Chapter 4

Tangent space, smooth points

4.1 Introduction

One definition of tangent space of a C^∞ manifold M at a point $x \in M$ is as the real vector space of derivations of the ring of C^∞ functions on M centered at x, or of the ring of germs of C^∞ functions at x. An analogue definition gives the definition of Zariski tangent space of an algebraic variety at a point. One needs to consider the analogue of the ring of germs of C^∞ functions at the point because if the algebraic variety is complete then global regular functions are locally constant. The advantage of such an abstract definition is that it is intrinsic by definition. On the other hand, we will identify the Zariski tangent space at a point a of a closed subset $X \subset \mathbb{A}^n$ with the classical embedded tangent space, defined by the common zeroes of the linear approximations at a of polynomials in a basis of the ideal $I(X)$.

A fundamental difference between quasi projective varieties and smooth manifolds is that the dimension of the tangent space at a point might depend on the point, even for an irreducible variety. The points where the dimension has a local minimum are the so-called smooth points of the variety. Smooth algebraic varieties resemble C^∞ manifolds, or even more closely complex manifolds.

In fact if the field is $\mathbb{C}$ then a smooth variety has a natural structure of complex manifold and regular maps between complex smooth varieties are holomorphic maps of complex manifolds.

4.2 Zariski's tangent and cotangent space

4.2.1 The local ring at a point

Definition 4.2.1. Let X be an algebraic variety, and let $x \in X$. Let (U, ϕ) and (V, ψ) be couples where U, V are open subsets of X containing x, and $\phi \in \mathbb{K}[U]$, $\psi \in \mathbb{K}[V]$. Then $(U, \phi) \sim (V, \psi)$ if there exists an open subset $W \subset X$ containing x such that $W \subset U \cap V$ and $\phi_{|W} = \psi_{|W}$.

One checks easily that $\sim$ is an equivalence relation: an equivalence class for the relation $\sim$ is a *germ of regular function of X at x*. We may define a sum and a product on the set of germs of regular functions of X at x by setting

$$[(U, \phi)] + [(V, \psi)] := [(U \cap V, \phi_{|U\cap V} + \psi_{|U\cap V})], \tag{4.2.1}$$

and

$$[(U, \phi)] \cdot [(V, \psi)] := [(U \cap V, \phi_{|U\cap V} \cdot \psi_{|U\cap V})]. \tag{4.2.2}$$

Of course one has to check that the equivalence class of the sum and product is independent of the choice of representatives: this is easy, we leave details to the reader. With these operations, the set of germs of regular functions of X at x is a ring.

Definition 4.2.2. Let X be an algebraic variety and let $x \in X$. The *local ring of X at x* is the ring of germs of regular functions of X at x, and is denoted $\mathscr{O}_{X,x}$.

Remark 4.2.3. Let X be an algebraic variety, and let $x \in X$. If $V \subset X$ is an open subset containing x then the homomorphism

$$\begin{array}{ccc} \mathscr{O}_{V,x} & \xrightarrow{\rho} & \mathscr{O}_{X,x} \\ [(U,\varphi)] & \mapsto & [(U,\varphi)] \end{array} \tag{4.2.3}$$

is an isomorphism. Since there exist many V which are affine, every local ring of a point on an algebraic variety is isomorphic to the local ring of a point on an affine variety.

There is a well-defined surjective homomorphism

$$\begin{array}{ccc} \mathscr{O}_{X,x} & \longrightarrow & \mathbb{K} \\ [(U,\phi)] & \mapsto & \phi(x) \end{array} \tag{4.2.4}$$

As a matter of notation we let $f(x)$ be the value of the above homomorphism on $f = [(U,\phi)]$. We have the natural homomorphism of rings

$$\begin{array}{ccc} \mathbb{K}[X] & \xrightarrow{\rho} & \mathscr{O}_{X,x} \\ f & \mapsto & [(X,f)] \end{array} \tag{4.2.5}$$

Let $\mathfrak{m}_x \subset \mathbb{K}[X]$ be the ideal defined by

$$\mathfrak{m}_x := \{f \in \mathbb{K}[X] \mid f(x) = 0\}\,. \tag{4.2.6}$$

If $f \notin \mathfrak{m}_x$ then $\rho(f)$ is invertible: in fact the open subset

$$X_f \coloneqq X \backslash V(f) \tag{4.2.7}$$

contains x and $[(X_f, 1/f)]$ is the inverse of $\rho(f)$. Let $\mathbb{K}[X]_{\mathfrak{m}_x}$ be the localization of $\mathbb{K}[X]$ at the maximal ideal $\mathfrak{m}_x$, and let $\varphi\colon \mathbb{K}[X] \to \mathbb{K}[X]_{\mathfrak{m}_x}$ be the localization homomorphism. By the universal property of the ring of fractions (see Proposition A.4.3) there exists a unique homomorphism

$$\overline{\rho}\colon \mathbb{K}[X]_{\mathfrak{m}_x} \to \mathscr{O}_{X,x} \tag{4.2.8}$$

such that $\rho = \overline{\rho} \circ \varphi$.

Proposition 4.2.4. *Keep notation as above, and suppose that X is an affine variety. Then*

$$\mathbb{K}[X]_{\mathfrak{m}_x} \xrightarrow{\overline{\rho}} \mathscr{O}_{X,x} \tag{4.2.9}$$

is an isomorphism.

Proof. Suppose that $\overline{\rho}(f/g) = 0$, where $f, g \in \mathbb{K}[X]$ with $g(x) \neq 0$. This means that there exists an open $U \subset X$ containing x such that $f_{|U} = 0$. Since the principal open affine subsets of X form a basis of the Zariski topology, there exists $h \in \mathbb{K}[X]$ such that $X_h \subset U$ and $x \in X_h$ (see Example 1.6.5). Thus $h \notin \mathfrak{m}_x$ and $h \cdot f = 0$: this gives that $f/g = 0$ in the localization $\mathbb{K}[X]_{\mathfrak{m}_x}$. This proves that $\overline{\rho}$ is injective.

Next we prove that $\overline{\rho}$ is surjective. Let $f \in \mathscr{O}_{X,x}$. Then, since principal open affine subsets of X form a basis of the Zariski topology, f is represented by a suitable (X_h, φ) where $h \in \mathbb{K}[X]$ does not vanish in x. By Example 1.6.5 we have $\varphi = g/h^N$ for suitable $g \in \mathbb{K}[X]$ and exponent N, and hence $f = [(X_h, \varphi)] = \overline{\rho}(g/h^N)$. □

By the above proposition and Proposition A.4.7 we get the following.

Corollary 4.2.5. *Let X be an algebraic variety, and let $x \in X$. Then $\mathscr{O}_{X,x}$ is a Noetherian local ring, and the homomorphism in* (4.2.4) *is the quotient map to its residue field.*

Abusing notation we let $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be the kernel of (4.2.4), i.e. the unique maximal ideal of germs of regular functions at x vanishing at x.

4.2.2 Zariski's tangent space

The homomorphism (4.2.4) equips $\mathbb{K}$ with a structure of $\mathscr{O}_{X,x}$-module. Moreover $\mathscr{O}_{X,x}$ is a $\mathbb{K}$-algebra. Thus it makes sense to consider $\mathbb{K}$-derivations of $\mathscr{O}_{X,x}$ to $\mathbb{K}$.

Definition 4.2.6. Let X be an algebraic variety, and let $x \in X$. The *Zariski tangent space to X at x* is $\mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{X,x}, \mathbb{K})$, and will be denoted by $\Theta_x X$.

Thus $\Theta_x X$ is an $\mathscr{O}_{X,x}$-module (see Section A.8), and since $\mathfrak{m}_x$ annihilates every derivation $\mathscr{O}_{X,x} \to \mathbb{K}$, it is a complex vector space. This corresponds to the $\mathbb{K}$ vector space structure of $\mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{X,x}, \mathbb{K})$ via the isomorphism $\mathbb{K} \xrightarrow{\sim} \mathscr{O}_{X,x}/\mathfrak{m}_x$.

Lemma 4.2.7. *Let $a \in \mathbb{A}^n$. The complex linear map*

$$\begin{array}{ccc} \Theta_a \mathbb{A}^n & \longrightarrow & \mathbb{K}^n \\ D & \mapsto & (D(z_1), \ldots, D(z_n)) \end{array} \tag{4.2.10}$$

is an isomorphism.

Proof. The formal partial derivative $\frac{\partial}{\partial z_m}$ defined by (A.8.7) defines an element of $\Theta_a \mathbb{A}^n$ by the familiar formula

$$\frac{\partial}{\partial z_m}\left(\frac{f}{g}\right)(a) := \frac{\frac{\partial f}{\partial z_m}(a) \cdot g(a) - f(a) \cdot \frac{\partial g}{\partial z_m}(a)}{g(a)^2}.$$

(See Example A.8.3.) Since $\frac{\partial}{\partial z_m}(z_j) = \delta_{mj}$, the map in (4.2.10) is surjective.

Let's prove that the map in (4.2.10) is injective. Assume that $D \in \Theta_{X,x}$ is mapped to 0 by the map in (4.2.10), i.e. $D(x_j) = 0$ for $j \in \{1, \ldots, n\}$. Let $f, g \in \mathbb{K}[z_1, \ldots, z_n]$, with $g(a) \neq 0$. Then

$$D\left(\frac{f}{g}\right) = \frac{D(f) \cdot g(a) - f(a) \cdot D(g)}{g(a)^2}.$$

(See Example A.8.3.) Hence it suffices to show that $D(f) = 0$ for every $f \in \mathbb{K}[z_1, \ldots, z_n]$. Consider the first-order expansion of f around a i.e. write

$$f = f(a) + \sum_{i=1}^{n} c_i (z_i - a) + R, \qquad R \in \mathfrak{m}_a^2. \tag{4.2.11}$$

Since D is zero on constants (because D is a $\mathbb{K}$-derivation) and $D(z_j) = 0$ for all j it follows that $D(f) = D(R)$, and the latter vanishes by Leibniz' rule and the hypothesis $D(z_j) = 0$ for all j. □

The differential of a regular map at a point of the domain is defined by the usual procedure. Explicitly, let $f \colon X \to Y$ be a regular map of quasi projective varieties, let $x \in X$ and $y := f(x)$. There is a well-defined pull-back homomorphism

$$\begin{array}{ccc} \mathscr{O}_{Y,y} & \xrightarrow{f^*} & \mathscr{O}_{X,x} \\ [(U, \phi)] & \mapsto & [(f^{-1}U, \phi \circ (f_{|f^{-1}U}))] \end{array} \tag{4.2.12}$$

The *differential of f at x* is the linear map of complex vector spaces

$$\begin{array}{ccc} \Theta_x X & \xrightarrow{df(x)} & \Theta_y Y \\ D & \mapsto & (\phi \mapsto D(f^* \phi)) \end{array} \tag{4.2.13}$$

Remark 4.2.8. Suppose that the pull-back homomorphism in (4.2.12) is a surjection. Then the differential $df(x)$ is injective. In particular if $j \colon X \hookrightarrow Y$ is the inclusion of a closed (or open) subset, then $dj(x) \colon \Theta_x X \to \Theta_x Y$ is injective for all $x \in X$.

The differential has the customary functorial properties. Suppose that we have

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3, \quad x_1 \in X_1, \quad x_2 = f_1(x_1). \tag{4.2.14}$$

Since $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$ we have

$$d(f_2 \circ f_1)(x_1) = df_2(x_2) \circ df_1(x_1). \tag{4.2.15}$$

Moreover $d\,\mathrm{Id}_X(x) = \mathrm{Id}_{\Theta_x X}$ for $x \in X$.

Remark 4.2.9. It follows from the above that if f is an isomorphism, then $df(x)\colon \Theta_x X \to \Theta_{f(x)} Y$ is an isomorphism, in particular $\dim \Theta_x X = \dim \Theta_y Y$.

4.2.3 Embedded tangent space

Let $\iota\colon X \hookrightarrow \mathbb{A}^n$ be the inclusion of a closed subset and $a \in X$. The differential $d\iota(a)\colon \Theta_a X \to \Theta_a \mathbb{A}^n$ is injective (see Remark 4.2.8), hence the differential $d\iota(a)$ identifies $\Theta_a X$ with a subspace of $\Theta_a \mathbb{A}^n$. The latter is identified with $\mathbb{K}^n$ via the isomorphism in (4.2.10). Thus we view $\Theta_a X$ as a vector subspace of $\mathbb{K}^n$.

Proposition 4.2.10. *Keeping notation as above, we have*

$$\Theta_a X = \left\{ v = (v_1, \ldots, v_n) \in \mathbb{K}^n \mid \sum_{i=1}^n \frac{\partial f}{\partial z_i}(a) \cdot v_i = 0 \quad \forall f \in I(X) \right\}. \tag{4.2.16}$$

Proof. The differential $d\iota(a)$ is injective because the pull-back $\iota^*\colon \mathscr{O}_{\mathbb{A}^n,a} \to \mathscr{O}_{X,a}$ is surjective. Let $D \in \mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{X,a}, \mathbb{K})$. If $f \in I(X) \subset \mathbb{K}[z_1, \ldots, z_n]$, then $d\iota(D)(f) = D(\iota^* f) = D(0) = 0$. Hence $\operatorname{im} d\iota(a)$ is contained in the right-hand side of (4.2.16). Let's prove that $\operatorname{im} d\iota(a)$ contains the right-hand side of (4.2.16). Let $\widetilde{D} \in \mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{\mathbb{A}^n,a}, \mathbb{K})$ belong to the right hand side of (4.2.16), i.e. $\widetilde{D}(f) = 0$ for all $f \in I(X)$. By Item (3) of Example A.8.3 it follows that $\widetilde{D}(\frac{f}{g}) = 0$ whenever $f, g \in \mathbb{K}[z_1, \ldots, z_n]$ and $f \in I(X)$ (of course we assume that $g(a) \neq 0$). Thus $\widetilde{D}$ descends to a $\mathbb{K}$-derivation $D \in \mathrm{Der}_{\mathbb{K}}(\mathscr{O}_{X,a}, \mathbb{K})$, and $\widetilde{D} = d\iota_*(a)(D)$. □

Remark 4.2.11. With the hypotheses of Proposition 4.2.10, suppose that $I(X)$ is generated by $f_1, \ldots, f_r$. Then

$$\Theta_a X = \left\{ v = (v_1, \ldots, v_n) \in \mathbb{K}^n \mid \sum_{i=1}^n \frac{\partial f_k}{\partial z_i}(a) \cdot v_i = 0 \quad k \in \{1, \ldots, r\} \right\}.$$

In fact, the right hand side of the above equation is equal to the right hand side of (4.2.16), because if $f = \sum_{j=1}^r g_j f_j$, then $\frac{\partial f}{\partial z_i}(a) = \sum_{j=1}^r g_j(a) \frac{\partial f_j(a)}{\partial z_i}$.

Remark 4.2.12. Since every point x of an algebraic variety X is contained in an open affine subset U, and $\Theta_x X = \Theta_x U$ (because restriction defines an identification $\mathscr{O}_{X,x} = \mathscr{O}_{U,x}$), the result above allows to compute the Zariski tangent space in general.

Definition 4.2.13. Let $X \subset \mathbb{A}^n$ be a closed subset and let $a \in X$. The *embedded tangent space of X at a* is the affine subspace $T_a X \subset \mathbb{A}^n$ containing a with vector space of translations given by $\Theta_a X$ (given by the equality in (4.2.16)).

Let $a = (a_1, \ldots, a_n)$, and suppose that $I(X) = (f_1, \ldots, f_r)$. By Remark 4.2.11 the embedded tangent space $T_a X$ is given by

$$T_a X = \left\{ z \in \mathbb{A}^n \mid \sum_{i=1}^n \frac{\partial f_k}{\partial z_i}(a) \cdot (z_i - a_i) = 0 \quad k \in \{1, \ldots, r\} \right\}. \tag{4.2.17}$$

Remark 4.2.14. Let $R \subset \mathbb{A}^n$ be a line containing a. Then R belongs to $T_a X$ if and only if the restriction to R of every $f \in I(X)$ has a zero of multiplicity at least 2 at a.

There is an analogous notion of embedded projective tangent space of a closed subset $X \subset \mathbb{P}^n$ at a point $[a] \in X$. First note that given any subspace $V \subset \Theta_a \mathbb{P}^n$ there exists a unique linear subspace $\Lambda \subset \mathbb{P}^n$ such that the tangent space $\Theta[a]\Lambda$ is equal to V.

Definition 4.2.15. Let $X \subset \mathbb{P}^n$ be a closed subset and let $[a] \in X$. The *embedded projective tangent space of X at* $[a]$ is the unique linear subsapce $\mathbb{T}_{[a]}X \subset \mathbb{P}^n$ containing $[a]$ such that $\Theta_{[a]}(\mathbb{T}_{[a]}X) = \Theta_{[a]}X$.

Proposition 4.2.16. *Let $X \subset \mathbb{P}^n$ be a closed subset and let $[a] = [a_0, \ldots, a_n] \in X$. Let $F_1, \ldots, F_r$ be homogeneous generators of the (homogeneous) ideal $I(X) \subset \mathbb{K}[Z_0, \ldots, Z_n]$. Then*

$$\mathbb{T}_{[a]}X = V\left(\sum_{i=0}^{n} \frac{\partial F_1(a)}{\partial Z_i} Z_i, \ldots, \sum_{i=0}^{n} \frac{\partial F_r(a)}{\partial Z_i} Z_i\right). \tag{4.2.18}$$

Proof. It suffices to prove that if $[a] \in \mathbb{P}^n_{Z_s}$ then the intersection of the two sides of (4.2.18) with $\mathbb{P}^n_{Z_s}$ are equal. The intersection of $\mathbb{T}_{[a]}X$ and $\mathbb{P}^n_{Z_s}$ is the embedded tangent space to $X \cap \mathbb{P}^n_{Z_s}$ at the point $(a_0/a_s, \ldots, a_{s-1}/a_s, a_{s+1}/a_s, \ldots, a_n/a_s)$, and hence can be computed by the equality in (4.2.17). We check that it is equal to the intersection of the right hand side of (4.2.18) with $\mathbb{P}^n_{Z_s}$. We may assume that $s = 0$. It suffices to prove that if $F \in I(Z)$ then

$$V\left(\frac{\partial F(a)}{\partial Z_0} + \sum_{i=1}^{n} \frac{\partial F(a)}{\partial Z_i} Z_i\right) = V\left(\sum_{i=1}^{n} \frac{\partial f(a_1, \ldots, a_n)}{\partial Z_i}(Z_i - a_i)\right). \tag{4.2.19}$$

In order to prove the equality in (4.2.19) we recall *Euler's identity*

$$(\deg F) \cdot F = \sum_{i=0}^{n} \frac{\partial F(a)}{\partial Z_i} Z_i.$$

Plugging in $Z_0 = 1$, $Z_i = a_i$ for $i \in \{1, \ldots, n\}$, and noting that $F(1, a_1, \ldots, a_n) = 0$ we get that

$$\frac{\partial F(a)}{\partial Z_0} = -\sum_{i=1}^{n} \frac{\partial F(a_1, \ldots, a_n)}{\partial Z_i} a_i. \tag{4.2.20}$$

This proves that the equality in (4.2.19) holds. □

Remark 4.2.17. Let $\Lambda \subset \mathbb{P}^n$ be a line containing $[a]$. Thus $\Lambda = \mathbb{P}(U)$ where $U \subset \mathbb{K}^{n+1}$ is a vector subspace of dimension 2. Then Λ belongs to $\mathbb{T}_a X$ if and only if for all homogeneous $F \in I(X)$ the restriction $F_{|U}$ has a zero of multiplicity at least 2 at a.

4.2.4 Zariski's cotangent space

Let X be an algebraic variety and let $x \in X$. The *cotangent space to X at x* is the dual complex vector space of the tangent space $\Theta_x X$, and is denoted $\Omega_X(x)$:

$$\Omega_X(x) := (\Theta_x X)^\vee. \tag{4.2.21}$$

We define a map

$$\mathscr{O}_{X,x} \xrightarrow{\delta_x} \Omega_X(x) \tag{4.2.22}$$

as follows. Let $f = [(U, \varphi)] \in \mathscr{O}_{X,x}$ and let $v \in \Theta_x X$ be a $\mathbb{K}$-derivation $\mathscr{O}_{X,x} \to \mathbb{K}$: then $\delta_x(f)$ acts on v by

$$\langle \delta_x(f), v \rangle := v(\varphi). \tag{4.2.23}$$

The action is independent of the representative of f.

Remark 4.2.18. We equip $\Omega_X(x)$ with a structure of $\mathscr{O}_{X,x}$-module by composing the evaluation map $\mathscr{O}_{X,x} \to \mathbb{K}$ given by (4.2.4) and scalar multiplication of the complex vector-space $\Omega_X(x)$. With this structure δ_x is a derivation over $\mathbb{K}$.

Let $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$ be the maximal ideal. Since δ_x is a derivation $\delta_x(f) = 0$ for $f \in \mathfrak{m}_x^2$, and hence we have an induced $\mathbb{K}$-linear map

$$\begin{array}{ccc} \mathfrak{m}_x/\mathfrak{m}_x^2 & \xrightarrow{\overline{\delta}_x} & \Omega_X(x) \\ [\phi] & \mapsto & d\phi(a) \end{array} \tag{4.2.24}$$

Proposition 4.2.19. *Keep notation as above. Then $\overline{\delta}_x$ is an isomorphism of $\mathbb{K}$ vector spaces.*

Proof. First we prove that $\overline{\delta}_x$ is surjective. If $X = \mathbb{A}^n$, surjectivity follows at once from Lemma 4.2.7. In general, we may assume that X is a closed subset of $\mathbb{A}^n$, and surjectivity follows from surjectivity for affine space.

In order to prove injectivity of $\overline{\delta}_x$, we must show that if $\phi \in \mathfrak{m}_x$ is such that $v(\phi) = 0$ for all $v \in \Theta_x X$, then $\phi \in \mathfrak{m}_x^2$. We may suppose that X is a closed subset of $\mathbb{A}^n$. In order to avoid confusion, we let $x = a = (a_1, \ldots, a_n)$. Let $(U, f/g)$ be a representative of ϕ, where $f, g \in \mathbb{K}[X]$, and $f(a) = 0$, $g(a) \neq 0$. It will suffice to prove that $f \in \mathfrak{m}_a^2$. Since $0 = v(\phi) = -g(a)^{-2} v(f)$ we have $v(f) = 0$. By Theorem 1.6.2 there exists $\tilde{f} \in \mathbb{K}[z_1, \ldots, z_n]$ such that $\tilde{f}_{|X} = f$. By Proposition 4.2.10 we may identify $\Theta_a X$ with the subspace of $\Theta_a \mathbb{K}^n = \mathbb{K}^n$ given by (4.2.16). In our new notation we have

$$\Theta_a X = \mathrm{Ann}(\{\delta_x(p) \mid p \in I(X)\}). \tag{4.2.25}$$

It follows that there exists $h \in I(X)$ such that $v(\tilde{f}) = v(h)$ fo all $v \in \Theta_a X$. Then $(\tilde{f} - h)_{|X} = f$ and $v(\tilde{f} - h) = 0$ for all $v \in \Theta_a X$. It follows (first-order Taylor expansion of $\tilde{f} - h$ at a) that

$$(\tilde{f} - h) \in (z_1 - a_1, \ldots, z_n - a_n)^2.$$

Since $h \in I(X)$ we get that $f \in \mathfrak{m}_a^2$. □

The following result is an immediate consequence of Corollary A.9.2.

Corollary 4.2.20. *Let X be a quasi-projective variety and let $x \in X$. Let $f_1, \ldots, f_n \in \mathfrak{m}_x \subset \mathscr{O}_{X,x}$, and suppose that $\delta_x(f_1), \ldots, \delta_x(f_n)$ generate $\Omega_X(x)$. Then $f_1, \ldots, f_n$ generate the maximal ideal $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$.*

One may also define a map

$$\mathscr{O}_{X,x} \xrightarrow{d_x} \Omega_X(x) \tag{4.2.26}$$

as follows. Let $f \in \mathscr{O}_{X,x}$ be represented by (U, ϕ). The codomain of the differential $d\phi(x) \colon \Theta_x U \to \Theta_{\phi(x)} \mathbb{K}$ is identified with with $\mathbb{K}$, because of the isomorphism in (4.2.10), and hence $d\phi(x) \in (\Theta_x U)^\vee$. Since $U \subset Z$ is an open subset containing x, the differential at x of the inclusion map defines an identification $\Theta_x U \xrightarrow{\sim} \Theta_x X$. Thus $d\phi(x) \in (\Theta_x X)^\vee = \Omega_X(x)$. One checks immediately that if (V, ψ) is another representative of f then $d\psi(x) = d\phi(x)$. We let

$$d_x(f) := d\phi(x), \qquad (U, \phi) \text{ any representative of } f. \tag{4.2.27}$$

Claim 4.2.21. *Let X be an algebraic variety and let $x \in X$. The map δ_x in* (4.2.22) *is equal to the map d_x in* (4.2.27).

Proof. This is because $\phi^*(z) = \phi$, where z is the coordinate on $\mathbb{K}$. □

Because of the above identification from now on we denote $\delta_x(f)$ by $df(x)$.

Remark 4.2.22. Let $f \in \mathbb{K}[z_1, \ldots, z_n]$ and $a \in \mathbb{A}^n$. Then the familiar formula

$$df(a) = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) dz_i(a)$$

holds. In fact this follows from the first-order Taylor expansion of f at a:

$$f = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a)(z_i - a_i) + \sum_{1 \leqslant i,j \leqslant n} m_{ij}(z_i - a_i)(z_j - a_j), \qquad m_{ij} \in \mathbb{K}[z_1, \ldots, z_n]. \tag{4.2.28}$$

Remark 4.2.23. Let $X \subset \mathbb{A}^n$ be closed, and let $a \in X$. Identify $\Theta_a \mathbb{A}^n$ with $\mathbb{K}^n$ via Lemma 4.2.7. By Remark 4.2.22 we have the identification

$$T_a X = \mathrm{Ann}\{df(a) \mid f \in I(X)\}.$$

Let $f \colon X \to Y$ be a regular map of algebraic varieties. Let $x \in X$, and let $y := f(x)$. The transpose of the differential $df(x) \colon \Theta_x X \to \Theta_y Y$ is also denoted by $f^*(x)$:

$$f^*(x) = df(x)^t \colon \Omega_Y(y) \longrightarrow \Omega_X(x). \tag{4.2.29}$$

The notation is justified by the following observation. By Proposition 4.2.19 and Claim 4.2.21 we have the identification

$$\begin{array}{ccc} \mathfrak{m}_y/\mathfrak{m}_y^2 = \Omega_Y(y) & \overset{df(x)^t}{\longrightarrow} & \Omega_X(x) = \mathfrak{m}_x/\mathfrak{m}_x^2 \\ [(U,\varphi)] \pmod{\mathfrak{m}_y^2} & \mapsto & [(f^{-1}(U), \varphi \circ (f_{|f^{-1}(U)})] \pmod{\mathfrak{m}_x^2} \end{array} \tag{4.2.30}$$

Proposition 4.2.24. *Let $f \colon X \to Y$ be a regular map of algebraic varieties, and let $\varphi \in \mathbb{K}[Y]$. Then for every $x \in X$ we have*

$$d(f^*\varphi)(x) = f^*(d\varphi(x)). \tag{4.2.31}$$

Proof. Follows from the description of the transpose of the differential in (4.2.30). In fact both left and right hand side are the class of $f^*(\varphi - \varphi(y))$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$.

□

4.3 Smooth points

4.3.1 Stratification by dimensions of the tangent spaces

Proposition 4.3.1. *Let X be a quasi projective variety. The function*

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{N} \\ x & \mapsto & \dim \Theta_x X \end{array} \tag{4.3.1}$$

is Zariski upper-semicontinuous, i.e. for every $k \in \mathbb{N}$

$$X_k := \{x \in X \mid \dim \Theta_x X \geqslant k\}$$

is closed in X.

Proof. Since X has an open affine covering, we may suppose that $X \subset \mathbb{A}^n$ is closed. Let $I(X) = (f_1, \ldots, f_r)$. For $x \in \mathbb{A}^n$ let

$$J(f_1, \ldots, f_s)(x) := \begin{pmatrix} \frac{\partial f_1}{z_1}(x) & \cdots & \frac{\partial f_1}{z_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{z_1}(x) & \cdots & \frac{\partial f_r}{z_n}(x) \end{pmatrix}$$

be the Jacobian matrix of $(f_1, \ldots, f_s)$ at x. By Proposition 4.2.10 we have that

$$X_k = \{x \in X \mid \operatorname{rk} J(f_1, \ldots, f_r)(x) \leqslant n - k\}. \tag{4.3.2}$$

Given multi-indices $I = \{1 \leqslant i_1 < \ldots < i_m \leqslant s\}$ and $J = \{1 \leqslant j_1 < \ldots < j_m \leqslant n\}$ let $J(f_1, \ldots, f_s)(x)_{I,J}$ be the $m \times m$ minor of $J(f_1, \ldots, f_r)(x)$ with rows corresponding to I and columns corresponding to J (if $m > \min\{r, n\}$ we set $J(f_1, \ldots, f_s)(x)_{I,J} = 0$). We may rewrite (4.3.2) as

$$X_k = X \cap V\left(\ldots, \det J(f_1, \ldots, f_r)(x)_{I,J}, \ldots\right)_{|I|=|J|=n-k+1}.$$

It follows that X_k is closed. □

Example 4.3.2. Let $X \subset \mathbb{A}^n$ be a hypersurface. Thus $I(X) = (f)$ where $f \in \mathbb{K}[z_1, \ldots, z_n]$ is a polynomial without multiple factors. Let $a \in X$. The tangent space to X at a is identified with the subspace of $\mathbb{K}^n$ defined by

$$\sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(a) \cdot v_i = 0.$$

Hence

$$\dim \Theta_a X = \begin{cases} n-1 & \text{if } (\frac{\partial f}{\partial z_1}(a), \ldots, \frac{\partial f}{\partial z_n}(a)) \neq 0, \\ n & \text{if } (\frac{\partial f}{\partial z_1}(a), \ldots, \frac{\partial f}{\partial z_n}(a)) = 0. \end{cases}$$

Let us show that

$$X \backslash V\left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right) \tag{4.3.3}$$

is an open *dense* subset of X (it is obviously open, the point is that it is dense), i.e. $\dim \Theta_a X = n-1$ for a in an open dense subset of X.

First assume that X is irreducible. As shown in Example 3.7.2 there exists $i \in \{1, \ldots, n\}$ such that $\partial f / \partial f z_i \neq 0$. Reordering the coordinates, we may assume that $i = n$. Write

$$f = a_0 z_n^d + a_1 z_n^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{K}[z_1, \ldots, z_{n-1}], \quad a_0 \neq 0, \quad d > 0.$$

Thus

$$\frac{\partial f}{z_n} = d a_0 z_n^{d-1} + (d-1) a_1 z_n^{d-2} + \cdots + a_{d-1} \neq 0.$$

The degree in z_n of f is d, i.e. f has degree d as element of $\mathbb{K}[z_1, \ldots, z_{n-1}][z_n]$. On the other hand, $\frac{\partial f}{z_n}$ is non zero and its degree in z_n is strictly smaller than d. Thus $f \nmid \frac{\partial f}{z_n}$, and hence the set in (4.3.3) is dense in X (recall that X is irreducible).

In general, let $f = f_1 \cdot \cdots \cdot f_r$ be the decomposition of f as product of (non associated) prime factors. Let $X_i = V(f_i)$. Then

$$X = X_1 \cup \cdots \cup X_r$$

is the irreducible decomposition of X. As shown above, for each $i \in \{1, \ldots, r\}$

$$X_j \backslash V\left(\frac{\partial f_j}{z_1}, \ldots, \frac{\partial f_j}{z_n}\right) \neq \emptyset.$$

Hence there exists $a \in X_j$ such that $\frac{\partial f_j}{z_h}(a) \neq 0$ for a certain $1 \leqslant h \leqslant n$. We may assume in addition that a does not belong to any other irreducible component of X. It follows that

$$\frac{\partial f}{z_h}(a) = \frac{\partial f_j}{z_h}(a) \cdot \prod_{k \neq j} f_k(a) \neq 0.$$

This proves that the open set in(4.3.3) has non empty intersection with every irreducible component of X, and hence is dense in X.

Notice also that if a belongs to more than one irreducible component of X, then all partial derivatives of f vanish at a. In other words, any point in the open dense subset of points a such that $\dim \Theta_a = n-1$ belongs to a single irreducible component of X.

4.3.2 Smooth points

Definition 4.3.3. Let X be an algebraic variety, and let $x \in X$. Then X is *smooth at* x if $\dim \Theta_x X = \dim_x X$, it is *singular* at x otherwise. The set of smooth points of X is denoted by X^{sm}, and if $X^{\mathrm{sm}} = X$ then X is *smooth.* The set of singular points of X is denoted by $\operatorname{sing} X$.

Example 4.3.4. Affine space is smooth. It follows that every algebraic variety which has an open covering by affine spaces is smooth. Examples of such varieties are Grassmannians.

Example 4.3.5. Let $X \subset \mathbb{A}^n$ be a hypersurface. Since the dimension of X is equal to $n-1$ (see Corollary 3.4.9), the set of smooth points of X is an open dense subset of X by Example 4.3.2. Notice that X may very well have many singular points. For example if Y, W are distinct irreducible components of X then every point of $Y \cap W$ is a singular point. Even an irreducible X may have many singular points, for example Whitneys' umbrella $V(x^2 - yw^2) \subset \mathbb{A}^3$ is singular along the line $V(x, w)$.

Proposition 4.3.6. *Let X be an algebraic variety. Then the following hold:*

1. *The set of smooth points of X contains an open dense subset of X.*
2. *For $x \in X$ we have* $\dim \Theta_x X \geqslant \dim_x X$.

Proof. (1): Suppose first that X is irreducible of dimension d. By Proposition 3.3.12 there is a birational map $g\colon X \dashrightarrow Y$, where $Y \subset \mathbb{A}^{d+1}$ is a hypersurface. By Proposition 3.2.11 there exist open dense subsets $U \subset X$ and $V \subset Y$ such that g is regular on U, and it defines an isomorphism $f\colon U \xrightarrow{\sim} V$. By Example 4.3.5, the set of smooth points Y^{sm} of Y is open and dense in Y. Since V is open and dense in Y the intersection $Y^{\mathrm{sm}} \cap V$ is open and dense dense in Y and hence $f^{-1}(Y^{\mathrm{sm}} \cap V)$ is an open dense subset of X. Since $f^{-1}(Y^{\mathrm{sm}} \cap V)$ is contained in U^{sm}, we have proved that the set of smooth points of X contains an open dense subset of X. We have proved that Item (1) holds if X is irreducible. In general, let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. Let

$$X_j^0 := (X \backslash \bigcup_{i \neq j} X_i) = (X_j \backslash \bigcup_{i \neq j} X_i)$$

By the result that was just proved, $(X_j^0)^{\mathrm{sm}}$ contains an open dense subset of smooth points. Every smooth point of X_j^0 is a smooth point of X, because X_j^0 is open in X. Thus $\bigcup_i (X_i^0)^{\mathrm{sm}}$ is an open dense subset of X, containing an open dense subset of X. This proves Item (1).

(2): Let $x_0 \in X$, and let X_0 be an irreducible component of X containing x_0 such that $\dim X_0 = \dim_{x_0} X$. By Item (1) X_0^{sm} contains an open dense subset of points x such that $\dim \Theta_x X_0 = \dim_x X_0$, and hence by Proposition 4.3.1 we have $\dim \Theta_x X_0 \geqslant \dim_x X_0$ for *all* $x \in X$. In particular $\dim \Theta_{x_0} X_0 \geqslant \dim_{x_0} X_0 = \dim_{x_0} X$. Since $\Theta_{x_0} X_0 \subset \Theta_{x_0} X$, it follows that $\dim \Theta_{x_0} X \geqslant \dim_{x_0} X$. □

4.4 The local ring of a smooth point is an integral domain

4.4.1 The main result

Theorem 4.4.1. *Let X be an algebraic variety, and let x be a smooth point of X. Then the local ring $\mathscr{O}_{X,x}$ is an integral domain.*

The proof of Theorem 4.4.1 is in Subsection 4.4.4. Here we note that $\mathscr{O}_{X,x}$ being an integral domain has a straightfoward geometric translation.

Claim 4.4.2. *Let X be an algebraic variety, and let $x \in X$. The local ring $\mathscr{O}_{X,x}$ is an integral domain if and only if there is only one irreducible component of X containing x.*

Proof. Suppose that there is only one irreducible component of X containing x. Then there is an open irreducible affine subset of X containing x. Hence we may assume that X is affine and irreducible. Thus $\mathscr{O}_{X,x}$ is isomorphic to $\mathbb{K}[X]_{\mathfrak{m}_x}$ by Proposition 4.2.4. Since X is irreducible the ring of regular functions $\mathbb{K}[X]$ is an integral domain. It follows that the localization $\mathbb{K}[X]_{\mathfrak{m}_x}$ is an integral domain. To prove the converse one argues as in the proof of Proposition 1.3.10. □

Next we show that from Theorem 4.4.1 we get a stronger version of Item (1) of Proposition 4.3.6.

Corollary 4.4.3. *Let X be an algebraic variety. Then the set X^{sm} of smooth points of X is an open dense subset of X.*

Proof. Let $X = \bigcup_{i \in I} X_i$ be the irreducible decomposition of X. By Theorem 4.4.1 we have

$$X^{\mathrm{sm}} \subset X \setminus \bigcup_{\substack{i,j \in I \\ i \neq j}} (X_i \cap X_j). \tag{4.4.1}$$

The right hand side of (4.4.1) is an open dense subset of X. Let X_i^0 be an irreducible component of the right hand side of (4.4.1). Thus $X_i^0 \subset X_i$ is the complement of the intersection of X_i with the other irreducible componets of X. The set of smooth points of X_i^0 is non empty by Proposition 4.3.6, and it is open by upper semicontinuity of the dimension of $\Theta_x X$ (Proposition 4.3.1), because $\dim_x X$ is independent of $x \in X_i^0$. Hence X^{sm} is an open dense subset of the open dense subset of X given by the right hand side of (4.4.1), and hence is open and dense in X.

□

4.4.2 Algebraic Implicit Function Theorem

We give an algebraic version of the (analytic) Implicit Function Theorem. The algebraic replacement for the ring of analytic functions defined in a neighborhood of $0 \in \mathbb{A}^n$ is the ring $\mathbb{K}[[z_1, \ldots, z_n]]$ of formal power series in $z_1, \ldots, z_n$ with complex coefficients. We have inclusions

$$\mathbb{K}[z_1, \ldots, z_n] \subset \mathscr{O}_{\mathbb{A}^n,0} \subset \mathbb{K}[[z_1, \ldots, z_n]]. \tag{4.4.2}$$

(The second inclusion is obtained by developing $\frac{f}{g}$ as convergent power series centered at 0, where $f, g \in \mathbb{K}[z_1, \ldots, z_n]$ and $g(0) \neq 0$.) We will need the following elementary results.

Lemma 4.4.4. *Let $\mathfrak{m} \subset \mathbb{K}[z_1, \ldots, z_n]$, $\mathfrak{m}' \subset \mathscr{O}_{\mathbb{A}^n,0}$ and $\mathfrak{m}'' \subset \mathbb{K}[[z_1, \ldots, z_n]]$ be the ideals generated by $z_1, \ldots, z_n$ in the corresponding ring. Then for every $i \geqslant 0$ we have $(\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^i$, and $(\mathfrak{m}')^i \cap \mathbb{K}[z_1, \ldots, z_n] = \mathfrak{m}^i$.*

Proof. By induction on i. For $i = 0$ the statement is trivially true. The proof of the inductive step is the same in both cases. For definiteness let us show that $(\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^{i+1}$, assuming that $(\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0} = (\mathfrak{m}')^i$. The non trivial inclusion is $(\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0} \subset (\mathfrak{m}')^{i+1}$. Assume that $f \in (\mathfrak{m}'')^{i+1} \cap \mathscr{O}_{\mathbb{A}^n,0}$. Then $f \in (\mathfrak{m}'')^i \cap \mathscr{O}_{\mathbb{A}^n,0}$, and hence $f \in (\mathfrak{m}')^i$ by the inductive hypothesis. Thus we may write

$$f = \sum_{|I|} \alpha_J z^J,$$

where the sum is over all multiindices $J = (j_1, \ldots, j_n)$ of weight $|J| = \sum_{s=1}^n j_s = i$, and $\alpha_J \in \mathscr{O}_{\mathbb{A}^n,0}$ for all J. Since $f \in (\mathfrak{m}'')^{i+1}$, we have $\alpha_J(0) = 0$ for all J. It follows that $\alpha_J \in \mathfrak{m}'$ for all J, and hence $f \in (\mathfrak{m}')^{i+1}$. □

Proposition 4.4.5 (Formal Implicit Function Theorem)**.** *Let $\varphi \in \mathbb{K}[[z_1, \ldots, z_n]]$, and suppose that*

$$\varphi = z_1 + \varphi_2 + \ldots + \varphi_d + \ldots, \quad \varphi_d \in \mathbb{K}[z_1, \ldots, z_n]_d. \tag{4.4.3}$$

Given $\alpha \in \mathbb{K}[[z_1, \ldots, z_n]]$, there exists a unique $\beta \in \mathbb{K}[[z_1, \ldots, z_n]]$ such that

$$(\alpha - \beta \cdot \varphi) \in \mathbb{K}[[z_2, \ldots, z_n]]. \tag{4.4.4}$$

Proof. Write $\beta = \beta_0 + \beta_1 + \ldots + \beta_d + \ldots$, where $\beta_d \in \mathbb{K}[z_1, \ldots, z_n]_d$, and the β_d's are the indeterminates. Expand the product $\beta \cdot \varphi$, and solve for β_0 by requiring that $\beta \cdot \varphi$ have the same linear term modulo $z_2, \ldots, z_n$ as α, then solve for β_1 by requiring that $\beta \cdot \varphi$ have the same quadratic term modulo $(z_2, \ldots, z_n)^2$ as α , etc. By (4.4.3) there is one and only one solution at each stage. □

Corollary 4.4.6. *With hypotheses as in Proposition 4.4.6, the natural map*

$$\mathbb{K}[[z_2, \ldots, z_n]] \to \mathbb{K}[[z_1, \ldots, z_n]]/(\varphi)$$

is an isomorphism.

4.4.3 Zero locus of polynomials with linearly independent differentials

Below is the main technical result needed to prove Theorem 4.4.1.

Proposition 4.4.7. *Let $f_1, \ldots, f_k \in \mathbb{K}[z_1, \ldots, z_n]$ and $a \in \mathbb{A}^n$. Suppose that*

(i) each f_i vanishes at a, and

(ii) the differentials $df_1(a), \ldots, df_k(a)$ are linearly independent.

Then $V(f_1, \ldots, f_k) = X \cup Y$, where

1. *X, Y are closed in $\mathbb{A}^n$, $a \in X$, while Y does not contain a;*
2. *X is irreducible of dimension $n-k$, it is smooth at a, and $T_a(X) = \mathrm{Ann}(\langle df_1(a), \ldots, df_k(a)\rangle)$ (as subspace of $T_a\mathbb{A}^n$).*

Moreover, there exists a principal open affine set $\mathbb{A}^n_g$ containing a such that $f_{1|\mathbb{A}^n_g}, \ldots, f_{k|\mathbb{A}^n_g}$ generate the ideal of $X \cap \mathbb{A}^n_g$.

Proof. By changing affine coordinates, if necessary, we may assume that $a = 0$, and that $df_i(0) = z_i$ for $i \in \{1, \ldots, k\}$. Let $J' \subset \mathscr{O}_{\mathbb{A}^n,0}$ be the ideal generated by $f_1, \ldots, f_k$ (to be consistent with our notation, we should write $J' = (\varphi(f_1), \ldots, \varphi(f_k))$), let $J := J' \cap \mathbb{K}[z_1, \ldots, z_n]$, and let $J'' \subset \mathbb{K}[[z_1, \ldots, z_n]]$ be the ideal generated by $f_1, \ldots, f_k$. Lastly, let $I \subset \mathbb{K}[z_1, \ldots, z_n]$ be the ideal generated by $f_1, \ldots, f_k$. We claim that

$$J \cdot g \subset I \subset J. \tag{4.4.5}$$

for a suitable $g \in \mathbb{K}[z_1, \ldots, z_n]$ with $g(0) \neq 0$. In fact, the second inclusion is trivially true. In order to prove the first inclusion, let $h_1, \ldots, h_r$ be generators of the ideal $J \subset \mathbb{K}[z_1, \ldots, z_n]$. By definition of J, there exist $a_i, g_i \in \mathbb{K}[z_1, \ldots, z_n]$, for $i \in \{1, \ldots, r\}$, such that $a_i \in I$, $g_i(0) \neq 0$, and $h_i = \frac{a_i}{g_i}$. Hence the second inclusion in (4.4.5) holds with $g = g_1 \cdot \ldots \cdot g_r$. This proves (4.4.5), and hence we have $V(J) \subset V(I) \subset (V(J) \cup V(g))$. It follows that, letting $X := V(J)$, there exists a closed $Y \subset V(g)$ such that

$$V(f_1, \ldots, f_k) = X \cup Y, \quad 0 \notin Y. \tag{4.4.6}$$

Let us prove that J is a prime ideal, so that in particular X is irreducible. First, we claim that

$$J'' \cap \mathscr{O}_{\mathbb{A}^n,0} = J'. \tag{4.4.7}$$

The non trivial inclusion to be proved is $J'' \cap \mathscr{O}_{\mathbb{A}^n,0} \subset J'$. Let $f \in J'' \cap \mathscr{O}_{\mathbb{A}^n,0}$. Then there exist $\alpha_1, \ldots, \alpha_k \in \mathbb{K}[[z_1, \ldots, z_n]]$ such that $f = \sum_{j=1}^k \alpha_j f_j$. Given $s \in \mathbb{N}$, let α_j^s be the MacLaurin polynomial of α_j of degree s, i.e. such that $(\alpha_j - \alpha_j^s) \in (\mathfrak{m}'')^{s+1}$, where $\mathfrak{m}''$ is as in Lemma 4.4.4. Then

$$f = \sum_{j=1}^k \alpha_j^{(s)} f_j + \sum_{j=1}^k (\alpha_j - \alpha_j^s) f_j.$$

Both addends are in $\mathscr{O}_{\mathbb{A}^n,0}$. In addition, the first addend belongs to J', and the second one belongs to $(\mathfrak{m}'')^{s+1}$. By Lemma 4.4.4, it follows that the second one belongs to $(\mathfrak{m}')^{s+1}$. Hence $f \in \bigcap_{s=0}^{\infty}(I' + (\mathfrak{m}')^{s+1})$. By Corollary A.10.2, it follows that $f \in I'$. This proves (4.4.7). By (4.4.7) and the definition of J, we have an inclusion

$$\mathbb{K}[z_1, \ldots, z_n]/J \subset \mathbb{K}[[z_1, \ldots, z_n]]/J''.$$

Hence, in order to prove that J is prime, it suffices to show that $\mathbb{K}[[z_1,\ldots,z_n]]/J''$ is an integral domain. In fact we will see that the natural map

$$\mathbb{K}[z_{k+1},\ldots,z_n] \longrightarrow \mathbb{K}[[z_1,\ldots,z_n]]/J'' \tag{4.4.8}$$

is an isomorphism of rings. This follows from the algebraic version of the Implicit Function Theorem, i.e. Proposition 4.4.6. In fact, by Proposition 4.4.6, the natural map $\mathbb{K}[[z_2,\ldots,z_n]] \to \mathbb{K}[[z_1,\ldots,z_n]]/(f_1)$ is an isomorphism. Let $i \in \{2,\ldots,k\}$. Given the identification $\mathbb{K}[[z_1,\ldots,z_n]]/(f_1) = \mathbb{K}[[z_2,\ldots,z_n]]$, the image of f_i under the quotient map $\mathbb{K}[[z_1,\ldots,z_n]] \to \mathbb{K}[[z_1,\ldots,z_n]]/(f_1)$ is an element $z_i + f_i'$, where $f_i' \in (\mathfrak{m}'')^2$ (notation as in Lemma 4.4.4). Iterating, we get that the map in (4.4.8) is an isomorphism of rings. As explained above, this proves that J is a prime ideal. In particular X is irreducible. Moreover, since $z_{k+1},\ldots,z_n \in \mathbb{K}[X]$, the isomorphism in (4.4.8) shows that $\mathbb{K}(X)$ has transcendence degree $n-k$, i.e. X has dimension $n-k$. Since $f_1,\ldots,f_k$ vanish on X, and their differentials are linearly independent, it follows that $\dim\Theta_0(X) \leqslant (n-k) = \dim_0 X$. Hence $\dim\Theta_0(X) = (n-k) = \dim_0 X$, by Item (2) of Proposition 4.3.6, i.e. X is smooth at 0, and $\Theta_0(X) \subset \Theta_0\mathbb{A}^n$ is the annihilator of $df_1(0),\ldots,df_k(0)$. This proves Items (1) and (2). The last statement in the proposition holds with the polynomial g appearing in (4.4.5). □

4.4.4 Proof that the local ring at a smooth point is an integral domain

First we prove the following result.

Proposition 4.4.8. *Let $X \subset \mathbb{A}^n$ be a Zariski closed subset. Let a be a smooth point of X, and let $k = n - \dim_a X$. Then following hold:*

1. *Only one irreducible component of X contains a.*

2. *There exist $f_1,\ldots,f_k \in \mathbb{K}[z_1,\ldots,z_n]$ with linerly independent differentials $df_1(a),\ldots,df_k(a)$, and a Zariski open affine subset $U \subset \mathbb{A}^n$ containing a, such that $I(X \cap U) = (f_{1|U},\ldots,f_{k|U})$;*

Proof. Since X is smooth at a, and $\dim_a X = n-k$, there exist $f_1,\ldots,f_k \in I(X)$ such that $df_1(a),\ldots,df_k(a)$ are linearly independent. Of course $X \subset V(f_1,\ldots,f_k)$. By Proposition 4.4.7 only one irreducible component of $V(f_1,\ldots,f_k)$ contains a, call it Y. Moreover $\dim Y = n-k$ by the same proposition. Every irreducible component of X containing a is contained in Y. Since $\dim_a X = n-k$, there exists (at least) one irreducible component of X containing a of dimension $n-k$. Let X_0 be such an irreducible component. By Proposition 3.4.8 we get that $X_0 = Y$. It follows that there is only one irreducible component of X containing a, and it is equal to the unique irreducible component of $V(f_1,\ldots,f_k)$ containing a. This prove Item (1). Item (2) follows from the last statement of Proposition 4.4.7. □

Theorem 4.4.1 follows at once from Proposition 4.4.8 because every point of an algebraic variety is contained in an affine open subset.

4.4.5 Smooth complex algebraic varieties and complex manifolds

4.5 Sard's Theorem for algebraic varieties in characteristic 0

4.6 Inverse function Theorem

4.6.1 Finite maps

Let $\varphi\colon A \to B$ be a homomorphism of rings. By setting $a \cdot b := \varphi(a)b$ we equip B with a structure of A-module: we say that B is *finite over A* if it is a finitely generated A-module. Let X, Y be affine varieties, and let $f\colon X \to Y$ be a regular map; the pull back $f^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ is a homomorphism of rings, hence (with f understood) it makes sense to state that $\mathbb{K}[X]$ is finite over $\mathbb{K}[Y]$.

Lemma 4.6.1. *Let $f\colon X \to Y$ be a projective map of quasi projective varieties. Let $y_0 \in Y$ and suppose that $f^{-1}(y_0)$ is finite. There exists an open affine $Y_0 \subset Y$ containing y_0 such that $X_0 := f^{-1}(Y_0)$ is affine and $\mathbb{K}[X_0]$ is finite over $\mathbb{K}[Y_0]$.*

Proof. By Definition 5.2.4 we may assume that $X \subset \mathbb{P}^n \times Y$ is closed and f is the restriction of the projection $\pi\colon \mathbb{P}^n \times Y \to Y$. Since $X \cap (\mathbb{P}^n \times y_0)$ is finite there exists homogeneous coordinates $[Z_0, \ldots, Z_n]$ on $\mathbb{P}^n$ such that $X \cap (V(Z_0) \times \{y_0\}) = \emptyset$. The intersection $X \cap (V(Z_0) \times Y)$ is a closed subset of $\mathbb{P}^n \times Y$. By Elimination Theory (i.e. Theorem 2.4.2) $C := \pi(X \cap (V(Z_0) \times Y))$ is closed in Y. Hence $(Y \backslash C)$ is an open subset of Y containing y_0. Let $Y_* \subset (Y \backslash C)$ be an open affine subset containing y_0. Then $X_* := X \cap (\mathbb{P}^n \times Y_*) = f^{-1}(Y_*)$ is a closed subset of the affine set $\mathbb{P}^n_{Z_0} \times Y_*$ and hence is affine. It remains to prove that $\mathbb{K}[X_*]$ is finite over $\mathbb{K}[Y_*]$. The proof is by induction on n. If $n = 0$ then $\mathbb{K}[X_*] = \mathbb{K}[Y_*]$ and there is nothing to prove. Let's prove the inductive step. Since X_* is closed in $\mathbb{P}^n \times Y_*$ there exist $F_i \in \mathbb{K}[X_*][Z_0, \ldots, Z_n]_{d_i}$ for $i = 1, \ldots, r$ such that

$$X_* = V(F_1, \ldots, F_r).$$

(See Claim 2.3.27.) Since $X_* \cap (V(Z_0) \times \{y_0\})$ is empty we have

$$V(F_1(y_0)(0, Z_1, \ldots, Z_n), \ldots, F_r(y_0)(0, Z_1, \ldots, Z_n)) = \emptyset.$$

By Hilbert's Nullstellensatz, there exists $M > 0$ such that

$$(Z_1, \ldots, Z_n)^M \subset (F_1(y_0)(0, Z_1, \ldots, Z_n), \ldots, F_r(y_0)(0, Z_1, \ldots, Z_n)).$$

It follows (see the proof of Theorem 2.4.2) that, shrinking Y_* around y_0, we may assume that

$$Z_1^M, \ldots, Z_n^M \in (F_1(0, Z_1, \ldots, Z_n), \ldots, F_r(0, Z_1, \ldots, Z_n)). \tag{4.6.1}$$

(Actually we may arrange so that (4.6.1) holds for the original Y_* - but we do not need this). Equation (4.6.1) gives that there exists

$$G = (Z_n^M + A_1 Z_n^{M-1} + \ldots + A_M) \in (F_1, \ldots, F_r), \qquad A_i \in \mathbb{K}[Y_*][Z_0, \ldots, Z_{n-1}]_i.$$

Thus $G|_{X_*} = 0$: dividing by Z_0^M and setting $z_i := Z_i/Z_0$, $a_i = A_i/Z_0^i \in \mathbb{C}[z_1, \ldots, z_{n-1}]$ we get that

$$(z_n^M + a_1 z_n^{M-1} + \ldots + a_M)|_{X_*} = 0. \tag{4.6.2}$$

Let $Q := [0, \ldots, 0, 1] \in \mathbb{P}^n$. The product of projection from Q and Id_{Y_*}

$$\begin{array}{ccc} (\mathbb{P}^n \backslash \{P\}) \times Y_* & \overset{\rho}{\longrightarrow} & \mathbb{P}^{n-1} \times Y_* \\ ([Z_0, \ldots, Z_n], p) & \mapsto & ([Z_0, \ldots, Z_{n-1}], p) \end{array}$$

is *not* projective but the restriction of ρ to X_* is projective. In fact locally over open sets of a covering $\bigcup_{j \in J} U_j$ of Y_* we may embed X_* as a closed subset of $\mathbb{P}^1 \times U_j$ so that ρ is the restriction of the projection $(\mathbb{P}^1 \times U_j) \to U_j$. Thus the image $\rho(X_*)$ is a closed subset of $\mathbb{P}^{n-1} \times Z_*$. Since the fiber of $\rho(X_*) \to Y_*$ over y_0 is finite we may assume (possibly after shrinking Y_* and X_*) that $\rho(X_*)$ is affine (we just proved it). The ring $\mathbb{K}[X_*]$ is obtained from $\mathbb{K}[\rho(X_*)]$ by adding z_n. Equation (4.6.2) gives that $\mathbb{K}[X_*]$ is finite over $\mathbb{K}[\rho(X_*)]$. By the inductive hypothesis $\mathbb{K}[\rho(X_*)]$ is finite over $\mathbb{K}[Y_*]$ (possibly after shrinking $\mathbb{K}[Y_*]$): it follows that $\mathbb{K}[X_*]$ is finite over $\mathbb{K}[Y_*]$. □

4.6.2 Proof of the algebraic inverse function Theorem

We prove the following analogue, in the category of algebraic varieties, of the local invertibility results valid for C^∞ or holomorphic maps.

Theorem 4.6.2. *Let $f\colon X \to Y$ be a projective map of algebraic varieties. Let $p \in X$ and suppose that the following hold:*

1. $f^{-1}(f(p)) = \{p\}$.

2. $df(p)\colon \Theta_p X \to \Theta_{f(p)} Y$ *is injective.*

Then there exists an open $U \subset Y$ containing $f(p)$ such that the restriction of f to $f^{-1}(U)$ is an isomorphism to a closed subset of U.

Proof. Since f is projective it has closed image: thus we may assume that f is surjective. By Lemma 4.6.1 we may assume that X and Y are affine and that $\mathbb{K}[X]$ is finite over $\mathbb{K}[Y]$. By surjectivity of f the pull-back defines an inclusion $f^*\colon \mathbb{K}[Y] \hookrightarrow \mathbb{K}[X]$. We will prove that there exists an open affine $\mathscr{U} \subset Y$ containing q such that $f^*|_{\mathscr{U}}\colon \mathbb{K}[\mathscr{U}] \hookrightarrow \mathbb{K}[f^{-1}\mathscr{U}]$ is surjective: that will give that $f|_{\mathscr{U}}\colon f^{-1}\mathscr{U} \to \mathscr{U}$ is an isomorphism. Let $q := f(p)$. By Item (1) and the Nullstellensatz we have

$$\mathfrak{m}_p = \sqrt{f^*\mathfrak{m}_q\mathbb{K}[X]}. \tag{4.6.3}$$

Here $f^*\mathfrak{m}_q\mathbb{K}[X]$ is the ideal of $\mathbb{K}[X]$ generated by $f^*\phi$ for $\psi \in \mathfrak{m}_q$ (we will use similar notation in the course of the proof). Let $\mathfrak{m}_p = (\phi_1, \ldots, \phi_n)$. Item (2) gives that for each $1 \leqslant i \leqslant n$ there exist an affine open U_i containing p and $\psi_i \in \mathbb{K}[Y]$ such that $(\phi_i - f^*\psi_i)|_{U_i} \in \mathfrak{m}_p^2\mathbb{K}[U_i]$. Since f is closed it follows that there exists a principal open affine Y_h neighborhood of q (thus $h \in \mathbb{K}[Y]$ with $h(q) \neq 0$) such that

$$(\phi_i - f^*\psi_i)|_{f^{-1}(Y_h)} \in \mathfrak{m}_p^2\mathbb{K}[f^{-1}(Y_h)] \quad \forall 1 \leqslant i \leqslant n. \tag{4.6.4}$$

Let's prove by "descending induction" on k that

$$\mathfrak{m}_p^k\mathbb{K}[f^{-1}(Y_h)] \subset f^*\mathfrak{m}_q\mathbb{K}[f^{-1}(Y_h)] \quad \forall 1 \leqslant k. \tag{4.6.5}$$

By (4.6.3) there exists $N > 0$ such that (4.6.5) holds for $k \geqslant N$. Let's prove the "inductive step": we assume that (4.6.5) holds with $k \geqslant 2$ and we prove that it holds with k replaced by $(k-1)$. Let

$$\varphi = \sum_{|L|=k-1} c_L\phi_1^{l_1}\ldots\phi_n^{l_n} \in \mathfrak{m}_p^{k-1}\mathbb{K}[f^{-1}(Y_h)]. \tag{4.6.6}$$

By (4.6.4) we may write $\phi_i = f^*\psi_i + \epsilon_i$ where $\epsilon_i \in \mathfrak{m}_p^2\mathbb{K}[f^{-1}(Y_h)]$ for $i = 1, \ldots, n$: substituting in (4.6.6) and invoking the inductive hypothesis we get that $\varphi \in f^*\mathfrak{m}_q\mathbb{K}[f^{-1}(Y_h)]$. We have proved (4.6.5). Since $\mathbb{K}[f^{-1}(Y_h)] = \mathbb{K}[Y]_{(f^*h^s)}$ (the localization of $\mathbb{K}[Y]$ with respect to the multiplicative system of powers of f^*h) we get that

$$I_p := \{\varphi \in \mathbb{K}[f^{-1}(Y_h)] \mid \varphi(p) = 0\} = f^*\mathfrak{m}_q\mathbb{K}[f^{-1}(Y_h)]. \tag{4.6.7}$$

Now notice that $\mathbb{K}[f^{-1}(Y_h)]$ is a finite $\mathbb{K}[Y_h]$-module because $\mathbb{K}[f^{-1}Y]$ is a finite $\mathbb{K}[Y]$-module. We will apply Nakayama's Lemma to the finitely generated $\mathbb{K}[Y_h]$-module

$$M := \mathbb{K}[f^{-1}(Y_h)]/f^*\mathbb{K}[Y_h]$$

and the ideal $\mathfrak{m}_q$. We claim that $M \subset \mathfrak{m}_q M$. In fact since $\mathbb{K} \subset f^*\mathbb{K}[Y_h]$ every element of M is represented by $\alpha \in I_p$ (notation as in (4.6.7)) and $\overline{\alpha} \in \mathfrak{m}_q M$ by (4.6.5). By Lemma A.9.2 there exists $\varphi \in \mathfrak{m}_q$ such that

$$(1+\varphi)\mathbb{K}[f^{-1}Y_h] \subset f^*\mathbb{K}[Y_h]. \tag{4.6.8}$$

The open affine $Y_{h(1+\varphi)} \subset Y$ contains q (because $\varphi(q) = 0$). By (4.6.8) we get that

$$\mathbb{K}[f^{-1}Y_{h(1+\varphi)}] = f^*\mathbb{K}[Y_{h(1+\varphi)}].$$

□

Example 4.6.3. Suppose that $X \subset \mathbb{P}^n$ is closed irreducible and $r \in (\mathbb{P}^n \backslash X)$. Let $H \subset \mathbb{P}^n$ be a hyperplane not containing r. Projection

$$\begin{array}{ccc} X & \xrightarrow{\pi} & H \\ p & \mapsto & \langle p, r \rangle \cap H \end{array}$$

is a projective map with finite fibers. Let $p \in X$ and suppose that the projective tangent space $\mathbf{T}_p X$ does *not* contain the line $\langle r, p \rangle$: then $df(p)$ is injective. Suppose in addition that $\pi^{-1}(\pi(p)) = \{p\}$: by Theorem 4.6.2 we get that π is birational onto its image. As long as $\dim \Theta_p(X) < n$, and X has codimension at least 2, there exists a point r such that the two conditions above hold. Iterating we get that if $\dim X = m$ we can choose a projection from a linear space of dimension $(n - m - 2)$ giving a birational map from $\varphi \colon X \to Y$ where $Y \subset \mathbb{P}^{m+1}$ is a hypersurface, and such that φ restricts to an isomorphism from a neighborood of p to a neighborhood of $\varphi(p)$.

4.7 The local ring of a smooth point is a unique factorization domain

4.7.1 Unique factorization of the local ring and codimension 1 closed subsets

4.7.2 Proof of unique factorization

4.7.3 Line bundles and divisors on locally factorial varieties

4.8 Tangent and cotangent bundle