in Example 3.2.10. Assume that $n \geqslant 3$, so that both non regular. Then

$$\mathrm{Reg}(f) = Q \backslash \{[0, 0, \dots, 0, 1]\}, \qquad \mathrm{Reg}(g) = \mathbb{P}^{n-1} \backslash V(T_0, G(T_0, \dots, T_{n-1})). \tag{3.2.4}$$

On the other hand open dense subsets which are isomorphic are strictly smaller than the regular loci. In fact f defines an isomorphism

$$Q \backslash V(Z_0) \xrightarrow{\sim} \mathbb{P}^{n-1} \backslash V(T_0). \tag{3.2.5}$$

If X, Y are algebraic varieties defined over a subfield $F \subset \mathbb{K}$, then one defines the notion of *rational map $f\colon X \dashrightarrow Y$ defined over F* by considering equivalence classes of couples (U, φ) where $U \subset X$ is an open subset defined over F and $\varphi\colon U \to Y$ is defined over F. As a consequence we have the notion of algebraic varieties defined over F which are *birational over F*. In particular we have the notion of an algebraic varieties defined over F which is *rational over F*.

Example 3.2.13. Let V_0 be an F vector space of dimension $n+1$, and let $P_0\colon V_0 \to F$ be a quadratic form of rank at least 3. Let $V \coloneqq V_0 \otimes_F \mathbb{K}$ and let $P\colon V \to \mathbb{K}$ be the quadratic form obtained from P_0 by extending scalars. Then $Q \coloneqq V(P)$ is a quadric defined over F. We claim that Q is rational over F if and only if $Q(F) \backslash \mathbb{P}(\ker P_0)$ is not empty. In fact suppose that there exists a birational map from a projective $\mathbb{P}^m$ (for some m) space to Q, and hence a regular dominant map $\varphi\colon U \to Q$ where $U \subset \mathbb{P}^m$ is open dense. There are plenty of points in U defined over F and their images are points in $Q(F)$. Moreover not all of these rational points are contained in $\mathbb{P}(\ker P_0)$ because φ is dominant. Hence $Q(F) \backslash \mathbb{P}(\ker P_0)$ is non empty. On the other hand, if there exists a point $[v_0]$ in $(Q(F) \backslash \mathbb{P}(\ker P_0))$, then the procedure described in Example 3.2.10 gives a birational map $f\colon Q \dashrightarrow \mathbb{P}(V/\langle v_0 \rangle)$ defined over F. In fact this holds because we can choose coordinates $Z_0, \dots, Z_n$ for V_0 such that $v_0 = (0, 0, \dots, 0, 1)$ and $F = Z_0 Z_n - G$, where $0 \neq G \in F[Z_0, \dots, Z_{n-1}]_2$.

Many natural invariants of complete algebraic varieties do not separate between birational varieties. This fact gives practical criteria that allow to establish that couples of complete varieties are not birational. On the other hand, it leads one to approach the classification of isomorphism classes of complete (or projective) varieties in two steps: first one classifies equivalence classes for birational equivalence, then one distinguishes isomorphim classes within each birational equivalence class.

3.3 The field of rational functions

If X is an affine variety then one can reconstruct X from the ring $\mathbb{K}[X]$ of regular functions on X. Actually there is a contravariant equivalence between the category of affine varieties and the category of finitely generated $\mathbb{K}$ algebras with no non zero nilpotents, see Section 1.8. On the other hand if X is proper then, since every regular function is locally constant, the ring $\mathbb{K}[X]$ gives very little information about X (unless X is a finite set, i.e. affine). One gets a rich algebraic object by associating to an irreducible algebraic variety the field of rational functions. From this field one reconstructs the algebraic variety modulo birational maps.

Let X be an irreducible algebraic variety. A *rational function on* X is a rational map $X \dashrightarrow \mathbb{K}(=\mathbb{A}^1)$. We define addition and multiplication of rational functions on X by adding and multiplying regular representatives. Let $f, g \colon X \dashrightarrow \mathbb{K}$ be represented by (U, φ) and (V, ψ) respectively. Then

$$\begin{aligned} f + g &:= [(U \cap V, \varphi_{|U\cap V} + \psi_{|U\cap V})], \\ f \cdot g &:= [(U \cap V, \varphi_{|U\cap V} \cdot \psi_{|U\cap V})]. \end{aligned}$$

The definition makes sense because changing representatives of f and g we get equivalent couples. We claim that with the above operations the set of rational functions on X is a field. It is obvious that it is a ring. To check that every non zero element has a multiplicative inverse let $f \colon X \dashrightarrow \mathbb{K}$ be a non zero rational function. Then $f = [(U, \varphi)]$ where $\varphi \neq 0$. Thus $V(\varphi) \subset U$ is a proper closed subset and therefore $U^0 := (U \backslash V(\varphi))$ is open dense in X. Then $g := [(U^0, \varphi^{-1})]$ is the multiplicative inverse of f.

Definition 3.3.1. Let X be an irreducible algebraic variety. The *field of rational functions on* X is the set of rational functions on X with the above operations. It is denoted by $\mathbb{K}(X)$.

Remark 3.3.2. Let X be an irreducible algebraic variety. We have a canonical embedding $\mathbb{K} \hookrightarrow \mathbb{K}(X)$ as the subfield of constant functions.

Remark 3.3.3. Let X be an irreducible algebraic variety. Let $U \subset X$ be a dense open subset. The map

$$\begin{array}{ccc} \mathbb{K}(U) & \xrightarrow{\alpha} & \mathbb{K}(X) \\ [(V,\varphi)] & \mapsto & [(V,\varphi)] \end{array} \tag{3.3.6}$$

is an isomorphism of extensions of $\mathbb{K}$, i.e. it is an isomorphism of fields and the composition $\mathbb{K} \hookrightarrow \mathbb{K}(U) \xrightarrow{\alpha} \mathbb{K}(X)$, where the first map is the the canonical embedding, equals the canonical embedding $\mathbb{K} \hookrightarrow \mathbb{K}(X)$. In particular $\mathbb{K}(X)$ is isomorphic (as extension of $\mathbb{K}$) to the field of rational functions of any of its dense open affine subsets.

The field of rational functions of an irreducible affine variety is isomorphic to the field of fractions of its ring of regular functions. To see this, first note that if X is an irreducible algebraic variety we have an inclusion of $\mathbb{K}$ extensions:

$$\begin{array}{ccc} (\text{field of fractions of } \mathbb{K}[X]) & \hookrightarrow & \mathbb{K}(X) \\ \frac{\alpha}{\beta} & \mapsto & [(X \backslash V(\beta), \frac{\alpha}{\beta})] \end{array} \tag{3.3.7}$$

Claim 3.3.4. *Let X be an affine irreducible variety. Then* (3.3.7) *is an isomorphism.*

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Proof. We must prove that the map in (3.3.7) is surjective. Let $f \in \mathbb{K}(X)$, and let (U, φ) represent f. By Example 1.6.5, there exists $0 \neq \gamma \in \mathbb{K}[X]$ such that the dense principal open subset X_γ is contained in U. Moreover, by Example 1.6.5 and Theorem 1.6.2, $\mathbb{K}[X_f]$ is generated as $\mathbb{K}$-algebra by $\mathbb{K}[X]$ and γ^{-1}, hence ϕ is represented by $(X_\gamma, \frac{\alpha}{\gamma^m})$ where $\alpha \in \mathbb{K}[X]$. Let $\beta := \gamma^m$. Since $X_\gamma = X_\beta$, we have proved that f belongs to the image of (3.3.7). □

Example 3.3.5. By Claim 3.3.4 the field $\mathbb{K}(\mathbb{A}^n)$ is the field of fractions of $\mathbb{K}[z_1, \ldots, z_n]$, i.e. $\mathbb{K}(z_1, \ldots, z_n)$. By Remark 3.3.3 we also have $\mathbb{K}(\mathbb{P}^n) \cong \mathbb{K}(z_1, \ldots, z_n)$.

Remark 3.3.6. If X is an irreducible algebraic variety then $\mathbb{K}(X)$ is finitely generated over $\mathbb{K}$. In fact by Remark 3.3.3 we may replace X by a dense open affine $Y \subset X$. Then $\mathbb{K}(Y)$ is the field of quotients of $\mathbb{K}[Y]$ by Claim 3.3.4. Let $Y \subset \mathbb{A}^n$ as a closed subset. By Theorem 1.6.2 the restriction of coordinate functions $z_{1|X}, \ldots, z_{n|X}$ generate $\mathbb{K}[Y]$ as $\mathbb{K}$-algebra and hence they generate $\mathbb{K}(Y)$ as extension of $\mathbb{K}$. In particular we can extract a transendence basis of $\mathbb{K}(Y)$ from $z_{1|X}, \ldots, z_{n|X}$.

Let $f\colon X \dashrightarrow Y$ be a dominant rational map of irreducible algebraic varieties. Since f is dominant the *pull-back* map

$$\begin{array}{ccc} \mathbb{K}(Y) & \xrightarrow{f^*} & \mathbb{K}(X) \\ \varphi & \mapsto & \varphi \circ f \end{array}$$

is well defined. The map f^* is an inclusion of fields and if $\mathbb{K} \hookrightarrow \mathbb{K}(Y)$ is the canonical inclusion then the composition $\mathbb{K} \hookrightarrow \mathbb{K}(Y) \xrightarrow{\varphi^*} \mathbb{K}(X)$ is the canonical inclusion. Thus f^* is a morphism of extensions of $\mathbb{K}$. Suppose that $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow W$ are dominant rational maps of irreducible algebraic varieties. Then $g \circ f\colon X \dashrightarrow W$ is dominant and

$$f^* \circ g^* = (g \circ f)^* . \tag{3.3.8}$$

Of course $\mathrm{Id}_X^*\colon \mathbb{K}(X) \to \mathbb{K}(X)$ is the identity map. This gives a contravariant functor

$$\begin{array}{ccc} \mathrm{RAT}/\mathbb{K} & \longrightarrow & \mathrm{FGF}/\mathbb{K} \\ X & \mapsto & \mathbb{K}(X) \\ X \overset{f}{\dashrightarrow} Y & \mapsto & f^* \end{array} \tag{3.3.9}$$

where $\mathrm{RAT}/\mathbb{K}$ is the category whose objects are irreducible algebraic varieties and $\mathrm{FGF}/\mathbb{K}$ is the category of finitely generated field extensions of $\mathbb{K}$ (with morphisms the morphisms as extensions of $\mathbb{K}$).

Proposition 3.3.7. *The functor in* (3.3.9) *is an equivalence between the category of irreducible algebraic varieties with homomorphisms dominant rational maps and the category of finitely generated field extensions of* $\mathbb{K}$.

Proposition 3.3.7 follows from Proposition 3.3.8, which proves that the functor in (3.3.9) is essentially surjective, and Proposition 3.3.9, which proves that it is fully faithful.

Proposition 3.3.8. *Let E be a finitely generated field extension of $\mathbb{K}$. There exist an irreducible algebraic variety X and an isomorphisms of $E \xrightarrow{\sim} \mathbb{K}(X)$ of extensions of $\mathbb{K}$.*

Proof. Let m be the transcendenece degree of E over $\mathbb{K}$. By Corollary A.5.9, there exist a prime polynomial $P \in \mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]$ and an isomorphism of extensions of $\mathbb{K}$

$$E \xrightarrow{\sim} \mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]/(P). \tag{3.3.10}$$

Write

$$P = z_{m+1}^d + c_1 z_{m+1}^{d-1} + \cdots + c_d, \quad c_i \in \mathbb{K}\left(z_1, \ldots, z_m\right).$$

Then, for $i \in \{1, \ldots, d\}$, we have $c_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{K}[z_1, \ldots, z_m]$ and $b_i \neq 0$. Let $\widetilde{P} \in \mathbb{K}[z_1, \ldots, z_{m+1}]$ be obtained from P by clearing denominators, i.e. $\widetilde{P} = (b_1 \cdot \ldots \cdot b_d)P$. Lastly, let $Q \in \mathbb{K}[z_1, \ldots, z_{m+1}]$ be obtained from $\widetilde{P}$ by factoring out the maximum common divisor of the coefficients of $\widetilde{P}$ as polynomial in z_{m+1} (recall that $\mathbb{K}[z_1, \ldots, z_m]$ is a UFD). Notice that Q is irreducible and hence prime. Write

$$Q = e_0 z_{m+1}^d + e_1 z_{m+1}^{d-1} + \cdots + e_d, \qquad e_i \in \mathbb{K}[z_1, \ldots, z_m], \quad e_0 \neq 0.$$

Then $X := V(Q) \subset \mathbb{A}^{m+1}$ is an irreducible hypersurface because Q is prime. Because of the isomorphism in (3.3.10) it suffices to prove that there is an isomorphism of extensions of $\mathbb{K}$

$$\mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]/(P) \xrightarrow{\sim} \mathbb{K}(X). \tag{3.3.11}$$

Let $\overline{z}_i := z_{i|X}$. We claim that the rational functions on X represented by $\{\overline{z}_1, \ldots, \overline{z}_m\}$ are algebraically independent over $\mathbb{K}$. In fact, suppose that $R \in \mathbb{K}[t_1, \ldots, t_m]$ and $R(\overline{z}_1, \ldots, \overline{z}_n) = 0$. By the fundamental Theorem of Algebra, for any $(\xi_1, \ldots, \xi_m) \in (\mathbb{A}^m \backslash V(e_0))$ there exists $\xi_{m+1} \in \mathbb{K}$ such that $(\xi_1, \ldots, \xi_m, \xi_{m+1}) \in X$. It follows that $R(\xi_1, \ldots, \xi_m) = 0$ for all $(\xi_1, \ldots, \xi_m) \in (\mathbb{A}^n \backslash V(e_0))$, and hence $R \cdot e_0$ vanishes identically on $\mathbb{A}^m$. Thus $R \cdot e_0 = 0$, and since $e_0 \neq 0$ it follows that $R = 0$. This proves that $\{\overline{z}_1, \ldots, \overline{z}_m\}$ are algebraically independent over $\mathbb{K}$. On the other hand $\overline{z}_{m+1}$ is algebraic over $\mathbb{K}(\overline{z}_1, \ldots, \overline{z}_m)$ and its minimal polynomial equals P. Thus by mapping z_i to $\overline{z}_i$ for $i \in \{1, \ldots, n+1\}$ (and mapping $\mathbb{K}$ to $\mathbb{K}$ by the identity map) we get an isomorphism of extensions of $\mathbb{K}$ as in (3.3.11). □

Proposition 3.3.9. *Let X and Y be irreducible algebraic varieties, and let $\alpha\colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$ is an inclusion of extensions of $\mathbb{K}$. There exists a unique dominant rational map $f\colon X \dashrightarrow Y$ such that $f^* = \alpha$.*

Proof. By remark 3.3.3 we may assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are closed. Hence by Claim 3.3.4 $\mathbb{K}(X)$, $\mathbb{K}(Y)$ are the fields of fractions of $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ respectively. By Theorem 1.6.2, $\mathbb{K}[X] = \mathbb{K}[z_1, \ldots, z_n]/I(X)$ and $\mathbb{K}[Y] = \mathbb{K}[w_1, \ldots, w_m]/I(Y)$. Given $p \in \mathbb{K}[z_1, \ldots, z_n]$ and $q \in \mathbb{K}[w_1, \ldots, w_m]$ we let $\overline{p} := p|_X$ and $\overline{q} := q|_Y$. We have

$$\alpha(\overline{w}_i) = \frac{\overline{f}_i}{\overline{g}_i}, \quad f_i, g_i \in \mathbb{K}[z_1, \ldots, z_n], \quad \overline{g}_i \neq 0.$$

Let $U := X \backslash (V(g_1) \cup \ldots \cup V(g_m))$. Then U is open and dense in X. Let

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{\varphi}} & \mathbb{A}^m \\ a & \mapsto & \left(\frac{f_1(a)}{g_1(a)}, \ldots, \frac{f_m(a)}{g_m(a)}\right) \end{array}$$

We claim that $\widetilde{\varphi}(U) \subset Y$. In fact let $h \in I(Y)$. Since α is an inclusion of extensions of $\mathbb{K}$,

$$h(\overline{f}_1/\overline{g}_1, \ldots, \overline{f}_m/\overline{g}_m) = h(\alpha(\overline{w}_1), \ldots, \alpha(\overline{w}_m)) = \alpha(h(\overline{w}_1, \ldots, \overline{w}_m)) = \alpha(0) = 0.$$

This proves that if $h \in I(Y)$ then h vanishes on $\widetilde{\varphi}(U)$, i.e. $\widetilde{\varphi}(U) \subset Y$. Thus $\widetilde{\phi}$ induces a regular map $\varphi\colon U \to Y$. If $b \in \mathbb{K}[Y] \subset \mathbb{K}(Y)$ then

$$\varphi^*(b) \in \mathbb{K}[U] \subset \mathbb{K}(U) = \mathbb{K}(X)$$

is equal to $\alpha(b)$. It follows that if $b \neq 0$ then $\varphi^*(b) \neq 0$. Thus φ is dominant. Let $f\colon X \dashrightarrow Y$ be the equivalence class of (U, ϕ). Then $f^* = \alpha$.

Moreover it is clear from the above construction that f is the unique rational (dominant) map such that $f^* = \alpha$. □

The result below follows at once from what has been proved above.

Corollary 3.3.10. *Irreducible algebraic varieties are birational if and only if their fields of rational functions are isomorphic as extensions of $\mathbb{K}$.*

Example 3.3.11. Let $p \in \mathbb{K}[z]$ be free of square factors (and $\deg p \geqslant 1$). Then $t^2 - p(z)$ is prime and hence $X := V\left(t^2 - p(z)\right) \subset \mathbb{A}^2$ is irreducible. Thus we have the extensions of fields $\mathbb{K}(X) \supset \mathbb{K}(z) \supset \mathbb{K}$ where the top extension is algebraic of degree 2. Then X is rational if and only if $\mathbb{K}(X)$ is a purely trascendental extension of $\mathbb{K}$. If $\deg p = 1$ then $\mathbb{K}(X)$ is a purely trascendental extension of $\mathbb{K}$ because it is generated (over $\mathbb{K}$) by t. Similarly it is a purely trascendental extension of $\mathbb{K}$ if $\deg p = 2$ by Example 1.5.9. If $\deg p \geqslant 3$ then X is not rational (the proof of this fact this requires new ideas) and hence $\mathbb{K}(X)$ is *not* a purely trascendental extension of $\mathbb{K}$.

The result below follows from the above corollary and the proof of Proposition **??**.

Proposition 3.3.12. *Let X be an irreducible algebraic variety and let $m := \mathrm{Tr.deg}_{\mathbb{K}} \mathbb{K}(X)$. Then X is birational to an irreducible hypersurface in $\mathbb{A}^{m+1}$.*

3.4 Dimension

Definition 3.4.1. 1. The *dimension* of an irreducible algebraic variety X is the transcendence degree of $\mathbb{K}(X)$ over $\mathbb{K}$.

2. Let X be an arbitrary quasi projective variety, and let $X = X_1 \cup \cdots \cup X_r$ be its irreducible decomposition. The *dimension* of X is the maximum of the dimensions of its irreducible components. We say that X has *pure dimension* n if every irreducible component of X has dimension n.

3. Let $p \in X$. The *dimension of X at p* is the maximum of the dimensions of the irreducible components of X containing p.

Remark 3.4.2. The dimension of an irreducible algebraic variety X is equal to the dimension of any open dense subset $U \subset X$. In fact, by definition it suffices to prove it for irreducible X, and in that case it holds because the fields of rational functions $\mathbb{K}(X)$ and $\mathbb{K}(U)$ are isomorphic extensions of $\mathbb{K}$.

Example 3.4.3. The dimension of $\mathbb{A}^n$ and of $\mathbb{P}^n$ is equal to n. In fact $\mathbb{K}(\mathbb{A}^n) = \mathbb{K}(\mathbb{P}^n) = \mathbb{K}(z_1, \ldots, z_n)$, and $\{z_1, \ldots, z_n\}$ is a transcendence basis of $\mathbb{K}(z_1, \ldots, z_n)$ over $\mathbb{K}$.

Example 3.4.4. The dimension of $\mathrm{Gr}(h, V)$ is equal to $h \cdot (\dim V - h)$, because it is irreducible and it contains an open dense subset isomorphic to an affine space of dimension $h \cdot (\dim V - h)$ (actually many such subsets), see Exercise 2.6.3.

Example 3.4.5. Let $X \subset \mathbb{A}^{n+1}$ be a hypersurface. We claim that X has pure dimension n. Since the irreducible components of X are hypersurfaces, in fact the zero loci of the prime factors of f, it suffices to show that if X is an irreducible hypersurface then it has dimensjon n. Let $I(X) = (f)$. Reordering the coordinates $(z_1, \ldots, z_n, z_{n+1})$ we may assume that

$$f = c_0 z_{n+1}^d + c_1 z_{n+1}^{d-1} + \cdots + c_d, \quad c_i \in \mathbb{K}[z_1, \ldots, z_n], \quad c_0 \neq 0, \quad d > 0. \tag{3.4.1}$$

For $i \in \{1, \ldots, n+1\}$ let $\overline{z}_i := z_{i|X}$. In the proof of Proposition 3.3.8 we showed that $\overline{z}_1, \ldots, \overline{z}_n$ are algebraically independent in $\mathbb{K}(X)$. Since $\mathbb{K}(X)$ is generated over $\mathbb{K}$ by $\overline{z}_1, \ldots, \overline{z}_n, \overline{z}_{n+1}$ and since $\overline{z}_{n+1}$ is algebraic over the subfield generated by $\overline{z}_1, \ldots, \overline{z}_n$ it follows that $\overline{z}_1, \ldots, \overline{z}_n$ is transcendence basis of $\mathbb{K}(X)$ over $\mathbb{K}$. Similarly, a hypersurface in $\mathbb{P}^{n+1}$ has pure dimension n. (Intersect with $\mathbb{P}^n_{Z_i}$ for $i \in \{0, 1, \ldots, n+1\}$.)

Remark 3.4.6. An algebraic variety has dimension 0 if and only if it is a finite set.

Remark 3.4.7. If $f\colon X \dashrightarrow Y$ is a dominant map of irreducible algebraic varieties then $\dim X \geqslant \dim X$ because we have the inclusion $f^*\colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$ of field extensions of $\mathbb{K}$.

Proposition 3.4.8. *Let X be an irreducible algebraic variety and let $Y \subset X$ be a proper closed subset. Then* $\dim Y < \dim X$.

Proof. We may assume that Y is irreducible. Since X is covered by open affine varieties, we may assume that X is affine. Thus we may assume that $X \subset \mathbb{A}^n$. Thus Y is also closed in $\mathbb{A}^n$. We may choose a transcendence basis $\{f_1, \ldots, f_d\}$ of $\mathbb{K}(Y)$, where each f_i is a regular function on Y, see Remark 3.3.6.

Let $\tilde{f}_1, \ldots, \tilde{f}_d \in \mathbb{K}[X]$ such that $\tilde{f}_{i|}W = f_i$. Since Y is a proper closed subset of X, there exists a non zero $g \in \mathbb{K}[X]$ such that $g_{|Y} = 0$. It suffices to prove that $\tilde{f}_1, \ldots, \tilde{f}_d, g$ are algebraically independent over. We argue by contradiction. Suppose that there exists $0 \neq P \in \mathbb{K}[S_1, \ldots, S_d, T]$ such that $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$. Since X is irreducible we may assume that P is irreducible. Restricting to Y the equality $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$, we get that $P(f_1, \ldots, f_d, 0) = 0$. Thus $P(S_1, \ldots, S_d, 0) = 0$, because $f_1, \ldots, f_d$ are algebraically independent. This means that T divides P. Since P is irreducible $P = cT$, $c \in \mathbb{K}^*$. Thus $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$ reads $g = 0$, and that is a contradiction. □

Corollary 3.4.9. *A (non empty) closed subset $X \subset \mathbb{A}^{n+1}$ has pure dimension n if and only if it is a hypersurface. Similarly, a closed subset $X \subset \mathbb{P}^{n+1}$ has pure dimension n if and only if it is a hypersurface.*

Proof. If $X \subset \mathbb{A}^{n+1}$ is a hypersurface then it has pure dimension n, see Eaxmple 3.4.1.

In order to prove the converse, suppose that $X \subset \mathbb{A}^{n+1}$ is a closed subset of pure dimension n. Thus every irreducible component of X is a closed subset of $\mathbb{A}^{n+1}$ of dimension n. Since the union of hypersurfaces in $\mathbb{A}^{n+1}$ is a hypersurface in $\mathbb{A}^{n+1}$, it suffices to prove that each irreducible component of X is a hypersurface. Thus we may assume that X is irreducible. Since $\dim X = n < \dim \mathbb{A}^{n+1}$, there exists a non zero $f \in I(X) \subset \mathbb{K}[z_1, \ldots, z_{n+1}]$. Since X is irreducible, the ideal $I(X)$ is prime, and hence there exists a prime factor g of f which vanishes on X. Thus $X \subset V(g)$ and $V(g)$ is irreducible. By Example 3.4.1 we have $\dim V(g) = n$, and hence $\dim X = \dim V(g)$. Since X is closed it follows from Proposition 3.4.8 that $X = V(g)$. This finishes the proof for closed subsets of $\mathbb{A}^{n+1}$.

The result for closed subsets of $\mathbb{P}^{n+1}$ follows by a smilar proof, or by intersecting with the standard open affine subsets $\mathbb{P}^n_{Z_i}$ for $i \in \{0, \ldots, n+1\}$. □

Proposition 3.4.10. *Let X, Y be algebraic varieties. Then* $\dim(X \times Y) = \dim X + \dim Y$.

Proof. We may assume that X and Y are irreducible affine varieties. There exist transcendence bases $\{f_1, \ldots, f_d\}$, $\{g_1, \ldots, g_e\}$ of $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ respectively given by regular functions. Let $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ be the projections. We claim that $\{\pi_X^*(f_1), \ldots, \pi_X^*(f_d), \pi_Y^*(g_1), \ldots, \pi_Y^*(g_e)\}$ is a transcendence basis of $\mathbb{K}(X \times Y)$.

First, by Proposition 2.3.6 $\mathbb{K}[X \times Y]$ is algebraic over the subring generated (over $\mathbb{K}$) by $\pi_X^*(f_1), \ldots, \pi_Y^*(g_e)$.

Secondly, let us show that $\pi_X^*(f_1), \ldots, \pi_Y^*(g_e)$ are algebraically independent. Suppose that there is a polynomial relation

$$\sum_{0 \leqslant m_1, \ldots, m_e \leqslant N} P_{m_1, \ldots, m_e}(\pi_X^*(f_1), \ldots, \pi_X^*(f_d)) \cdot \pi_Y^*(g_1)^{m_1} \cdot \ldots \cdot \pi_Y^*(g_e)^{m_e} = 0,$$

where each $P_{m_1, \ldots, m_e}$ is a polynomial. Since $g_1, \ldots, g_e$ are algebraically independent we get that $P_{m_1, \ldots, m_e}(f_1(a), \ldots, f_d(a)) = 0$ for every $a \in X$. Since $f_1, \ldots, f_d$ are algebraically independent, it follows that $P_{m_1, \ldots, m_e} = 0$ for every $0 \leqslant m_1, \ldots, m_e \leqslant N$, and hence $P = 0$. This proves that $\pi_X^*(f_1), \ldots, \pi_Y^*(g_e)$ are algebraically independent. □

3.5 Dimension and intersection

3.5.1 Closed subsets of $\mathbb{P}^n$: dimension and intersection with linear subspaces

Let $X \subset \mathbb{P}^n$ be a hypersurface. Thus $X = V(F)$ where $F \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ with $d > 0$ and $F \neq 0$. Let $\Lambda = \mathbb{P}(U)$ be a linear subspace of $\mathbb{P}^n$, i.e. $U \subset \mathbb{K}^{n+1}$ is a $\mathbb{K}$ vector subspace. Then $\Lambda \cap X = V(F_{|U})$. It follows that if $\dim \Lambda \geqslant 1$ then Λ has non empty intersection with X. If, on the other hand, $\dim \Lambda = 0$ i.e. Λ is a point, then $\Lambda \cap X$ is empty for all points in the dense open subset $\mathbb{P}^n \backslash X$. An analogous characterization of the dimension of a closed subset of $\mathbb{P}^n$ holds in general. In order to formulate the relevant result we introduce a definition and a classical piece of terminology.

Definition 3.5.1. Let X be an irreducible algebraic variety, and let $Y \subset X$ be a closed subset. The *codimension of Y in X* is equal to $\dim X - \dim Y$, and is denoted by $\operatorname{cod}(Y, X)$.

Terminology 3.5.2. Let X be an algebraic variety, and let $\mathscr{P}$ be a property that each point of X might or might not have (formally "the subset of points of X having the property $\mathscr{P}$"). Then *a general point of X has property $\mathscr{P}$* if there is a dense open subset of X of points having property $\mathscr{P}$.

Proposition 3.5.3. *Let $X \subset \mathbb{P}^n$ be closed.*

(a) Let $k < \operatorname{cod}(X, \mathbb{P}^n)$. Then for a general $\Lambda \in \operatorname{Gr}(k, \mathbb{P}^n)$ we have $\Lambda \cap X = \varnothing$ (i.e. there exists a dense open $U \subset \operatorname{Gr}(k, \mathbb{P}^n)$ such that $\Lambda \cap X = \varnothing$ for all $\Lambda \in U$).

(b) Let $\Lambda \subset \mathbb{P}^n$ be a linear subspace such that $\dim \Lambda \geqslant \operatorname{cod}(X, \mathbb{P}^n_{\mathbb{C}})$. Then $\Lambda \cap X \neq \varnothing$.

The proof of Proposition 3.5.3 is given after a few preliminary results.

Definition 3.5.4. Let $X \subset \mathbb{P}^n$ be closed. For $k \in \{0, \dots, n\}$ let $\Gamma_X(k) \subset X \times \mathbb{Gr}(k, \mathbb{P}^n)$ be given by

$$\Gamma_X(k) = \{(p, \Lambda) \in X \times \mathbb{Gr}(k, \mathbb{P}^n) \mid p \in \Lambda\}.$$

Proposition 3.5.5. *Let $X \subset \mathbb{P}^n$ be closed. The following hold:*

(a) $\Gamma_X(k)$ *is closed in* $X \times \mathbb{Gr}(k, \mathbb{P}^n)$

(b) $\dim \Gamma_X(k) = \dim X + k(n-k)$.

(c) If X is irreducible then $\Gamma_X(k)$ is irreducible.

Proof. Let us show that $\Gamma_{\mathbb{P}^n}(k)$ is closed. Let $A := (a_{i,j}) \in M_{k+1,n+1}(\mathbb{K})$ be a matrix of maximal rank, i.e. of rank $k+1$. Thus the rows of A span a subspace $U_A \subset \mathbb{K}^{n+1}$ of dimension $k+1$, and hence $\mathbb{P}(U_A) \in \mathbb{Gr}(k, \mathbb{P}^n)$. Let $[Z] \in \mathbb{P}^n$. Then $([Z], \mathbb{P}(U_A)) \in \Gamma_{\mathbb{P}^n}(k)$ if and only if the $(k+2) \times (n+1)$ matrix obtained by adding the row Z to A has rank less than $k+2$, i.e. if and only if for all $0 \leqslant j_0 < j_1 < \dots < j_{k+1} \leqslant (n+1)$ we have

$$\operatorname{Det} \begin{bmatrix} X_{j_0} & X_{j_1} & \cdots & \cdots & X_{j_{k+1}} \\ a_{0,j_0} & a_{0,j_1} & \cdots & \cdots & a_{0,j_{k+1}} \\ a_{1,j_0} & a_{1,j_1} & \cdots & \cdots & a_{1,j_{k+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k,j_0} & a_{k,j_1} & \cdots & \cdots & a_{k,j_{k+1}} \end{bmatrix} = 0$$

Expanding the determinant on the left hand side we get that $([Z], \mathbb{P}(U_A)) \in \Gamma_{\mathbb{P}^n}(k)$ if and only if

$$\sum_{s=0}^{k+1} p_{j_0, j_1, \dots, j_{k+1}} X_{j_s} = 0 \tag{3.5.1}$$

for all $0 \leqslant j_0 < j_1 < \dots < j_{k+1} \leqslant (n+1)$, where $[\dots, p_{j_0,j_1,\dots,j_{k+1}}, \dots]$ are the Plücker coordinates of $\mathscr{P}(U_A)$ (see Exercise 2.6.4) with respect to the basis of $\bigwedge^{k+1} \mathbb{K}^{n+1}$ associated to the standard basis of $\mathbb{K}^{n+1}$. This proves that $\Gamma_{\mathbb{P}^n}(k)$ is closed.

Now we show that $\Gamma_X(k)$ is closed for $X \subset \mathbb{P}^n$ closed. Let $\pi \colon \mathbb{P}^n \times \mathbb{Gr}(k, \mathbb{P}^n) \to \mathbb{P}^n$ be the projection. Then $\Gamma_X(k) = \pi^{-1}(X) \cap \Gamma_{\mathbb{P}^n}(k)$. Since X is closed in $\mathbb{P}^n$ and π is regular $\pi^{-1}(X)$ is closed in $\mathbb{P}^n \times \mathbb{Gr}(k, \mathbb{P}^n)$ and hence $\Gamma_X(k)$ is closed in $\mathbb{P}^n \times \mathbb{Gr}(k, \mathbb{P}^n)$ because $\Gamma_{\mathbb{P}^n}(k)$ is closed. Of course this gives that $\Gamma_X(k)$ is closed in $X \times \mathbb{Gr}(k, \mathbb{P}^n)$. This finishes the proof of Item (a).

Next we note that if $X = X_1 \cup \dots \cup X_r$ is the irreducible decomposition of X then

$$\Gamma_X(k) = \Gamma_{X_1}(k) \cup \dots \cup \Gamma_{X_r}(k). \tag{3.5.2}$$

From this we get that it suffices to prove that (b) and (c) hold for X irreducible. For $i \in \{0, \dots, n\}$ we have the isomorphism

$$\begin{array}{ccc} X_{Z_i} \times \operatorname{Gr}(k, \mathbb{K}^n) & \xrightarrow{\alpha_i} & \Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathbb{Gr}(k, \mathbb{P}^n)) \\ (p, W) & \mapsto & (p, \overline{p+W}) \end{array} \tag{3.5.3}$$

where W is a k-dimensional vector subspace of $\mathbb{K}^n$ viewed as the vector space acting on the affine space $\mathbb{P}^n_{Z_i} \simeq \mathbb{A}^n$, and $\overline{p+W}$ denotes the closure in $\mathbb{P}^n$ of the affine subspace $p + W \subset \mathbb{P}^n_{Z_i} \simeq \mathbb{A}^n$. Suppose that $\Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathbb{Gr}(k, \mathbb{P}^n))$ is non empty. Then by the isomorphism in (3.5.3) it is irreducible, and

$$\dim \left(\Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathbb{Gr}(k, \mathbb{P}^n))\right) = \dim X_{Z_i} \times \operatorname{Gr}(k, \mathbb{K}^n) = \dim X + \dim \operatorname{Gr}(k, \mathbb{K}^n) = \dim X + k(n-k).$$

(See Exercise 3.8.2 for the dimension of $\operatorname{Gr}(k, \mathbb{K}^n)$.) Thus $\Gamma_X(k)$ is covered by the open non empty irreducible subsets $\Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathbb{Gr}(k, \mathbb{P}^n))$. Since any two (non empty) such subsets have non empty intersection (because X is irreducible), Items (b) and (c) follow. $\square$

Corollary 3.5.6. *Let $X \subset \mathbb{P}^n$ be closed. If $k \leqslant \operatorname{cod}(X, \mathbb{P}^n)$ then*

$$\dim \Gamma_X(k) \leqslant \dim \mathbb{G}\mathrm{r}(k, \mathbb{P}^n), \tag{3.5.4}$$

and equality holds if and only if $k = \operatorname{cod}(X, \mathbb{P}^n)$.

Proof. By Exercise 3.8.2 we have

$$\dim \mathbb{G}\mathrm{r}(k, \mathbb{P}^n) = n - k + k(n-k). \tag{3.5.5}$$

The corollary follows at once from the above equality and Proposition 3.5.5. □

Proposition 3.5.7. *Let $X \subset \mathbb{P}^n$ be closed. Suppose that $p \in \mathbb{P}^n \backslash X$ and that $H \subset \mathbb{P}^n$ is a hyperplane not containing p. Let*

$$\begin{array}{ccc} (\mathbb{P}^n \backslash \{p\}) & \xrightarrow{\pi_p} & H \\ q & \mapsto & \langle p, q \rangle \cap H \end{array}$$

be projection from p. Then $\pi_p(X)$ is a closed subset of H and $\dim \pi_p(X) = \dim X$.

Proof. We may assume that X is irreducible. Since $\pi_{p|X}$ is regular and X is projective $\pi_p(X)$ is closed. It remains to prove that $\dim \pi_p(X) = \dim X$. We may assume that $p = [0, \ldots, 0, 1]$ and $H = V(X_n)$. We have

$$\pi_p([Z_0, \ldots, Z_n]) = [Z_0, \ldots, Z_{n-1}].$$

Let $Y := \pi_p(X)$. The map π_p defines a regular surjective map $\rho \colon X \to Y$ between irreducible (projective) varieties. We have the injection of fields $\rho^* \colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$. It suffices to prove that $\mathbb{K}(X)$ is algebraic over $\rho^* \mathbb{K}(Y)$.

One of $V(Z_0), \ldots, V(Z_{n-1})$ does *not* contain Y, say $V(Z_0)$, and hence $\mathbb{K}(Y)$ is generated over $\mathbb{K}$ by

$$(Z_1/Z_0)|_Y, \ldots, (Z_{n-1}/Z_0)|_Y.$$

On the other hand $\mathbb{K}(X)$ is generated by

$$(Z_1/Z_0)|_X = \rho^* \left((Z_1/Z_0)|_Y \right), \ldots, (Z_{n-1}/Z_0)|_X = \rho^* \left((Z_{n-1}/Z_0)|_Y \right)$$

and $(Z_n/Z_0)|_X$. There exists $F \in I(X)$ such that $F(p) \neq 0$ because $p \notin X$. Since $p = [0, \ldots, 0, 1]$ we get that

$$F = a_0 Z_n^d + a_1 Z_n^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{K}[Z_0, \ldots, Z_{n-1}]_i, \quad a_0 \neq 0. \tag{3.5.6}$$

Dividing by Z_0^d and restricting to X we get that

$$\overline{a}_0 \cdot ((Z_n/Z_0)_{|X})^d + \overline{a}_1 \cdot ((Z_n/Z_0)_{|X})^{d-1} + \cdots + \overline{a}_d = 0$$

where for $0 \leqslant j \leqslant d$

$$\overline{a}_j := (a_j/Z_0^j)_{|X} \in \mathbb{K}\left(\rho^* \left((Z_1/Z_0)|_Y \right), \ldots, \rho^* \left((Z_{n-1}/Z_0)|_Y \right)\right). \tag{3.5.7}$$

Since $\overline{a}_0 \neq 0$ this proves that $(Z_n/Z_0)|_X$ is algebraic over $\rho^* \mathbb{K}(Y)$.

□

Proof of Proposition 3.5.3. By considering an irreducible component of X of maximum dimension we may assume that X is irreducible (see (3.5.2)). Let $\rho \colon \Gamma_X(k) \to \mathbb{G}\mathrm{r}(k, \mathbb{P}^n)$ be the restriction of the projection map $\mathbb{P}^n \times \mathbb{G}\mathrm{r}(k, \mathbb{P}^n) \to \mathbb{G}\mathrm{r}(k, \mathbb{P}^n)$. Then $\Lambda \in \mathbb{G}\mathrm{r}(k, \mathbb{P}^n)$ has non empty intersection with X if and only if it belongs to $\operatorname{im}(\rho)$. The map ρ is closed because $\Gamma_X(k)$ is projective, hence $\operatorname{im}(\rho)$ is closed. Moreover $\operatorname{im}(\rho)$ is irreducible because X is irreducible. Thus ρ defines a dominant map $\Gamma_X(k) \to \operatorname{im}(\rho)$ of irreducible varieties. It follows that $\dim(\operatorname{im}(\rho)) \leqslant \Gamma_X(k)$. Now suppose that $k < \operatorname{cod}(X, \mathbb{P}^n)$. By Corollary 3.5.6 we get that $\dim(\operatorname{im}(\rho)) < \dim \mathbb{G}\mathrm{r}(k, \mathbb{P}^n)$ and hence $\mathbb{G}\mathrm{r}(k, \mathbb{P}^n) \backslash \operatorname{im}(\rho)$ is an open dense subset of $\dim \mathbb{G}\mathrm{r}(k, \mathbb{P}^n)$. Item (a) follows because any $\Lambda \in (\mathbb{G}\mathrm{r}(k, \mathbb{P}^n) \backslash \operatorname{im}(\rho))$ does not intersect X.

Next we prove Item (b). The proof is by induction on $\mathrm{cod}(X, \mathbb{P}^n)$. If $\mathrm{cod}(X, \mathbb{P}^n) = 0$ the result is trivial (if you don't like to start from $\mathrm{cod}(X, \mathbb{P}^n) = 0$ you may begin from $\mathrm{cod}(X, \mathbb{P}^n) = 1$, i.e. X is a hypersurface). Let's prove the inductive step. Let $p \in \Lambda$. If $p \in X$ there is nothing to prove; thus we may assume that $p \notin X$. Choose a hyperplane $H \subset \mathbb{P}^n$ not containing p and let π_p be projection from p to H as in (3.7.1). Then $\pi_p(X) \subset H \simeq \mathbb{P}^{n-1}$ is closed because X is projective, and $\dim \pi_p(X) = \dim X$ by Proposition 3.5.7. Thus

$$\mathrm{cod}(\pi_p(X), \mathbb{P}^{n-1}) = \mathrm{cod}(X, \mathbb{P}^n) - 1. \tag{3.5.8}$$

Let $\overline{\Lambda} := \pi(\Lambda \backslash \{p\}) = \Lambda \cap H$. Thus $\overline{\Lambda} \subset H$ is a linear subspace and $\dim \overline{\Lambda} = (\dim \Lambda - 1)$. By the equality in (3.5.8) it follows that $\dim \overline{\Lambda} \geqslant \mathrm{cod}(\pi_p(X), \mathbb{P}^{n-1})$. Hence $\overline{\Lambda} \cap \pi_p(X)$ is non empty by the inductive hypothesis. Let $q \in \overline{\Lambda} \cap \pi_p(X)$. Since $q \in \pi_p(X)$ there exists $\tilde{q} \in X$ such that $\pi_p(\tilde{q}) = q$. But $\tilde{q} \in \Lambda$ because $q \in \overline{\Lambda}$. Thus $\tilde{q} \in X \cap \Lambda$. $\square$

3.5.2 Dimension of intersections

The result below is a remarkable generalization of the well-known result in linear algebra stating that the set of solutions of a system of m homogeneous linear equations in $n \geqslant m$ unknowns has dimension at least $n - m$.

Proposition 3.5.8. *Let $X, Y \subset \mathbb{P}^n$ be closed and suppose that $(\dim X + \dim Y) \geqslant n$. Then $X \cap Y$ is non empty and it has dimension at least $\dim X + \dim Y - n$. If X and Y are irreducible then each of the irreducible components of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$.*

Remark 3.5.9. It is clear that one needs the hypothesis that X, Y be closed for the thesis of Proposition 3.5.8 to hold. The hypothesis that the ambient algebraic variety is $\mathbb{P}^n$ is also a key hypothesis. As soon as one replaces $\mathbb{P}^n$ by other complete algebraic varieties the thesis fails to hold. As a test consider replacing $\mathbb{P}^n$ by a product of projective spaces, or by a Grassmannian.

We prove Proposition 3.5.8 after going through a series of preliminary results. The result below proves the special case of Proposition 3.5.8 that one gets by letting Y be a hyperplane.

Proposition 3.5.10. *Let $X \subset \mathbb{P}^n$ be closed, irreducible of strictly positive dimension. Let $H \subset \mathbb{P}^n$ a hyperplane not containing X. Then $X \cap H$ is non empty and it has pure dimension equal to $\dim X - 1$.*

Proof. Since $X \cap H \subsetneq X$ we have $\dim X \cap H < \dim X$ by Proposition 3.4.8. Let $c := \mathrm{cod}(X, \mathbb{P}^n)$. Let $\Lambda \subset H$ be a linear subspace such that $\dim \Lambda = c$. Note that such subspaces exist because by hypothesis $c \leqslant (n-1) = \dim H$. By Proposition 3.5.3 applied to $X \subset \mathbb{P}^n$ we have $\Lambda \cap X \neq \emptyset$, and since $\Lambda \subset H$ we have $\Lambda \cap X \subset \Lambda \cap (X \cap H)$. This proves that $X \cap H$ is non empty and also, by Proposition 3.5.3, that $\mathrm{cod}(X \cap H, H) \leqslant c$. The latter inequality gives that

$$\dim(X \cap H) \geqslant \dim H - c = n - 1 - c = \dim X - 1. \tag{3.5.9}$$

This proves that $X \cap H$ is non empty and $\dim(X \cap H) = \dim X - 1$.

The proposition states that in addition $X \cap H$ has pure dimension. This result is not needed for the proof of Proposition 3.5.8, but is very important. The proof is by induction on $\mathrm{cod}(X, \mathbb{P}^n)$. If $\mathrm{cod}(X, \mathbb{P}^n) = 0$ then $X = \mathbb{P}^n$ and the statement of the proposition is trivially true. If $\mathrm{cod}(X, \mathbb{P}^n) = 1$ then X is a hypersurface by Corollary 3.4.9, hence $X \cap H$ is a hypersurface in H and hence every irreducible component of $X \cap H$ has codimension one in H by Corollary 3.4.9. This proves the validity of the proposition if $\mathrm{cod}(X, \mathbb{P}^n) = 1$. Now we prove the inductive step. Assume that $\mathrm{cod}(X, \mathbb{P}^n) = c \geqslant 2$. Let Y be an irreducible component of $X \cap H$. Pick a point $p \in H \backslash X$ and a hyperplane L not containing p and different from H. Let

$$\begin{array}{ccc} \mathbb{P}^n \backslash \{p\} & \xrightarrow{\pi_p} & L \\ q & \mapsto & \langle p, q \rangle \cap L \end{array}$$

be the projection from p. Let $H_0 := \pi_p(H \backslash \{p\})$. Note that $H_0 \subset L$ is a hyperplane. We consider $\pi_p(X) \cap H_0$. Let $X \cap H = Y \cup Y_1 \cup \cdots \cup Y_r$ be the irreducible decomposition of $X \cap H$. We have

$$\pi_p(X) \cap H_0 = \pi_p(Y) \cup \pi_p(Y_1) \cup \ldots \cup \pi_p(Y_r),$$

and, since $p \notin X$, each of $\pi_p(Y), \pi_p(Y_1), \ldots, \pi_p(Y_r)$ is closed by Proposition 3.5.7. We claim that there exists p such that

$$\pi_p(Y) \not\subset \pi_p(Y_i) \quad \forall i \in \{1, \ldots, r\}. \tag{3.5.10}$$

In fact let $q \in Y \backslash \bigcup_{i=1}^{r} Y_i$. By Claim 3.5.12 $J(q, Y_i)$ is closed, irreducible, and

$$\dim J(q, Y_i) = \dim Y_i + 1. \tag{3.5.11}$$

Since $\dim Y_i \leqslant \dim X - 1$ and since $\operatorname{cod}(X, \mathbb{P}^n) \geqslant 2$ we have $\dim Y_i \leqslant \dim H - 2$. Thus (3.5.11) gives that $J(q, Y_i) \neq H$. Hence there exists

$$p \in H \backslash \bigcup_{i=1}^{r} J(q, Y_i). \tag{3.5.12}$$

For such a p the statement in (3.5.10) holds, and hence $\pi_p(Y)$ is an irreducible component of $\pi_p(X) \cap H_0$.

By the inductive hypothesis we get that $\dim \pi_p(Y) = \dim \pi_p(X) - 1$. Since $\dim \pi_p(Y) = \dim Y$ and $\dim \pi_p(X) = \dim X$ (by Proposition 3.5.7) we are done. □

Let $X, Y \subset \mathbb{P}^N$ be two closed subsets. Let $\langle X \rangle \subset \mathbb{P}^N$ and $\langle Y \rangle \subset \mathbb{P}^N$ be the linear subspaces generated by X and Y respectively.

Definition 3.5.11. Suppose that

$$\langle X \rangle \cap \langle Y \rangle = \varnothing. \tag{3.5.13}$$

The *join* $J(X, Y)$ *of* X *and* Y is the subset of $\mathbb{P}^N$ swept out by the lines joining a point of X to a point of Y, i.e.

$$J(X, Y) := \bigcup_{p \in X, q \in Y} \langle p, q \rangle. \tag{3.5.14}$$

Claim 3.5.12. *Let $X, Y \subset \mathbb{P}^N$ be closed and assume that* (3.5.13) *holds.*

1. *$J(X, Y)$ is closed in $\mathbb{P}^N$.*
2. *If X and Y are irreducible then $J(X, Y)$ is irreducible.*
3. $\dim J(X, Y) = \dim X + \dim Y + 1$.

Proof. Let $m := \dim\langle X \rangle$ and $n := \dim\langle Y \rangle$. There exist homogeneous coordinates

$$[S_0, \ldots, S_m, T_0, \ldots, T_n, U_0, \ldots, U_p]$$

on $\mathbb{P}^N$ such that $\langle X \rangle = \{[S_0, \ldots, S_m, 0, \ldots, 0]\}$ and $\langle Y \rangle = \{[0, \ldots, 0, T_0, \ldots, T_n, 0, \ldots, 0]\}$. Then

$$J(X, Y) = \{[S_0, \ldots, S_m, T_0, \ldots, T_n, 0, \ldots, 0] \mid [S_0, \ldots, S_m] \in X, \quad [T_0, \ldots, T_n] \in Y\}. \tag{3.5.15}$$

Item (1) follows at once.

Let $r \in (J(X, Y) \backslash X \backslash Y)$. By (3.5.13) there is *unique* couple $(\varphi_1(r), \varphi_2(r)) \in X \times Y$ such that $r \in \langle \varphi_1(r), \varphi_2(r) \rangle$. Thus we have a map

$$\begin{array}{ccc} (J(X, Y) \backslash X \backslash Y) & \xrightarrow{\varphi} & X \times Y \\ r & \mapsto & (\varphi_1(r), \varphi_2(r)) \end{array} \tag{3.5.16}$$

As is easily checked φ is regular. The fibers of φ are isomorphic to $\mathbb{K}^\times$. Moreover for any $i \in \{0, \ldots, m\}$ and $j \in \{0, \ldots, n\}$ we have

$$\varphi^{-1}(X_{S_i} \times Y_{T_j}) \cong X_{S_i} \times Y_{T_j} \times \mathbb{K}^\times. \tag{3.5.17}$$

Items (2) and (3) follow from this. □

Proof of Proposition 3.5.8. It suffices to prove the statement for X, Y irreducible. Let $[S_0, \ldots, S_n, T_0, \ldots, T_n]$ be homogeneous coordinates on $\mathbb{P}^{2n+1}$. We have the two embeddings

$$\begin{array}{ccccccc} \mathbb{P}^n & \xrightarrow{i} & \mathbb{P}^{2n+1} & \qquad & \mathbb{P}^n & \xrightarrow{j} & \mathbb{P}^{2n+1} \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0, \ldots, Z_n, 0, \ldots, 0] & & [Z_0, \ldots, Z_n] & \mapsto & [0, \ldots, 0, Z_0, \ldots, Z_n] \end{array} \tag{3.5.18}$$

Since the images of i and j are disjoint linear subspaces of $\mathbb{P}^{2n+1}$ the join $J(i(X), j(Y))$ is defined. Let $\Lambda \subset \mathbb{P}^{2n+1}$ be the linear subspace given by

$$\Lambda := V(S_0 - T_0, \ldots, S_n - T_n). \tag{3.5.19}$$

We have the isomorphism

$$\begin{array}{ccc} X \cap Y & \xrightarrow{\sim} & \Lambda \cap J(i(X), j(Y)) \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0, \ldots, Z_n, Z_0, \ldots, Z_n] \end{array} \tag{3.5.20}$$

By Claim 3.5.12 the closed subset $J(i(X), j(Y)) \subset \mathbb{P}^{2n+1}$ is irreducible of (pure) dimension equal to $\dim X + \dim Y + 1$. On the other hand Λ is a codimension-$(n+1)$ linear subspace of $\mathbb{P}^{2n+1}$, hence by repeated application of Proposition 3.5.10 we get that $\Lambda \cap J(i(X), j((Y))$ is non empty and each of its irreducible components has dimension at least equal to $(\dim X + \dim Y - n)$. By the isomorphism in (3.5.20) the proposition follows. $\square$

3.5.3 Dimension and chains of closed subsets

Let X be an algebraic variety. By Noetherianity there is no infinite chain of closed subsets

$$X = X_0 \supsetneq X_1 \supsetneq \ldots X_{n-1} \supsetneq X_n \supsetneq \tag{3.5.21}$$

Thus every such chain is finite. The following result characterizes the dimension of X via the length of such chains.

Proposition 3.5.13. *Let X be an algebraic variety.*

1. *The dimension of X is equal to the maximum of the set of n for which there exists a chain*

$$X = X_0 \supsetneq X_1 \supsetneq \ldots X_{n-1} \supsetneq X_n \tag{3.5.22}$$

   of closed subsets with X_i irreducible for all $i \in \{1, \ldots, n\}$.

2. *Let $p \in X$. The dimension of X at p is equal to the maximum of the set of n for which there exists a chain*

$$X = X_0 \supsetneq X_1 \supsetneq \ldots X_{n-1} \supsetneq X_n \ni p \tag{3.5.23}$$

   of closed subsets with X_i irreducible for all $i \in \{1, \ldots, n\}$.

Proof. We may assume that X is irreducible. In fact, in proving Item (a) it suffices to replace X by an irreducible component computing $\dim X$ (i.e. whose dimension is equal to $\dim X$), in proving Item (b) it suffices to replace X by an irreducible component containing p and computing $\dim_p X$. Replacing X by an open dense affine subset (containing p if we are proving Item (b)) we may assume that X is affine and irreducible. Let $d := \dim X$. Let N be the maximum of the set of n for which there exists a chain as in (3.5.22) if we are are proving Item (a), respectively a chain as in (3.5.23) if we are are proving Item (b). If we have (3.5.22) then $\dim \dim X_i > X_{i+1}$ for all $i \in \{0, \ldots, n-1\}$. Since $\dim X_n \geqslant 0$ it follows that $n \leqslant d$. Similarly, if we have (3.5.22) then $n \leqslant d$. Thus $N \leqslant d$, both when proving Item (a) and when proving Item (b). We prove that $N \geqslant d$ by induction on d. If $d = 0$ then X is a singleton and the statement is trivially true. We prove the inductive step. Thus $d > 0$. Since X is affine we may assume that $X \subset \mathbb{A}^m = \mathbb{P}^m_{Z_0}$. Let $\overline{X} \subset \mathbb{P}^m$ be the closure of X. Then X is an open dense subset of $\overline{X}$ and $\dim X = \dim \overline{X}$. Let $p \in X$, and let $H \subset \mathbb{P}^m$ be a hyperplane containing p but not containing X.

By Proposition 3.5.10 the intersection $H \cap \overline{X}$ has pure dimension $d-1$. Since H contains p there exists an irreducible component of $H \cap \overline{X}$ containing p, call it X_1. Of course $\dim X_1 = d-1$. By inductive hypothesis there exists a chain $X_1 \supsetneq \ldots X_{d-2} \supsetneq X_{d-1} \ni p$ of closed irreducible subsets. Adding $X_0 = X$ we get the desired chain (this finishes the proof of both items). □

Remark 3.5.14. The *Krull dimension* of a (commutative unitary) ring R is the supremum of n such that there is a chain

$$P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_{n-1} \subsetneq P_n$$

of prime ideals $P_i \subset R$. Proposition 3.5.13 shows that if X is an affine variety then its dimension equals the Krull dimension of the ring $\mathbb{K}[X]$ of regular functions on X.

3.5.4 Dimensions of fibers

The problems that we discuss are the following. Let $f\colon X \to Y$ be a regular map of algebraic varieties.

1. How does the dimension of fibers $f^{-1}(y)$ vary as $y \in Y$ varies?
2. What kind of subset of Y is the image of f?

Note that there is a relation between the two questions, since the image of f is the set of $y \in Y$ such that $\dim f^{-1}(y) \geqslant 0$.

Example 3.5.15. Let V be a finitely generated $\mathbb{K}$ vector space. Let $X \subset \mathbb{P}(V^\vee)^m \times \mathbb{P}(V)$ be given by

$$\{[\varphi_1], \ldots, [\varphi_m], [v]) \mid \varphi_1(v) = \ldots = \varphi_m(v) = 0\}. \tag{3.5.24}$$

As is easily checked X is a closed subset of $\mathbb{P}(V^\vee)^m \times \mathbb{P}(V)$. Let $f\colon X \to \mathbb{P}(V^\vee)^m$ be the restriction of the projection map $\mathbb{P}(V^\vee)^m \times \mathbb{P}(V) \to \mathbb{P}(V^\vee)^m$. Let $y := ([\varphi_1], \ldots, [\varphi_m]) \in \mathbb{P}(V^\vee)^m$: then $f^{-1}(y)$ is identified with the projectivization of the kernel of the map $V \to \mathbb{K}^m$ defined by $(\varphi_1, \ldots, \varphi_m)$. From this we get that the dimension of $f^{-1}(y)$ is an upper semicontinuous function of y, i.e. for every $k \in \mathbb{N}$ the set $D_k \subset Y$ of y such that $\dim f^{-1}(y) \geqslant k$ is (Zariski) closed. In fact D_k is the locus of $y = ([\varphi_1], \ldots, [\varphi_m])$ such that the linear map $(\varphi_1, \ldots, \varphi_m)$ has rank at most $\dim V - k - 1$ and thus, once a basis of V has been chosen, D_k is the zero locus of determinants of all $(\dim V - k) \times (\dim V - k)$ minors of the matrix associated to the linear map $(\varphi_1, \ldots, \varphi_m)$ by the choice of a basis of V. In particular the image of f equals D_0 and hence is closed.

Example 3.5.16. Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be defined by $f(w,z) := (w, wz)$. We have

$$\dim f^{-1}(a,b) = \begin{cases} 1 & \text{if } a = b = 0, \\ 0 & \text{if } a \neq 0, \\ -\infty & \text{if } a = 0,\ b \neq 0. \end{cases} \tag{3.5.25}$$

In particular the image of f is not closed nor open.

Thus in Example 3.5.15 the dimensions of fibers vary more nicely than in Example 3.5.16. There is a hypotheses that guarantees a behaviour similar to that of Example 3.5.15.

Definition 3.5.17. A regular map $i\colon X \to Y$ of algebraic varieties is a *closed embedding* if it factors as $i = j \circ f$, where $f\colon X \xrightarrow{\sim} W$ is an isomorphism between X and a closed subvariety $W \subset Y$ and $j\colon W \hookrightarrow Y$ is the inclusion map.

Definition 3.5.18. A regular map $f\colon X \to Y$ of algebraic varieties is a *projective* map if there exists a closed embedding $i\colon X \hookrightarrow \mathbb{P}^N \times Y$ such that $f = p_Y \circ i$ where $p_Y : \mathbb{P}^N \times Y \to Y$ is the projection.

Remark 3.5.19. Let $f\colon X\to Y$ be regular map of algebraic varieties, and suppose that X is projective. Then f is a projective map. In fact, assuming that $X\subset\mathbb{P}^N$ is closed, let

$$\begin{array}{ccc} X & \overset{i}{\longrightarrow} & \mathbb{P}^N\times Y \\ x & \mapsto & (x,f(x)) \end{array} \tag{3.5.26}$$

Then i is a closed embedding because it defines an isomorphism between X and the graph $\Gamma_f\subset X\times Y\subset\mathbb{P}^N\times Y$ which is closed in $X\times Y$ (Lemma 2.4.4) and hence closed in $\mathbb{P}^N\times Y$. Since $f=p_Y\circ i$ where $p_Y\colon\mathbb{P}^N\times Y\to Y$ is the projection, f is a projective map.

The result below states that if a regular map is projective (with irreducible domain) then the dimensions of its fibers behave as in Example 3.5.15.

Theorem 3.5.20. *Let $f\colon X\to Y$ be a (regular) projective map of algebraic varieties.*

(a) The function

$$\begin{array}{ccc} Y & \overset{\varphi}{\longrightarrow} & \mathbb{N} \\ y & \mapsto & \dim f^{-1}(y) \end{array} \tag{3.5.27}$$

is upper semicontinuous, i.e. for every $k\in\mathbb{N}$ the set $\varphi^{-1}([k,+\infty))$ is (Zariski) closed.

(b) If X,Y are irreducible and f is dominant then

$$\{y\in Y\mid f^{-1}(y)\text{ has pure dimension equal to }\dim X-\dim Y\} \tag{3.5.28}$$

is an open dense subset of Y.

We prove Theorem 3.5.20 after a few preliminary results.

Proposition 3.5.21. *Let X be an irreducible algebraic variety, and let $f\colon X\to\mathbb{K}$ be a non constant regular function such that $V(f):=f^{-1}(0)$ is non empty. Then $V(f)$ has pure dimension equal to* $\dim X-1$.

Proof. Since X is a (finite) union of open affine subsets we may assume that X is affine. Thus $X\subset\mathbb{A}^n$ is a closed irreducible subset. By Theorem 1.6.2 there exists $\tilde f\in\mathbb{K}[z_1,\ldots,z_n]$ such that $f=\tilde f_{|X}$. Let $Y:=V(\tilde f)$. By hypothesis $X\cap Y$ is non empty: let W be one of its irreducible components. We must prove that $\dim W=\dim X-1$. We have $\mathbb{A}^n=\mathbb{P}^n_{Z_0}\subset\mathbb{P}^n$ as open dense subset. Let $\overline{X},\overline{Y},\overline{W}\subset\mathbb{P}^n$ be the closures of X, Y and W respectively. Then $\overline{Y}\subset\mathbb{P}^n$ is a hypersurface. Let $P\in\mathbb{K}[Z_0,\ldots,Z_n]$ be a homogeneous polynomial such that $\overline{Y}=V(P)$, and let d be its degree. Let $N:=\binom{d+n}{n}-1$, and let

$$\begin{array}{ccc} \mathbb{P}^n & \overset{\nu_d^n}{\longrightarrow} & \mathbb{P}^N \\ [Z_0,\ldots,Z_n] & \mapsto & [Z_0^d,Z_0^{d-1}X_1,\ldots,Z_n^d] \end{array}$$

be the Veronese map. Since $\overline{Y}=V(P)$ and P has degree d, there exists a hyperplane $H\subset\mathbb{P}^N$ such that $(\nu_d^n)^{-1}(H)=\overline{Y}$. Thus ν_d^n defines an isomorphism $\overline{X}\cap\overline{Y}\overset{\sim}{\longrightarrow}\nu_d^n(\overline{X})\cap H$, and $\nu_d^n(\overline{W})$ is an irreducible component of $\nu_d^n(\overline{X})\cap H$. Since f is not constant (and X is irreducible) H does not contain $\nu_d^n(\overline{X})$.

By Proposition 3.5.10 we have

$$\dim W=\dim\overline{W}=\dim\nu_d^n(\overline{W})=\dim\nu_d^n(\overline{X})-1=\dim\overline{X}-1=\dim X-1.$$

$\square$

Corollary 3.5.22. *Let $f\colon X\to Y$ be a regular map of algebraic varieties. Let $p\in X$. Every irreducible component of $f^{-1}(f(p))$ has dimension at least equal to* $\dim X-\dim_{f(p)}Y$.

Proof. Since X and Y are covered by open affine subsets, we may assume that X and Y are affine. Let $q := f(p)$ and let $m := \dim_q Y$. We claim that there exist $\varphi_1, \ldots, \varphi_m \in \mathbb{K}[Y]$ such that q is an irreducible component of $V(\varphi_1, \ldots, \varphi_m)$. In fact one argues by induction on m (see the proof of Proposition 3.5.13). If $m = 0$ the statement is trivially true. Let $m > 0$ and assume that the claim holds for lower values of m. Since $\dim_q Y > 0$ there exists $\varphi_m \in \mathbb{K}[Y]$ vanishing at q and not vanishing on any irreducible component of Y containing q. Then $V(\varphi_m)$ contains q, and by Proposition 3.5.21 its dimension at q is equal to $m - 1$. By the inductive hypothesis there exist $\psi_1, \ldots, \psi_{m-1} \in \mathbb{K}[V(\varphi_1)]$ such that q is an irreducible component of $V(\psi_1, \ldots, \psi_{m-1}) \subset V(\varphi_1)$. Since $V(\varphi_1)$ is a closed affine subset of the affine variety Y, there exist $\varphi_1, \ldots, \varphi_{m-1} \in \mathbb{K}[Y]$ whose restrictions to $V(\varphi_1)$ are equal to $\psi_1, \ldots, \psi_{m-1}$ respectively. Then q is an irreducible component of $V(\varphi_1, \ldots, \varphi_m)$. Thus we have

$$V(f^*(\varphi_1), \ldots, f^*(\varphi_m)) = f^{-1}(q) \sqcup W,$$

where W is closed in X, i.e. $f^{-1}(q)$ is a union of irreducible components of $V(f^*(\varphi_1), \ldots, f^*(\varphi_m))$. By repeated application of Proposition 3.5.21 every irreducible component of $V(f^*(\varphi_1), \ldots, f^*(\varphi_m))$ has dimension at least equal to $\dim X - m = \dim X - \dim Y$. □

Proof of Theorem 3.5.20. (a): By definition of projective morphism we may assume that $X \subset \mathbb{P}^N \times Y$ is closed and that f is the restriction to X of the projection map $p_Y : X \subset \mathbb{P}^N \times Y \to Y$. Let $y \in Y$. By Proposition 3.5.3 $\dim f^{-1}(y) \geqslant k$ if and only if for all $\Lambda \in \mathbb{G}\mathrm{r}(N-k, \mathbb{P}^N)$ we have $\Lambda \times \{y\} \cap X \neq \varnothing$. Thus we have

$$\varphi^{-1}([k, +\infty)) = \bigcap_{\Lambda \in \mathbb{G}\mathrm{r}(N-k, \mathbb{P}^N)} p_Y(\Lambda \times Y \cap X). \tag{3.5.29}$$

Now $\Lambda \times Y \cap X$ is closed in $\mathbb{P}^N \times Y$ because by hypotheses X is closed in $\mathbb{P}^N \times Y$. Since p_Y is closed (by the Main Theorem of Elimination Theory), it follows that $\varphi^{-1}([k, +\infty))$ is closed.

(b): Since f is closed $\mathrm{im}(f) = Y$. Let $y \in Y$. Then $y \in \mathrm{im}(f)$ and hence by Corollary 3.5.22 every irreducible component of $f^{-1}(y)$ has dimension at least $r := \dim X - \dim Y$. It follows that

$$\{y \in Y \mid f^{-1}(y) \text{ has pure dimension equal to } \dim X - \dim Y\} = Y \backslash \varphi^{-1}([r+1, +\infty)). \tag{3.5.30}$$

By Item (a) the right hand side in (3.5.30) is open in Y. It remains to prove that it is non empty. Replacing Y by an open dense affine subset we may assume that Y is affine. Thus $Y \subset \mathbb{A}^m = \mathbb{P}^m_{T_0}$. Let $\overline{Y} \subset \mathbb{P}^m_{T_0}$ be the closure of Y. Let

$$\mathbb{P}^N \times \mathbb{P}^m \overset{\sigma_{N,m}}{\longrightarrow} \Sigma_{N,m} \subset \mathbb{P}^{Nm+N+m} \tag{3.5.31}$$

be the Segre map, which is an isomorphism. Then $\sigma_{N,m}(X)$ is locally closed in $\mathbb{P}^{Nm+N+m}$ because X is locally closed in $\mathbb{P}^N \times \overline{Y}$ which is closed in $\mathbb{P}^N \times \mathbb{P}^m$. Of course the restriction of $\sigma_{N,m}$ to X defines an isomorphism $X \overset{\sim}{\longrightarrow} \sigma_{N,m}(X)$. Applying $r+1$ times Proposition 3.5.10, we get that if $H_1, \ldots, H_{r+1}$ are $r+1$ general hyperplanes in $\mathbb{P}^{Nm+N+m}$ then $\sigma_{N,m}(X) \cap H_1 \cap \ldots \cap H_{r+1}$ is either empty (recall that $\sigma_{N,m}(X)$ is only locally closed) or it has pure dimension equal to $\dim X - r - 1 < \dim Y$. Since $\sigma_{N,m}$ is an isomorphism this means that

$$\dim(X \cap \sigma^{-1}_{N,m}(H_1) \cap \ldots \cap \sigma^{-1}_{N,m}(H_{r+1})) < \dim Y. \tag{3.5.32}$$

Since f is closed it follows that

$$f(X \cap \sigma^{-1}_{N,m}(H_1) \cap \ldots \cap \sigma^{-1}_{N,m}(H_{r+1})) \subsetneq Y \tag{3.5.33}$$

is a (proper) closed subset. On the other hand if $y \in Y$ then $H_i \cap (\mathbb{P}^N \times \{y\})$ is a hyperplane, and hence $\varphi^{-1}([r+1, +\infty))$ is contained in the set on the left hand side of (3.5.33). This finishes the proof that the set in (3.5.28) is an open dense subset of Y. □

Corollary 3.5.23. *Let $f : X \to Y$ be a (regular) projective map of irreducible algebraic varieties. Suppose that there exists $y_0 \in Y$ such that $f^{-1}(y_0)$ is non empty and $\dim f^{-1}(y_0) = \dim X - \dim Y$. Then f is surjective, and for a general $y \in Y$ the fiber $f^{-1}(y)$ has pure dimension equal to $\dim X - \dim Y$.*

Proof. Since f is projective it is closed. Suppose that f is not surjective. Then the image $\mathrm{im}(f)$ is a proper closed subset of Y, and hence $\dim \mathrm{im}(f) < \dim Y$ because Y is irreducible. By Corollary 3.5.22 it follows that

$$\dim f^{-1}(y) \geqslant (\dim X - \dim \mathrm{im}(f)) > \dim X - \dim Y \tag{3.5.34}$$

for every $y \in Y$. This contradicts the hypothesis that $\dim f^{-1}(y_0) = \dim X - \dim Y$. We have proved that f is surjective. By Theorem 3.5.20 it follows that for general $y \in Y$ the fiber $f^{-1}(y)$ has pure dimension equal to $\dim X - \dim Y$. □

Corollary 3.5.24. *Let $f\colon X \to Y$ be a (regular) projective map of algebraic varieties. Suppose that Y is irreducible and that the fibers $f^{-1}(y)$ of f are all irreducible of the same dimension. Then X is irreducible.*

Proof. Let $X = X_1 \cup \ldots \cup X_r$ be the decomposition into irreducibles. The map f is surjective because the fibers $f^{-1}(y)$ of f are all irreducible of the same dimension (hence there are no empty fibers), and thus $Y = f(X_1) \cup \ldots \cup f(X_r)$. Write $\{1, \ldots, r\} = S \sqcup N$ where S is the set of i such that $f(X_i) = Y$ and N is the set of i such that $f(X_i) \subsetneq Y$. Since $f(X_i)$ is closed in Y for each i (because f is closed) and Y is irreducible, S is non empty. Let $i_0 \in S$ such that $\dim X_{i_0} \geqslant \dim X_i$ for all $i \in S$. We claim that $X_{i_0} = X$ and hence$X$ is irreducible. If $y \in Y$ does not belong to the proper closed subset $\bigcup_{i \in N} f(X_i)$ then $\dim f^{-1}(y) = \dim X_{i_0} - \dim Y$ by Theorem 3.5.20 and our equidimensionality hypothesis. Thus $\dim f^{-1}(y) = \dim X_{i_0} - \dim Y$ for all $y \in Y$ by the equidimensionality hypothesis. Moreover, letting $f_{i_0}\colon X_{i_0} \to Y$ be the restriction of f, we have $\dim f_{i_0}^{-1}(y) = \dim X_{i_0} - \dim Y$ for all $y \in Y$. Thus, for every $y \in Y$, $f_{i_0}^{-1}(y) \subset f^{-1}(y)$ is a closed subset and $\dim f_{i_0}^{-1}(y) = \dim f^{-1}(y)$. Since $f^{-1}(y)$ is irreducible it follows that $f_{i_0}^{-1}(y) = f^{-1}(y)$. This shows that $X_{i_0} = X$ as claimed. □

Lastly we prove that the image of a regular map of algebraic varieties is very large in its closure.

Proposition 3.5.25. *Let $f\colon X \to Y$ be a regular map of algebraic varieties. The image of f contains an open dense subset of its closure.*

Proof. It suffices to prove the proposition for X irreducible. We claim that in fact it suffices to prove the result for an irreducible affine variety. In fact we may write $X = X_0 \cup X_1 \cup \ldots \cup X_m$ where X_0 is an open dense (irreducible) affine subset of X, X_1 is an open dense (irreducible) affine subset of one of the irreducible components of $X \backslash X_0$ (unless $X_0 = X$ in which case we stop), and so on. Of course the process stops after a finite set of steps by Noetherianity. Thus we may suppose that X is irreducible and affine. Thus we may assume that $X \subset \mathbb{A}^N = \mathbb{P}^N_{Z_0}$ is closed. Let $\Gamma_f \subset X \times Y$ be the graph of f, and let $\overline{\Gamma}_f \subset \mathbb{P}^N \times Y$ be the closure in $\mathbb{P}^N \times Y$. Let $F\colon \overline{\Gamma}_f \to \times Y$ be the restriction of the projection $\mathbb{P}^N \times Y \to Y$. The isomorphism $X \xrightarrow{\sim} \Gamma_f$ defined by $x \mapsto (x, f(x))$ allows us to identify X with Γ_f which is an open dense subset of the irreducible algebraic variety $\overline{\Gamma}_f$. Note that the restriction of F to $X = \Gamma_f$ is equal to f. Note also that by construction the map $F\colon \overline{\Gamma}_f \to \times Y$ is projective. In particular $\mathrm{im}(F) \subset Y$ is closed (and irreducible because $\overline{\Gamma}_f$ is irreducible, being the closure of the irreducible Γ_f). Since $\mathrm{im}(f) \subset \mathrm{im}(F)$ it suffices to show that $\mathrm{im}(f)$ contains an open dense subset of $\mathrm{im}(F)$. Let $\overline{\Gamma}_f \backslash \Gamma_f = W_1 \cup \ldots \cup W_r$ be the decomposition into irreducible components. Each W_i is closed in $\overline{\Gamma}_f$ hence closed in $\mathbb{P}^N \times Y$ and thus $F(W_i) \subset \mathrm{im}(F)$ is closed. Let

$$S := \{i \in \{1, \ldots, r\} \mid F(W_i) = \mathrm{im}(F)\}, \qquad N := \{i \in \{1, \ldots, r\} \mid F(W_i) \subsetneq \mathrm{im}(F)\}. \tag{3.5.35}$$

For each $i \in S$ let $\mathscr{U}_i \subset \mathrm{im}(F)$ be the open dense subset of y such that $f^{-1}(y) \cap W_i$ has pure dimension equal to $\dim W_i - \dim \mathrm{im}(F)$ (such a $\mathscr{U}_i$ exists by Theorem 3.5.20 applied to the restriction of F to W_i). Let $\mathscr{U} \subset \mathrm{im}(F)$ be the open dense subset of y such that $f^{-1}(y)$ has pure dimension equal to $\dim \overline{\Gamma}_f - \dim \mathrm{im}(F)$. Lastly let

$$\mathscr{W} := \mathscr{U} \cap \bigcap_{i \in S} \mathscr{U}_i \cap \bigcap_{i \in N} (\mathrm{im}(F) \backslash F(W_i)). \tag{3.5.36}$$

Then $\mathscr{W}$ is an open dense subset of $\mathrm{im}(F)$. We claim that $\mathscr{W}$ is contained in the image of f. In fact let $y \in \mathscr{W}$. We must show that $F^{-1}(y) \cap \Gamma_f \neq \varnothing$. If $i \in N$ then $F^{-1}(y) \cap W_i = \varnothing$. If $i \in S$ then $f^{-1}(y) \cap W_i$ has pure dimension equal to $\dim W_i - \dim \mathrm{im}(F)$ which is smaller than $\dim \overline{\Gamma}_f - \dim \mathrm{im}(F)$ which is the (pure) dimension of $F^{-1}(y)$. It follows, as claimed, that $F^{-1}(y) \cap \Gamma_f \neq \varnothing$. □

Remark 3.5.26. If $f \colon M \to N$ is a smooth map of C^∞ manifold it might very well be that $f(M)$ does not contain any non-empty open subset of $\overline{f(M)}$. For example, let

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2 \\ t & \mapsto & [(t, \sqrt{2}t)] \end{array}$$

We have $\overline{f(\mathbb{R})} = \mathbb{T}^2$ but $f(\mathbb{R})$ does not contain any non-empty open subset of $\mathbb{T}^2$ because it is a subset of measure 0. Notice also that the analogue of Proposition 3.5.25 does not hold if we consider real quasi-projective sets with the Zariski topology and real regular maps: consider the projection

$$\begin{array}{ccc} \mathbb{A}^2_{\mathbb{R}} \supset V(x^2 + y^2 - 1) & \longrightarrow & \mathbb{A}^1_{\mathbb{R}} \\ (x, y) & \mapsto & x \end{array}$$

3.5.5 Linear subspaces contained in a hypersurface

Let $X \subset \mathbb{P}^n$ be closed. If r is a natural number let $F_r(X) \subset \mathbb{G}\mathrm{r}(r, \mathbb{P}^n)$ be the subset of r-dimensional linear subspaces contained in X, i.e.

$$F_r(X) \coloneqq \{\Lambda \in \mathbb{G}\mathrm{r}(r, \mathbb{P}^n) \mid \Lambda \subset X\}. \tag{3.5.37}$$

Then $F_r(X)$ is a closed subset of $\mathbb{G}\mathrm{r}(r, \mathbb{P}^n)$ (see Exercise 2.6.6), and hence it is a projective variety.

Question 3.5.27. Is $F_r(X)$ non empty? If it is non empty, what is its dimension?

Consider the case of a hypersurface $X \subset \mathbb{P}^n$. If X is a hyperplane then $F_r(X)$ is a Grassmannian. If X is a non degenerate quadric hypersurface, i.e. $X = V(q)$ where $q \in \mathbb{K}[Z_0, \ldots, Z_n]_2$ is a non degenerate quadratic form, then $F_r(X)$ is an interesting variety, see Exercises 3.8.7, 3.8.8 and 3.8.9, but it does not vary as X varies (all non degenerate quadric hypersurfaces are projectively equivalent). If the ideal of X is generated by a homogeneous polynomial of degree greater than 2 then $F_r(X)$ varies with X and is often a very interesting variety.

The results of the present section can be used to give (partial) answers to the above question. In order to apply the results about dimensions of fibers we must parametrize hypersurfaces with an algebraic variety. Recall that the homogeneous ideal of a hypersurface in a projective space is principal.

Definition 3.5.28. Let $X \subset \mathbb{P}^n$ be a hypersurface. The *degree of* X is the degree of any generator of the (homogeneous) ideal $I(X) \subset \mathbb{K}[Z_0, \ldots, Z_n]$.

Let $d > 0$. Then hypersurfaces in $\mathbb{P}^n$ of degree d are parametrized by the subset of $\mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d)$ whose elements are the points $[P]$ with $P \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ a square-free polynomial (so that P generates a radical ideal). This subset turns out to be open, but as soon as $d > 1$ it is not the whole projective space $\mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d)$. Since it is much better to deal with complete varieties than with non complete varieties, we would rather have $\mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d)$ as parameter space. This forces us to consider hypersurfaces with "multiplicities". This is the higher dimensional version of "roots of polynomials in one variable counted with multiplicities". The relevant definitions go as follows. Let $\mathrm{Div}(\mathbb{P}^n)$ be the abelian group with generators the irreducible hypersurfaces in $\mathbb{P}^n$. Thus an element of $\mathrm{Div}(\mathbb{P}^n)$ is a *formal* finite sum $D = \sum_{i \in I} m_i D_i$, where each m_i is an integer, and the D_i's are pairwise distinct irreducible hypersurface in $\mathbb{P}^n$. We have the degree homomorphism (of abelian groups)

$$\begin{array}{ccc} \mathrm{Div}(\mathbb{P}^n) & \xrightarrow{\deg} & \mathbb{Z} \\ \sum_{i \in I} m_i D_i & \mapsto & \sum_{i \in I} m_i \deg D_i \end{array} \tag{3.5.38}$$

The divisor $\sum_{i\in I} m_i D_i$ is *effective* if $m_i > 0$ for all $i \in I$ (the divisor 0 is effective, it correpsonds to the empty index set I). Let $\mathrm{Div}_+(\mathbb{P}^n) \subset \mathrm{Div}(\mathbb{P}^n)$ be the monoid of effective divisors.

Let $P \in \mathbb{K}[Z_0, \dots, Z_n]_d$ be non zero, and let $P = \prod_{i=1}^{r} P_i^{m_i}$ be the decomposition into prime factors, where for $i \neq j$ the factors P_i and P_j are not associated. The *divisor of* P is the element of $\mathrm{Div}(\mathbb{P}^n)$ defined by

$$\mathrm{div}(P) := \sum_{i=1}^{r} m_i V(P_i). \tag{3.5.39}$$

Note that $\mathrm{div}(P)$ is effective. Let $\mathrm{Div}_+^d(\mathbb{P}^n) := \mathrm{Div}_+(\mathbb{P}^n) \cap \deg^{-1}(d) \subset \mathrm{Div}(\mathbb{P}^n)$ be the subset of effective divisors of degree d. The map

$$\begin{array}{ccc} \mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d) & \xrightarrow{\mathrm{div}} & \mathrm{Div}_+^d(\mathbb{P}^n) \\ [P] & \mapsto & \mathrm{div}(P) \end{array} \tag{3.5.40}$$

is a bijection. This gives a geometric interpretation of $\mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d)$. From now on we identify $\mathrm{Div}_+^d(\mathbb{P}^n)$ with $\mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d)$ via the bijection in (3.5.40). If $D = \sum_{i\in I} m_i D_i$ is an effective divisor, i.e. $m_i > 0$ for each $i \in I$, the *support of* D is the union of the D_i's and is denoted by $\mathrm{supp}\, D$. Let $D \in \mathrm{Div}_+^d(\mathbb{P}^n)$ with $D = \mathrm{div}(P)$. Then

$$F_r(D) := \{\Lambda \in \mathbb{G}\mathrm{r}(r, \mathbb{P}^n) \mid \Lambda \subset \mathrm{supp}\, D\} = \{U \in \mathrm{Gr}(r+1, \mathbb{K}^{n+1}) \mid P_{|U} = 0\}. \tag{3.5.41}$$

In order to answer Question 3.5.27 we let $\Gamma_{r,d}(\mathbb{P}^n) \subset \mathbb{G}\mathrm{r}(r, \mathbb{P}^n) \times \mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d)$ be the incidence subset given by

$$\Gamma_{r,d}(\mathbb{P}^n) := \{(\Lambda, D) \mid \Lambda \subset \mathrm{supp}\, D\}. \tag{3.5.42}$$

Claim 3.5.29. *$\Gamma_{r,d}(\mathbb{P}^n)$ is a closed subset of $\mathbb{G}\mathrm{r}(r, \mathbb{P}^n) \times \mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d)$.*

We leave the proof of Claim 3.5.29 as an exercise.

The restrictions of the projections to $\Gamma_{r,d}(\mathbb{P}^n)$ give us two regular projective maps

$$\begin{tikzcd} & \Gamma_{r,d}(\mathbb{P}^n) \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ \mathbb{G}\mathrm{r}(r, \mathbb{P}^n) & & \mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d) \end{tikzcd} \tag{3.5.43}$$

Let $D \in \mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d)$: then $F_r(D)$ is identified with $\rho^{-1}(D)$. In other words the fibers of ρ are what we are after. It follows that knowing the dimension of $\Gamma_{r,d}(\mathbb{P}^n)$ helps in answering Question 3.5.27. While the fibers of ρ can be quite mysterious, the fibers of π are extremely simple. This allows, by applying the results of the present section, to compute the dimension of $\Gamma_{r,d}(\mathbb{P}^n)$. The following claim is an extremely easy result.

Claim 3.5.30. *Keep notation as above, and suppose that $r \in \{0, \dots, n-1\}$. Let $\Lambda \in \mathbb{G}\mathrm{r}(r, \mathbb{P}^n)$. Then $\pi^{-1}(\Lambda)$ is a linear subspace of $\mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d)$ of codimension equal to $\binom{r+d}{r}$.*

Applying Corollary 3.5.24 and Theorem 3.5.20 to the (projective) regular map π it follows that $\Gamma_{r,d}(\mathbb{P}^n)$ is irreducible and that

$$\mathrm{cod}(\Gamma_{r,d}(\mathbb{P}^n),\, \mathbb{G}\mathrm{r}(r, \mathbb{P}^n) \times \mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d)) = \binom{r+d}{r}. \tag{3.5.44}$$

Hence we get that

$$\dim \rho(\Gamma_{r,d}(\mathbb{P}^n)) \leqslant \dim \Gamma_{r,d}(\mathbb{P}^n) = \dim \mathbb{P}(\mathbb{K}[Z_0, \dots, Z_n]_d)) + \dim \mathbb{G}\mathrm{r}(r, \mathbb{P}^n) - \binom{r+d}{r}. \tag{3.5.45}$$

In particular we get the following negative result.

Proposition 3.5.31. *Keep notation as above, and assume that* $(r+1)(n-r) < \binom{r+d}{r}$. *If* $D \in \mathrm{Div}^d_+(\mathbb{P}^n)$ *is general then* $F_r(D)$ *is empty.*

On the other hand Corollaries 3.5.22 and 3.5.23 give the following positive result.

Proposition 3.5.32. *Keep notation as above, and assume that* $(r+1)(n-r) \geqslant \binom{r+d}{r}$. *Suppose that there exists (one)* $D \in \mathrm{Div}^d_+(\mathbb{P}^n)$ *such that* $F_r(D)$ *is non empty and that*

$$\dim F_r(D) = (r+1)(n-r) - \binom{r+d}{r}. \tag{3.5.46}$$

Then $F_r(D)$ *is non empty for every* $D \in \mathrm{Div}^d_+(\mathbb{P}^n)$, *and for general* D *the equality in* (3.5.46) *holds.*

3.6 Degree of maps

3.6.1 Degree of a map

Definition 3.6.1. Let $f\colon X \dashrightarrow Y$ be a rational map of irreducible algebraic varieties. The *degree of* f, denoted by $\deg f$, is given by

$$\deg f := \begin{cases} 0 & \text{if } f \text{ is not dominant,} \\ [\mathbb{K}(X) : f^*\mathbb{K}(Y)] & \text{if } f \text{ is dominant.} \end{cases}$$

Thus $0 < \deg f < \infty$ if and only if f is dominant and $\dim X = \dim Y$.

Definition 3.6.2. Let $f\colon X \dashrightarrow Y$ be a finite degree rational map of irreducible algebraic varieties. The *separable degree of* f, denoted by $\deg_s f$, is given by

$$\deg_s f := \begin{cases} 0 & \text{if } f \text{ is not dominant,} \\ [\mathbb{K}(X)^s : f^*\mathbb{K}(Y)] & \text{if } f \text{ is dominant,} \end{cases}$$

where $\mathbb{K}(X)^s \subset \mathbb{K}(X)$ is the maximal separable extension of $f^*\mathbb{K}(Y)$ (see Theorem A.5.3).

Note that $\deg_s f$ divides $\deg f$, and that if $\mathbb{K}$ has characteristic 0 then $\deg_s f = \deg f$. As a matter of notation we denote $f^*\mathbb{K}(Y)$ by $\mathbb{K}(Y)$ whenever there is no possibility of confusion.

Remark 3.6.3. $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow W$ be rational dominant maps of irreducible algebraic varieties. Then (see Remark A.5.6) we have

$$\deg(g \circ f) = \deg(g) \cdot \deg(f), \qquad \deg_s(g \circ f) = \deg_s(g) \cdot \deg_s(f). \tag{3.6.1}$$

Example 3.6.4. Let $(z_1, \ldots, z_n, w)$ be affine coordinates on $\mathbb{A}^{n+1}$. Let $X \subset \mathbb{A}^{n+1}$ be an irreducible hypersurface and let $I(X) = P$. Write

$$P = a_0 w^d + a_1 w^{d-1} + \cdots + a_d, \qquad a_i \in \mathbb{K}[z_1, \ldots, z_n], \quad a_0 \neq 0$$

Let

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{A}^n \\ (z_1, \ldots, z_n, w) & \mapsto & (z_1, \ldots, z_n) \end{array}$$

Then $\deg f = d$. In fact if $d = 0$ then $\operatorname{im} f = V(a_0) \subsetneq \mathbb{A}^n$ and hence f is not dominant. If $d > 0$ then

$$\mathbb{K}(X) = \mathbb{K}(z_1, \ldots, z_n)[w]/(P)$$

and hence $[\mathbb{K}(X) : \mathbb{K}(z_1, \ldots, z_n)] = d$. Suppose that $\operatorname{char} \mathbb{K} = p > 0$. Let m be the maximum integer such that $p^m \mid (d-i)$ for all $i \in \{0, \ldots, d\}$ such that $a_i \neq 0$. Then $\deg_s f = d/p^m$. In fact let $R \in \mathbb{K}[z_1, \ldots, z_n, w]$ be the polynomial such that

$$P(z_1, \ldots, z_n, w) = R(z_1, \ldots, z_n, w^{p^m}),$$

and let $u := w^{p^m}$. Then

$$\mathbb{K}(z_1, \ldots, z_n)[u] \cong \mathbb{K}(z_1, \ldots, z_n)[t]/(R(z_1, \ldots, z_n, t))$$

is the maximal separable extension of $\mathbb{K}(z_1, \ldots, z_n)$ in $\mathbb{K}(X)$.

Definition 3.6.5. A rational map $f\colon X \dashrightarrow Y$ of irreducible algebraic varieties is *generically separable* if it is dominant and $\mathbb{K}(X)$ is an algebraic separable extension of $f^*\mathbb{K}(Y)$.

In other words f is generically separable if it has finite non zero degree and $\deg f = \deg_s f$.
Below is the main result of the present section.

Proposition 3.6.6. *Let $f\colon X \to Y$ be a finite degree regular map of irreducible algebraic varieties. Then for a general $y \in Y$ we have*

$$|f^{-1}(y)| = \deg_s f. \tag{3.6.2}$$

Example 3.6.7. Let us check the statement of Proposition 3.6.6 for the map $f\colon X \to \mathbb{A}^n$ of Example 3.6.4. Let $P \in \mathbb{K}[z_1, \ldots, z_n, w]$ be as in that example. Let $R \in \mathbb{K}[z_1, \ldots, z_n, w]$ be as in Example 3.6.4. Let $g\colon X \to V(R)$ be defined by $g(z, w) := (z, w^{p^m})$, and let $h\colon V(R) \to \mathbb{A}^n$ be defined by $h(z, w) := z$. The regular map $f\colon X \to \mathbb{A}^n$ factorizes as the composition

$$X \xrightarrow{g} V(R) \xrightarrow{h} \mathbb{A}^n. \tag{3.6.3}$$

Note that the corresponding pull-back maps of rational functions give the chain of extensions

$$\mathbb{K}(X) \supset \mathbb{K}(z_1, \ldots, z_n)[u] \supset \mathbb{K}(z_1, \ldots, z_n).$$

The map g is bijective, hence the statement of Proposition 3.6.6 for the map $f\colon X \to \mathbb{A}^n$ holds if and only if $|h^{-1}(\overline{z})| = d$ for a general $\overline{z} \in \mathbb{A}^n$. The polynomial $\frac{\partial R}{\partial w}$ is non zero (the corresponding extension of fields is separable) and its degree in w is strictly smaller than the degree in w of R. Hence $\frac{\partial R}{\partial w}$ is not a multiple of R. Since P is prime (because it generates the ideal of the irreducible hypersurface X) also R is prime. It follows that $\frac{\partial R}{\partial w}$ does not vanish on $V(R)$ and hence $V(R, \partial R/\partial w)$ is a proper closed subset of $V(R)$. Thus $V(R, \partial R/\partial w)$ has dimension smaller than n, and hence also $\Delta := \overline{h(V(R, \partial R/\partial w))}$. In other words Δ is a proper closed subset of $\mathbb{A}^n$, and thus $U := (\mathbb{A}^n \backslash V(a_0) \backslash \Delta)$ is an open dense subset $\mathbb{A}^n$. Let $\overline{z} \in U$. Then $R(\overline{z}, w) \in \mathbb{K}[w]$ is a polynomial of degree $\deg_s f$ with simple roots and hence $|h^{-1}(\overline{z})| = \deg_s f$.

Remark 3.6.8. Let $f\colon X \to Y$ be a finite degree regular map of irreducible algebraic varieties. Let $U \subset X$ be open and dense. Then for general $y \in Y$ we have

$$f^{-1}(y) \subset U. \tag{3.6.4}$$

In fact if $f(X)$ is not dense in Y, i.e. $\deg f = 0$, then (3.6.4) holds for y an element of the open dense subset $(Y \backslash \overline{f(X)})$ because $f^{-1}(y)$ is empty, and if $\deg f > 0$ then $\dim(X \backslash U) < \dim X = \dim Y$ (the last equality holds because $0 < \deg f < \infty$), hence $\dim \overline{f(X \backslash U)} < \dim Y$ and clearly (3.6.4) holds for $y \in (Y \backslash \overline{f(X \backslash U)})$. Sligthly more generally, suppose that we have a commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, dotted, "\varphi"] \arrow[d, "f_1"'] & X_1 \arrow[d, "f_2"] \\ Y_1 \arrow[r, dotted, "\psi"] & Y_2 \end{tikzcd} \tag{3.6.5}$$

where f_1, f_2 are finite degree regular maps of irreducible algebraic varieties, and φ, ψ are birational maps. By Proposition 3.2.11 there exist open dense subsets $U_i \subset X_i$ and $V_i \subset Y_i$ for $i \in \{1, 2\}$ such that φ, ψ are regular on U_1, V_1 respectively, and they define isomorphisms $\varphi_{|U_1}\colon U_1 \xrightarrow{\sim} U_2$, $\psi_{|V_1}\colon V_1 \xrightarrow{\sim} V_2$. Then for general $y \in V_1$ we have $f_1^{-1}(y) \subset V_1$ and $f_2^{-1}(\psi(y)) \subset V_2$. In particular we get that the equality in (3.6.12) holds for f_1 if and only it holds for f_2.

3.6.2 Degree and fibers of a generically separable map

Here we prove that the equality in (3.6.12) holds if $f\colon X \to Y$ is a generically separable map, i.e. the following result.

Proposition 3.6.9. *Let $f\colon X \to Y$ be a finite degree generically separable regular map of irreducible algebraic varieties. Then for a general $y \in Y$ we have*

$$|f^{-1}(y)| = \deg_s f = \deg f. \tag{3.6.6}$$

Before giving the proof we consider the following more general version of Example 3.6.4. Let Y be an irreducible affine variety. Let $P \in \mathbb{K}(Y)[w]$ be an irreducible polynomial:

$$P = w^d + a_1 w^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{K}(Y).$$

Since Y is affine $\mathbb{K}(Y)$ is the field of fractions of $\mathbb{K}[Y]$. Thus there exists $0 \neq b \in \mathbb{K}[Y]$ such that $b \cdot a_i \in \mathbb{K}[Y]$ for all $1 \leqslant i \leqslant d$. Let $c_0 := b$, $c_i := b \cdot a_i$, $1 \leqslant i \leqslant d$ and

$$\widetilde{P} := c_0 w^d + c_1 w^{d-1} + \cdots + c_d \in \mathbb{K}[Y][w]. \tag{3.6.7}$$

If $\mathbb{K}[Y]$ is a UFD we may factor out the gcd $\{c_0, \ldots, c_d\}$ and hence by renaming the c_i's we may assume that gcd $\{c_0, \ldots, c_d\} = 1$. It follows that $V(\widetilde{P})$ is irreducible (the proof is the same as the one for hypersurfaces in $\mathbb{A}^n$). In general $\mathbb{K}[Y]$ is not a UFD and hence there might be no way of "reducing" the polynomial of (3.6.7) in order to get that $V(\widetilde{P})$ is irreducible. (An example of this phenomenon: $Y := V(z_1z_2 - z_3z_4)$ and $\widetilde{P} = z_1 w - z_3$. Then $V(\widetilde{P}) = X \cup Y$ where X is the closure of the locus of $(z, w) \in V(\widetilde{P})$ with $z_1 \neq 0$ and $Y := V(z_1, z_3)$. Each of the irreducible components X, Y has dimension 3.) Let $\pi\colon Y \times \mathbb{A}^1 \to Y$ be the projection map. An irreducible component V_i of $V(\widetilde{P})$ *dominates* Y if $\overline{\pi(V_i)} = Y$.

Claim 3.6.10. *Keep hypotheses and notation as above. There is one and only one irreducible component of $V(\widetilde{P})$ which dominates Y, call it V_0. We have an isomorphism*

$$\mathbb{K}(V_0) \cong \mathbb{K}(Y)[w]/(\widetilde{P}) \tag{3.6.8}$$

such that, letting $\pi_0\colon V_0 \to Y$ be the restriction of π, the inclusion of fields $\pi_0^*\colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(V_0)$ is the obvious one given the above isomorphism.

Proof. To simplify notation let $V := V(\widetilde{P})$. Since $\pi(V)$ contains $Y \backslash V(c_0)$, which is dense in Y, there exists at least one irreducible component V_0 of V such that $\overline{\pi(V_0)} = Y$. Let $g \in I(V_0)$. We claim that

$$\widetilde{P} \mid g \text{ in } \mathbb{K}(Y)[w]. \tag{3.6.9}$$

(Note: we do not claim that $\widetilde{P}|g$ in $\mathbb{K}[Y][w]$.) In fact suppose that $\widetilde{P}$ does not divide g. Then $\widetilde{P}$ and g are coprime in $\mathbb{K}(Y)[w]$ because $\widetilde{P}$ is prime, and hence there exist $\alpha, \beta \in \mathbb{K}(Y)[w]$ such that

$$\alpha \cdot \widetilde{P} + \beta \cdot g = 1.$$

Multiplying by $0 \neq \gamma \in \mathbb{K}[Y][w]$ such that $\alpha \cdot \gamma$, $\beta \cdot \gamma$ belong to $\mathbb{K}[Y][w]$ we get that

$$(\alpha \cdot \gamma)\widetilde{P} + (\beta \cdot \gamma) g = \gamma.$$

Let $q \in V_0$: then $g(q) = 0$, and since $V \supset V_0$ we get that $\gamma(q) = 0$. Since $\gamma \neq 0$ it follows that $\pi(V_0)$ is not dense in Y, and that is a contradiction. This proves that (3.6.9) holds.

Let $I(V_0) = (g_1, \ldots, g_r)$. From (3.6.9) we get that there exist $h_1, \ldots, h_r \in \mathbb{K}[Y][w]$ and $m_1, \ldots, m_r \in \mathbb{K}[Y]$ such that

$$m_i \cdot g_i = \widetilde{P} \cdot h_i, \quad m_i \neq 0, \quad i = 1, \ldots, r.$$

Set $m = m_1 \cdot \cdots \cdot m_r$. By the above equation we get that $V \backslash V(m) = V_0 \backslash V(m)$ and hence V_0 is the unique irreducible component of V dominating Y. The last statement of the claim is clealry true. □

Proof of Proposition 3.6.9. Let $d := \deg f > 0$. Since Y is covered by open affine sets we may assume that Y itself is affine. By definition we have an inclusion $f^*\colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$. Since $\mathbb{K}(X)$ is a separable extension of $\mathbb{K}(Y)$ there exists $\xi \in \mathbb{K}(X)$ primitive over $\mathbb{K}(Y)$. Let

$$P = w^d + a_1 w^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{K}(Y) \tag{3.6.10}$$

be the minimal polynomial of ξ. Let $V(\widetilde{P}) \subset Y \times \mathbb{A}^1$ - notation as in Claim 3.6.10. Let $V_0 \subset V(\widetilde{P})$ be the unique irreducible component dominating Y. We have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & V_0 \arrow[dl, "\pi_0"] \\ & Y & \end{tikzcd}$$

with ϕ birational. By Remark 3.6.8 it suffices to prove that for general $y \in Y$ we have

$$|\pi_0(y)| = \deg \pi_0. \tag{3.6.11}$$

Arguing as in Example 3.6.7 we immediately get the above equality. □

3.6.3 Degree and fibers: purely inseparable extensions of fields

Here we prove that the equality in (3.6.12) holds if $\mathbb{K}(X) \supset f^*\mathbb{K}(Y)$ is a purely inseparable extension, i.e. the following result.

Proposition 3.6.11. *Let $f\colon X \to Y$ be a finite degree regular map of irreducible algebraic varieties such that $\mathbb{K}(X) \supset \mathbb{K}(Y)$ is a purely inseparable extension. Then for a general $y \in Y$ we have*

$$|f^{-1}(y)| = \deg_s f = 1. \tag{3.6.12}$$

Proof. The charachteristic of $\mathbb{K}$ is positive because in charachteristic 0 every algebraic extension is separable. Let $\operatorname{char} \mathbb{K} = p > 0$. Since X and Y contain (many) dense open affine subsets we may assume that X and Y are affine, see Remark 3.6.8. Let $\varphi_1, \ldots, \varphi_r$ be generators of $\mathbb{K}[X]$ as $\mathbb{K}[Y]$ algebra. Let $i \in \{1, \ldots, r\}$. Since $\mathbb{K}(X) \supset \mathbb{K}(Y)$ is a purely inseparable extension there exist $a_i, b_i \in \mathbb{K}[Y]$ with $a_i \neq 0$ and $m_i \in \mathbb{N}_+$ such that (see Theorem A.5.3)

$$a_i \varphi_i^{m_i} + b_i = 0. \tag{3.6.13}$$

Let $U := Y \backslash V(a_1, \ldots, a_r)$. Thus U is open dense in Y. By Proposition 3.5.25 the image of f contains an open dense subset $V \subset Y$. The open subset $U \cap V$ is dense in Y. If $y \in U \cap V$ then $f^{-1}(y)$ is a singleton by the equations in (3.6.13) (recall that $\operatorname{char} \mathbb{K} = p$). This finishes the proof. □

3.6.4 Degree and fibers: the general case

Here we prove Proposition 3.6.6. If $\deg f = 0$ then $\overline{f(X)} \neq Y$ and $f^{-1}(y)$ is empty for y an element of the open dense subset $Y \backslash \overline{f(X)}$. If $\mathbb{K}(X)$ is a separable extension of $\mathbb{K}(Y)$, then Proposition 3.6.6 holds by Proposition 3.6.9. Lastly, assume that $\mathbb{K}(X)$ is not a separable extension of $\mathbb{K}(Y)$. Then we have the chain of inclusions $\mathbb{K}(X) \supset \mathbb{K}(X)^s \supset \mathbb{K}(Y)$ where $\mathbb{K}(X)^s$ is the maximal separable extension of $\mathbb{K}(Y)$ in $\mathbb{K}(X)$. Let W be an algebraic variety such that $\mathbb{K}(W) \cong \mathbb{K}(X)^s$, see Proposition 3.3.8. Correspondingly, see Proposition 3.3.9, we have have a factorization of f as $f = h \circ g$ as a product of dominant rational maps of irreducible algebraic varieties of finite degrees

$$X \overset{g}{\dashrightarrow} W \overset{h}{\dashrightarrow} Y.$$

Let $W_0 \subset W$ be the open dense subset of points where h is regular, and let $X_0 \subset X$ be the open dense subset of points x such that g is regular and $g(x) \in W_0$. Let $g_0\colon X_0 \to W_0$, $h_0\colon W_0 \to Y$ and

$f_0 \colon X_0 \to Y$ be the restrictions of g, h and f to X_0, W_0 and X_0 respectively. We have the factorization $f_0 = h_0 \circ g_0$, and by Remark 3.6.8 it suffices to prove that $|f_0^{-1}(y)| = \deg f_s$. By Proposition 3.6.11 we may further shrink W_0 and X_0 so that $g_0(w)$ is a singleton for every $w \in W_0$.

Let $y \in Y$ be general. Since h_0 is separable we have $|h_0^{-1}(y)| = \deg_s(h_0) = \deg_s h$ by Proposition 3.6.9. Moreover, since $y \in Y$ is general, each point of $h_0^{-1}(y)$ belongs to W_0 (see Remark 3.6.8) and hence

$$|g_0^{-1}(h_0^{-1}(y))| = |h_0^{-1}(y)| = \deg_s h = \deg_s f. \tag{3.6.14}$$

3.7 Degree of a closed subset of a projective space

3.7.1 Definition

Let $X \subset \mathbb{P}^n$ be closed, and let c be its codimension. Suppose that X is irreducible and let π be the forgetful regular map

$$\begin{array}{ccc} \Gamma_X(c) & \xrightarrow{\pi} & \mathbb{G}\mathrm{r}(c, \mathbb{P}^n) \\ (p, \Lambda) & \mapsto & \Lambda \end{array} \tag{3.7.1}$$

Since $\Gamma_X(c)$ and $\mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ are irreducible we have a well-defined $\deg \pi$. By Corollary 3.5.6 we have $\dim \Gamma_X(c) = \dim \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$. Thus $\deg \pi < \infty$. The *degree of* X is defined to be the separable degree

$$\deg X := \deg_s(\Gamma_X(c) \xrightarrow{\pi} \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)). \tag{3.7.2}$$

In general let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. The *degree of* X is defined to be the sum of the degrees of irreducible components of X which realize the dimension of X:

$$\deg X := \sum_{\dim X_i = \dim X} \deg X_i. \tag{3.7.3}$$

Proposition 3.7.1. *Let $X \subset \mathbb{P}^n$ be closed of codimension c. There exists an open dense $U \subset \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ with the following property: if $\Lambda \in U$ then $X \cap \Lambda$ is finite non empty of cardinality equal to* $\deg X$.

Proof. If X is irreducible the first statement follows from Proposition 3.6.6 applied to the map π in (3.7.1), and the positivity of $\deg X$ follows from Proposition 3.5.3. In general let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. If $\Lambda \in \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ is general then by Proposition 3.5.3

$$\Lambda \cap X_i = \emptyset \text{ if } \dim X_i < \dim X, \quad \Lambda \cap (X_i \cap X_j) = \emptyset \text{ if } i \neq j. \tag{3.7.4}$$

It follows that if $\Lambda \in \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ is general then

$$\Lambda \cap X = \bigsqcup_{\dim X_i = \dim X} \Lambda \cap X_i, \tag{3.7.5}$$

and hence the claim follows from the case when X is irreducible. □

Example 3.7.2. If $X \subset \mathbb{P}^n$ is a hypersurface we have given another definition of the degree of X, namely in Definition 3.5.28. That definition agrees with the definition given above. To see why we may assume that X is irreducible because the irreducible components of a hypersurface are hypersurfaces. We show that the map $\Gamma_X(1) \xrightarrow{\pi} \mathbb{G}\mathrm{r}(1, \mathbb{P}^n)$ has degree equal to $\deg F$.

Let $I(X) = (F)$. Thus F is prime. There exists $s \in \{0, \ldots, n\}$ such that $\partial F/\partial Z_s \neq 0$. In fact suppose the contrary. Then $\mathrm{char}\, \mathbb{K} = p > 0$ and $F = \sum_{|I|=d} a_I Z^{pI}$. For each I let $b_I \in \mathbb{K}$ be the unique p-th root of a_I. Then $F = (\sum_{|I|=d} b_I Z^I)^p$, and this contradicts the hypothesis that F is prime. Reordering the indices we may assume that

$$\partial F/\partial Z_1 \neq 0. \tag{3.7.6}$$

Let $\mathscr{U} \subset \mathbb{G}\mathrm{r}(1, \mathbb{P}^n)$ be the open dense subset parametrizing lines which do not meet the codimension 2 linear subspace $V(Z_0, Z_1)$. It suffices to prove that the map

$$\Gamma_X(1) \cap (X \times \mathscr{U}) \xrightarrow{\pi} \mathscr{U} \tag{3.7.7}$$

has degree equal to $\deg F$. We have the isomorphism

$$\begin{array}{ccc} \mathbb{A}^{2n-2} & \xrightarrow{\sim} & \mathscr{U} \\ (a_2, \ldots, a_n, b_2, \ldots, b_n) & \mapsto & \mathrm{span}([1, 0, a_2, \ldots, a_n],\ [0, 1, b_2, \ldots, b_n]) \end{array} \tag{3.7.8}$$

Let $Y \subset \mathscr{U} \times \mathbb{P}^1$ be the closed subset defined by (we let $a := a_2, \ldots, a_n$, $b := b_2, \ldots, b_n$)

$$Y := \{(a, b, [T_0, T_1]) \mid F(T_0, T_1, a_2T_0 + b_2T_1, \ldots, a_nT_0 + b_nT_1) = 0\}. \tag{3.7.9}$$

We have the isomorphism

$$\begin{array}{ccc} Y & \xrightarrow{\sim} & \Gamma_X(1) \cap (X \times \mathscr{U}) \\ (a, b, [T_0, T_1]) & \mapsto & [T_0, T_1, a_2T_0 + b_2T_1, \ldots, a_nT_0 + b_nT_1] \end{array} \tag{3.7.10}$$

and via this isomorphism the map in (3.7.7) is identified with the forgetful map

$$\begin{array}{ccc} Y & \xrightarrow{f} & \mathscr{U} \\ (a, b, [T_0, T_1]) & \mapsto & (a_2, \ldots, a_n, b_2, \ldots, b_n) \end{array} \tag{3.7.11}$$

The polynomial $G := F(T_0, T_1, a_2T_0 + b_2T_1, \ldots, a_nT_0 + b_nT_1) \in \mathbb{K}(a_2, \ldots, a_n, b_2, \ldots, b_n)[T_0, T_1]$, which is homogeneous in T_0, T_1 of degree equal to $\deg F$ is prime because F is prime, and moreover $\partial G / \partial T_1 \neq 0$ because of (3.7.6). It follows that f is generically separable of degree equal to $\deg F$, see Example 3.6.4.

Remark 3.7.3. The map in (3.7.2) is separable in general, but we do not show this here.

Example 3.7.4. Let $\mathcal{C}_n \subset \mathbb{P}^n$ be the *rational normal curve*, i.e. the image of the Veronese map

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\nu_n^1} & \to \mathbb{P}^n \\ [S, T] & \mapsto & [S^n, S^{n-1}T, \ldots, T^n] \end{array} \tag{3.7.12}$$

Then $\deg \mathcal{C}_n = n$. In fact let $[\xi_0, \ldots, \xi_n]$ be homogeneous coordinates on $\mathbb{P}^n$. Let $H := V(\sum_{i=0}^n a_i \xi_i) \subset \mathbb{P}^n$ be a hyperplane. Then

$$H \cap \mathscr{C}_n = \nu_n^1(\{[S, T] \in \mathbb{P}^1 \mid \sum_{i=0}^n a_i S^{n-i} T^i = 0\}). \tag{3.7.13}$$

Hence for a general hyperplane H the cardinality of $H \cap \mathscr{C}_n$ is equal to n, and by Proposition 3.7.1 we get that $\deg \mathscr{C}_n = n$.

Remark 3.7.5. Let $X \subset \mathbb{P}^n$ be closed of dimension d. Then $\deg X$ is equal to the cardinality of $X \cap H_1 \cap \ldots \cap H_d$ for $H_1, \ldots, H_d \subset \mathbb{P}^n$ generic hyperplanes. (This means that there exists an open dense subset $U \subset (\mathbb{P}^n)^\vee \times \ldots \times (\mathbb{P}^n)^\vee$ with the property that $|X \cap H_1 \cap \ldots \cap H_d| = \deg X$ for $(H_1, \ldots, H_d) \in U$.) In fact if $H_1, \ldots, H_d \subset \mathbb{P}^n$ are generic hyperplanes then $H_1 \cap \ldots \cap H_d$ is a generic linear subspace of dimension equal to $n - d$, i.e. the codimension of X.

Proposition 3.7.6. *Let $X \subset \mathbb{P}^n$ be closed of pure dimension, and let c be its codimension. Let $\Lambda \in \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ be such that $\Lambda \cap X$ is zero dimensional. Then $\Lambda \cap X$ has cardinality at most equal to $\deg X$.*

Proof. *** □

Corollary 3.7.7. *A closed, pure dimensional $X \subset \mathbb{P}^n$ has degree* 1 *if and only if it is a linear subspace.*

Proof. *** □

3.7.2 Minimal degree of non degenerate closed subsets of projective spaces

Definition 3.7.8. A subset $X \subset \mathbb{P}^n$ is *degenerate* if it contained in a proper linear subspace, it is *non degenerate* if it is not degenerate.

Proposition 3.7.9. *Let $X \subset \mathbb{P}^n$ be a closed irreducible non degenerate subset. Then*

$$\deg X + \dim X \geqslant n + 1. \tag{3.7.1}$$

Proof. *** □

The following result describes all the 1 dimensional (curves) of $\mathbb{P}^n$ for which the inequality in (3.7.1) is an equality.

Proposition 3.7.10. *Let $X \subset \mathbb{P}^n$ be a closed irreducible non degenerate curve. Then* $\deg X = n$ *if and only if X is projectively equivalent to the rational normal curve.*

Proof. *** □

Definition 3.7.11. A closed irreducible non degenerate subset $X \subset \mathbb{P}^n$ has *extremal degree* if $\deg X + \dim X \geqslant n + 1$, i.e. the inequality in (3.7.1) is an equality.

The closed irreducible non degenerate subsets of $\mathbb{P}^n$ of extremal degree have been classified. Before giving the classification we discuss a few constructions. Let $\mathbf{d} := (d_1, \ldots, d_r)$ be a sequence of r positive integers, let $N(\mathbf{d}) := r - 1 + \sum_{i=1}^{r} d_i$ and let $\Lambda_1, \ldots, \Lambda_r \subset \mathbb{P}^{N(\mathbf{d})}$ be linear subspaces of dimensions $d_1, \ldots, d_r$ spanning $\mathbb{P}^{N(\mathbf{d})}$. In other words there is a direct sum decomposition $\mathbb{K}^{N(\mathbf{d})+1} = V_1 \oplus \ldots \oplus V_r$ and $\Lambda_i = \mathbb{P}(V_i)$ for $i \in \{1, \ldots, r\}$. For $i \in \{1, \ldots, r\}$ let $\nu^1_{d_i} \colon \mathbb{P}^1 \to \Lambda_i$ be "the" Veronese map.

Definition 3.7.12. Keeping notation as above, the *rational normal scroll* $S(d_1, \ldots, d_r) \subset \mathbb{P}^{N(\mathbf{d})}$ is defined by

$$S(d_1, \ldots, d_r) := \bigcup_{p \in \mathbb{P}^1} \langle \nu^1_{d_1}(p), \ldots, \nu^1_{d_i}(p), \ldots, \nu^1_{d_r}(p) \rangle. \tag{3.7.2}$$

Remark 3.7.13. Note that $S(d_1)$ is the rational normal curve of degree d_1.

Example 3.7.14. One checks that $S(d_1, \ldots, d_r)$ is irreducible of dimension r, that it is a closed non degenerate subset of $\mathbb{P}^N(\mathbf{d})$, and that

$$\deg S(d_1, \ldots, d_r) = \sum_{i=1}^{r} d_i. \tag{3.7.3}$$

Hence $S(d_1, \ldots, d_r)$ has extremal degree. We prove the equality in (3.7.3) later on. See Exercise 3.8.14 for a proof for $d_1 = \ldots = d_r = 1$.

Example 3.7.15. Let $v_2^2 \colon \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ be the Veronese map given by monomials of degree 2, and let $X := \operatorname{im}(v_2^2)$ (the *Veronese surface in* $\mathbb{P}^5$). Then $\deg X = 4$ (see Exercise 3.8.15) and hence X has extremal degree.

Remark 3.7.16. Let $\Lambda \subset \mathbb{P}^n$ be a hyperplane, and let $X \subset \Lambda$ be irreducible and non degenerate (in Λ of course). If $p_0 \in (\mathbb{P}^n \backslash \Lambda)$ the cone on X with vertex p_0 is

$$Y := \bigcup_{x \in X} \langle p_0, x \rangle. \tag{3.7.4}$$

One checks easily that Y is irreducible, $\dim Y = \dim X + 1$ and Y and non degenerate in $\mathbb{P}^n$. Moreover we have $\deg Y = \deg X$ (intersect Y with $\dim X + 1$ general hyperplanes, see Remark 3.7.5). In particular, if X has extremal degree then so does Y. Of course we ca iterate the above procedure and we get an iterated cone. If X has extremal degree then so does an iterated cone over X.

Theorem 3.7.17 (del Pezzo-Bertini)**.** *A closed irreducible non degenerate subset $X \subset \mathbb{P}^n$ has extremal degree if and only if it is projectively equivalent to one of the following:*

1. *$\mathbb{P}^n \subset \mathbb{P}^n$ (trivial case).*
2. *A hypersurface $X \subset \mathbb{P}^n$ of degree 2 (semitrivial case).*
3. *The rational normal scroll $S(d_1, \ldots, d_r) \subset \mathbb{P}^{N(\mathbf{d})}$.*
4. *The Veronese surface in $\mathbb{P}^5$.*

An iterated cone over one of the above varieties.

3.8 Exercises

Exercise 3.8.1. The Veronese map is

$$\begin{array}{ccc} \mathbb{P}^2 & \overset{f}{\dashrightarrow} & \mathbb{P}^2 \\ [Z_0, Z_1, Z_2] & \mapsto & [Z_1 Z_2, Z_0 Z_2, Z_0 Z_1] \end{array} \tag{3.8.5}$$

1. Prove that f is a birational map.
2. Determine $\mathrm{Reg}(f)$.
3. Describe maximal open sets $U, V \subset \mathbb{P}^2$ such that f induecs an isomorphism $U \xrightarrow{\sim} V$.

Exercise 3.8.2. Prove that $\dim \mathrm{Gr}(h, V)$ has dimension equal to $h \cdot (\dim V - h)$.

Exercise 3.8.3. An *algebraic group* is an algebraic variety G equipped with a group structure such that the map

$$\begin{array}{ccc} G \times G & \longrightarrow & G. \\ (x, y) & \mapsto & xy^{-1} \end{array} \tag{3.8.6}$$

is regular. For example $\mathrm{GL}_n(\mathbb{K})$ with matrix multiplication is an algebraic group. Prove that the irreducible components of an algebraic groups are pairwise disjoint and they all have the same dimension.

Exercise 3.8.4. Let $M_{n,n}(\mathbb{K})$ be the vector-space of $n \times n$ matrices with entries in $\mathbb{K}$. If $\mathrm{char}\,\mathbb{K} \neq 2$ define $\mathrm{O}_n(\mathbb{K})$ and $\mathrm{SO}_n(\mathbb{K})$ as usual:

$$\mathrm{O}_n(\mathbb{K}) := \{A \in M_{n,n}(\mathbb{K}) \mid A^t \cdot A = 1_n\}, \qquad \mathrm{SO}_n(\mathbb{K}) := \{A \in \mathrm{O}_n(\mathbb{K}) \mid \mathrm{Det}\, A = 1\}, \tag{3.8.7}$$

where $1_n \in M_{n,n}(\mathbb{K})$ is the unit matrix.

1. Let $Q := V(z_1^2 + z_2^2 + \ldots + z_n^2 - 1) \subset \mathbb{A}^n$, and let $f\colon \mathrm{SO}_n(\mathbb{K}) \to Q$ be the map associating to $A \in \mathrm{SO}_n(\mathbb{K})$ its first column. Prove that $f^{-1}(z)$ is isomorphic to $\mathrm{SO}_{n-1}(\mathbb{K})$ for every $z \in Q$.
2. Let X be an irreducible component of $\mathrm{SO}_n(\mathbb{K})$. Prove that $f(X)$ contains an open dense subset of Q.
3. Prove by induction on n that $\mathrm{SO}_n(\mathbb{K})$ is irreducible.
4. Prove that $\mathrm{O}_n(\mathbb{K})$ has two irreducible components.

Exercise 3.8.5. Let

$$U_n(\mathbb{K}) := \{Z \in M_{n,n}(\mathbb{K}) \mid \mathrm{Det}(1_n - Z) \neq 0\}.$$

The *Cayley map* is given by

$$\begin{array}{ccc} U_n(\mathbb{K}) & \xrightarrow{\varphi} & M_{n,n}(\mathbb{K}) \\ Z & \mapsto & (1_n + Z) \cdot (1_n - Z)^{-1} \end{array} \tag{3.8.8}$$

1. Prove that φ defines a birational map $f\colon M_{n,n}(\mathbb{K}) \dashrightarrow M_{n,n}(\mathbb{K})$. Determine the rational inverse $f^{-1}\colon M_{n,n}(\mathbb{K}) \dashrightarrow M_{n,n}(\mathbb{K})$
2. Assume that $\mathrm{char}\,\mathbb{K} \neq 2$. Let $\mathfrak{o}_n(\mathbb{K}) \subset M_{n,n}(\mathbb{K})$ be the subspace of anti-symmetric matrices and let $\mathrm{SO}_n(\mathbb{K}) \subset M_{n,n}(\mathbb{K})$ be the group of special orthogonal matrices. Prove that if $Z \in \mathfrak{o}_n(\mathbb{K}) \cap U_n(\mathbb{K})$ then $\varphi(Z) \in \mathrm{SO}_n(\mathbb{K})$. Let $\psi\colon \mathfrak{o}_n(\mathbb{K}) \cap U_n(\mathbb{K}) \to \mathrm{SO}_n(\mathbb{K})$ be the restriction of φ.

3. Prove that the image of ψ is dense in $\mathrm{SO}_n(\mathbb{K})$, and hence ψ defines a dominant rational map $g\colon \mathfrak{o}_n(\mathbb{K}) \dashrightarrow \mathrm{SO}_n(\mathbb{K})$.
4. Prove that $\mathrm{Reg}(f^{-1})$ contains an open dense subset of $\mathrm{SO}_n(\mathbb{K})$ and hence g is a birational map.
5. Notice that g is defined over the prime field. Produce many matrices in $\mathrm{SO}_3(\mathbb{Q})$.

Exercise 3.8.6. Let V be a finitely generated $\mathbb{K}$ vector space, and let $Q \subset \mathbb{P}(V)$ be a quadric hypersurface, i.e. $Q = V(q)$ where $q\colon V \to \mathbb{K}$ is a non zero quadratic form (if $q = \ell^2$ where $\ell\colon V \to \mathbb{K}$ is a linear form then $V(q) = V(\ell)$ is a hyperplane, the "quadric hypersurface" should be understood to mean the degree 2 effective divisor $2V(\ell)$). The *kernel of* q, denoted by $\ker q$, is the set of $u \in V$ such that $q(v+u) = q(v)$ for all $v \in V$. Note that $\ker q$ is a vector subspace of V. The quadric is *degenerate* if $\ker q \neq \{0\}$, and *non degenerate* if $\ker q = \{0\}$.

1. Prove that if Q is a degenerate quadric hypersurface then Q is a *cone with vertex* $\mathbb{P}(\ker q)$, i.e. there exists a closed subset $X_Q \subset \mathbb{P}(V/\ker q)$ such that Q is the union of the linear subspaces $\Lambda \subset \mathbb{P}(V)$ containing $\mathbb{P}(\ker q)$ parametrized by X (we say that Q is a cone with vertex $\mathbb{P}(\ker q)$ *over* X). (Note: this makes sense because the projective space $\mathbb{P}(V/\ker q)$ is in a natural bijective correspondence with vector subspaces $W \subset V$ containing $\ker q$ whose dimension is equal to $\dim(\ker q) + 1$.) Show that X_Q is a non degenerate quadric in $\mathbb{P}(V/\ker q)$.
2. Let $Q_1, Q_2 \subset \mathbb{P}(V)$ be quadric hypersurfaces, given by quadratic forms q_1, q_2 respectively. Prove that Q_1, Q_2 are *projectively equivalent*, i.e. there exists $g \in \mathrm{PGL}_{n+1}(\mathbb{K})$ such that $g(Q_1) = Q_2$, if and only if $\ker q_1$ and $\ker q_2$ have the same dimensions.
3. Now suppose that $\operatorname{char}\mathbb{K} \neq 2$. Let $Q \subset \mathbb{P}(V)$ be a non degenerate quadric hypersurface, given by the quadratic form q. Since $\operatorname{char}\mathbb{K} \neq 2$ there exists a unique bilinear symmetric form $b\colon V \times V \to \mathbb{K}$ such that $q(v) = b(v,v)$ for all $v \in V$. Let $p_0 = [v_0] \in \mathbb{P}(V)$. The *polar hypersurface of* p_0 is given by
$$p_0^{\perp} := \mathbb{P}(v_0^{\perp}) = \mathbb{P}(\{v \in V \mid b(v_0, v) = 0\}). \tag{3.8.9}$$
Prove that $p_0^{\perp} \cap Q$ is a quadric hypersurface in $p_0^{\perp}$, that it is degenerate if and only if $p_0 \in Q$, and that if the latter holds then it is a cone with vertex p_0 over a non degenerate quadric hypersurface in $\mathbb{P}(v_0^{\perp}/\langle v_0\rangle)$.

Exercise 3.8.7. Assume that $\operatorname{char}\mathbb{K} \neq 2$. Let $Q^n \subset \mathbb{P}^{n+1}$ be a non degenerate quadric hypersurface.

1. Prove that if $\Lambda \subset Q$ is a linear space then $\dim\Lambda \leqslant \frac{n}{2}$.
2. Let $r := \lfloor \frac{n}{2} \rfloor$. Describe $F_r(Q^n)$ in terms of well-known varieties for $n \in \{1,2,3,4\}$.

Exercise 3.8.8. Assume that $\operatorname{char}\mathbb{K} \neq 2$. Let $Q^{2r+1} \subset \mathbb{P}^{2r+2}$ be a non degenerate odd dimensional quadric hypersurface.

1. Let $\Gamma_r(Q^{2r+1}) \subset Q^{2r+1} \times \mathbb{G}\mathrm{r}(r, \mathbb{P}^{2r+2})$ be the incidence subset defined by
$$\Gamma_r(Q^{2r+1}) := \{(p, \Lambda) \mid p \in \Lambda \subset Q^{2r+1}\}. \tag{3.8.10}$$
The restrictions to $\Gamma_r(Q^{2r+1})$ of the projection maps of $Q^{2r+1} \times \mathbb{G}\mathrm{r}(r, \mathbb{P}^{2r+2})$ define regular projective maps
$$\begin{tikzcd} & \Gamma_r(Q^{2r+1}) \arrow[dl, "\pi_r"'] \arrow[dr, "\rho_r"] & \\ Q^{2r+1} & & \mathbb{G}\mathrm{r}(r, \mathbb{P}^{2r+2}) \end{tikzcd} \tag{3.8.11}$$
Note that $\mathrm{im}(\rho_r) = F_r(Q^{2r+1})$. Prove that $\pi_r^{-1}(p) \cong F_{r-1}(Q^{2r-1})$ for every $p \in Q^{2r+1}$.
2. Prove, using Item (2) and arguing by induction on r, that $F_r(Q^{2r+1})$ is irreducible and
$$\dim F_r(Q^{2r+1}) = \binom{r+2}{2}. \tag{3.8.12}$$

Exercise 3.8.9. Let $Q^{2r} \subset \mathbb{P}^{2r+1}$ be a non degenerate even dimensional quadric hypersurface. The purpose of this exercise is to prove that

(I) $F_r(Q^{2r})$ has two irreducible components $F_r(Q^{2r})_+$ and $F_r(Q^{2r})_-$, each of dimension $\binom{r+1}{2}$, and they are disjoint.

(II) $\Lambda_1, \Lambda_2 \in F_r(Q^{2r})$ belong to the same irreducible component if and only if

$$\dim(\Lambda_1 \cap \Lambda_2) \equiv r \pmod 2.$$

(Here we agree that $\dim \varnothing = -1$.)

1. Prove that each irreducible component of $F_r(Q^{2r})$ has dimension at least $\binom{r+1}{2}$. (Hint: this amounts to the statement that each irreducible component of $F_r(Q^{2r})$ has codimension at most $\dim \mathbb{K}[T_0, \ldots, T_r]_2$.)
2. Let $H \subset \mathbb{P}^{2r+1}$ be a hyperplane such that $Q^{2r} \cap H$ is non degenerate, i.e. $H = p^{\perp}$ where $p \in (\mathbb{P}^{2r+1} \backslash Q^{2r})$. Show that there is a well-defined regular map

$$\begin{array}{ccc} F_r(Q^{2r}) & \longrightarrow & F_{r-1}(Q^{2r-1}) \\ \Lambda & \mapsto & \Lambda \cap H \end{array} \tag{3.8.13}$$

and use this, together with Item (1) to prove that $F_r(Q^{2r})$ has pure dimension $\binom{r+1}{2}$, and that it has at most two connected components.
3. Given $W \in F_r(Q^{2r})$ let $\mathscr{A}_W \subset F_r(Q^{2r})$ be defined by

$$\mathscr{A}_W := \{U \in F_r(Q^{2r}) \mid U \cap W = \{0\}\}.$$

Prove that $\mathscr{A}_W$ is an affine space of dimension equal to $\binom{r+1}{2}$, and that if $W_1, W_2 \in \mathscr{A}_W$ then

$$\dim W_1 \cap W_2 \equiv r + 1 \pmod 2.$$

(Hint: Fix $W_0 \in \mathscr{A}_W$. The symmetric bilinear form b induces an isomorphism $W \simeq W_0^{\vee}$. Given $U \in \mathscr{A}_W$ the condition $U \cap W = \{0\}$ gives that $U \subset \mathbb{K}^{2r+2} = W_0 \oplus W$ is the graph of a linear map $\phi_U : W_0 \to W \simeq W_0^{\vee}$. Lastly note that ϕ_U is skew-symmetric, i.e. $\phi_U^t = -\phi_U$, because W_0 is isotropic for a quadratic form defining Q^{2r}.)
4. Prove that $F_r(Q^{2r}) = \bigcup_{W \in F_r(Q^{2r})} \mathscr{A}_W$ and that if $\dim W_1 \cap W_2 \not\equiv r + 1 \pmod 2$ then $\mathscr{A}_{W_1} \cap \mathscr{A}_{W_2} = \varnothing$.
5. Prove that (I) and (II) hold.

Exercise 3.8.10. Prove Claims 3.5.29 and 3.5.30.

Exercise 3.8.11. Prove Propositions 3.5.31 and 3.5.32.

Exercise 3.8.12. The goal of the exercise is to prove the following result:

$$\text{Every cubic surface in } \mathbb{P}^3 \text{ contains a line.} \tag{3.8.14}$$

Let $F = Z_3 \cdot (Z_0 Z_1 - Z_2^2) + L_1 \cdot L_2 \cdot L_3$ where $L_i \in \mathbb{K}[Z_0, Z_1, Z_2]_1$ are linear functions such that the following hold.

- The intersection in $\mathbb{P}^2$ of $V(L_i)$ and $V(Z_0 Z_1 - Z_2^2)$ consists of two distinct points.
- If $i, j \in \{1, 2, 3\}$ are distinct then $V(L_i) \cap V(L_j) \cap V(Z_0 Z_1 - Z_2^2)$ (intersection in $\mathbb{P}^2$) is empty.

It follows that the intersection in $\mathbb{P}^2$ of $V(L_1 \cdot L_2 \cdot L_3)$ and $V(Z_0 Z_1 - Z_2^2)$ consists of six distinct points:

$$V(L_1 \cdot L_2 \cdot L_3) \cap V(Z_0 Z_1 - Z_2^2) = \{q_1, \ldots, q_6\}. \tag{3.8.15}$$

Let $X := V(F) \subset \mathbb{P}^3$.

1. Prove that F is prime and hence X is an irreducible cubic surface.
2. Let $p_0 = [0, 0, 0, 1] \in X$, and let

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & \mathbb{P}^2 \\ [Z_0, Z_1, Z_2, Z_3] & \mapsto & [Z_0, Z_1, Z_2] \end{array} \tag{3.8.16}$$

Show that f is birational.
3. Let $R \subset \mathbb{P}^3$ be a line. Prove that $R \subset X$ if and only if one of the following hold.

(3a) $R = \langle p_0, q_j \rangle$ where q_j is one of the points appearing in (3.8.15).

(3b) $R = \langle q_j, q_k \rangle$ where q_j, q_k are two distinct pointsappearing in (3.8.15).

4. Prove that the statement in (3.8.14) holds.

Exercise 3.8.13. Let $\mathscr{P}_d^n \subset \mathrm{Div}_+^d(\mathbb{P}^n)$ be the set of *prime divisors*, i.e. the set of irreducible hypersurface of degree d. Prove that if $n \geqslant 2$ and $d \geqslant 2$ then the codimension of the complement of $\mathscr{P}_d^n$ in $\mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d)$ is equal to

$$\binom{d+n-1}{n-1} - n. \tag{3.8.17}$$

In particular $\mathscr{P}_d^n$ is a dense open subset of $\mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d)$. (Hint: For $0 < a < d$ let $\Gamma_a^d(\mathbb{P}^n) \subset \mathrm{Div}_+^a(\mathbb{P}^n) \times \mathrm{Div}_+^d(\mathbb{P}^n)$ be the subset defined by

$$\Gamma_a^d(\mathbb{P}^n) := \{(A, D) \mid D = A + B,\ B \in \mathrm{Div}_+^{d-a}(\mathbb{P}^n)\}. \tag{3.8.18}$$

Show that $\Gamma_a^d(\mathbb{P}^n)$ is closed subset. Consider the two regular maps $\Gamma_a^d(\mathbb{P}^n) \to \mathrm{Div}_+^a(\mathbb{P}^n)$ and $\Gamma_a^d(\mathbb{P}^n) \to \mathrm{Div}_+^d(\mathbb{P}^n)$ given by the restrictions of the projections of $\mathrm{Div}_+^a(\mathbb{P}^n) \times \mathrm{Div}_+^d(\mathbb{P}^n)$.)

Exercise 3.8.14. Let

$$\mathbb{P}^1 \times \mathbb{P}^n \xrightarrow{\sigma} \mathbb{P}^{2n+1}$$

be the Segre embedding, and let $X_n \subset \mathbb{P}^{2n+1}$ be its image. Prove that $\deg X_n = n+1$, i.e. $\deg X_n + \dim X_n = 2n+2$. Next show that X_n is projectively equivalent to the rational normal scroll $S(1, 1, \ldots, 1)$, see Example 3.7.14.

Exercise 3.8.15. Let $v_2^2 \colon \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ be the Veronese map given by monomials of degree 2, and let $X := \mathrm{im}(v_2^2)$. Prove that $\deg X = 4$ by following the steps below.

1. By Remark 3.7.5 it suffices to prove that for general hyperplanes $H_1, H_2 \subset \mathbb{P}^5$ the intersection $X \cap H_1 \cap H_2$ has cardinality 4.
2. By Item (1) we are reduced to showing that if $C_1, C_2 \subset \mathbb{P}^2$ are generic curves of degree 2 then $C_1 \cap C_2$ has cardinality 4. Prove it.