An introduction to Algebraic Geometry - Varieties

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21st April 2024

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Chapter 0

Introduction

Motivation

We will describe some problems and results in order to whet your appetite. Some (or most) of the statements below might leave you puzzled, do not worry, they will become clear later on. In fact one of the goals of reading the book is to be able to understand what is written in the paragraphs below.

We start from the following well known indefinite integral:

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x.$$

What if we ask

$$\int \frac{dx}{\sqrt{1-x^3}} = ?$$

Note that one gets the first integral by writing out the formula for the length of arcs of a circle. Similarly, one gets the second integral, or more generally integrals of functions $p(x)^{-1/2}$, where p is a polynomial of degree 3 (or 4), if one sets out to compute the length of arcs of ellipses. There is no way to express the second integral starting from elementary functions. What Fagnano discovered for similar integrals, and what Euler amplified, is that, although we cannot express the integral via elementary functions, there is a rational addition formula, i.e. there exists a rational function F of four variables such that for fixed l_0 and varying a, b we have

$$\int_{l_0}^{a} \frac{dx}{\sqrt{1-x^3}} + \int_{l_0}^{b} \frac{dx}{\sqrt{1-x^3}} = \int_{l_0}^{c} \frac{dx}{\sqrt{1-x^3}} + \text{const},$$

where

$$c = F(a, b, \sqrt{1-a^3}, \sqrt{1-b^3}).$$

Let us sketch a geometric explanation of the addition formula. First of all it is convenient to allow x, y to be complex numbers. Since couples $(x, \sqrt{1-x^3})$ are solutions of the equation $x^3 + y^2 = 1$, we consider the curve $C_0 \subset \mathbb{A}^2(\mathbb{C})$ whose equation is $x^3 + y^2 = 1$, where $\mathbb{A}^2(\mathbb{C}) = \mathbb{C}^2$ is the standard complex affine plane. Now C_0 is a complex submanifold of $\mathbb{A}^2(\mathbb{C})$, hence a 1-dimensional complex manifold. Since it is not compact, we consider its closure $C \subset \mathbb{P}^2(\mathbb{C})$ in the projective complex plane. This means adding a single point "at infinity", namely $[0, 0, 1]$ (we let $[T, X, Y]$ be homogeneous coordinates, and $x = X/T$, $y = Y/T$). Note that by integrating the 1-form dx/y on C (as we will do) we do not have to pay attention to which of the two square roots of $1 - x^3$ we choose. A fundamental observation is that dx/y is holomorphic on all of C_0, including the points $(e^{2\pi m i/3}, 0)$ where the denominator vanishes), and moreover it extends to a holomorphic 1-form on all of C. In order to show that there is an addition formula we fix a line $R_0 \subset \mathbb{P}^2(\mathbb{C})$ intersecting C in 3 points $\overline{p}_1, \overline{p}_2, \overline{p}_3$ and, given another line R intersecting C in 3 points p_1, p_2, p_3, we let

$$\int_{R_0}^{R} \frac{dx}{y} := \int_{\overline{p}_1}^{p_1} \frac{dx}{y} + \int_{\overline{p}_2}^{p_2} \frac{dx}{y} + \int_{\overline{p}_3}^{p_3} \frac{dx}{y}.$$

Of course in order to make sense of the right hand side one needs to choose paths starting at $\overline{p}_i$ and ending at p_i for $i \in \{1, 2, 3\}$. By Goursat's Theorem the integrals do not vary if the paths are homotopically equivalent. Hence if we let R move in a small open subset of $\mathbb{P}^2(\mathbb{C})^\vee$ we may choose well defined homotopy classes of such paths and the integral above defines a well defined holomorphic function on the open set. There is no way to define a holomorphic function

$$R \overset{\Phi}{\mapsto} \int_{R_0}^{R} \frac{dx}{y}.$$

on all of $\mathbb{P}^2(\mathbb{C})^\vee$: if we define it locally and then we move around, when we come back the value of the function will change by an additive constant. Since it changes by an additive constant, the differential $d\Phi$ is a well defined holomorphic 1-form ω on all of $\mathbb{P}^2(\mathbb{C})^\vee$ although Φ is only well defined locally. Since every holomorphic 1-form on a complex projective space is zero, we get that $\omega = 0$, i.e. the (locally defined) function Φ is constant. Now notice that the given points $p_1, p_2 \in C$ there is a unique line R containing p_1, p_2 (if $p_1 = p_2$ we let R be the tangent to C at p_1), and that the coordinates of the third point of intersection of R and C, i.e. p_3, are rational functions of the coordinates of the first two points. This gives the validity of the formula

$$\int_{l_0}^{a} \frac{dx}{\sqrt{1-x^3}} + \int_{l_0}^{b} \frac{dx}{\sqrt{1-x^3}} = -\int_{l_0}^{c} \frac{dx}{\sqrt{1-x^3}} + \text{const},$$

where c is a rational function of $(a, b, \sqrt{1-a^3}, \sqrt{1-b^3})$. With a little more work one gets from this the addition formula as formulated above.

Next we ask more in general what can be said about integrals of the form

$$\int \frac{dx}{\sqrt{D(x)}}, \tag{0.0.1}$$

where $D(x)$ is a polynomial. For simplicity we assume that $D(x)$ has no multiple roots. If $D(x)$ has degree 3, then the arguments above apply verbatim to give an addition formula. In general, the first step is to consider the curve $C_0 \subset \mathbb{A}^2(\mathbb{C})$ whose equation is $y^2 = D(x)$. This is a 1-dimensional complex submanifold of $\mathbb{A}^2(\mathbb{C})$. Since it is not compact it is convenient to compactify. The closure of C_0 in $\mathbb{P}^2(\mathbb{C})$ is compact, but if the degree of $D(x)$ is greater than 3 then the closure of C_0 is not a submanifold of $\mathbb{P}^2(\mathbb{C})$ at its unique "point at infinity"(i.e. $[0, 0, 1]$). Nonetheless there is 1-dimensional complex manifold C containing C_0 as an open dense subset, in fact $C \backslash C_0$ consists of a single point if $D(x)$ has odd degree, and consists of two points if $D(x)$ has even degree. The qualitative behaviour of the integral that we set out to study is determined by the topology of C. The C^∞ manifold underlying C is connected, compact and orientable surface. By the classification compact surfaces it is homeomorphic to a connected sum of g tori. In fact one show that

$$g = \left\lfloor \frac{\deg D - 1}{2} \right\rfloor. \tag{0.0.2}$$

For example, if D has degree 3 then $g = 1$, i.e. C is a torus. Suppose that $g > 1$. Then there exists an addition formula, but it involves the addition of vectors in $\mathbb{C}^g$ obtained by integrating the g linearly independent holomorphic 1-forms

$$\frac{dx}{y}, \frac{xdx}{y}, \ldots, \frac{x^{g-1}dx}{y}. \tag{0.0.3}$$

Lastly we discuss how the topological quantity g (the genus of C) controls the arithmetic of C. Suppose that the polynomial $p(x)$ has integer coefficients. If p is a prime we let $\overline{D}(x) \in \mathbb{F}_p[x]$ be the polynomial whose coefficients are the equivalence classes of the coefficients of D - we say that $\overline{D}(x)$ is obtained from D reducing modulo p. We suppose that $\overline{D}(x)$ has the same degree as D (i.e. p does not divide the leading coefficient of D), and that $\overline{D}(x)$ does not have multiple roots in the algebraic closure of $\mathbb{F}_p$. We also assume that $p \neq 2$. For $n \geqslant 1$ let $\mathbb{F}_{p^n}$ be the finite field of cardinality p^n, and let $C(\mathbb{F}_{p^n})$ be the set of solutions in $\mathbb{F}_{p^n}$ of the equation $y^2 = \overline{D}(x)$. We view the points at infinity (there is one if $\deg D$ is odd and two if $\deg D$ is even) as solutions "in $\mathbb{F}_{p^n}$". A convenient generating function for the cardinalities $|C(\mathbb{F}_{p^n})|$ is given by Weil's zeta function

$$Z(C,T) \coloneqq \exp\left(\sum_{n=1}^{\infty} \frac{|C(\mathbb{F}_{p^n})|}{n} T^n\right). \tag{0.0.4}$$

A famous theorem of Weil states that

$$Z(C,T) = \frac{\prod_{i=1}^{2g}(1-a_iT)}{(1-T)(1-pT)}, \tag{0.0.5}$$

where each a_i is an algebraic integer of modulus $p^{1/2}$ (the last statement is an analogue of Riemann's hypothesis). This shows that the topological genus g can be extracted from the number of solutions $(x,y) \in \mathbb{A}^2(\mathbb{F}_{p^n})$ of the equation $y^2 = \overline{D}(x)$. We also see that there is an explicit formula giving the cardinality $|C(\mathbb{F}_{p^n})|$ for all n once we know the cardinalities $|C(\mathbb{F}_p)|, |C(\mathbb{F}_{p^2})|, \ldots, |C(\mathbb{F}_{p^{2g}})|$. The function of s obtained by making the substitution $T = p^{-s}$, i.e. $Z(C, p^{-s})$, is a precise analogue of Riemann's zeta function $\zeta(s)$, and the statement that each a_i has modulus $p^{1/2}$ is the analogue of the Riemann Hypothesis. It is very compelling evidence in favour of the validity of the Riemann Hypothesis.

Chapter 1

Quasi projective varieties

Throughout the book $\mathbb{K}$ is an algebraically closed field, e.g. $\mathbb{K} = \mathbb{C}$ or $\overline{\mathbb{Q}}$, the algebraic closure of the rational field $\mathbb{Q}$, or $\overline{\mathbb{F}_p}$, the algebraic closure of the finite field $\mathbb{F}_p$ where p is a prime. We are interested in understanding the set of solutions $(z_1, \ldots, z_n) \in \mathbb{K}^n$ of a family of polynomial equations

$$f_1(z_1, \ldots, z_n) = 0, \ldots, f_r(z_1, \ldots, z_n) = 0.$$

"Polynomial equations" means each f_i is an element of the polynomial ring $\mathbb{K}[z_1, \ldots, z_n]$.

In order to understand the geometry of a set of solutions of polynomial equations, it is convenient to replace affine space $\mathbb{A}^n(\mathbb{K})$ by projective space $\mathbb{P}^n(\mathbb{K})$, and consider the set of points in $\mathbb{P}^n(\mathbb{K})$ which are solutions of homogeneous polynomial equations in the homogeneous coordinates. As motivation for this step we recall that results in projective geometry are usually cleaner than in affine geometry - for example two distinct lines in a projective plane have exactly one point of intersection, while two distinct lines in an affine line may intersect in one point or be disjoint. If $\mathbb{K} = \mathbb{C}$ we may guess that passing to projective space makes life simpler because $\mathbb{P}^n(\mathbb{C})$ with the classical topology is compact, while $\mathbb{A}^n(\mathbb{C})$ is not (unless $n = 0$).

Whenever there is no possibility of a misunderstanding we omit $\mathbb{K}$ from the notation for affine and projective space, i.e. $\mathbb{A}^n$ is $\mathbb{A}^n(\mathbb{K})$ and $\mathbb{P}^n$ is $\mathbb{P}^n(\mathbb{K})$.

1.1 Zariski's topology on affine space

If $f_1, \ldots, f_r \in \mathbb{K}[z_1, \ldots, z_n]$, we let

$$V(f_1, \ldots, f_r) := \{z \in \mathbb{A}^n \mid f_i(z) = 0 \ \ \forall\, i \in \{1, \ldots, r\}\}. \tag{1.1.1}$$

More generally, if $I \subset \mathbb{K}[z_1, \ldots, z_n]$ is an ideal (note: the inclusion sign $\subset$ does not mean strict inclusion, and similarly for $\supset$) we let

$$V(I) := \{z \in \mathbb{A}^n \mid f(z) = 0 \ \ \forall\, f \in I\}. \tag{1.1.2}$$

Unless $n = 0$ or $I = 0$ an ideal I of $\mathbb{K}[z_1, \ldots, z_n]$ has an infinite number of elements so that $V(I)$ is the set of solutions of an infinite set of polynomial equations. However I has a finite set of generators $f_1, \ldots, f_r$ by Hilbert's basis Theorem A.3.6, and it follows that $V(I) = V(f_1, \ldots, f_r)$. In fact it is clear that $V(I) \subset V(f_1, \ldots, f_r)$. For the reverse inclusion $V(f_1, \ldots, f_r) \subset V(I)$ notice that if $z \in V(f_1, \ldots, f_r)$ and $f \in I$, then $f = \sum_{i=1}^{r} g_i f_i$ for suitable $g_1, \ldots, g_r \in \mathbb{K}[z_1, \ldots, z_n]$ and hence $f(z) = \sum_{i=1}^{r} g_i(z) f_i(z) = 0$.

An elementary observation is that passing from ideals to their zero sets reverses inclusion, i.e. if $I, J \subset \mathbb{K}[z_1, \ldots, z_n]$ are ideals then

$$I \subset J \text{ implies that } V(I) \supset V(J). \tag{1.1.3}$$

Proposition 1.1.1. *The collection of subsets $V(I) \subset \mathbb{A}^n$, where I runs through the collection of ideals of $\mathbb{K}[z_1, \ldots, z_n]$, satisfies the axioms for the closed subsets of a topological space.*

Proof. We have $\varnothing = V((1))$, $\mathbb{A}^n = V((0))$.

Let $I, J \subset \mathbb{K}[z_1, \ldots, z_n]$ be ideals. We claim that $V(I) \cup V(J) = V(I \cap J)$. We have $V(I), V(J) \subset V(I \cap J)$, because $I, J \supset I \cap J$. Thus $V(I) \cup V(J) \subset V(I \cap J)$. Hence it suffices to show that if $z \in V(I \cap J)$ and $z \notin V(I)$, then $z \in V(J)$. Since $x \notin V(I)$, there exists $f \in I$ such that $f(z) \neq 0$. If $g \in J$, then $f \cdot g \in I \cap J$, and thus $(f \cdot g)(z) = 0$ because $z \in V(I \cap J)$. Since $f(z) \neq 0$, it follows that $g(z) = 0$. This proves that $z \in V(J)$.

Lastly, let $\{I_t\}_{t \in T}$ be a family of ideals of $\mathbb{K}[z_1, \ldots, z_n]$. Then

$$\bigcap_{t \in T} V(I_t) = V(\langle \{I_t\}_{t \in T} \rangle),$$

where $\langle \{I_t\}_{t \in T} \rangle$ is the ideal generated by the collection of the I_t's. □

Definition 1.1.2. The *Zariski topology* of $\mathbb{A}^n$ is the topology whose closed sets are the sets $V(I)$, where I runs through the collection of ideals of $\mathbb{K}[z_1, \ldots, z_n]$. The Zariski topology of a subset $A \subset \mathbb{A}^n$ is the topology induced by the Zariski topology of $\mathbb{A}^n$.

Remark 1.1.3. If $\mathbb{K} = \mathbb{C}$, the Zariski topology is weaker than the classical topology of $\mathbb{A}^n$. In fact, unless $n = 0$, the Zariski is much weaker than the classical topology, in particular it is *not* Hausdorff.

Example 1.1.4. A subset $X \subset \mathbb{A}^n$ is a *hypersurface* if it is equal to $V(f)$, where f is a non constant homogeneous polynomial.

A picture of a hypersurface in $\mathbb{A}^2$ is in Figure 1.1. Notice that (x, y) are the affine coordinates - in general, whenever we consider affine or projective space of small dimension, we will denore affine or homogeneous coordinates by letters $x, y, z, \ldots$ and $X, Y, Z, \ldots$ respectively.

What is the field $\mathbb{K}$? The picture shows points with real coordinates. We can view the picture as a "slice" of the corresponding hypersurface over $\mathbb{C}$, or as the closure (either in the Zariski or the classical topology) of the corresponding hypersurface over the algebriac closure of the rationals $\overline{\mathbb{Q}}$.

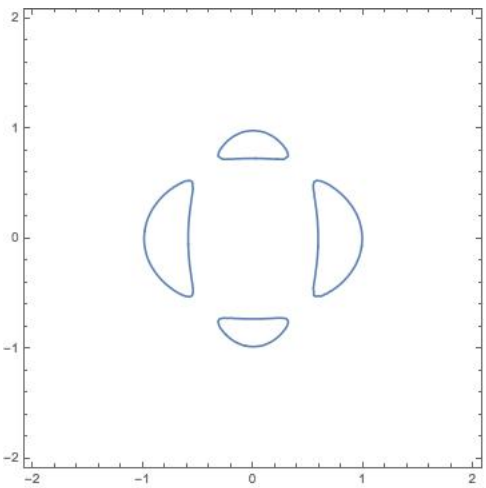


Figure 1.1: $(x^2 + 2y^2 - 1)(3x^2 + y^2 - 1) + \frac{3}{100} = 0$

Given a subset $X \subset \mathbb{A}^n$, let

$$I(X) := \{f \in \mathbb{K}[z_1, \ldots, z_n] \mid f(z) = 0 \text{ for all } z \in X\}. \tag{1.1.4}$$

Clearly $I(X)$ is an ideal of $\mathbb{K}[z_1, \ldots, z_n]$ and X is contained in the closed set $V(I(X))$. Moreover $V(I(X))$ is the closure of X in the Zariski topology. In fact suppose that $V(J) \subset \mathbb{A}^n$ is a closed subset containing X. Then $f(z) = 0$ for all $f \in J$ and $z \in X$, and hence $J \subset I(X)$. This shows that $V(J) \supset V(I(X))$ (recall (1.1.3)).

Remark 1.1.5. Let $\mathscr{A}$ be a finite dimensional affine space over $\mathbb{K}$ of dimension n. Then the Zariski topology on $\mathscr{A}$ may be defined by analogy with the case of $\mathbb{A}^n$, simply replacing $\mathbb{K}[z_1, \ldots, z_n]$ by the $\mathbb{K}$ algebra of polynomial functions on $\mathscr{A}$ (which is isomorphic to $\mathbb{K}[z_1, \ldots, z_n]$). Another way of putting it is that an affine transformation of $\mathbb{A}^n$ is a homemorphism for the Zariski topology.

1.2 Zariski's topology on projective space

Let $F \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ be homogeneous of degree d (to be correct we should say that F belongs to the homogeneous summand of degree d, because the degree of 0 is $-\infty$). Let $x = [Z] \in \mathbb{P}^n$. Then $F(Z) = 0$ if and only if $F(\lambda Z) = 0$ for every $\lambda \in \mathbb{K}^*$, because $F(\lambda Z) = \lambda^d F(Z)$. Hence, although $F(x)$ is not defined, it makes to state that $F(x) = 0$ or $F(x) \neq 0$. Thus if $F_1, \ldots, F_r \in \mathbb{K}[Z_0, \ldots, Z_n]$ are homogeneous (of possibly different degrees) it makes sense to let

$$V(F_1, \ldots, F_r) := \{x \in \mathbb{P}^n \mid F_1(x) = \ldots = F_r(x) = 0\}. \tag{1.2.1}$$

As in the case of affine space, it is convenient to consider the zero locus of ideals, but we need to consider homogeneous ideals. An ideal $I \subset \mathbb{K}[Z_0, \ldots, Z_n]$ is *homogeneous* if

$$I = \bigoplus_{d=0}^{\infty} I \cap \mathbb{K}[Z_0, \ldots, Z_n]_d, \tag{1.2.2}$$

i.e. if it is generated by homogeneous elements. Let $I \subset \mathbb{K}[Z_0, \ldots, Z_n]$ be a homogeneous ideal; we let

$$V(I) := \{x \in \mathbb{P}^n \mid F(x) = 0 \ \ \forall \text{ homogeneous } F \in I\}.$$

By Hilbert's basis Theorem A.3.6 I is generated by a finite set of homogeneous polynomials $F_1, \ldots, F_r$, and hence $V(I) = V(F_1, \ldots, F_r)$. Notice that if $I \subset \mathbb{K}[Z_0, \ldots, Z_n]$ is a homogeneous ideal we have two different meanings for $V(I)$, namely the subset of $\mathbb{P}^n$ defined above and the subset of $\mathbb{A}^{n+1}$ defined in (1.1.2). The context will indicate which of the two we mean.

Proceeding as in the proof of Proposition 1.1.1 one gets the following result.

Proposition 1.2.1. *The collection of subsets $V(I) \subset \mathbb{P}^n$, where I runs through the collection of homogeneous ideals of $\mathbb{K}[Z_0, \ldots, Z_n]$, satisfies the axioms for the closed subsets of a topological space.*

Definition 1.2.2. The *Zariski topology* of $\mathbb{P}^n$ is the topology whose closed sets are the sets $V(I) \subset \mathbb{P}^n$, where I runs through the collection of homogeneous ideals of $\mathbb{K}[Z_0, \ldots, Z_n]$. The Zariski topology of a subset $A \subset \mathbb{P}^n$ is the topology induced by the Zariski topology of $\mathbb{P}^n$.

Remark 1.2.3. Let $\pi\colon (\mathbb{K}^{n+1}\backslash\{0\}) \longrightarrow \mathbb{P}^n$ be the map defined by $\pi(Z) = [Z]$, so that $\mathbb{P}^n$ is identified as the quotient of $\mathbb{K}^{n+1}\backslash\{0\}$ for the action by homotheties. The Zariski topology of $\mathbb{P}^n$ is the quotient of the Zariski topology on $\mathbb{K}^{n+1}\backslash\{0\}$.

Remark 1.2.4. If $F \in \mathbb{K}[Z_0, \ldots, Z_n]$ is homogeneous we let

$$\mathbb{P}^n_F := \mathbb{P}^n \backslash V(F). \tag{1.2.3}$$

Thus $\mathbb{P}^n_F$ is an open subset of $\mathbb{P}^n$.

From now on we make the identification

$$\begin{array}{ccc} \mathbb{A}^n & \longleftrightarrow & \mathbb{P}^n_{Z_0} \\ (z_1, \ldots, z_n) & \mapsto & [1, z_1, \ldots, z_n] \end{array}$$

The Zariski topology of $\mathbb{A}^n$ induced by the Zariski topology on $\mathbb{P}^n$ is the same as the Zariski topology of Definition 1.1.2. In fact let $X \subset \mathbb{A}^n$. Suppose first that X is closed for the topology induced from the Zariski topology of $\mathbb{P}^n$, i.e. $X = (\mathbb{P}^n_{Z_0}) \cap V(F_1, \ldots, F_r)$, where each $F_j \in \mathbb{K}[Z_0, Z_1, \ldots, Z_n]$ is homogeneous. Then $X = V(f_1, \ldots, f_r)$, where

$$f_j(z_1, \ldots, z_n) := F(1, z_1, \ldots, z_n).$$

Next suppose that X is closed for the Zariski topology of Definition 1.1.2, i.e. $X = V(f_1, \ldots, f_r)$ where $f_1, \ldots, f_r \in \mathbb{K}[z_1, \ldots, x_n]$. We may assume that all f_j are non zero because $\mathbb{A}^n$ is clearly closed for the induced topology, and hence each f_j has a well defined degree d_j . For $j \in \{1, \ldots, r\}$ let

$$F_j(Z_0, \ldots, Z_n) := Z_0^{d_j} f\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right).$$

Then F_j is a homogeneous polynomial of degree d_j and hence $V(F_1, \ldots, F_r) \subset \mathbb{P}^n$ is a closed subset. Since

$$V(f_1, \ldots, f_r) = (\mathbb{P}^n_{Z_0}) \cap V(F_1, \ldots, F_r),$$

we get that $V(f_1, \ldots, f_r)$ is closed for the induced topology.

Example 1.2.5. A subset $X \subset \mathbb{P}^n$ is a *hypersurface* if it is equal to $V(F)$, where F is a non constant homogeneous polynomial. Notice that $V(F) \cap \mathbb{A}^n$ is a hypersurface unless $F = cZ_0^d$ for some $c \in \mathbb{K}^*$.

Given a subset $A \subset \mathbb{P}^n$, let

$$I(A) := \langle F \in \mathbb{K}[Z_0, \ldots, Z_n] \mid F \text{ is homogeneous and } F(p) = 0 \text{ for all } p \in A\rangle, \tag{1.2.4}$$

where $\langle , \rangle$ means "the ideal generated by". Clearly $I(A)$ is a homogeneous ideal of $\mathbb{K}[Z_0, \ldots, Z_n]$, and $V(I(A))$ is the closure of A in the Zariski topology.

Definition 1.2.6. A *quasi-projective* variety is a Zariski locally closed subset of a projective space, i.e. $X \subset \mathbb{P}^n$ such that $X = U \cap Y$, where $U, Y \subset \mathbb{P}^n$ are Zariski open and Zariski closed respectively.

Example 1.2.7. By Remark 1.2.4, every closed subset of $\mathbb{A}^n$ is a quasi projective variety.

Remark 1.2.8. If V is a finite dimensional complex vector space, the Zariski topology on $\mathbb{P}(V)$ is defined by imitating what was done for $\mathbb{P}^n$: one associates to a homogeneous ideal $I \subset \operatorname{Sym} V^\vee$ the set of zeroes $V(I)$, etc. Everything that we do in the present chapter applies to this situation, but for the sake of concreteness we formulate it for $\mathbb{P}^n$.

1.3 Decomposition into irreducibles

A proper closed subset $X \subset \mathbb{P}^1$ (or $X \subset \mathbb{A}^1$) is a finite set of points. In general, a quasi projective variety is a finite union of closed subsets which are irreducible, i.e. are not the union of proper closed subsets. In order to formulate the relevant result, we give a few definitions.

Definition 1.3.1. Let X be a topological space. We say that X is *reducible* if either $X = \emptyset$ or there exist proper closed subsets $Y, W \subset X$ such that $X = Y \cup W$. We say that X is *irreducible* if it is not reducible.

Example 1.3.2. A subset $A \subset \mathbb{R}^n$ with the euclidean (classical) topology is irreducible if and only if it is a singleton.

Example 1.3.3. Projective space $\mathbb{P}^n$ with the Zariski topology is irreducible. In fact suppose that $\mathbb{P}^n = X \cup Y$ with X and Y proper closed subsets. Then there exist homogeneous $F \in I(X)$ and $G \in I(Y)$ such that $F(y) \neq 0$ for one (at least) $y \in Y$ and $G(x) \neq 0$ for one (at least) $x \in X$. In particular both F and G are non zero, and hence $FG \neq 0$ because $\mathbb{K}[Z_0, \ldots, Z_n]$ is an integral domain. On the other hand $FG = 0$ because $\mathbb{P}^n = Y \cup W$. This is a contradiction, and hence $\mathbb{P}^n$ is irreducible.

Remark 1.3.4. Since the field $\mathbb{K}$ is algebraically closed it is infinite, and hence there is no distinction between the polynomial ring $\mathbb{K}[z_1, \ldots, z_n]$ and the ring of polynomial functions in $z_1, \ldots, z_n$. That is implicit in the argument given in Example 1.3.3, and it will appear repeatedly.

Definition 1.3.5. Let X be a topological space. An *irreducible decomposition of* X consists of a decomposition (possibly empty)

$$X = X_1 \cup \cdots \cup X_r \tag{1.3.1}$$

where each X_i is a closed irreducible subset of X (irreducible with respect to the induced topology) and moreover $X_i \not\subset X_j$ for all $i \neq j$.

We will prove the following result.

Theorem 1.3.6. *Let $A \subset \mathbb{P}^n$ with the (induced) Zariski topology. Then A admits an irreducible decomposition, and such a decomposition is unique up to reordering of components.*

The key step in the proof of Theorem 1.3.6 is the following remarkable consequence of Hilbert's basis Theorem A.3.6.

Proposition 1.3.7. *Let $A \subset \mathbb{P}^n$, and let $A \supset X_0 \supset X_1 \supset \ldots \supset X_m \supset \ldots$ be a descending chain of Zariski closed subsets of A, i.e $X_m \supset X_{m+1}$ for all $m \in \mathbb{N}$. Then the chain is stationary, i.e. there exists $m_0 \in \mathbb{N}$ such that $X_m = X_{m_0}$ for $m \geqslant m_0$.*

Proof. Let $\overline{X}_i$ be the closure of X_i in $\mathbb{P}^n$. Then $X_i = A \cap \overline{X}_i$, because X_i is closed in A. Hence we may replace X_i by $\overline{X}_i$, or equivalently we may suppose that the X_i are closed in $\mathbb{P}^n$. Let $I_m = I(X_m)$. Then $I_0 \subset I_1 \subset \ldots \subset I_m \subset \ldots$ is an ascending chain of (homogeneous) ideals of $\mathbb{K}[Z_0, \ldots, Z_n]$. By Hilbert's basis Theorem and Lemma A.3.3 the ascending chain of ideals is stationary, i.e. there exists $m_0 \in \mathbb{N}$ such that $I_{m_0} = I_m$ for $m \geqslant m_0$. Thus $X_{m_0} = V(I_{m_0}) = V(I_m) = X_m$ for $m \geqslant m_0$. □

Proof of Theorem 1.3.6. If A is empty, then it is the empty union (of irreducibles). . Next, suppose that A is not empty and that it does not admit an irreducible decomposition; we will arrive at a contradiction. First A in reducible, i.e. $A = X_0 \cup W_0$ with $X_0, W_0 \subset A$ proper closed subsets. If both X_0 and W_0 have an irreducible decomposition, then A is the union of the irreducible components of X_0 and W_0, contradicting the assumption that A does not admit an irreducible decomposition. Hence one of X_0, W_0, say X_0, does *not* have an irreducible decomposition. In particular X_0 is reducible. Thus $X_0 = X_1 \cup W_1$ with $X_1, W_1 \subset X_0$ proper closed subsets, and arguing as above, one of X_1, W_1, say X_1, does not admit a decomposition into irredicbles. Iterating, we get a strictly descending chain of closed subsets

$$A \supsetneq X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_m \supsetneq X_{m+1} \supsetneq \cdots$$

This contradicts Proposition 1.3.7. This proves that X has a decomposition into irreducibles $X = X_1 \cup \ldots \cup X_r$.

By discarding X_i's which are contained in X_j with $i \neq j$, we may assume that if $i \neq j$, then X_i is not contained in X_j.

Lastly, let us prove that such a decomposition is unique up to reordering, by induction on r. The case $r = 1$ is trivially true. Let $r \geqslant 2$. Suppose that $X = Y_1 \cup \ldots \cup Y_s$, where each Y_j is Zariski closed irreducible, and $Y_j \not\subset Y_k$ if $j \neq k$. Since Y_s is irreducible, there exists i such that $Y_s \subset X_i$. We may assume that $i = r$. By the same argument, there exists j such that $X_r \subset Y_j$. Thus $Y_s \subset X_r \subset Y_j$. It follows that $j = s$, and hence $Y_s = X_r$. It follows that $X_1 \cup \ldots \cup X_{r-1} = Y_1 \cup \ldots \cup Y_{s-1}$, and hence the decomposition is unique up to reordering by the inductive hypothesis. □

Definition 1.3.8. Let X be a quasi projective variety, and let

$$X = X_1 \cup \ldots \cup X_r$$

be an irreducible decomposition of X. The X_i's are the *irreducible components of* X (this makes sense because, by Theorem 1.3.6, the collection of the X_i's is uniquely determined by X).

We notice the following consequence of Proposition 1.3.7.

Corollary 1.3.9. *A quasi projective variety X (with the Zariski topology) is quasi compact, i.e. every open covering of X has a finite subcover.*

The following result makes a connection between irreducibility and algebra.

Proposition 1.3.10. *A subset $X \subset \mathbb{P}^n$ is irreducible if and only if $I(X)$ is a prime ideal.*

Proof. The proof has essentially been given in Example 1.3.3. Suppose that X is irreducible. In particular $X \neq \varnothing$ (by definition), and hence $I(X)$ is a proper ideal of $\mathbb{K}[Z_0, \ldots, Z_n]$. We must prove that $\mathbb{K}[Z_0, \ldots, Z_n]/I(X)$ is an integral domain. Suppose the contrary. Then there exist

$$F, G \in \mathbb{K}[Z_0, \ldots, Z_n], \quad F \notin I(X), \ G \notin I(X), \tag{1.3.2}$$

such that

$$F \cdot G \in I(X). \tag{1.3.3}$$

By (1.3.2) both $X \cap V(F)$ and $X \cap V(G)$ are proper closed subsets of X, and by (1.3.3) we have $X = (X \cap V(F)) \cup (X \cap V(G))$. This is a contradiction, hence $I(X)$ is a prime ideal.

Next, assume that X is reducible; we must prove that $I(X)$ is not prime. If $X = \varnothing$, then $I(X) = \mathbb{K}[Z_0, \ldots, Z_n]$ and hence $I(X)$ is not prime. Thus we may assume that $X \neq \varnothing$, and hence there exist proper closed subset $Y, W \subset X$ such that $X = Y \cup W$. Since $Y \not\subset W$ and $W \not\subset Y$, there exist $F \in (I(Y) \backslash I(W))$ and $G \in (I(W) \backslash I(Y))$. It follows that both (1.3.2) and (1.3.3) hold, and hence $I(X)$ is not prime. □

Remark 1.3.11. Let $I := (Z_0^2) \subset \mathbb{K}[Z_0, Z_1]$. Then $V(I) = \{[0, 1]\}$ is irreducible although I is *not* prime. Of course $I(V(I))$ is prime, it equals (Z_0).

Remark 1.3.12. Let $X \subset \mathbb{A}^n$. Let $I(X) \subset \mathbb{K}[z_1, \ldots, z_n]$ be the ideal of polynomials vanishing on X. Then X is irreducible if and only if $I(X)$ is a prime ideal. The proof is analogous to the proof of Proposition 1.3.10. One may also directly relate $I(X)$ with the ideal $J \subset \mathbb{K}[Z_0, \ldots, Z_n]$ generated by homogeneous polynomials vanishing on X (as subset of $\mathbb{P}^n$), and argue that $I(X)$ is prime if and only if J is.

1.4 The Nullstellensatz

Let an ideal I in a ring R. The *radical* of I, denoted by $\sqrt{I}$, is the set of elements $a \in R$ such that $a^m \in I$ for some $m \in \mathbb{N}$. As is easily checked, $\sqrt{I}$ is an ideal. It is clear that $\sqrt{I} \subset I(V(I))$. The Nullstellensatz states that we have equality.

Theorem 1.4.1 (Hilbert's Nullstellensatz)**.** *Let $I \subset \mathbb{K}[z_1, \ldots, z_n]$ be an ideal. Then $I(V(I)) = \sqrt{I}$.*

Before discussing the proof of the Nullstellensatz, we introduce some notation. For $a = (a_1, \ldots, a_n)$ an element of $\mathbb{A}^n$, let

$$\mathfrak{m}_a := (z_1 - a_1, \ldots, z_n - a_n) = \{f \in \mathbb{K}[z_1, \ldots, z_n] \mid f(a_1, \ldots, a_n) = 0\}. \tag{1.4.1}$$

Notice that $\mathfrak{m}_a$ is the kernel of the surjective homomorphism

$$\begin{array}{ccc} \mathbb{K}[z_1, \ldots, z_n] & \xrightarrow{\phi} & \mathbb{K} \\ f & \mapsto & f(a_1, \ldots, a_n), \end{array}$$

and hence is a maximal ideal. The Nullstellensatz is a consequence of the following result.

Proposition 1.4.2. *An ideal $\mathfrak{m} \subset \mathbb{K}[z_1, \ldots, z_n]$ is maximal if and only if there exists $(a_1, \ldots, a_n) \in \mathbb{A}^n$ such that $\mathfrak{m} = \mathfrak{m}_a$.*

Proof. We have shown that $\mathfrak{m}_a$ is maximal. Now suppose that $\mathfrak{m} \subset \mathbb{K}[z_1, \ldots, z_n]$ is a maximal ideal. Let $F := \mathbb{K}[z_1, \ldots, z_n]/\mathfrak{m}$. Then F is an algebraic extension of $\mathbb{K}$ by Corollary A.5.2. Since $\mathbb{K}$ is algebraically closed $F = \mathbb{K}$, and hence the quotient map is

$$\mathbb{K}[z_1, \ldots, z_n] \xrightarrow{\phi} \mathbb{K}[z_1, \ldots, z_n]/\mathfrak{m} = \mathbb{K}.$$

For $i \in \{1, \ldots, n\}$ let $a_i := \phi(z_i)$. Then $(z_i - a_i) \in \ker \phi$. Since $\mathfrak{m}_a$ is generated by $(z_1 - a_1), \ldots, (z_n - a_n)$ it follows that $\mathfrak{m}_a \subset \mathfrak{m}$. Since both $\mathfrak{m}_a$ and $\mathfrak{m}$ are maximal it follows that $\mathfrak{m} = \mathfrak{m}_a$. □

Corollary 1.4.3 (Weak Nullstellensatz)**.** *Let $I \subset \mathbb{K}[z_1, \ldots, z_n]$ be an ideal. Then $V(I) = \varnothing$ if and only if $I = (1)$.*

Proof. If $I = (1)$, then $V(I) = \varnothing$. Assume that $V(I) = \varnothing$. Suppose that $I \neq (1)$. Then there exists a maximal ideal $\mathfrak{m} \subset \mathbb{K}[z_1, \ldots, z_n]$ containing I. Since $I \subset \mathfrak{m}$, $V(I) \supset V(\mathfrak{m})$. By Proposition 1.4.2 there exists $a \in \mathbb{K}^n$ such that $\mathfrak{m} = \mathfrak{m}_a$ and hence $V(\mathfrak{m}) = V(\mathfrak{m}_a) = \{(a_1, \ldots, a_n)\}$. Thus $a \in V(I)$ and hence $V(I) \neq \varnothing$. This is a contradiction, and hence $I = (1)$. □

Proof of Hilbert's Nullsetellensatz (Rabinowitz's trick). Let $f \in I(V(I))$. By Hilbert's basis theorem $I = (g_1, \ldots, g_s)$ for $g_1, \ldots, g_s \in \mathbb{K}[z_1, \ldots, z_n]$. Let $J \subset \mathbb{K}[z_1, \ldots, z_n, w]$ be the ideal

$$J := (g_1, \ldots, g_s, f \cdot w - 1).$$

Since $f \in I(V(I))$ we have $V(J) = \varnothing$ and hence by the Weak Nullstellensatz $J = (1)$. Thus there exist $h_1, \ldots, h_s, h \in \mathbb{K}[x_1, \ldots, x_n, y]$ such that

$$\sum_{i=1}^{s} h_i g_i + h\,(f \cdot w - 1) = 1.$$

Replacing w by $1/f(z)$ in the above equality we get

$$\sum_{i=1}^{s} h_i \left(z, \frac{1}{f(z)}\right) g_i(z) = 1. \tag{1.4.2}$$

Let $d >> 0$: multiplying both sides of (1.4.2) by f^d we get that

$$\sum_{i=1}^{s} \overline{h}_i\,(z)\, g_i(z) = f^d(z), \quad \overline{h}_i \in \mathbb{K}[z_1, \ldots, z_n].$$

Thus $f \in \sqrt{I}$. □

Example 1.4.4. Let $V(F) \subset \mathbb{P}^n$ be a hypersurface, and let $F_1, \ldots, F_r$ be the distinct prime factors of the decomposition of F into a products of primes (recall that $\mathbb{K}[Z_0, \ldots, Z_n]$ is a UFD, by Corollary A.2.2). The irreducible decomposition of $V(F)$ is

$$V(F) = V(F_1) \cup \ldots \cup V(F_r).$$

In fact, each $V(F_i)$ is irreducible by Proposition 1.3.10. What is not obvious is that $V(F_i) \not\subset V(F_j)$ if F_i, F_j are non associated primes. This follows from Hilbert's Nullstellensatz.

1.5 Regular maps

Let $U \subset \mathbb{P}^n$ be a locally closed subset. Suppose that $F_0, \ldots, F_m \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ are homogeneous polynomials of the same degree, and that for all $[Z] \in U$ we have $(F_0(Z), \ldots, F_m(Z)) \neq (0, \ldots, 0)$. Let $[Z] \in U$. Then $[F_0(Z), \ldots, F_m(Z)] \in \mathbb{P}^m$ and if $\lambda \in \mathbb{K}^*$ we have

$$[F_0(\lambda Z), \ldots, F_m(\lambda Z)] = [\lambda^d F_0(Z), \ldots, \lambda^d F_m(Z)] = [F_0(Z), \ldots, F_m(Z)].$$

Hence we may define

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{P}^m \\ [Z] & \to & [F_0(Z), \ldots, F_m(Z)] \end{array} \tag{1.5.1}$$

Maps as above are the local models for regular maps between quasi projective varieties.

Definition 1.5.1. Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be locally closed subsets (hence X and Y are quasi projective varieties), and let $\varphi\colon X \to Y$ be a map. Then φ is *regular at* $a \in X$ if there exist an open $U \subset X$ containing a such that the restriction of φ to U is described as in (1.5.1). (We assume that $(F_0(Z), \ldots, F_m(Z)) \neq (0, \ldots, 0)$ for all $[Z] \in U$.) The map φ is *regular* if it is regular at each point of X.

Remark 1.5.2. Let $\varphi\colon X \to Y$ be a map between quasi projective varieties. Suppose that $Y = \bigcup_{i \in I} U_i$ is an open cover, that $\varphi^{-1}U_i$ is open in X for each $i \in I$ and that the restriction

$$\begin{array}{ccc} \varphi^{-1}(U_i) & \longrightarrow & U_i \\ x & \mapsto & \varphi(x) \end{array}$$

is regular for each $i \in I$. Then φ is regular. In other words regularity of a map is a local notion.

Proposition 1.5.3. *A regular map of quasi projective varieties is Zariski continuous.*

Proof. Let $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ be Zariski locally closed, and let $\varphi\colon X \to Y$ be a regular map. We must prove that if $C \subset Y$ is Zariski closed, then $\varphi^{-1}(C)$ is Zariski closed in X. Let $U \subset W$ be an open subset such that (1.5.1) holds. Let us show that $\varphi^{-1}(C) \cap U$ is closed in U. Since C is closed $C = V(I) \cap Y$ where $I \subset \mathbb{K}[T_0, \ldots, T_m]$ is a homogeneous ideal. Thus

$$\varphi^{-1}(C) \cap U = \{[Z] \in U \mid P(F_0(Z), \ldots, F_m(Z)) = 0 \; \forall P \in I\}.$$

Since each $P(F_0(Z), \ldots, F_m(Z))$ is a homogeneous polynomial, we get that $\varphi^{-1}(C) \cap U$ is closed in U.

By definition of regular map X can be covered by Zariski open sets U_α such that (1.5.1) holds with U replaced by U_α. We have proved that $C_\alpha := \varphi^{-1}(C) \cap U_\alpha$ is closed in U_α for all α. It follows that $\varphi^{-1}(C)$ is closed. In fact let $\overline{C}_\alpha \subset X$ be the closure of C_α and $D_\alpha := X \backslash U_\alpha$. Since C_α is closed in U_α we have

$$\overline{C}_\alpha \cap U_\alpha = C_\alpha = \varphi^{-1}(C) \cap U_\alpha. \tag{1.5.2}$$

Moreover D_α is closed in X because U_α is open. By (1.5.2) we have

$$\varphi^{-1}(C) = \bigcap_\alpha \left(\overline{C}_\alpha \cup D_\alpha\right).$$

Thus $\varphi^{-1}(C)$ is an intersection of closed sets and hence is closed. □

It is convenient to unravel the condition of being regular for maps with domain a subset of an affine space or both domain and codomain subsets of an affine space.

Example 1.5.4. Let $X \subset \mathbb{A}^n$ $(= \mathbb{P}^n_{Z_0})$ and $Y \subset \mathbb{P}^m$ be locally closed subsets, and let $\varphi\colon X \to Y$ be a map. Then φ is a regular map if and only if, given any $a \in X$, there exist $f_0, \ldots, f_m \in \mathbb{K}[z_1, \ldots, z_n]$ (in general *not* homogeneous) such that on an open subset $U \subset X$ containing a we have

$$\varphi(z) = [f_0(z), \ldots, f_m(z)]. \tag{1.5.3}$$

(This includes the statement that $V(f_1, \ldots, f_m) \cap U = \emptyset$.) In fact, if φ is regular there exist homogeneous $F_0, \ldots, F_m \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ such that $\varphi([1, z]) = [F_0(1, z), \ldots, F_m(1, z)]$, and it suffices to let $f_j(z) := F_j(1, z)$. Conversley, if (1.5.3) holds, then

$$\varphi([Z_0, Z_1, \ldots, Z_n]) = [Z_0^d, Z_0^d f_1\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right), \ldots, Z_0^d f_m\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_n}{Z_0}\right)], \tag{1.5.4}$$

and for d is large enough, each of the rational functions appearing in (1.5.4) is actually a homogeneous polynomial of degree d.

Example 1.5.5. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be locally closed subsets and let $\varphi\colon X \to Y$ be a map. Recall that $\mathbb{A}^n = \mathbb{P}^n_{Z_0}$ and $\mathbb{A}^m = \mathbb{P}^m_{T_0}$. Then φ is regular if and only if locally there exist $f_0, \ldots, f_m \in \mathbb{K}[z_1, \ldots, z_n]$ (in general *not* homogeneous) such that

$$f(z) = \left(\frac{f_1(z)}{f_0(z)}, \ldots, \frac{f_m(z)}{f_0(z)}\right). \tag{1.5.5}$$

Here it is understood that $f_0(z) \neq 0$ for all z in the relevant open subset U of X. In fact this follows from (1.5.3) if we divide the homogeneous coordinates of $\varphi(z)$ by $f_0(z)$ (by hypothesis it does not vanish for $z \in U$).

The identity map of a quasi projective variety is regular (choose $F_j(Z) = Z_j$). If $\varphi\colon X \to Y$ and $\psi\colon Y \to W$ are regular maps of quasi projective varieties, the composition $\psi \circ \varphi\colon X \to W$ is regular because the composition of homogeneous polynomial functions is a homogeneous polynomial function. Thus we have the *category of quasi projective varieties*. In particular we have the notion of isomorphism between quasi projective varieties.

Definition 1.5.6. A quasi projective variety is

- an *affine variety* if it is isomorphic to a closed subset of an affine space (as usual $\mathbb{A}^n = \mathbb{P}^n_{Z_0} \subset \mathbb{P}^n$),
- a *projective variety* if it is isomorphic to a closed subset of a projective space.

Remark 1.5.7. Let X be an affine variety. If $Y \subset X$ is closed then it is an affine variety. In fact by hypothesis there exist a closed subset $W \subset \mathbb{A}^n$ and an isomorphism $\varphi\colon X \xrightarrow{\sim} W$. Since φ is an isomorphism it is a homeomorphism (see Proposition 1.5.3), and hence $\varphi(Y)$ is a closed subset of W. Since W is closed in $\mathbb{A}^n$, it follows that $\varphi(Y)$ is a closed subset of $\mathbb{A}^n$. The isomorphism $Y \xrightarrow{\sim} \varphi(Y)$ shows that Y is an affine variety. Similarly one shows that if X is a projective variety and $Y \subset X$ is closed, then Y is a projective variety.

The example below gives open (and non closed) subsets of an affine space which are affine varieties.

Example 1.5.8. Let $f \in \mathbb{K}[z_1, \ldots, z_n]$. We let

$$\mathbb{A}^n_f := \mathbb{A}^n \backslash V(f). \tag{1.5.6}$$

Let $Y := V(f(z_1, \ldots, z_n) \cdot w - 1) \subset \mathbb{A}^{n+1}$. The regular map

$$\begin{array}{ccc} \mathbb{A}^n_f & \xrightarrow{\varphi} & Y \\ (z_1, \ldots, z_n) & \mapsto & (z_1, \ldots, z_n, \frac{1}{f(z_1,\ldots,z_n)}) \end{array}$$

is an isomorphism. In fact the inverse of φ is given by

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & \mathbb{A}^n_f \\ (z_1, \ldots, z_n, w) & \mapsto & (z_1, \ldots, z_n) \end{array}$$

Example 1.5.9. Let

$$\mathcal{C}_d = \left\{[\xi_0, \ldots, \xi_d] \in \mathbb{P}^d \mid \operatorname{rk}\begin{pmatrix} \xi_0 & \xi_1 & \cdots & \xi_{d-1} \\ \xi_1 & \xi_2 & \cdots & \xi_d \end{pmatrix} \leqslant 1\right\}. \tag{1.5.7}$$

Since a matrix has rank at most 1 if and only if all the determinants of its 2×2 minors vanish it follows that $\mathcal{C}_d$ is closed. We have a regular map

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\varphi_d} & \mathcal{C}_d \\ [s, t] & \mapsto & [s^d, s^{d-1}t, \ldots, t^d] \end{array} \tag{1.5.8}$$

Let us prove that φ_d is an isomorphism. Let $\psi_d\colon \mathcal{C}_d \to \mathbb{P}^1$ be defined as follows:

$$\psi_d([\xi_0, \ldots, \xi_d]) = \begin{cases} [\xi_0, \xi_1] & \text{if } [\xi_0, \ldots, \xi_d] \in \mathcal{C}_d \cap \mathbb{P}^d_{\xi_0} \\ [\xi_{d-1}, \xi_d] & \text{if } [\xi_0, \ldots, \xi_d] \in \mathcal{C}_d \cap \mathbb{P}^d_{\xi_d} \end{cases}$$

Of course in order for this to make sense one has to check the following:

1. The subset $\mathscr{C}_d$ is the union of the open subsets $\mathcal{C}_d \cap \mathbb{P}^d_{\xi_0}$ and $\mathcal{C}_d \cap \mathbb{P}^d_{\xi_d}$.

2. The two expressions for ψ_d coincide for points in $\mathscr{C}_d \cap \mathbb{P}^d_{\xi_0} \cap \mathbb{P}^d_{\xi_d}$.

To prove (1) suppose that $[\xi] \in \mathscr{C}_d$ and $\xi_0 = 0$. By the equations defining $\mathscr{C}_d$ it follows that $\xi_1 = 0$, $\xi_2 = 0$, etc. up to $\ldots = \xi_{d-1}$. Hence if $\xi_0 = 0$ then $\xi_d \neq 0$, and this prove that Item (1) holds. To prove Item (2) suppose that $[\xi] \in \mathscr{C}_d \cap \mathbb{P}^d_{\xi_0} \cap \mathbb{P}^d_{\xi_d}$. By the equations defining $\mathscr{C}_d$ it follows that $\xi_0 \cdot \xi_n - \xi_1\xi_{n-1} = 0$ and hence $[\xi_0, \xi_1] = [\xi_{d-1}, \xi_d]$. This prove that Item (2) holds.

One checks easily that $\psi_d \circ \varphi_d = \mathrm{Id}_{\mathbb{P}^1}$ and $\varphi_d \circ \psi_d = \mathrm{Id}_{\mathscr{C}_d}$. Thus φ_d is an isomorphism, as claimed.

Definition 1.5.10. The closed subset $\mathscr{C}_d \subset \mathbb{P}^d$ defined in (1.5.7) or any $X \subset \mathbb{P}^d$ projectively equivalent to $\mathscr{C}_d$ (i.e. given by $g(\mathscr{C}_d)$ where $g \in \mathrm{PGL}_n(\mathbb{K})$) is a *rational normal curve in* $\mathbb{P}^d$.

In the above definition "rational" refers to the fact that $\mathscr{C}_d$ (and hence also any X projectively equivalent to $\mathscr{C}_d$) is isomorphic to $\mathbb{P}^1$, "curve" refers to the fact that $\mathbb{P}^1$ (and hence also $\mathscr{C}_d$) has dimension 1 (we will define the dimension of a quasi projective variety later on), the attribute "normal" will be explained later in the book.

The remark below shows that, in the definition of regular map, we cannot require that φ is given globally by homogeneous polynomials.

Remark 1.5.11. Unless we are in the trivial case $d = 1$, it is *not* possible to define ψ_d *globally* as

$$\psi_d([\xi_0, \ldots, \xi_d]) = [P(\xi_0, \ldots, \xi_d), Q(\xi_0, \ldots, \xi_d)], \tag{1.5.9}$$

with $P, Q \in \mathbb{K}[\xi_0, \ldots, \xi_d]_e$ not vanishing simultaneously on $\mathscr{C}_d$. In fact suppose that (1.5.9) holds, and let

$$p(s,t) := P(s^d, \ldots, t^d), \quad q(s,t) := Q(s^d, \ldots, t^d).$$

The polynomials $p(s,t), q(s,t)$ are homogeneous of degree de, they do not vanish simultaneously on a non zero $(s_0, t_0) \in \mathbb{K}^2$, and for all $[s,t] \in \mathbb{P}^1$ we have $[p(s,t), q(s,t)] = [s,t]$. The last equality means that $tp(s,t) = sq(s,t)$. It follows that $p(s,t) = s \cdot r(s,t)$ and $q(s,t) = t \cdot r(s,t)$ where $r(s,t)$ has no non trivial zeroes. Thus $r(s,t)$ is constant. In particular $de = \deg p = \deg q = 1$, and hence $d = 1$.

The example below extends Example 1.5.9 to arbitrary dimension.

Example 1.5.12. We recall the formula

$$\dim \mathbb{K}[Z_0, \ldots, Z_n]_d = \binom{d+n}{n}. \tag{1.5.10}$$

(See Exercise 1.9.9 for a proof.) Let $N(n;d) := \binom{d+n}{n} - 1$. Let

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\nu_d^n} & \mathbb{P}^{N(n;d)} \\ [Z] & \mapsto & [Z_0^d, Z_0^{d-1}Z_1, \ldots, Z_n^d] \end{array} \tag{1.5.11}$$

be defined by all homogeneous monomials of degree d - this is a *Veronese map*. Clearly ν_d^n is regular. Note that for $n = 1$ we get back the map φ_d in (1.5.8).

The homogeneous coordinates on $\mathbb{P}^{N(n;d)}$ appearing in (1.5.11) are indiced by length $n+1$ multiindices $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ such that $\deg I := i_0 + \ldots + i_n = d$; we denote them by $[\ldots, \xi_I, \ldots]$. Let $\mathscr{V}_d^n \subset \mathbb{P}^{N(n;d)}$ be the closed subset defined by

$$\mathscr{V}_d^n := V(\ldots, \xi_I \cdot \xi_J - \xi_K \cdot \xi_L, \ldots),$$

where I, J, L, K run through all multiindices such that $I + J = K + L$. Clearly $\nu_d^n(\mathbb{P}^n) \subset \mathscr{V}_d^n$. Let us show that ν_d^n is an isomorphism onto $\mathscr{V}_d^n$.

Let $s \in \{0, \ldots, n\}$, and let $H \in \mathbb{N}^{n+1}$ be a multiindex of degree $(d-1)$. We let $e_s \in \mathbb{N}^{n+1}$ be the element all of whose entries are equal to 0 except for the entry at place $s+1$, which is equal to 1, and $H_s := H + e_s$. Also let

$$\begin{array}{ccc} \mathscr{V}_d^n \backslash V(\xi_{H_0}, \ldots, \xi_{H_n}) & \xrightarrow{\varphi_d^n(H)} & \mathbb{P}^n \\ [\ldots, \xi_I, \ldots] & \mapsto & [\xi_{H_0}, \ldots, \xi_{H_n}] \end{array}$$

Clearly $\varphi_d^n(H)$ is regular. Moreover if $[\ldots, \xi_I, \ldots] \in \mathscr{V}_d^n$ then there exist a multiindex $H \in \mathbb{N}^{n+1}$ of degree $(d-1)$ such that x belongs to $\mathscr{V}_d^n \backslash V(\xi_{H_0}, \ldots, \xi_{H_n})$ for $H \in \mathbb{N}^{n+1}$ (there exists $I \in \mathbb{N}^{n+1}$ of degree d such that $\xi_I \neq 0$ and $I = H + e_s$ where s is such that $i_s \neq 0$). Moreover we claim that if $[\ldots, \xi_I, \ldots] \in \mathscr{V}_d^n$ belong both to the domain of $\varphi_d^n(H)$ and to the domain of $\varphi_d^n(H')$, then

$$\varphi_d^n(H)([\ldots, \xi_I, \ldots]) = [\xi_{H_0}, \ldots, \xi_{H_n}] = [\xi_{H'_0}, \ldots, \xi_{H'_n}] = \varphi_d^n(H')([z]). \tag{1.5.12}$$

In fact for $s, t \in \{0, \ldots, n\}$ we have $H_s + H'_t = H + H' + e_s + e_t = H_t + H'_s$, thus $\xi_{H_i} \cdot \xi_{H'_j} - \xi_{H_j} \cdot \xi_{H'_i} = 0$ by the equations defining $\mathscr{V}_d^n$, and this proves that the equality in (1.5.12) holds. This shows that the maps $\varphi_d^n(H)$'s define a regular map

$$\mathscr{V}_d^n \xrightarrow{\varphi_d^n} \mathbb{P}^n. \tag{1.5.13}$$

We claim that

$$\varphi_d^n \circ \nu_d^n = \mathrm{Id}_{\mathbb{P}^n} \tag{1.5.14}$$

$$\nu_d^n \circ \varphi_d^n = \mathrm{Id}_{\mathscr{V}_d^n}. \tag{1.5.15}$$

The first equality is easily checked. In order to check the second equality it suffices to show that ν_d^n is surjective. One may proceed as follows. Let $x = [\ldots, \xi_I, \ldots] \in \mathscr{V}_d^n$ be a point such that $\xi_{de_s} \neq 0$ for some $s \in \{0, \ldots, n\}$. Thus $x \in (\mathscr{V}_d^n \backslash V(\xi_{H_0}, \ldots, \xi_{H_n}))$ where $H = (d-1)e_0$. It is not difficult to show that $x = \nu_d^n([\xi_{H_0}, \ldots, \xi_{H_n}])$. Hence it suffices to prove that if $x = [\ldots, \xi_I, \ldots] \in \mathscr{V}_d^n$, then there exists $s \in \{0, \ldots, n\}$ such that $\xi_{de_s} \neq 0$. Equivalently, we must show that the following statement holds: if $\xi := (\ldots, \xi_I, \ldots)$ is such that $\xi_{de_s} = 0$ for all $s \in \{0, \ldots, n\}$ and $\xi_I \cdot \xi_J = \xi_K \cdot \xi_L$ whenever $I + J = K + L$, then $\xi_I = 0$ for all multiindices I. This is easily proved by "descending induction" on the maximum of $i_0, \ldots, i_n$. If the maximum is d, then $\xi_I = 0$ by hypothesis. Suppose that the maximum is at least $d/2$, i.e. that there exists $s \in \{0, \ldots, n\}$ be such that $2i_s \geqslant d$. Then $2I = de_s + J$ where $J \in \mathbb{N}^{n+1}$ is a multiindex of degree d and hence $\xi_I^2 = \xi_{de_s} \cdot \xi_J = 0$ by by the equations defining $\mathscr{V}_d^n$. Thus $\xi_I = 0$. This proves that if the maximum is at least $d/2$ then $\xi_I = 0$. Iterating the argument we get that if the maximum is at least $d/4$ then $\xi_I = 0$ etc.

The Veronese map allows us to show that the open affine subsets of a quasi projective variety form a basis for the Zariski topology. First we need a definition.

Definition 1.5.13. Let $X \subset \mathbb{P}^n$ be a closed subset. A *principal open subset* of X is an open $U \subset X$ which is equal to

$$X_F := X \backslash V(F),$$

where $F \in \mathbb{K}[Z_0, \ldots, Z_n]$ is a homogeneous polynomial of *strictly positive degree.*

Claim 1.5.14. *Let $X \subset \mathbb{P}^n$ be closed. A principal open subset of X is an affine variety.*

Proof. First we prove the claim for $X = \mathbb{P}^n$. Let $F \in \mathbb{K}[Z_0, \ldots, Z_n]$ be a homogeneous polynomial of strictly positive degree d. In order to prove that $\mathbb{P}_F^n$ is affine we consider the Veronese map $\nu_d^n \colon \mathbb{P}^n \longrightarrow \mathbb{P}^{N(n,d)}$, see (1.5.11). Let $\mathscr{V}_d^n := \mathrm{im}(\nu_d^n)$ be the corresponding Veronese variety. As shown in Example 1.5.12 the map $\mathbb{P}^n \to \mathscr{V}_d^n$ defined by ν_d^n is an isomorphism. Let $F = \sum_I a_I Z^I$, and let $H \subset \mathbb{P}^{N(n,d)}$ be the hyperplane $H = V(\sum_I a_I \xi_I$. Then we have the isomorphism

$$\begin{array}{ccc} \mathbb{P}_F^n & \xrightarrow{\sim} & (\mathscr{V}_d^n \backslash H) \\ x & \mapsto & \nu_d^n(x) \end{array} \tag{1.5.16}$$

But $\mathbb{P}^{N(n,d)} \backslash H$ is the affine space $\mathbb{A}^{N(n,d)}$, and hence $(\mathscr{V}_d^n \backslash H)$ is a closed subset of $\mathbb{A}^{N(n,d)}$. Hence the map in (1.5.16) is an isomorphism between $\mathbb{P}^n_F$ and closed subset of $\mathbb{A}^{N(n,d)}$, and therefore $\mathbb{P}^n_F$ is an affine variety.

In general, let $X \subset \mathbb{P}^n$ be closed, and let F be as above. Then X_F is a closed subset of the affine variety $\mathbb{P}^n_F$, and hence it is an affine variety, see Remark rmk:trapano. □

Proposition 1.5.15. *The open affine subsets of a quasi projective variety form a basis of the Zariski topology.*

Proof. Since a quasi-projective variety is an open subset of a projective variety, it suffices to prove the result for projective varieties. Let $X \subset \mathbb{P}^n$ be closed. Let $U \subset X$ be open. If $U = X$ then

$$U = X = X_{Z_0} \cup X_{Z_1} \cup \ldots \cup X_{Z_n}, \tag{1.5.17}$$

and each of the X_{Z_i}'s is an open affine subset by Claim 1.5.14.

Next asssume that $U \neq X$. Then $U = X \backslash V(F_1, \ldots, F_r)$, where each F_j is a non constant homogeneous polynomial, and $r \geqslant 1$. Then

$$U = X_{F_1} \cup \ldots \cup X_{F_r},$$

and each of the X_{F_j}'s is an open affine subset by Claim 1.5.14. □

1.6 Regular functions on affine varieties

Definition 1.6.1. A *regular function* on a quasi projective variety X is a regular map $X \to \mathbb{K}$.

Let X be a non empty quasi projective variety. The set of regular functions on X with pointwise addition and multiplication is a $\mathbb{K}$-algebra, named the *ring of regular functions* of X. We denote it by $\mathbb{K}[X]$.

If X is a projective variety, then it has few regular functions. In fact we will prove (see Corollary 2.4.8) that every regular function on X is locally constant. On the other hand, affine varieties have plenty of functions. In fact if $X \subset \mathbb{A}^n$ is closed we have an inclusion

$$\mathbb{K}[z_1, \ldots, z_n]/I(X) \hookrightarrow \mathbb{K}[X]. \tag{1.6.1}$$

Theorem 1.6.2. *Let $X \subset \mathbb{A}^n$ be closed. Then the homomorphism in* (1.6.1) *is an isomorphism, i.e. every regular function on X is the restriction of a polynomial function on $\mathbb{A}^n$.*

Theorem 1.6.2 follows from the Nullstellensatz. Before giving the proof we discusse a particular instance of Theorem 1.6.2, which shows the relation with the Nullstellensatz. Let $X \subset \mathbb{A}^n$ be closed. Suppose that $g \in \mathbb{K}[z_1, \ldots, z_n]$ and that $g(a) \neq 0$ for all $a \in Z$. Then $1/g \in \mathbb{K}[X]$ and hence Theorem 1.6.2 predicts the existence of $f \in \mathbb{K}[z_1, \ldots, z_n]$ such that $g^{-1} = f_{|X}$. Such an f exists by the Nullstellensatz. In fact let $X = V(g_1, \ldots, g_r)$ where $g_1, \ldots, g_r \in \mathbb{K}[z_1, \ldots, z_n]$. By our hypothesis on g we have $V(g_1, \ldots, g_r, g) = \emptyset$, and hence $(g_1, \ldots, g_r, g) = (1)$ by the Nullstellensatz. Hence there exist $f_1, \ldots, f_r, f \in \mathbb{K}[z_1, \ldots, z_n]$ such that

$$f_1 \cdot g_1 +, \ldots, f_r \cdot g_r + f \cdot g = 1.$$

Restricting to X we get that $f(x) = g(x)^{-1}$ for all $x \in X$, as claimed.

Before proving Theorem 1.6.2, we notice that, if $X \subset \mathbb{A}^n$ is closed, the Nullstellensatz for $\mathbb{K}[z_1, \ldots, z_n]$ implies a Nullstellensatz for $\mathbb{K}[z_1, \ldots, z_n]/I(X)$. First a definition: given an ideal $J \subset (\mathbb{K}[z_1, \ldots, z_n]/I(X))$ we let

$$V(J) := \{a \in X \mid f(a) = 0 \quad \forall f \in J\}.$$

The following result follows at once from the Nullstellensatz.

Proposition 1.6.3 (Nullstellensatz for a closed subset of $\mathbb{A}^n$). *Let $X \subset \mathbb{A}^n$ be closed, and let $J \subset (\mathbb{K}[z_1, \ldots, z_n]/I(X))$ be an ideal. Then*

$$\left\{f \in (\mathbb{K}[z_1, \ldots, z_n]/I(X)) \mid f_{|V(J)} = 0\right\} = \sqrt{J}.$$

(The radical $\sqrt{J}$ is taken inside $\mathbb{K}[z_1, \ldots, z_n]/I(X)$.) In particular $V(J) = \varnothing$ if and only if $J = (1)$.

We introduce notation that is useful in the proof of Theorem 1.6.2. Given a quasi projective variety X, and $f \in \mathbb{K}[X]$, let

$$X_f := X \backslash V(f), \tag{1.6.2}$$

where $V(f) := \{x \in X \mid f(x) = 0\}$. Note the similarity with the notation for principal open subsets of projective varieties.

Remark 1.6.4. Assume that X is affine, hence we may assume that $X \subset \mathbb{A}^n$ is closed. The collection of open subsets $\{X_f\}$ is a basis for the Zariski topology of X. In fact let U be an open subset of X. Then $U = X \backslash V(g_1, \ldots, g_r)$ where $g_i \in \mathbb{K}[z_1, \ldots, z_n]$ for $i \in \{1, \ldots, r\}$. Let $f_i := g_{i|X}$. Then $U = X_{g_1} \cup \ldots \cup X_{g_r}$.

Proof of Theorem 1.6.2. The proof is simpler if X is irreducible. We first give the proof under this hypothesis. Let $\varphi \in \mathbb{K}[X]$. We claim that there exist $f_i, g_i \in \mathbb{K}[z_1, \ldots, z_n]$ for $1 \leqslant i \leqslant d$ with $g_i \notin I(X)$ such that

(a) $X = \bigcup_{1 \leqslant i \leqslant d} X_{g_i}$, i.e. $V(g_1, \ldots, g_d) \cap X = \varnothing$,

(b) for all $x \in X_{g_i}$ we have $\varphi(x) = \frac{f_i(x)}{g_i(x)}$,

In fact by definition of regular function (see Example 1.5.5) there exist an open cover $X = \bigcup_{\alpha \in A} U_\alpha$ and $f_\alpha, g_\alpha \in \mathbb{K}[z_1, \ldots, z_n]$ for each $\alpha \in A$ such that $U_\alpha \subset X_{g_\alpha}$ and $\varphi(x) = \frac{f_\alpha(x)}{g_\alpha(x)}$ for each $x \in U_\alpha$. Since the Zariski topology is quasi compact (see Corollary 1.3.9) we may assume that index set A is finite, say $A = \{1, \ldots, d\}$. Of course we may asssume that $g_i \not\equiv 0$ for all $i \in \{1, \ldots, d\}$. Since X is irreducible so is X_{g_i} and hence U_i is dense in X_{g_i}. This imples that $\varphi(x) = \frac{f_i(x)}{g_i(x)}$ on all of X_{g_i} because regular functions are Zariski continuous (see Proposition 1.5.3). This proves the claim.

In the rest of the proof we adopt the following notation: for $f \in \mathbb{K}[z_1, \ldots, z_n]$ we let $\overline{f} := f_{|X}$.

For $i = 1, \ldots, d$ the equality $\overline{g}_i \varphi = \overline{f}_i$ holds on X_{g_i} by Item (2). Since X is irreducible and X_{g_i} is a non empty subset of X it is dense in X, and hence $\overline{g}_i \varphi = \overline{f}_i$ on all of X (this is where the hypothesis that X is irreducible simplifies the proof). By Proposition 1.6.3 we have that $(\overline{g}_1, \ldots, \overline{g}_d) = (1)$, i.e. there exist $h_1, \ldots, h_d \in \mathbb{K}[z_1, \ldots, z_n]$ such that

$$1 = \overline{h}_1 \overline{g}_1 + \cdots + \overline{h}_d \overline{g}_d.$$

where $\overline{h}_i := h_{i|X}$. Multiplying by φ both sides of the above equality we get that

$$\varphi = \overline{h}_1 \overline{g}_1 \varphi + \cdots + \overline{h}_d \overline{g}_d \varphi = \overline{h}_1 \overline{f}_1 + \ldots + \overline{h}_1 \overline{f}_d = (h_1 f_1 + \cdots + h_d f_d)_{|X}. \tag{1.6.3}$$

This shows that φ is the restriction to X of a polynomial function on $\mathbb{A}^n$.

Now we give the proof for arbitrary (closed) X. Let $\varphi \in \mathbb{K}[X]$. This time we claim that there exist $f_i, g_i \in \mathbb{K}[z_1, \ldots, z_n]$ for $i \in \{1, \ldots, d\}$ such that

1. $X = \bigcup_{1 \leqslant i \leqslant d} X_{g_i}$, i.e. $V(g_1, \ldots, g_d) \cap X = \varnothing$,
2. for all $a \in X_{g_i}$ we have $\varphi(a) = \frac{f_i(a)}{g_i(a)}$,
3. for $1 \leqslant i \leqslant j$ we have $(g_j f_i - g_i f_j)|_X = 0$.

We start proving the claim as in the case of X irreducible. There is a finite open cover $X = \bigcup_{\alpha\in A} U_\alpha$ and $f_\alpha, g_\alpha \in \mathbb{K}[z_1,\ldots,z_n]$ for each $\alpha \in A$ such that $U_\alpha \subset X_{g_\alpha}$ and $\varphi(x) = \frac{f_\alpha(x)}{g_\alpha(x)}$ for each $x \in U_\alpha$. We may cover U_α by open affine sets $X_{\gamma_{\alpha,1}},\ldots,X_{\gamma_{\alpha,r}}$, see Remark 1.6.4. Since $V(\overline{g}_\alpha) \subset \bigcap_{j=1}^{r} V(\overline{\gamma}_{\alpha,j})$ (recall that $\overline{g}_\alpha$ and $\overline{\gamma}_{\alpha,j}$ are the restrictions to X of g_α and $\gamma_{\alpha,j}$ respectively), the Nullstellensatz for X gives that, for each α, j, there exist $N_{\alpha,j} > 0$ and $\mu_{\alpha,j} \in \mathbb{K}[z_1,\ldots,z_n]$ such that $\overline{\gamma}_{\alpha,j}^{N_{\alpha,j}} = \overline{\mu_{\alpha,j}} \cdot \overline{g}_\alpha$. Hence $\varphi(x) = \mu_{\alpha,j}(x) f_\alpha(x)/\gamma_{\alpha,j}(x)^{N_{\alpha,j}}$ for all $x \in X_{\gamma_{\alpha,j}}$. Since $V(\gamma_{\alpha,j}) = V(\gamma_{\alpha,j}^{N_{\alpha,j}})$ it follows that there exist $f_i', g_i' \in \mathbb{K}[z_1,\ldots,z_n]$ for $i \in \{1,\ldots,d\}$ such that $X = \bigcup_{i=1}^d X_{g_i'}$ and $\varphi(x) = f_i'(x)/g_i'(x)$ for all $x \in X_{g_i'}$. For $i \in \{1,\ldots,d\}$ let

$$f_i := f_i' g_i', \qquad g_i := (g_i')^2.$$

Clearly Items (1) and (2) hold. In order to check Item (3) we write

$$(g_j f_i - g_i f_j)|_X = ((g_j')^2 f_i' g_i' - (g_i')^2 f_j' g_j')|_X = ((g_i' g_j')(f_i' g_j' - f_j' g_i'))|_X.$$

Since $\varphi(z) = f_i'(x)/g_i'(x) = f_j'(x)/g_j'(x)$ for all $x \in X_{g_i'} \cap X_{g_j'}$ the last term vanishes on $X_{g_i'} \cap X_{g_j'}$. On the other hand the last term vanishes also on $(X \backslash X_{g_i'} \cap X_{g_j'}) = X \cap V(g_i' g_j')$ because of the factor $(g_i' g_j')$. This finishes the proof that there exist $f_i, g_i \in \mathbb{K}[z_1,\ldots,z_n]$ for $i \in \{1,\ldots,d\}$ such that (1), (2) and (3) hold.

Next, for $i = 1,\ldots,d$ let $\overline{g}_i := g_{i|X}$ and $\overline{f}_i := f_{i|X}$. Then

$$\overline{g}_i \varphi = \overline{f}_i. \tag{1.6.4}$$

In fact by Item (1) it suffices to check that (1.6.4) holds on X_{g_j} for $j = 1,\ldots,d$. For $j = i$ it holds by Item (2), for $j \neq i$ it holds by Item (3). Given the equalities in (1.6.4), one finishes the proof proceeding as in the case when X is irreducible. □

Example 1.6.5. Let X be an affine variety, thus we may assume that $X \subset \mathbb{A}^n$ is closed. If $f \in \mathbb{K}[X]$ then X_f is a principal open subset of $\overline{X}$. In fact by Theorem 1.6.2 there exists $g \in \mathbb{K}[z_1,\ldots,z_n]$ such that $f = g_{|X}$. If $d \gg 0$ then

$$G(Z_0,\ldots,Z_n) := Z_0^d g\left(\frac{Z_1}{Z_0},\ldots,\frac{Z_1}{Z_0}\right)$$

is a homogeneous polynomial whose zero locus (in $\mathbb{P}^n$) is equal to the union of $V(Z_0)$ and $V(g)$ (which is contained in $\mathbb{A}^n$). Hence $\overline{X}_G = (\overline{X}\backslash V(G)) = (X\backslash V(g)) = X_f$. An explicit isomorphism between X_f and a closed subset of an affine space is obtained as follows. Let $Y := V(J) \subset \mathbb{A}^{n+1}$ where J is the ideal generated by $I(X)$ and the polynomial $g(z_1,\ldots,z_n)\cdot z_{n+1} - 1$. Then the map

$$\begin{array}{ccc} X_f & \longrightarrow & Y \\ (z_1,\ldots,z_n) & \mapsto & \left(z_1,\ldots,z_n,\frac{1}{f(z_1,\ldots,z_n)}\right) \end{array}$$

is an isomorphism (see Example 1.5.8). Note that by Theorem 1.6.2 every regular function on X_f is given by the restriction to X_f of $\frac{h}{f^m}$, where $h \in \mathbb{K}[X]$ and $m \in \mathbb{N}$.

1.7 Quasi-projective varieties defined over a subfield of $\mathbb{K}$

Let $F \subset \mathbb{K}$ be a subfield, for example $\mathbb{R} \subset \mathbb{C}$, $\mathbb{Q} \subset \mathbb{C}$ or $\mathbb{F}_q \subset \overline{\mathbb{F}}_q$ where $q = p^r$ with p a prime.

Definition 1.7.1. A locally closed subset $X \subset \mathbb{P}^n(\mathbb{K})$ is *defined over* F if both the homogeneous ideals $I(\overline{X}) \subset \mathbb{K}[Z_0,\ldots,Z_n]$ and $I(\overline{X}\backslash X) \subset \mathbb{K}[Z_0,\ldots,Z_n]$ admit sets of generators belonging to $F[Z_0,\ldots,Z_n]$.

Trivially $\mathbb{P}^n(\mathbb{K})$ and $\mathbb{A}^n(\mathbb{K}) = \mathbb{P}^n(\mathbb{K})_{Z_0}$ are defined over the prime field, i.e. over $\mathbb{Q}$ if $\operatorname{char}\mathbb{K} = 0$ and over $\mathbb{F}_p$ if $\operatorname{char}\mathbb{K} = p$.

Remark 1.7.2. A locally closed subset $X \subset \mathbb{A}^n(\mathbb{K}) = \mathbb{P}^n(\mathbb{K})_{Z_0}$ is defined over F if both the ideals $I(\overline{X}) \subset \mathbb{K}[z_1, \ldots, z_n]$ and $I(\overline{X}\backslash X) \subset \mathbb{K}[z_1, \ldots, z_n]$ (in general non homogeneous) admit sets of generators which belong to $F[z_1, \ldots, z_n]$. This is so because a polynomial $p \in \mathbb{K}[z_1, \ldots, z_n]$ of degree d vanishes on X if and only if the homogeneous polynomial $P := Z_0^d \cdot f(Z_1/Z_0, \ldots, Z_n/Z_0)$ vanishes on $\overline{X}$, and conversely a homogeneous $P \in \mathbb{K}[Z_0, \ldots, Z_n]$ vanishes on $\overline{X}$ if and only if $P(1, z_1, \ldots, z_n) \in \mathbb{K}[z_1, \ldots, z_n]$ vanishes on X.

Example 1.7.3. Let $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$. If a_i belongs to F for all $i \in \{1, \ldots, n\}$ then $\{a\}$ is defined over F because its ideal is generated by $(z_1 - a_1, \ldots, z_n - a_n)$. The converse is true if we make a hypothesis on the field extension $F \subset \mathbb{K}$. Let $\mathrm{Aut}(\mathbb{K}, F)$ be the group of automorphisms of $\mathbb{K}$ fixing every element of F. Assume that the field of elements of $\mathbb{K}$ fixed by $\mathrm{Aut}(\mathbb{K}, F)$ is equal to F. (Since $\mathbb{K}$ is algebraically closed this holds if $\mathrm{char}\,\mathbb{K} = 0$ or, in case $\mathrm{char}\,\mathbb{K} = p$ if F is perfect, i.e. every element of F has a p-th root in F (necessarily unique).) With this hypothesis, suppose that $\{a\}$ is defined over F, and let $p_1, \ldots, p_r \in F[z_1, \ldots, z_n]$ be generators of $I(\{a\}) \subset \mathbb{K}[z_1, \ldots, z_n]$. For $j \in \{1, \ldots, r\}$ let $p_j = \sum_I c_{j,I} z^I$ where $c_{j,I} \in F$ for each multiindex I. If $\sigma \in \mathrm{Aut}(\mathbb{K}, F)$ we have

$$0 = \sigma(0) = \sigma(p_j(a)) = p_j(\sigma(a_1), \ldots, \sigma(a_n)) = \sum_I c_{j,I} \sigma(a_1)^{i_1} \ldots \sigma(a_n)^{i_n} = p_j(\sigma(a)). \tag{1.7.1}$$

(The third equality holds because p_j has coefficients in F.) Since the above equality holds for generators of the ideal of $\{a\}$, we get that $(\sigma(a_1), \ldots, \sigma(a_n)) = (a_1, \ldots, a_n)$ for all $\sigma \in \mathrm{Aut}(\mathbb{K}, F)$. By our hypothesis on $\mathrm{Aut}(\mathbb{K}, F)$ it follows that $a_i \in F$ for all i.

Example 1.7.4. Let $Q \in \mathbb{R}[Z_0, \ldots, Z_n]_2$ be a non zero quadratic form. Then $Z := V(Q) \subset \mathbb{P}^n(\mathbb{C})$ is a projective variety defined over $\mathbb{R}$. In fact if Q has rank at least 2 then Q generates $I(Z)$, and if Q has rank 1, i.e. $Q = L^2$ for $L \in \mathbb{C}[Z_0, \ldots, Z_n]_1$ then either $L \in \mathbb{R}[Z_0, \ldots, Z_n]_1$ or $\sqrt{-1}L \in \mathbb{R}[Z_0, \ldots, Z_n]_1$.

Example 1.7.5. The Fermat hypersurface $X := V(\sum_{i=0}^n Z_i^d)$ is defined over the prime field. In order to check this one must show that $I(X)$, i.e. the radical of $(\sum_{i=0}^n Z_i^d)$ is generated by a polynomial with coefficients in the prime field. If $\mathrm{char}\,\mathbb{K}$ does not divide d then the polynomial $\sum_{i=0}^n Z_i^d$ generates a radical ideal in $\mathbb{K}[Z_0, \ldots, Z_n]$ (to see this take the formal partial derivative with respect to one of its variables), and hence it generates $I(X)$. Since the coefficients of $\sum_{i=0}^n Z_i^d$ belong to the prime field we are done. If $\mathrm{char}\,\mathbb{K} = p > 0$ write $d = p^r d_0$ where p does not divide d_0. Then $\sum_{i=0}^n Z_i^d = (\sum_{i=0}^n Z_i^{d_0})^{p^r}$ and hence $I(X)$ is generated by $\sum_{i=0}^n Z_i^{d_0}$ (see above). Since the coefficients of $\sum_{i=0}^n Z_i^{d_0}$ belong to the prime field we are done.

Remark 1.7.6. Let $F \subset F' \subset \mathbb{K}$ be an inclusion of fields, and let $X \subset \mathbb{P}^n(\mathbb{K})$ be a locally closed subset defined over F. Then X is also defined over F'. In particular if X is defined over the prime field it is defined over every subfield of $\mathbb{K}$.

Definition 1.7.7. Let $X \subset \mathbb{P}^n(\mathbb{K})$ be a locally closed subset defined over F. We let $X(F) \subset X$ be the set of points represented by $(n+1)$-tuples $(Z_0, Z_1, \ldots, Z_n) \in F^{n+1}\backslash\{(0, \ldots, 0)\}$.

Remark 1.7.8. Let $X \subset \mathbb{A}^n(\mathbb{K})$ be a locally closed subset defined over F. Then $X(F) \subset X$ is equal to $X \cap \mathbb{A}^n(F)$.

Remark 1.7.9. Let $F \subset F' \subset \mathbb{K}$ be an inclusion of fields, and let $X \subset \mathbb{P}^n(\mathbb{K})$ be a locally closed subset defined over F. Then X is also defined over F' and hence $X(F')$ is also defined. In particular $X(\mathbb{K})$ is defined and equals X.

Remark 1.7.10. Let p be a prime, and suppose that $\mathbb{F}_q \subset \mathbb{K}$ where $q = p^r$. Let $X \subset \mathbb{P}^n(\overline{\mathbb{F}}_p)$ be a locally closed subset defined over F_q. For each $m \in \mathbb{N}_+$ there is a unique inclusion $\mathbb{F}_q \subset \mathbb{F}_{q^m} \subset \mathbb{K}$, and hence we have $X(\mathbb{F}_{q^m})$. Clearly $X(\mathbb{F}_{q^m})$ is a finite set.

Definition 1.7.11. Let $X \subset \mathbb{P}^n(\overline{\mathbb{F}}_p)$ be a locally closed subset defined over F_q, where $q = p^r$. The *Weil Zeta function of* X is defined to be formal power series in the variable T given by

$$Z(X, T) := \exp\left(\sum_{m=1}^{\infty} \frac{|X(\mathbb{F}_{q^m})|}{m} T^m\right) \tag{1.7.2}$$

Definition 1.7.12. Let $X \subset \mathbb{P}^n(\mathbb{K})$ and $Y \subset \mathbb{P}^m(\mathbb{K})$ be locally closed subset, both defined over a subfield $F \subset \mathbb{K}$. A map $\varphi := X \to Y$ is *defined over* F if for each $a \in X$ there exist an open $U \subset X$ containing a and $P_j \in F[Z_0, \ldots, Z_n]_d$ for $j \in \{0, \ldots, m\}$ (d depends on U), such that the restriction of φ to U is

$$\begin{array}{ccc} U & \longrightarrow & \mathbb{P}^m \\ [Z] & \to & [P_0(Z), \ldots, P_m(Z)] \end{array} \tag{1.7.3}$$

(of course $(P_0(Z), \ldots, P_m(Z)) \neq (0, \ldots, 0)$ for all $[Z] \in U$).

Let $F \subset \mathbb{K}$ be a subfield. If $X \subset \mathbb{P}^n(\mathbb{K})$ is a locally closed subset defined over F then the identity map $\mathrm{Id}_X : X \to X$ is clearly defined over F. If $X \subset \mathbb{P}^n(\mathbb{K})$, and $Y \subset \mathbb{P}^m(\mathbb{K})$, $W \subset \mathbb{P}^l(\mathbb{K})$ are locally closed subsets defined over F and $\varphi : X \to Y$, $\psi : Y \to W$ are regular maps defined over F then the composition $\psi \circ \varphi : X \to W$ is also defined over F. In fact this holds because if $P \in F[Z_0, \ldots, Z_m]_d$ and $Q_0, \ldots, Q_m \in F[T_0, \ldots, T_n]_e$ then $P(Q_0, \ldots, Q_m) \in F[T_0, \ldots, T_n]_{de}$.

Hence we have the category of quasi projective varieties defined over F. In particular we have the notion of isomorphism over F of varieties defined over F.

Remark 1.7.13. Let $X \subset \mathbb{P}^n(\mathbb{K})$ and $Y \subset \mathbb{P}^m(\mathbb{K})$ be locally closed subsets defined over F. If $\varphi : X \to Y$ is a regular map defined over F then $\varphi(X(F)) \subset Y(F)$ because the value of a polynomial with coefficients in F at $(A_0, \ldots, A_n) \in F^{n+1}$ belongs to F.

Example 1.7.14. Let $Q_1, Q_2 \in \mathbb{R}[Z_0, \ldots, Z_n]_2$ be non degenerate quadratic forms, and let $X_i := V(Q_i)$ for $i \in \{1, 2\}$. Then $X_i \subset \mathbb{P}^n(\mathbb{C})$ is a projective variety defined over $\mathbb{R}$. Since Q_i is diagonalizable in suitable coordinates, there exists a projectivity $\varphi : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$ whose restriction to X_1 defines an isomorphism $X_1 \xrightarrow{\sim} X_2$. In particular X_1 is isomorphic to X_2 (over $\mathbb{C}$). On the other hand X_1 is not necessarily isomorphic to X_2 over $\mathbb{R}$. In fact let $Q_1 := \sum_{j=0}^n Z_j^2$ and $Q_2 := Z_0^2 - \sum_{j=1}^n Z_j^2$. Thus $X_1(\mathbb{R})$ is empty while $X_2(\mathbb{R})$ is not empty. Since a regular map $\varphi : X_1 \to X_2$ defined over $\mathbb{R}$ maps $X_1(\mathbb{R})$ to $X_2(\mathbb{R})$ it follows that X_1 is not isomorphic to X_2 over $\mathbb{R}$ (we assume that $n \geqslant 1$).

Under a suitable hypothesis we can avoid computing the radical of ideals if we wish to decide whether a locally closed subset $X \subset \mathbb{P}^n(\mathbb{K})$ is defined over a subfield $F \subset \mathbb{K}$. Let $\mathrm{Aut}(\mathbb{K}/F)$ be the group of automorphisms of K which are the identity on F.

Proposition 1.7.15. *Suppose that the fixed field of* $\mathrm{Aut}(\mathbb{K}/F)$ *is equal to* F. *Let* $X \subset \mathbb{P}^n(\mathbb{K})$ *be a locally closed subset given by* $V(I) \backslash V(J)$ *where* $I, J \subset \mathbb{K}[Z_0, \ldots, Z_n]$ *are homogeneous ideals generated by polynomials in* $F[Z_0, \ldots, Z_n]$. *Then* X *is defined over* F.

Before proving Proposition 1.7.15 we go through a few preliminaries. The group $\mathrm{Aut}(\mathbb{K})$ of field automorphisms of $\mathbb{K}$ acts on $\mathbb{P}^n$ as follows: for $\sigma \in \mathrm{Aut}(\mathbb{K})$

$$\begin{array}{ccc} \mathrm{Aut}(\mathbb{K}) \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ (\sigma, [Z_0, \ldots, Z_n]) & \mapsto & [\sigma(Z_0), \ldots, \sigma(Z_n)] \end{array} \tag{1.7.4}$$

Note that if $X \subset \mathbb{A}^n$ $(= \mathbb{P}^n_{Z_0})$ then $\sigma(z_1, \ldots, z_n) = (\sigma(z_1), \ldots, \sigma(z_n))$.

Remark 1.7.16. In general the map $\mathbb{P}^n \to \mathbb{P}^n$ that one gets by fixing a non trivial $\sigma \in \mathrm{Aut}(\mathbb{K})$ in (1.7.4) is not regular. For example if $F = \mathbb{R} \subset \mathbb{C}$ and σ is complex conjugation the map is not regular.

Proposition 1.7.17. *Let* $X \subset \mathbb{P}^n(\mathbb{K})$ *be a locally closed subset given by* $V(I) \backslash V(J)$ *where* $I, J \subset \mathbb{K}[Z_0, \ldots, Z_n]$ *are homogeneous ideals generated by polynomials in* $F[Z_0, \ldots, Z_n]$. *If* $\sigma \in \mathrm{Aut}(\mathbb{K}/F)$ *then* $\sigma(X) = X$.

Proof. It suffices to prove that $\sigma(X) = X$ for $X = V(I) \subset \mathbb{P}^n(\mathbb{K})$ where $I \subset \mathbb{K}[Z_0, \ldots, Z_n]$ is a homogeneous ideal generated by polynomials in $F[Z_0, \ldots, Z_n]$. Let $P \in F[Z_0, \ldots, Z_n] \cap I(X)$ be homogeneous. Thus $P = \sum_I c_I Z^I$ where each c_I belongs to F. If $[A_0, \ldots, A_n] \in X$ then $P(A_0, \ldots, A_n) = 0$ and hence

$$0 = \sigma(P(A_0, \ldots, A_n)) = \sum_I \sigma(c_I) \sigma(A_0)^{i_0} \ldots \sigma(A_n)^{i_n} = \sum_I c_I \sigma(A_0)^{i_0} \ldots \sigma(A_n)^{i_n} = P(\sigma(A_0), \ldots, \sigma(A_n)).$$

This proves that $\sigma(X) \subset X$ because the ideal $I(X) \subset \mathbb{K}[Z_0, \ldots, Z_n]$ is generated by homogeneous elements in $F[Z_0, \ldots, Z_n]$. Thus we also have $\sigma^{-1}(X) \subset X$ and hence $X \subset \sigma(X)$. □

Proof of Proposition 1.7.15. The group $\mathrm{Aut}(\mathbb{K}/F)$ acts on $\mathbb{K}[Z_0, \ldots, Z_n]$ by acting on the coefficients of polynomials. We claim that $\mathrm{Aut}(\mathbb{K}/F)$ maps $I(X)$ to itself. In fact let $\sigma \in \mathrm{Aut}(\mathbb{K}/F)$ and let $P \in I(X)$ be a homogeneous polynomial, $P = \sum_I c_I Z^I$. By Proposition 1.7.17 we have $\sigma^{-1}(X) = X$, hence

$$\sigma(P)(A) = \sum_I \sigma(c_I) A^I = \sigma\left(\sum_I c_I \sigma^{-1}(A_0)^{i_0} \ldots \sigma^{-1}(A_n)^{i_n}\right) = \sigma(P(\sigma^{-1}(A)) = 0.$$

We have an obvious isomorphism $\mathbb{K}[Z_0, \ldots, Z_n] \cong \mathbb{K} \otimes_F F[Z_0, \ldots, Z_n]$ and the action of $\mathrm{Aut}(\mathbb{K}/F)$ that we have just defined matches the action considered in Section A.6. By Proposition A.6.3 it follows that $I(X)$ is generated (as $\mathbb{K}$ vector space by its intersection with $F[Z_0, \ldots, Z_n]$. This proves Proposition 1.7.15. □

Example 1.7.18. Assume that $\mathrm{char}\,\mathbb{K} = p > 0$. Let $F \colon \mathbb{K} \to \mathbb{K}$ be the Frobenius automorphism: $F(a) := a^p$. Let r be a positive natural number. Of course F^r is also an automorphism of $\mathbb{K}$. Note that $F^r(a) = a^q$ and that $F^r \in \mathrm{Aut}(\mathbb{K}/\mathbb{F}_q)$. There exists a unique embedding $\mathbb{F}_q \subset \mathbb{K}$. Suppose that $X \subset \mathbb{P}^n$ is a locally closed subset defined over $\mathbb{F}_q$. Proposition 1.7.17 gives that we have the bijective map

$$\begin{array}{ccc} X & \xrightarrow{\pi} & X \\ [Z] & \mapsto & [Z_0^q, \ldots, Z_n^q]. \end{array}$$

This is the *Frobenius map of* X. Note the exceptional feature of the Frobenius map: it is regular (see remark 1.7.16) and even defined over the prime field. Note also Note also that $X(\mathbb{F}_q)$ is equal to the fixed locus of π:

$$X(\mathbb{F}_q) = \mathrm{Fix}(\pi). \tag{1.7.5}$$

1.8 Geometry and Algebra

Below is a remarkable consequences of Theorem 1.6.2.

Proposition 1.8.1. *Let R be a finitely generated $\mathbb{K}$ algebra with no non zero nilpotents. There exists an affine variety X such that $\mathbb{K}[X] \cong R$ (as $\mathbb{K}$ algebras).*

Proof. Let $\alpha_1, \ldots, \alpha_n$ be generators (over $\mathbb{K}$) of R, and let $\varphi \colon \mathbb{K}[z_1, \ldots, z_n] \to R$ be the surjection of algebras mapping z_i to α_i. The kernel of φ is an ideal $I \subset \mathbb{K}[z_1, \ldots, z_n]$, which is radical because R has no nilpotents. Let $X := V(I) \subset \mathbb{A}^n$. Then $\mathbb{K}[X] \cong R$ by Theorem 1.6.2. □

The Nullstellensatz allows one to construct X abstractly from the $\mathbb{K}$ algebra as follows. Let

$$\mathrm{Spec}_m(R) := \{\mathfrak{m} \subset R \mid \mathfrak{m} \text{ is a maximal ideal of } R\}$$

be the *maximal spectrum* of R. Hilbert's Nullstellensatz gives a bijection

$$\begin{array}{ccc} X & \leftrightarrow & \mathrm{Spec}_m(R) \\ p & \mapsto & \{f \in R \mid f(p) = 0\} \end{array}$$

Thus X may be identified with $\mathrm{Spec}_m(R)$. Moreover $f \in R$ defines a function $\mathrm{Spec}_m(R) \to \mathbb{K}$ by setting $f(\mathfrak{m}) := f \pmod{\mathfrak{m}}$. This makes sense because the composition

$$\mathbb{K} \longrightarrow R \longrightarrow R/\mathfrak{m} \tag{1.8.1}$$

is an isomorphism.

Actually we get a contravariant equivalence between the category of affine varieties over $\mathbb{K}$ with no non zero nilpotents and that of finitely generated $\mathbb{K}$-algebras. First we give a definition.

Definition 1.8.2. Let $\varphi\colon X \to Y$ be a regular map of non empty quasi projective varieties. The pull-back $\varphi^*\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ is the homomorphism of $\mathbb{K}$ algebras defined by

$$\begin{array}{ccc} \mathbb{K}[Y] & \xrightarrow{\varphi^*} & \mathbb{K}[X] \\ f & \mapsto & f \circ \varphi \end{array} \tag{1.8.2}$$

Proposition 1.8.3. *Let Y be an affine variety, and let X be a quasi projective variety. The map*

$$\begin{array}{ccc} \{X \xrightarrow{\varphi} Y \mid \varphi \text{ regular}\} & \longrightarrow & \{\mathbb{K}[Y] \xrightarrow{\alpha} \mathbb{K}[X] \mid \alpha \text{ homom. of } \mathbb{K}\text{-algebras}\} \\ \varphi & \mapsto & \varphi^* \end{array} \tag{1.8.3}$$

is a bijection.

Proof. We may assume that $Y \subset \mathbb{A}^n$ is closed; for $i \in \{1, \ldots, n\}$ let $\overline{z}_i \coloneqq z_{i|X}$. Suppose that $f, g\colon X \to Y$ are regular maps, and that $f^* = g^*$. Then $f^*(\overline{z}_i) = g^*(\overline{z}_i)$ for $i \in \{1, \ldots, n\}$, and hence $f = g$. This proves injectivity of the map in (1.8.3).

In order to prove surjectivity, let $\alpha\colon \mathbb{K}[Y] \to \mathbb{K}[X]$ be a homomorphism of $\mathbb{K}$ algebras. Let $f_i := \alpha(\overline{z}_i)$, and let $\varphi\colon X \to \mathbb{A}^n$ be the regular map defined by $\varphi(x) := (f_1(x), \ldots, f_n(x))$ for $x \in X$. We claim that $\varphi(x) \in Y$ for all $x \in X$. In fact, since Y is closed, it suffices to show that $g(\varphi(x)) = 0$ for all $g \in I(X)$. Now

$$g(\varphi(x)) = g(f_1(x), \ldots, f_n(x)) = g(\alpha(\overline{z}_1), \ldots, \alpha(\overline{z}_n)) = \alpha(g(\overline{z}_1), \ldots, \overline{z}_n) = \alpha(0) = 0.$$

(The third equality holds because α is a homomorphism of $\mathbb{K}$-algebras.) Thus φ is a regular map $f\colon X \to Y$ such that $\varphi^*(\overline{z}_i)) = \alpha(\overline{z}_i)$ for $i \in \{1, \ldots, n\}$. By Theorem 1.6.2 the $\mathbb{K}$-algebra $\mathbb{K}[Y]$ is generated by $\overline{z}_1, \ldots, \overline{z}_n$; it follows that $\varphi^* = \alpha$. $\square$

Corollary 1.8.4. *In Proposition 1.8.1, the affine variety X such that $\mathbb{K}[X] \cong R$ is unique up to isomorphism.*

Proposition 1.8.3 shows that by associating to an affine variety over $\mathbb{K}$ the $\mathbb{K}$-algebra of its regular functions we get a contravariant equivalence between the category of affine varieties over $\mathbb{K}$ (with maps the regular maps) and the category of finitely generated $\mathbb{K}$-algebras with no non-zero nilpotent elements. Note that if $\varphi\colon S \to R$ is a morphism of finitely generated $\mathbb{K}$-algebras with no non-zero nilpotent elements the corresponding map (in the reverse direction) between the associated affine varieties is given by

$$\begin{array}{ccc} \operatorname{Spec}_m(R) & \longrightarrow & \operatorname{Spec}_m(S) \\ \mathfrak{m} & \mapsto & \varphi^{-1}(\mathfrak{m}) =: \mathfrak{m}^c \end{array}$$

(notice that $\varphi^{-1}(\mathfrak{m})$ is maximal because φ is a morphism of $\mathbb{K}$-algebras).

1.9 Exercises

Exercise 1.9.1. Which of the following subsets of $\mathbb{A}^2$ are locally closed? Which are closed?

(a) $X \coloneqq \left\{(x, y) \mid \exp\left(2\pi\sqrt{-1}x\right) = 1\right\} \subset \mathbb{A}^2(\mathbb{C})$.

(b) $Y \coloneqq \left\{\left(t, t^2\right) \mid t \in \mathbb{K}\right\} \subset \mathbb{A}^2(\mathbb{K})$.

(c) $W \coloneqq \left\{\left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right) \mid t \in \mathbb{C} \setminus \left\{\pm\sqrt{-1}\right\}\right\} \subset \mathbb{A}^2(\mathbb{C})$.

(d) $V \coloneqq \left\{(t, tu) \mid (t, u) \in \mathbb{K}^2\right\} \subset \mathbb{A}^2(\mathbb{K})$.

Exercise 1.9.2. Compute $I(Z)$ for

1. $Z = V\left(x^2 + 1\right) \subset \mathbb{A}^1(\mathbb{K})$,

2. $Z = \mathbb{Z}^2 \subset \mathbb{A}^2(\mathbb{C})$,

3. $Z = V\left(x^2 - y^2, x^2 - xy\right) \subset \mathbb{A}^2(\mathbb{K})$.

Exercise 1.9.3. Let $M_{2,2}(\mathbb{C})$ be the complex vector-space of 2×2 complex matrices. Let $n > 0$ and let $U_n \subset M_{2,2}(\mathbb{C})$ be the set of matrices T such that $T^n = 1$ (here $1 \in M_{2,2}(\mathbb{C})$ is the unit matrix).

1. Prove that U_n is a closed subset (for the Zariski Topology) of $M_{2,2}(\mathbb{C})$.

2. Describe the irreducible components of U_n and show that there are $\binom{n+1}{2}$ of them.

Exercise 1.9.4. Let $f_1, \ldots, f_r \in \mathbb{K}[x, y]$ and suppose that

$$\gcd\{f_1, \ldots, f_r\} = 1.$$

Show that $V(f_1, \ldots, f_r) \subset \mathbb{A}^2(\mathbb{K})$ is finite.

Exercise 1.9.5. Let $X \subset \mathbb{A}^2(\mathbb{K})$ be a proper closed irreducible subset. Show that Z is either a singleton or an irreducible hypersurface.

Exercise 1.9.6. Let $M_n(\mathbb{K})$ be the vector-space of $n \times n$ matrices with entries in $\mathbb{K}$, and let $M_n(\mathbb{K})_- \subset M_n(\mathbb{K})$ be the subspace of skew-symmetric matrices. Let $X \in M_n(\mathbb{K})_-$: then

$$X = \begin{bmatrix} 0 & x_{1,2} & \cdots & \cdots & x_{1,n} \\ -x_{1,2} & 0 & x_{2,3} & \cdots & x_{2,n} \\ -x_{1,3} & -x_{1,3} & 0 & \cdots & x_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{1,n} & -x_{2,n} & \cdots & \cdots & 0 \end{bmatrix}$$

Thus $\{x_{1,2}, \ldots, x_{1,n}, x_{2,3}, \ldots, x_{n-1,n}\}$ is a basis of the dual of $M_n(\mathbb{K})_-$, and hence $\mathbb{K}[x_{1,2}, \ldots, x_{1,n}, x_{2,3}, \ldots, x_{n-1,n}]$ is the $\mathbb{K}$ algebra of. polynomial functions on $M_n(\mathbb{K})_-$. Let $\Delta_n \subset M_n(\mathbb{K})_-$ be the set of $n \times n$ singular skew-symmetric matrices, and let δ_n be the polynomial on $M_n(\mathbb{K})_-$ given by $\delta_n(X) := \det X$. Then Δ_n is closed in $M_n(\mathbb{K})_-$ because $\Delta_n = V(\delta_n)$. Prove the following:

(1.9.6a) If n is odd then $\Delta_n = M_n(\mathbb{K})_-$.

(1.9.6b) If n is even then Δ_n is a hypersurface and $I(\Delta_n) \neq (\delta_n)$.

Exercise 1.9.7. An affine map

$$\begin{array}{ccc} \mathbb{A}^n & \longrightarrow & \mathbb{A}^n \\ Z & \mapsto & A \cdot Z + B \end{array}$$

(here Z, B are column vectors with n entries and $A \in \mathrm{GL}_n(\mathbb{K})$) is an automorphism of $\mathbb{A}^n$.

(1.9.7a) Show that *every* automorphism of $\mathbb{A}^1$ is an affine map.

(1.9.7b) Let $n \geqslant 2$. Show that if $f \in \mathbb{K}[z_1, \ldots, z_{n-1}]$ then

$$\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{\Phi_f} & \mathbb{A}^n \\ z & \mapsto & (z_1, \ldots, x_{n-1}, z_n + f(z_1, \ldots, z_{n-1}) \end{array} \tag{1.9.4}$$

is an automorphism. Prove that Φ_f is an affine map if and only if $\deg f \leqslant 1$.

Exercise 1.9.8. Show that one can prove the validity of Theorem 1.6.2 for $\mathbb{A}^n$ by invoking unique factorization in $\mathbb{K}[z_1, \ldots, z_n]$, without using the Nullstellensatz.

Exercise 1.9.9. Let K be a field. Given a finite-dimensional K-vector space V define the formal power series $p_V \in \mathbb{Z}[[t]]$ as

$$P_V := \sum_{d=0}^{\infty} (\dim_k \operatorname{Sym}^d V) t^d$$

where $\operatorname{Sym}^d V$ is the symmetric product of V. Thus if $V = K[x_1, \ldots, x_n]_1$ then $S^d(K[x_1, \ldots, x_n]_1) = K[x_1, \ldots, x_n]_d$.

1. Prove that if $V = U \oplus W$ then $P_V = P_U \cdot P_W$.
2. Prove that if $\dim_K V = n$ then $P_V = (1-t)^{-n}$ and hence the equality in (1.5.10) holds.

Exercise 1.9.10. The purpose of the present exercise is to give a different proof of the properties of the Veronese map ν_d^n discussed in Example 1.5.12, *valid if* $\operatorname{char} \mathbb{K} = 0$*, or more generally* $\operatorname{char} \mathbb{K}$ *does not divide* $d!$. Let

$$\begin{array}{ccc} \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1) & \xrightarrow{\mu_d^n} & \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_d) \\ [L] & \mapsto & [L^d] \end{array} \tag{1.9.5}$$

and let $\mathscr{W}_d^n = \operatorname{im}(\mu_d^n)$. The above map can be identified with the Veronese map ν_d^n. In fact, writing $L \in \mathbb{K}[T_0, \ldots, T_n]_1$ as $L = \sum_{i=0}^n \alpha_i T_i$, we see that $[\alpha_0, \ldots, \alpha_n]$ are coordinates on $\mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1)$, and they give an identification $\mathbb{P}^n \xrightarrow{\sim} \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1)$. Moreover, let

$$\begin{array}{ccc} \mathbb{P}^{\binom{d+n}{n}-1} & \xrightarrow{\sim} & \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_d), \\ [\ldots, \xi_I, \ldots] & \mapsto & \sum\limits_{\substack{I=(i_0,\ldots,i_n) \\ i_0+\ldots+i_n=d}} \frac{d!}{i_0! \cdot \ldots \cdot i_n!} \xi_I T^I \end{array}$$

where $T^I = T_0^{i_0} \cdot \ldots \cdot T_n^{i_n}$. By Newton's formula $(\sum_{i=0}^n \alpha_i T_i)^d = \sum_I \frac{d!}{i_0! \cdot \ldots \cdot i_n!} \alpha^I T^I$, we see that, modulo the above isomorphisms, the Veronese map ν_d^n is identified with μ_d^n, and hence $\mathscr{V}_d^n$ is identified with $\mathscr{W}_d^n$.

Now let us show that $\mathscr{W}_d^n$ is closed. The key observation is that $[F] \in \mathscr{W}_d^n$ if and only if $\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n}$ span a 1-dimensional subspace of $\mathbb{K}[Z_0, \ldots, Z_n]$. This may be proved by induction on $\deg F$ and *Euler's identity*

$$\sum_{j=0}^n Z_j \frac{\partial F}{\partial Z_j} = (\deg F) \cdot F, \tag{1.9.6}$$

valid for F *homogeneous.* Now, the condition that $\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n}$ span a 1-dimensional subspace of $\mathbb{K}[Z_0, \ldots, Z_n]$ is equivalent to the vanishing of determinants of all 2×2 minors of the matrix whose entries are the coordinates of $\frac{\partial F}{\partial Z_0}, \ldots, \frac{\partial F}{\partial Z_n}$; thus $\mathscr{W}_d^n$ is closed.

In order to show that μ_d^n is an isomorphism, we notice that if $F = L^d$, where $L \in \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1$ is non zero, then for each $i \in \{0, \ldots, n\}$ the partial derivative $\frac{\partial^{n-1} F}{\partial Z_i^{n-1}}$ is a multiple of L (eventually equal to 0 if $\frac{\partial L}{\partial Z_i} = 0$), and that one at least of such $(n-1)$-th partial derivative is non zero. Thus, the inverse of μ_d^n is the regular map $\theta_d^n \colon \mathscr{W}_d^n \longrightarrow \mathbb{P}(\mathbb{K}[T_0, \ldots, T_n]_1)$ defined by

$$\theta_d^n([F]) := \begin{cases} [\frac{\partial^{n-1} F}{\partial Z_0^{n-1}}] & \text{if } \frac{\partial^{n-1} F}{\partial Z_0^{n-1}} \neq 0, \\ \cdots\cdots & \cdots\cdots \\ [\frac{\partial^{n-1} F}{\partial Z_n^{n-1}}] & \text{if } \frac{\partial^{n-1} F}{\partial Z_n^{n-1}} \neq 0. \end{cases} \tag{1.9.7}$$

Exercise 1.9.11. Let $X \subset \mathbb{P}^n(\mathbb{C})$ and $Y \subset \mathbb{P}^m(\mathbb{C})$ be complex quasi projective varieties defined over $\mathbb{R}$, and let $\varphi \colon X \to Y$ be a regular map defined over $\mathbb{R}$. Note that the map $X(\mathbb{R}) \to Y(\mathbb{R})$ defined by the restriction of φ to $X(\mathbb{R})$ is continuous for the euclidean topologies of $X(\mathbb{R})$ and $Y(\mathbb{R})$. Using this prove that the real quadrics

$$V(Z_0^2 - Z_1^2 - Z_2^2 - Z_3^2) \subset \mathbb{P}^3(\mathbb{C}), \quad V(Z_0^2 + Z_1^2 - Z_2^2 - Z_3^2) \subset \mathbb{P}^3(\mathbb{C}) \tag{1.9.8}$$

are not isomorphic over $\mathbb{R}$ although they are isomorphic (actually projectively equivalent) over $\mathbb{C}$.

Exercise 1.9.12. We recall that if $\phi \colon B \to A$ is a homomorphism of rings, and $I \subset A$, $J \subset B$ are ideals, the *contraction* $I^c \subset B$ and the *extension* $J^e \subset A$ are the ideals defined as follows:

$$I^c := \phi^{-1}(I), \quad J^e := \left\{ \sum_{i=1}^r \lambda_i \phi(b_i) \mid \lambda_i \in A,\ b_i \in J\ \forall i = 1, \ldots, r \right\} \tag{1.9.9}$$

(In other words, J^e is the ideal of A generated by $\phi(J)$.)

Let $f\colon X \to Y$ be a regular map between affine varieties and suppose that $f^*\colon \mathbb{K}[Y] \longrightarrow \mathbb{K}[X]$ is injective.

1. Let $p \in X$. Prove that $\mathfrak{m}_p^c = \mathfrak{m}_{f(p)}$, in particular it is maximal.
2. Let $q \in Y$. Prove that
$$f^{-1}(q) = \left\{p \in X \mid \mathfrak{m}_p \supset \mathfrak{m}_q^e\right\},$$
and conclude, by the Nulstellensatz, that $f^{-1}(q)$ is not empty if and only if $\mathfrak{m}_q^e \neq \mathbb{K}[X]$.

Exercise 1.9.13. The left action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathbb{A}^n$ defines a left action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathbb{K}[z_1, \ldots, z_n]$ as follows. Let $\phi \in \mathbb{K}[z_1, \ldots, z_n]$ and $g \in \mathrm{GL}_n(\mathbb{K})$. Let z be the column vector with entries $z_1, \ldots, z_n$: we define $g\phi \in \mathbb{K}[z_1, \ldots, z_n]$ by letting
$$g\phi(X) := \phi(g^{-1} \cdot z).$$
Now let $G < \mathrm{GL}_n(\mathbb{K})$ be a subgroup. The *algebra of G-invariant polynomials* is
$$\mathbb{K}[z_1, \ldots, z_n]^G := \{\phi\mathbb{K}[z_1, \ldots, z_n] \in \mid g\phi = \phi \;\forall g \in G\}.$$
(it is clearly a $\mathbb{K}$-algebra). Now suppose that G is finite. One identifies $\mathbb{A}^n/G$ with an affine variety proceeding as follows.

1. Define the *Reynolds operator* as
$$\begin{array}{ccc} \mathbb{K}[z_1, \ldots, z_n] & \longrightarrow & \mathbb{K}[z_1, \ldots, z_n]^G \\ \phi & \mapsto & \frac{1}{|G|}\sum_{g\in G} g\phi. \end{array}$$
Prove the *Reynolds identity*
$$R(\phi\psi) = \phi R(\psi) \quad \forall \phi \in \mathbb{K}[z_1, \ldots, z_n]^G.$$
2. Let $I \subset \mathbb{K}[z_1, \ldots, z_n]$ be the ideal generated by *homogeneous* $\phi \in \mathbb{K}[z_1, \ldots, z_n]^G$ of strictly positive degree (i.e. non-constant). By Hilbert's basis theorem there exists a finite basis $\{\phi_1, \ldots, \phi_d\}$ of I; we may assume that each ϕ_i is homogeneous and G-invariant. Prove that $\mathbb{K}[z_1, \ldots, z_n]^G$ is generated *as $\mathbb{K}$-algebra* by $\phi_1, \ldots, \phi_d$. Since $\mathbb{K}[z_1, \ldots, z_n]^G$ is an integral domain with no nilpotents it follows that there exist an affine variety X (well-defined up to isomorphism) such that $\mathbb{K}[X] \xrightarrow{\sim} \mathbb{K}[z_1, \ldots, z_n]^G$. One sets $\mathbb{A}^n/G =: X$.
3. Let $\iota\colon \mathbb{K}[z_1, \ldots, z_n]^G \hookrightarrow \mathbb{K}[z_1, \ldots, z_n]$ be the inclusion map. By Proposition 1.8.3, there exist a unique regular map
$$\mathbb{A}^n \xrightarrow{\pi} X = \mathbb{A}^n/G. \tag{1.9.10}$$
such that $\iota = \pi^*$. Prove that
$$\pi(p) = \pi(q) \quad \text{if and only if} \quad q = gp \text{ for some } g \in G,$$
and that π is surjective. [*Hint:* Let $J \subset \mathbb{K}[z_1, \ldots, z_n]^G$ be an ideal. Show that $J^e \cap \mathbb{K}[z_1, \ldots, z_n]^G = J$ where J^e is the extension relative to the inclusion ι.]

Exercise 1.9.14. Keep notation and hypotheses as in Exercise 1.9.13. Describe explicitly $\mathbb{A}^n/G$ and the quotient map $\pi\colon \mathbb{A}^n \to \mathbb{A}^n/G$ for the following groups $G < \mathrm{GL}_n(\mathbb{K})$:

1. $n = 2$, $G = \{\pm 1_2\}$.
2. $n = 2$, $G = \left\langle \begin{pmatrix} \omega_k & 0 \\ 0 & \omega_k^{-1} \end{pmatrix} \right\rangle$ where ω_k is a primitive k-th rooth of 1.
3. $G = \mathcal{S}_n$, the group of permutation of n elements viewed in the obvious way as a subgroup of $\mathrm{GL}_n(\mathbb{K})$ (group of permutations of coordinates).

Chapter 2

Algebraic varieties

2.1 Introduction

The definition of quasi projective variety that we have given sounds very classical when compared to the definition of smooth manifold that one learns in a first course in Differential Geometry. In the present chapter we provide a definition of algebraic variety along the lines of the definition of smooth manifold. Quasi projective varieties are examples of algebraic varieties. ***********

2.2 Algebraic prevarieties

Definition of algebraic prevariety

Definition 2.2.1. Let X be a topological space. An *algebraic atlas* of X defined over $\mathbb{K}$ consists of an open covering $\mathscr{A} = \{A_i\}_{i\in I}$ of X, and for each $i \in I$ an affine variety V_i defined over $\mathbb{K}$ (with the Zariski topology) together with a homeomorphism $\varphi_i \colon V_i \longrightarrow A_i$ (an *affine chart*), such that for each $i, j \in I$ the transition map

$$\begin{array}{ccc} V_i \cap \varphi_i^{-1}(A_i \cap A_j) & \xrightarrow{\varphi_{j,i}} & V_j \cap \varphi_j^{-1}(A_j \cap A_i) \\ p & \mapsto & \varphi_j^{-1}(\varphi_i(p)) \end{array} \tag{2.2.1}$$

is a regular map of quasi projective varieties.

Example 2.2.2. Let X be a quasi projective variety. The collection $\mathscr{A} := \{A_i\}_{i\in I}$ of open affine subsets of X is a basis for the Zariski topology of X, see Proposition 1.5.15. Choosing for every $i \in I$ the identity affine chart $\mathrm{Id}_{A_i} \colon A_i \xrightarrow{\sim} A_i$ we get the *canonical algebraic atlas of* X.

Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be topological spaces with algebraic atlases over $\mathbb{K}$. Thus $\mathscr{A} = \{A_i\}_{i\in I}$ and $\mathscr{B} = \{B_j\}_{j\in J}$ are open coverings of X and Y respectively, and we are given homeomorphisms $\varphi_i \colon V_i \xrightarrow{\sim} A_i$ and $\psi_j \colon W_j \xrightarrow{\sim} B_j$ for all $i \in I$ and $j \in J$, where V_i and W_j are affine varieties.

Definition 2.2.3. A *regular map* $(X, \mathscr{A}) \to (Y, \mathscr{B})$ of topological spaces with algebraic atlases defined over $\mathbb{K}$ is a continuous map $f \colon X \to Y$ such that for all $i \in I$ and $j \in J$ the composition

$$\varphi_i^{-1}(A_i \cap f^{-1}B_j) \xrightarrow{\varphi_i|_{\dots}} A_i \cap f^{-1}B_j \xrightarrow{f|_{\dots}} B_j \xrightarrow{\psi_j^{-1}} W_j \tag{2.2.2}$$

is a regular map of (quasi projective) varieties. As a matter of notation we denote the map by $f \colon (X, \mathscr{A}) \to (Y, \mathscr{B})$ or simply by $f \colon X \to Y$.

Example 2.2.4. Let X, Y be quasi projective varieties and let $\mathscr{A}, \mathscr{B}$ be their canonical atlases, see Example 2.2.2. If $f \colon X \to Y$ is a regular map, then it is a regular map of topological spaces with atlases.

Note that the composition of regular maps between topological spaces with algebraic atlases is regular, and the identity map $(X, \mathscr{A}) \to (X, \mathscr{A})$ is regular.

Definition 2.2.5. Let X be a topological space. An algebraic atlas $\mathscr{A}$ on X is *equivalent* to an algebraic atlas $\mathscr{B}$ on X (both atlases defined over $\mathbb{K}$) if the identity maps $\mathrm{Id}_X := (X, \mathscr{A}) \to (X, \mathscr{B})$ and $\mathrm{Id}_X := (X, \mathscr{B}) \to (X, \mathscr{A})$ are both regular.

Note that $\mathscr{A}$ is equivalent to itself, $\mathscr{A}$ equivalent to $\mathscr{B}$ implies that $\mathscr{B}$ equivalent to $\mathscr{A}$, and that if $\mathscr{A}$ is equivalent to $\mathscr{B}$ and $\mathscr{B}$ is equivalent to $\mathscr{C}$, then $\mathscr{A}$ is equivalent to $\mathscr{C}$. This justifies the use of the word "equivalent".

Definition 2.2.6. An *algebraic prevariety defined over* $\mathbb{K}$ (or simply a prevariety) is a couple $(X, [\mathscr{A}])$ where X is a topological space and $[\mathscr{A}]$ is an equivalence class of algebraic atlases. It is of *finite type* it there exists a representative of the equivalence class of $\mathscr{A}$ with a finite set of indices. Let $(X, [\mathscr{A}])$ and $(Y, [\mathscr{B}])$ be algebraic prevarieties over $\mathbb{K}$; a map $f\colon X \to Y$ is *regular* if it is regular as map $(X, \mathscr{A}) \to (Y, \mathscr{B})$ (this makes sense because if it is regular for one choice of representative atlases then it is regular for any choice).

Whenever the equivalence class of finite algebraic atlases $[\mathscr{A}]$ is understood (or when we are too lazy to write it out) we denote $(X, [\mathscr{A}])$ by X. The topology of an algebraic prevariety $(X, [\mathscr{A}])$ is called (for obvious reasons) the Zariski topology of X.

Remark 2.2.7. A quasi projective variety with the equivalence class of its canonical atlas is a prevariety. In fact it is a prevariety of finite type because the Zariski topology is quasi-compact, see Corollary 1.3.9. Let X, Y be quasi projective varieties viewed as prevarieties (via their canonical atlases). A map $f\colon X \to Y$ is regular (as map of prevarieties) if and only if it is a regular map of quasi projective varieties.

Example 2.2.8. A finite algebraic atlas for $\mathbb{P}^n$ is as follows. Let $A_i := \mathbb{P}^n_{Z_i} \cong \mathbb{A}^n$ for $i \in \{0, \ldots, n\}$. Let $z_0(i), \ldots, z_{i-1}(i), z_{i+1}(i), \ldots, z_n(i)$ (there is no $z_i(i)$) be the affine coordinates on A_i given by $z_s(i) := Z_s/Z_i$. We can think of the coordinates $z_s(i)$ as giving the map $\varphi_i\colon \mathbb{A}^n \to A_i$. Thus $\varphi_i^{-1}(A_i \cap A_j) = \mathbb{A}^n \backslash V(z_j(i))$ and $\varphi_j^{-1}(A_j \cap A_i) = \mathbb{A}^n \backslash V(z_i(j))$. The transition map $\varphi_{j,i}$ is determined by the formulae

$$\varphi^*_{j,i}(z_s(j)) := \begin{cases} z_j(i)^{-1} \cdot z_s(i) & \text{if } s \neq i \\ z_j^{-1}(i) & \text{if } s = i \end{cases} \tag{2.2.3}$$

Example 2.2.9. Let X be a prevariety. An open subset $U \subset X$ can be given the structure of a prevariety so that the inclusion $U \hookrightarrow X$ is regular. In fact let $\{A_i\}_{i \in I}$ be an algebraic atlas, with affine charts $\varphi_i\colon V_i \to A_i$. For $i \in I$ let $W_i := \varphi_i^{-1}(A_i \cap U)$. Then W_i is an open subset of V_i, and it is the union of its open affine subsets $U_{i,j}$ where $j \in J(i)$ for an index set $J(i)$ which depends on $i \in I$. As algebraic atlas of U we take the collection $\{\varphi_i(U_{i,j})\}_{i \in I, j \in J(i)}$ with affine charts $\varphi_{i|U_{i,j}}\colon U_{i,j} \to \varphi_i(U_{i,j})$. Similarly, a closed subset $Y \subset X$ can be given the structure of a prevariety so that the inclusion $Y \hookrightarrow X$ is regular. We leave details to the reader. Lastly, if $Y \subset X$ is a locally closed subset, say $Y = U \cap W$ where $U \subset X$ is open and $W \subset X$ is closed, then Y is a closed subset of the prevariety U, and hence it inherits a structure of prevariety.

Prevarieties of finite type have an irreducible decomposition. First we prove the following result.

Lemma 2.2.10. *Let X be a prevariety of finite type, and let*

$$X \supset X_0 \supset X_1 \supset \ldots \supset X_n \supset X_{n+1} \ldots \tag{2.2.4}$$

be a descending chain of closed subsets indexed by $\mathbb{N}$. Then the chain is stationary, i.e. there exists $m \in \mathbb{N}$ such that $X_n = X_{n+1}$ for all $n \geqslant m$.

Proof. Let $\{A_i\}_{i\in I}$ be a finite algebraic atlas, with affine charts $\varphi_i\colon V_i \to A_i$. For each $i \in I$ the descending chain of closed subsets

$$V_i \supset \varphi_i^{1}(X_0) \supset \varphi_i^{1}(X_1) \supset \ldots \supset \varphi_i^{1}(X_n) \supset \varphi_i^{1}(X_{n+1}) \ldots \tag{2.2.5}$$

is stationary by Proposition 1.3.7. Thus there exists $m_i \in \mathbb{N}$ such that $X_n = X_{n+1}$ for all $n \geqslant m_i$. The proposition holds with $m \coloneqq \max\{m_i\}_{i\in I}$ (which exists because I is finite). □

Proposition 2.2.11. *If X is a prevariety of finite type it has an irreducible decomposition.*

Proof. Since Lemma 2.2.10 holds, one can repeat word-by-word the proof of Theorem 1.3.6. □

Prevarieties defined over a subfield

Let $F \subset \mathbb{K}$ be a subfield. Then one can repeat all the definitions above restricting to affine varieties and regular maps defined over F in order to define prevarieties defined over F. An algebraic atlas $\mathscr{A} = \{A_i\}_{i\in I}$ on a topological space X with affine charts $\varphi_i\colon V_i \to A_i$ is *defined over F* if

1. for all $i \in I$ the affine variety V_i is defined over F,

2. for all $i, j \in I$ the quasi projective variety $V_i \cap \varphi_i^{-1}(A_i \cap A_j)$ is defined over F and the transition map in (2.2.1) is regular.

Let $(X, [\mathscr{A}])$ and $(Y, [\mathscr{B}])$ be topological spaces X with algebraic atlases defined over F. A regular map $f\colon (X, [\mathscr{A}]) \to (Y, [\mathscr{B}])$ is *defined over F* if the maps in (2.2.2) are defined over F for every i, j. This said it is clear how to mimick the definitions that we have given in order to define what are prevarieties defined over F and what are regular maps defined over F. Note that if $(X, [\mathscr{A}])$ is a prevariety defined over F then $X(F)$ makes sense, it consists of all the points $\varphi_i(a)$ where $a \in V_i(F)$. This makes sense because if $\varphi_i(a) \in A_j$ then $\varphi_i(a) = \varphi_j(\varphi_j^{-1}(\varphi_i(a)))$ and since the map appearing in (2.2.1) is defined over F we have $\varphi_j^{-1}(\varphi_i(a)) \in V_j(F)$. Moreover if $f\colon (X, [\mathscr{A}]) \to (Y, [\mathscr{B}])$ is is a regular map defined over F then $f(X(F)) \subset Y(F)$.

Gluing affine varieties

A method for producing a topological space with an algebraic atlas is to glue affine varieties along open subsets via regular maps. The simplest case is the following: let V, W be affine varieties, with isomorphic open subsets $A \subset V$ and $B \subset W$, and let $f\colon A \xrightarrow{\sim} B$ be an isomorphism. Let $\sim$ be the equivalence relation on $V \sqcup W$ generated by letting $p \sim f(p)$ for $p \in A \subset V$ (and $f(p) \in B \subset W$). Let

$$X := V \sqcup W/\sim$$

be the quotient topological space. Let $\pi\colon (V \sqcup W) \to X$ be the quotient map. The associated algebraic atlas of X is given by the open covering $\{\pi(V), \pi(W)\}$ and the homeomorphisms $V \xrightarrow{\sim} \pi(V)$, $W \xrightarrow{\sim} \pi(W)$ obtained by restricting π.

Example 2.2.12. Let $V = W = \mathbb{A}^1$, $A = B = \mathbb{A}^1\backslash\{0\}$, and let

$$\begin{array}{ccc} A \supset \mathbb{A}^1\backslash\{0\} & \xrightarrow{f} & \mathbb{A}^1\backslash\{0\} \subset B \\ z & \mapsto & z^{-1} \end{array} \tag{2.2.6}$$

and

$$\begin{array}{ccc} A \supset \mathbb{A}^1\backslash\{0\} & \xrightarrow{g} & \mathbb{A}^1\backslash\{0\} \subset B \\ z & \mapsto & z \end{array} \tag{2.2.7}$$

Let X be the quotient topological space for the identification in (2.2.6), and let $\mathscr{A}$ be the corresponding atlas. The prevariety $(X, [\mathscr{A}])$ is isomorphic to $\mathbb{P}^1$ with its canonical algebraic atlas. In fact let $\widetilde{\varphi}\colon V \sqcup W \longrightarrow \mathbb{P}^1$ be the map defined by

$$\widetilde{\varphi}(z) := \begin{cases} [1, z] & \text{if } z \in V, \\ [z, 1] & \text{if } z \in W. \end{cases} \tag{2.2.8}$$

Then $\widetilde{\varphi}$ descends to a regular map $\varphi := (X, \mathscr{A}) \longrightarrow \mathbb{P}^1$ which is an isomorphism. We will come back later to the prevariety corresponding to the identification in (2.2.7).

A more general version of the gluing construction is as follows. Suppose that we are given

- a family of affine varieties $\{V_i\}_{i \in I}$,
- for all $i, j \in I$ open subsets $A_{i,j} \subset V_i$ and $B_{i,j} \subset V_j$ and a (gluing) regular map $\varphi_{j,i}\colon A_{i,j} \to B_{i,j}$,

subject to the following conditions:

Hypothesis 2.2.13. *1. For all $i \in I$ we have $A_{i,i} = B_{i,i} = V_i$ and $\varphi_{i,i} = \mathrm{Id}_{V_i}$.*

2. For all $i, j \in I$ we have $A_{j,i} = B_{i,j}$ (and of course $B_{j,i} = A_{i,j}$) $\varphi_{i,j} = \varphi_{j,i}^{-1}$.

3. For all $i, j, k \in I$ and $p \in A_{i,j}$ such that $\varphi_{j,i}(p) \in A_{jk}$ we have

$$\varphi_{k,j}(\varphi_{j,i}(p)) = \varphi_{k,i}(p). \tag{2.2.9}$$

Gluing Construction 2.2.14. Let $\sim$ be the relation on $\bigsqcup_{i \in I} V_i$ defined as follows. Let $p \in V_i$ and $q \in V_j$ for $i, j \in I$: then $p \sim q$ if $p \in A_{i,j}$, $q \in B_{i,j}$, and $q = \varphi_{j,i}(p)$. Then $\sim$ is an equivalence relation. In fact the relation is reflexive by Item (1), it is symmetric by Item (2), and it is transitive by Item (3). Let

$$X := \bigsqcup_{i \in I} V_i / \sim$$

be the quotient topological space. Let $\pi\colon \bigsqcup_{i \in I} V_i \to X$ be the quotient map. The associated algebraic atlas of X is given by the open covering $\{\pi(V_i)\}_{i \in I}$ and the homeomorphisms $V_i \xrightarrow{\sim} \pi(V_i)$ obtained by restricting π.

Example 2.2.15. Let $I := \{0, 1, \ldots, n\}$ and let $V_i = \mathbb{A}^n$ for all $i \in I$. Let $(z_0(i), \ldots, z_{i-1}(i), z_{i+1}(i), \ldots, z_n(i))$ be affine coordinates on V_i (note that there is no coordinate $z_i(i)$). Let $A_{i,j} := \mathbb{A}^n \backslash V(z_j(i))$ and $B_{i,j} := \mathbb{A}^n \backslash V(z_i(j))$. We define $\varphi_{j,i}\colon A_{i,j} \to B_{i,j}$ by letting

$$\varphi_{j,i}^*(z_s(j)) := \begin{cases} z_j(i)^{-1} \cdot z_s(i) & \text{if } s \neq i \\ z_j^{-1}(i) & \text{if } s = i \end{cases} \tag{2.2.10}$$

One checks that Items (1), (2) and (3) above hold. The corresponding prevariety $(X, [\mathscr{A}])$ is isomorphic to $\mathbb{P}^n$, see Example 2.2.8. Explicitely, let $\widetilde{\varphi}\colon V_0 \sqcup \ldots \sqcup V_n \longrightarrow \mathbb{P}^n$ be the map defined by setting

$$\begin{array}{ccc} V_i & \longrightarrow & \mathbb{P}^n \\ (z_0(i), \ldots, z_{i-1}(i), z_{i+1}(i), \ldots, z_n(i)) & \mapsto & [z_0(i), \ldots, z_{i-1}(i), 1, z_{i+1}(i), \ldots, z_n(i)] \end{array} \tag{2.2.11}$$

Then $\widetilde{\varphi}$ descends to a regular map $\varphi := (X, \mathscr{A}) \longrightarrow \mathbb{P}^n$ which is an isomorphism.

Example 2.2.16. Let $(Y, [\mathscr{A}])$ be a prevariety, with affine charts $\psi_i := V_i \to A_i$. For $i, j \in I$ let $A_{i,j} := \psi_i^{-1}(A_i \cap A_j)$ and $B_{i,j} := \psi_j^{-1}(A_j \cap A_i)$. Let

$$\begin{array}{ccc} A_{i,j} & \xrightarrow{\varphi_{j,i}} & B_{i,j} \\ p & \mapsto & \varphi_j^{-1}(\varphi_i(p)) \end{array} \tag{2.2.12}$$

Then Hypothesis 2.2.13 holds, hence there is a corresponding prevariety $(X, [\mathscr{B}])$, where $\mathscr{B}$ is the algebraic atlas $\{\pi(V_i)\}_{i \in I}$. Clearly $(X, [\mathscr{B}])$ is isomorphic to $(Y, \mathscr{A}])$ - this generalizes Example 2.2.15.

As shown by the example above, the gluing construction is at the heart of the definition of prevariety. In fact they are two different point of views of the same objects. In the definition of a prevariety we are given a topological space and a collection of affine charts, in the gluing construction we are given a collection of affine varieties and gluing data $\varphi_{j,i}$ and we define a topological space.

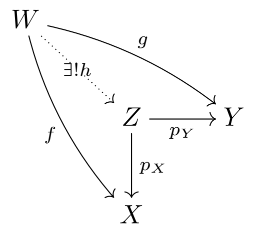
2.3 Products, algebraic varieties

Let X be a prevariety. The Zarisky Topology of X is not Hausdorff unless X is finite. Nonethless X might share key properties of Hausdorff topological spaces. In fact suppose that X is an affine variety. Thus we may assume that $X \subset \mathbb{A}^n$ is closed. The square $X \times X \subset \mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ is closed, so it is an affine variety. Moreover the diagonal $\Delta_X \subset X \times X$ is closed in the Zariski topology. In fact let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be the obvious affine coordinates on $\mathbb{A}^n \times \mathbb{A}^n$: then Δ_X is the intersection of $X \times X$ and the closed subset $V(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$. Recall that a topological space X is Hausdorff if and only if the diagonal in $X \times X$ (wuth the product topology) is closed. So apparently we have a contradiction: if X is an affine variety which is not finite then it is not Hausdorff but its diagonal is closed in $X \times X$. In fact this is not a contradiction because if X is not finite the Zarsiki topology on $X \times X$ si much finer that the product topology. The conclusion is that the right version of Hausdorfness for an algebraic prevariety X is that the diagonal be closed in $X \times X$. Thus our first step is to define the product of prevarieties.

Products in a category

We start by recalling the definition of product of two objects in a category.

Definition 2.3.1. Let $\mathscr{C}$ be a category, and let $X, Y \in \mathrm{Ob}(\mathscr{C})$ be objects of $\mathscr{C}$. A *product of X and Y* consists of an object $Z \in \mathrm{Ob}(\mathscr{C})$ and morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$ (the projections) which have the following universal property. Assume that $W \in \mathrm{Ob}(\mathscr{C})$ and that $f : W \to X$, $g : W \to Y$ are morphisms. Then there exists a unique morphism $h : W \to Z$ such that the following is a commutative diagram



(2.3.1)

Suppose that a product W of X and Y exists. If W' is another product of X and Y (with projections $p'_X : W' \to X$ and $p'_Y : W' \to Y$), then there exists a unique morphism $h : W \to W'$ commuting with the projections, i.e. $p'_X \circ \beta = p_X$ and $p'_Y \circ \beta = p_Y$. Of course we also have the corresponding morphism $h' : W' \to W$. By the unicity requirement in the definition of product the compositions $h' \circ h$ and $h \circ h'$ are equal to the identities of W and W'. Thus we have a well defined isomorphism between any two products of X and Y (assuming a product exists). Since the product is well defined up to (unique) isomorphism it makes sense to talk of "the" product of X and Y. One denotes it by $X \times Y$. We denote by (f, g) the unique morphism h appearing in (2.3.1).

Example 2.3.2. Let **Sets** be the category of sets (one has to be careful with definitions or one runs into Russell's paradox, but we ignore this point here). If $X, Y \in \mathrm{Ob}(\mathbf{Sets})$ i.e. X, Y are sets, then the Cartesian product $X \times Y$ with projections $p_X(x, y) := x$ and $p_Y(x, y) := y$ is the product of X and Y in the category **Sets**.

Example 2.3.3. Let **Grps** be the category of groups. If $G, H \in \mathrm{Ob}(\mathbf{Grps})$ i.e. G, H are groups, then the direct product $G \times H$ with projections $p_G(g, h) := g$ and $p_H(g, h) := h$ is the product of G and H in the category **Grps**. **Sets**.

Example 2.3.4. Let S be a set, and let $\mathbf{Sets}/S$ be the category whose objects are maps $f\colon X \to S$ from a set X to S, and morphisms from a map $f\colon X \to S$ to a map $g\colon Y \to S$ are morphisms $\varphi\colon X \to Y$ which commute with f and g, i.e. a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array} \tag{2.3.2}$$

The product of $f\colon X \to S$ and $g\colon Y \to S$ in the category $\mathbf{Sets}/S$ is given by the object

$$\begin{array}{ccc} X \times_S Y := \{(x,y) \in X \times Y \mid f(x) = g(y)\} & \longrightarrow & S \\ (x,y) & \mapsto & f(x)\,(= g(y)) \end{array} \tag{2.3.3}$$

(the *fiber product* of X and Y over S) with projections given by the restrictions of the projections $X \times Y \to X$ and $X \times Y \to Y$.

Products of affine varieties

Let X, Y be affine varieties. Thus, we may assume that $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are closed subsets. Then $X \times Y \subset \mathbb{A}^m \times \mathbb{A}^n \cong \mathbb{A}^{m+n}$ is a closed subset, and the projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ given by the two projections are regular.

Proposition 2.3.5. *Keeping notation as above, $X \times Y$ with projections p_X, p_Y is the product of X and Y in the category of prevarieties.*

Proof. Let W be a prevariety and let $f\colon W \to X$, $g\colon W \to Y$ be regular maps. We must prove that there exists a regular map $h\colon W \to X \times Y$ such that $p_X \circ h = f$, $p_y \circ h = g$, and that h is unique. Since prevarieties are sets (with extra structure) and regular maps between prevarieties are maps between the underlying sets (satisfying suitable conditions), if h exists it is necessarily given by

$$\begin{array}{ccc} W & \overset{(f,g)}{\longrightarrow} & X \times Y \\ p & \mapsto & (f(p), g(p)) \end{array} \tag{2.3.4}$$

Thus all we need to prove is that (f,g) is regular. As we showed (see Example 2.2.16) any prevariety is obtained by the gluing construction in 2.2.14. Thus W is obtained by gluing affine varieties $\{V_i\}_{i\in I}$ as in 2.2.14. To simplify notation denote $\pi(V_i) \subset W$ by V_i. It suffices to show that the restriction of (f,g) to V_i is regular. Since f and g are regular both the restrictions of f and g to V_i are regular. It follows at once that the restriction of (f,g) to V_i is regular. □

The $\mathbb{K}$ algebra of regular functions of $X \times Y$ is constructed from $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ as follows. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projections. The $\mathbb{K}$-bilinear map

$$\begin{array}{ccc} \mathbb{K}[X] \times \mathbb{K}[Y] & \longrightarrow & \mathbb{K}[X \times Y] \\ (f,g) & \mapsto & \pi_X^*(f) \cdot \pi_Y^*(g) \end{array} \tag{2.3.5}$$

induces a linear map

$$\mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[Y] \longrightarrow \mathbb{K}[X \times Y]. \tag{2.3.6}$$

Proposition 2.3.6. *The map in* (2.3.6) *is an isomorphism.*

Proof. We may assume that $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are closed subsets. Since $X \times Y \subset \mathbb{A}^{m+n}$ is closed the map in (2.3.6) is surjective by Theorem 1.6.2. It remains to prove injectivity, i.e. the following: if $A \subset \mathbb{K}[X]$ and $B \subset \mathbb{K}[Y]$ are finite-dimensional complex vector subspaces, then the map $A \otimes B \to \mathbb{K}[X \times Y]$ obtained by restriction of (2.3.6) is injective. Let $\{f_1, \ldots, f_a\}$, $\{g_1, \ldots, g_b\}$ be bases of A and B. By considering the maps

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{K}^a \\ z & \mapsto & (f_1(z), \ldots, f_a(z)) \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & \mathbb{K}^b \\ z & \mapsto & (g_1(z), \ldots, g_b(z)) \end{array} \tag{2.3.7}$$

we get that there exist $p_1, \ldots, p_a \in X$ and $q_1, \ldots, q_b \in Y$ such that the square matrices $(f_i(p_j))$ and $(g_i(q_j))$ are non-singular. By change of bases, we may assume that $f_i(p_j) = \delta_{ij}$ and $g_k(q_h) = \delta_{kh}$. Computing the values of $\pi_X^*(f_i) \cdot \pi_Y^*(g_j)$ on (p_s, q_t) for $1 \leqslant i, s \leqslant a$ and $1 \leqslant j, t \leqslant b$ we get that the functions $\ldots, \pi_X^*(f_i) \cdot \pi_Y^*(g_j), \ldots$ are linearly independent. Thus $A \otimes B \to \mathbb{K}[W \times Z]$ is injective. □

Products of prevarieties

Proposition 2.3.7. *Let X, Y be prevarieties. There exists a product of X and Y in the category of prevarieties.*

Proof. By Example 2.2.16 X is obtained by gluing affine varieties $\{V_i\}_{i \in I}$ as in 2.2.14, and Y is obtained by gluing affine varieties $\{W_j\}_{j \in J}$. More precisely for each $i_1, i_2 \in I$ we have regular maps $\varphi^X_{i_2,i_1} : A^X_{i_1,i_2} \to B^X_{i_1,i_2}$, where $A^X_{i_1,i_2} \subset V_{i_1}$ and $B^X_{i_1,i_2} \subset V_{i_2}$ are open subsets, and they are the gluings defining X. Analogously, for each $j_1, j_2 \in J$ we have regular maps $\varphi^Y_{j_2,j_1} : A^Y_{j_1,j_2} \to B^Y_{j_1,j_2}$, where $A^Y_{j_1,j_2} \subset W_{j_1}$ and $B^Y_{j_1,j_2} \subset W_{j_2}$ are open subsets, and they are the gluings defining Y. Then we can glue the collection of affine varieties $\{V_i \times W_j\}_{(i,j) \in I \times J}$ as follows. For $(i_1, j_1), (i_2, j_2) \in I \times J$ let

$$A_{(i_1,j_1),(i_2,j_2)} := A^X_{i_1,i_2} \times A^Y_{j_1,j_2} \subset V_{i_1} \times W_{j_1}, \qquad B_{(i_1,j_1),(i_2,j_2)} := B^X_{i_1,i_2} \times B^Y_{j_1,j_2} \subset V_{i_2} \times W_{j_2} \tag{2.3.8}$$

These are open subsets of $V_{i_1} \times W_{j_1}$ and $V_{i_2} \times W_{j_2}$ respectively. We let

$$\begin{array}{ccc} A_{(i_1,j_1),(i_2,j_2)} & \xrightarrow{\varphi_{(i_1,j_1),(i_2,j_2)}} & B_{(i_1,j_1),(i_2,j_2)} \\ (p, q) & \mapsto & (\varphi^X_{i_2,i_1}(p), \varphi^Y_{j_2,j_1}(q)) \end{array} \tag{2.3.9}$$

This collection of affine varieties and gluing maps satisfy the conditions in Hypothesis 2.2.13. Let Z be the prevariety obtained by gluing the $\{V_i \times W_j\}_{(i,j) \in I \times J}$'s as specified above. We have obvious maps $p_X : Z \to X$ and $p_Y : Z \to Y$. In fact let $z \in Z$. Then $z = (p, q) \in V_i \times W_j$ for some $(i, j) \in I \times J$ (by no means unique). Here, in order to simplify notation, we denote $\pi^X(V_i) \subset X$ and $\pi^Y(W_j) \subset Y$ by V_i and W_j respectively. Then we let $p_X(p, q) := p$ and $p_Y(p, q) := q$. As is easily checked the maps p_X, p_Y are regular. We claim that Z with the regular maps p_X and p_Y is the categorical product of X and Y. First note that the map of sets $(p_X, p_W) : Z \to X \times Y$ is bijective. Hence, given regular maps $f : U \to X$ and $g : U \to Y$, there is a unique map $h : U \to Z$ of sets commuting with the projections. In fact if $u \in U$ we let $h(u)$ be the unique $z \in Z$ such that $p_X(z) = f(u)$ and $p_Y(z) = g(u)$. Arguing as in the proof of Proposition 2.3.5 one shows that h is a regular map. □

Remark 2.3.8. We stress that the categorical product of prevarieties X, Y is canonically identified, as a set, with the Cartesian product of X and Y.

Remark 2.3.9. Let X, Y be prevarieties, and let $X_0 \subset X$, $Y_0 \subset Y$ be locally colsed subsets. Then $X_0 \times Y_0 \subset X \times Y$ is a locally closed subset. The restrictions of the projections $X \times Y \to X$ and $X \times Y \to Y$ to $X_0 \times Y_0$ define regular maps $p_{X_0} : X_0 \times Y_0 \to X_0$ and $p_{Y_0} : X_0 \times Y_0 \to Y_0$. As is easily checked, $X_0 \times Y_0$ with the regular maps p_{X_0} and p_{Y_0} is the product of X_0 and Y_0.

Remark 2.3.10. If $F \subset \mathbb{K}$ is a subfield and X, Y are prevarieties over F, then $X \times Y$ is defined over F. We leave the reader check this fact.

Separated prevarieties

Let X be a prevariety. The *diagonal* $\Delta_X \subset X \times X$ is defined to be

$$\Delta_X := \{(x,x) \mid x \in X\}. \tag{2.3.10}$$

This makes sense because as a set $X \times X$ is identified with the Cartesian square of X.

Example 2.3.11. Let X be an affine variety. Thus we may assume that $X \subset \mathbb{A}^n$ is closed. Then $X \times X \subset \mathbb{A}^{2n}$ is closed. Letting $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be the standard affine coordinates on $\mathbb{A}^{2n}$, we have

$$\Delta_X \cap (X \times X) = V(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n) \cap (X \times X). \tag{2.3.11}$$

hence the diagonal of an affine variety is closed.

Remark 2.3.12. Let X be an algebraic prevariety. The diagonal Δ_X is a locally closed subset of $X \times X$. In fact by Example 2.2.16 X is obtained by gluing affine varieties $\{V_i\}_{i \in I}$ as in 2.2.14. The open subsets $V_i \times V_j$ for $(i,j) \in I$ cover X (to simplify notation we denote $\pi(V_i)$, $\pi(A_{ij})$ and $\pi(B_{ij})$ by V_i, A_{ij} and B_{ij} respectively). Thus it suffices to show that the intersection of Δ_X with each open subset $V_i \times V_j$ is locally closed in $V_i \times V_j$. We have

$$\Delta_X \cap (V_i \times V_j) = \{(x,y) \in A_{ij} \times B_{ij} \mid y = \varphi_{ji}(x)\}. \tag{2.3.12}$$

Arguing as in Example 2.3.11 we get that $\Delta_X \cap (V_i \times V_j)$ is a closed subset of the open set $A_{ij} \times B_{ij}$, and hence Δ_X is a closed subset of the open subset of $X \times X$ given by the union of all the $A_{ij} \times B_{ij}$'s.

Example 2.3.13. Let $(Y, [\mathscr{B}])$ be the prevariety defined by the second atlas (given by the regular map g) in Example 2.2.12. Then the diagonal is not closed in $Y \times Y$. In fact denote by V, W the open subsets $\pi(V), \pi(W)$ respectively. Then $V \times W \cong \mathbb{A}^2$ and

$$\Delta_Y \cap (V \times W) = \{(z,z) \in \mathbb{A}^2 \mid z \neq 0\}, \tag{2.3.13}$$

which is not closed.

Definition 2.3.14. An algebraic prevariety X is *separated* if the diagonal $\Delta_X \subset X \times X$ is closed.

Example 2.3.15. By Example 2.3.11 an affine variety X with its canonical structure of prevariety is separated.

Remark 2.3.16. Let X be a prevariety. We may assume that X is obtained by gluing affine varieties $\{V_i\}_{i \in I}$ as in 2.2.14. As usual we denote $\pi(V_i)$, $\pi(A_{ij})$ and $\pi(B_{ij})$ by V_i, A_{ij} and B_{ij} respectively. Since $\{V_i \times V_j\}_{(i,j) \in I^2}$ is an open covering of $X \times X$ the diagonal Δ_X is closed in $X \times X$ if and only if $\Delta_X \cap (V_i \times V_j)$ is closed for all $(i,j) \in I^2$. Since $\Delta_X \cap (V_i \times V_i)$ is closed, see Example 2.3.15, it suffices to check that $\Delta_X \cap (V_i \times V_j)$ is closed for all couples $i \neq j$. We can halve the verifications needed because $\Delta_X \cap (V_i \times V_j)$ is closed if and only if $\Delta_X \cap (V_j \times V_i)$ is closed. Moreover, since $\Delta_X \cap (A_{ij} \times B_{ij})$ is closed (see Remark 2.3.12), in order to show that $\Delta_X \cap (V_i \times V_j)$ is closed it suffices to show that there exists a subset $C \subset A_{ij} \times B_{ij}$ containing $\Delta_X \cap (V_i \times V_j)$ which is closed in $V_i \times V_j$.

Example 2.3.17. Let $(X, [\mathscr{A}])$ be the prevariety defined by the first atlas (given by the regular map f) in Example 2.2.12. Then $(X, [\mathscr{A}])$ is separated. In fact denote by V, W the open subsets $\pi(V), \pi(W)$ respectively. Then $V \times W \cong \mathbb{A}^2$ and

$$\Delta_Y \cap (V \times W) = V(wz - 1). \tag{2.3.14}$$

Since $(X, [\mathscr{A}])$ is isomorphic to $\mathbb{P}^1$, we get that $\mathbb{P}^1$ is separated.

Example 2.3.18. Let $(Y, [\mathscr{B}])$ be the prevariety defined by the second atlas (given by the regular map g) in Example 2.2.12. The diagonal Δ_Y is not closed in $Y \times Y$, see Example 2.3.13. Hence $(Y, [\mathscr{B}])$ is not separated.

The following result shows that separated prevarieties enjoy a key property of Hausdorff topological spaces.

Proposition 2.3.19. *Let X, Y be prevarieties, and assume that Y is separated. If $f, g\colon X \to Y$ are regular maps, then the subset of X defined by*

$$\{x \in X \mid f(x) = g(x)\} \tag{2.3.15}$$

is closed in X.

Proof. By the universal property of the product $Y \times Y$ (we let p_1, p_2 be the projections to Y) we have the regular map $(f, g)\colon X \to Y \times Y$ such that $p_1 \circ (f, g) = f$ and $p_2 \circ (f, g) = g$. Let W be the subset of X appearing in (2.3.15). Then $W = (f, g)^{-1}(\Delta_Y)$. Since Y is separated Δ_Y is closed and hence W is closed. □

A useful result valid for separated varieties is the following.

Proposition 2.3.20. *Let X be a separated prevariety. If $U, V \subset X$ are open affine subsets then the intersection $U \cap V$ is affine.*

Proof. The map

$$\begin{array}{ccc} U \cap V & \longrightarrow & (U \times V) \cap \Delta_X \\ x & \mapsto & (x, x) \end{array} \tag{2.3.16}$$

is an isomorphism. Since $U \times V$ is affine and Δ_X is closed in $X \times X$, it follows that $U \cap V$ is isomorphic to a closed subset of an affine variety, and hence is affine. □

Algebraic varieties

Definition 2.3.21. An algebraic prevariety is an *algebraic variety* if it is of finite type and separated.

An affine variety is an algebraic variety by Remark 2.2.7 and Example 2.3.15. Also $\mathbb{P}^1$ is an algebraic variety by Remark 2.2.7 and Example 2.3.17. More generally, a quasi projective variety is an algebraic variety.

Proposition 2.3.22. *A quasi projective variety (with its canonical structure of prevariety) is an algebraic variety.*

Proof. We have already noticed that a quasi projective variety is of finite type, see Remark 2.2.7. It remains to show that it is separated. First we consider $\mathbb{P}^n$ (the key case). Following Example 2.2.15, $\mathbb{P}^n$ is obtained by gluing $(n+1)$ copies $\{V_0, \ldots, V_n\}$ of affine space $\mathbb{A}^n$. As usual we use the same symbol V_i to denote $\pi(V_i)$. It suffices to check that $\Delta_{\mathbb{P}^n} \cap (V_i \times V_j)$ is closed in $(V_i \times V_j)$ for all $i \neq j$. By the formulae for the gluing maps in (2.2.10) we get that $\Delta_{\mathbb{P}^n} \cap (V_i \times V_j)$ is contained in the closed subset $V(x_j(i) \cdot x_i(j) - 1) \subset (V_i \times V_j)$. Since this closed subset is contained in $A_{ij} \times B_{ij}$ we are done, see the last sentence of Remark 2.3.16. Now let $X \subset \mathbb{P}^n$ be a locally closed subset. Then $X \times X$ is a locally closed subset of $\mathbb{P}^n \times \mathbb{P}^n$ and $\Delta_{\mathbb{P}^n} \cap (X \times X) = \Delta_X$. Since $\Delta_{\mathbb{P}^n}$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$, it follows that Δ_X is closed in $X \times X$. □

Next we consider constructions which, starting from an algebraic variety (or algebraic varieties) produce another algebraic variety.

Let X be an algebraic prevariety, with algebraic atlas $\mathscr{A} = \{A_i\}_{i \in I}$ and affine charts $\varphi_i\colon V_i \to A_i$. If $U \subset X$ is an open subset then we define an algebraic atlas on U as follows. For $i \in I$ the open subset $\varphi_i^{-1}(V_i \cap U) \subset V_i$ is the union of its open affine subsets. The restriction of φ_i to an open affine subset $W_{i,k} \subset \varphi_i^{-1}(V_i \cap U)$ defines a homeomorphism $\psi_{i,k}\colon W_{ik} \to \varphi_i(W_{i,k})$, and $\varphi_i(W_{i,k})$ is open in U. Thus U is covered by the open subsets $\varphi_i(W_{i,k})$ and the maps $\psi_{i,k}$ give affine charts. If the algebraic atlas $\mathscr{A}$ is replaced by an equivalent one we get an equivalent atlas of U. Thus we have equipped U with a canonical structure of prevariety. Note that the inclusion map $U \hookrightarrow X$ is regular.

Proposition 2.3.23. *Let X be an algebraic variety. If $U \subset X$ is an open subset with its canonical structure of algebraic prevariety (see Example 2.2.9), then U is an algebraic variety.*

Proof. We must prove that U is of finite type and separated. Since X is of finite type there exists a finite algebraic atlas $\mathscr{A} = \{A_1, \ldots, a_n\}$ with affine charts $\varphi_i \colon V_i \to A_i$. The open subset $\varphi_i^{-1}(V_i \cap U)$ is the union of its affine open subsets, and since the Zariski topology of a quasi projective variety is quasi compact it is the union of a finite family of open subsets. From this it follows that U has a finite algebraic atlas, and hence is of finite type. The subset $(U \times U) \subset (X \times X)$ is the (categorical) square of U, this follows from the universal property of the product $X \times X$ and the fact that the ninclusion $U \hookrightarrow X$ is regular. Since the diagonal Δ_U is equal to the intersection $(U \times U) \cap \Delta_X$ in $X \times X$ and Δ_X is closed in $X \times X$, it follows that Δ_U is closed in $U \times U$, and hence U is separated. □

Arguing as above one proves the following result.

Proposition 2.3.24. *Let X be an algebraic variety. If $Y \subset X$ is a locally closed subset with its canonical structure of algebraic prevariety (see Example 2.2.9), then Y is an algebraic variety.*

Proposition 2.3.25. *If X, Y are algebraic varieties, then the product $X \times Y$ is an algebraic variety.*

Proof. By hypothesis there exists finite algebraic atlases $\mathscr{A} = \{A_i\}_{i \in I}$, $\mathscr{B} = \{B_j\}_{j \in J}$ of X and Y respectively, with affine charts $\varphi_i \colon V_i \to A_i$ and $\psi_j \colon W_j \to B_j$. Then $\mathscr{A} \times \mathscr{B} \coloneqq \{A_i \times B_j\}_{(i,j) \in I \times J}$, with affine charts $\varphi_i \times \psi_j \colon V_i \times W_j \to A_i \times B_j$ is a finite algebraic atlas of $X \times Y$. Thus $X \times Y$ is of finite type. The projection maps $f \colon (X \times Y) \times (X \times Y) \to X \times X$ and to $Y \times Y$

$$\begin{array}{ccc} (X \times Y) \times (X \times Y) & \xrightarrow{f} & X \times X \\ ((x_1, y_1), (x_2, y_2)) & \mapsto & (x_1, x_2) \end{array}, \quad \begin{array}{ccc} (X \times Y) \times (X \times Y) & \xrightarrow{g} & Y \times Y \\ ((x_1, y_1), (x_2, y_2)) & \mapsto & (y_1, y_2) \end{array}, \tag{2.3.17}$$

are regular (by the universal property of products) and hence continuous. Thus $\Delta_{X \times Y} = f^{-1}(\Delta_X) \cap g^{-1}(\Delta_Y)$ is closed. This proves that $X \times Y$ is separated. □

Products of quasi projective varieties

In the present subsubsection we prove the following result.

Proposition 2.3.26. *If X and Y are quasi projective varieties, then $X \times Y$ is a quasi projective variety.*

Before proving Proposition 2.3.26 we go through a few preliminary results. A polynomial $F(W; Z) \in \mathbb{K}[W_0, \ldots, W_m, Z_0, \ldots, Z_n]$ is *bihomogeneous of degree* (d, e) if $F(\lambda W; \mu Z) = \lambda^d \mu^e F(W; Z)$ for all $\lambda, \mu \in \mathbb{K}$. Let $F_i \in \mathbb{K}[W_0, \ldots, W_m, Z_0, \ldots, Z_n]$ for $i \in \{1, \ldots, r\}$ be a bihomogeneous polynomial of degree (d_i, e_i). Then it makes sense to let

$$V(F_1, \ldots, F_r) \coloneqq \{([W], [Z]) \in \mathbb{P}^m \times \mathbb{P}^n \mid F_1(W; Z) = \ldots = F_r(W; Z) = 0\}. \tag{2.3.18}$$

Claim 2.3.27. *A subset $X \subset \mathbb{P}^m \times \mathbb{P}^n$ is closed if and only if there exist bihomogeneous polynomials $F_1, \ldots, F_r \in \mathbb{K}[W_0, \ldots, W_m, Z_0, \ldots, Z_n]$ such that $X = V(F_1, \ldots, F_r)$.*

Proof. We have

$$\mathbb{P}^m \times \mathbb{P}^n = \bigcup_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant n}} \mathbb{P}^m_{W_i} \times \mathbb{P}^n_{Z_j} \tag{2.3.19}$$

and each of the open subsets $\mathbb{P}^m_{W_i} \times \mathbb{P}^n_{Z_j}$ is isomorphic to $\mathbb{A}^{m+n}$. If $F_1, \ldots, F_r$ are as above, then $V(F_1, \ldots, F_r) \cap \mathbb{P}^m_{W_i} \times \mathbb{P}^n_{Z_j}$ is clearly closed. It follows that $V(F_1, \ldots, F_r)$ is closed. Now suppose that $X \subset \mathbb{P}^m \times \mathbb{P}^n$ is closed. Then $X \cap \mathbb{P}^m_{W_i} \times \mathbb{P}^n_{Z_j}$ is closed for every i, j, and hence there exists $f^{i,j}_1, \ldots, f^{i,j}_s \in \mathbb{K}[\frac{W_0}{W_i}, \ldots, \frac{W_m}{W_i}, \frac{Z_0}{Z_j}, \ldots, \frac{Z_n}{Z_j}]$ such that

$$X \cap \mathbb{P}^m_{W_i} \times \mathbb{P}^n_{Z_j} = V(f^{i,j}_1, \ldots, f^{i,j}_s). \tag{2.3.20}$$

If $d \gg 0$ and $e \gg 0$ are natural numbers then $F_l^{i,j} := W_i^d \cdot Z_j^e \cdot f_l^{i,j}\left(\frac{W_0}{W_i}, \dots, \frac{W_m}{W_i}, \frac{Z_0}{Z_j}, \dots, \frac{Z_n}{Z_j}\right)$ is bihomogeneous of degree (d, e) and $V(F_l^{i,j}) = \overline{V(f_l^{i,j})} \cup V(W_i) \cup V(Z_j)$. Thus X is the zero locus of all the bihomogeneous polynomials $F_l^{i,j}$'s.

$\square$

Remark 2.3.28. In the statement of Claim 2.3.27 we may require, if we wish, that all the F_i's are bihomogeneous of degrees (d_i, d_i), i.e. of the same degrees in both variables.

Next we introduce Segre varieties and Segre maps. Let $M_{m+1,n+1}(\mathbb{K})$ be the $\mathbb{K}$ vector space of complex $(m+1) \times (n+1)$ matrices. Row and column indices for matrices in $M_{m+1,n+1}(\mathbb{K})$ start from 0. Thus we denote them by

$$T = \begin{pmatrix} T_{00} & T_{01} & \dots & T_{0n} \\ T_{10} & T_{11} & \dots & T_{1n} \\ \dots & \dots & \dots & \dots \\ T_{m0} & T_{m1} & \dots & T_{mn} \end{pmatrix} \tag{2.3.21}$$

Let

$$\Sigma_{m,n} := \{[T] \in \mathbb{P}(M_{m+1,n+1}(\mathbb{K})) \mid \operatorname{rk} T \leqslant 1\}.$$

Then $\Sigma_{m,n}$ is a projective variety in $\mathbb{P}(M_{m+1,n+1}(\mathbb{K})) = \mathbb{P}^{mn+m+n}$. In fact $[T] \in \mathbb{P}(M_{m+1,n+1}(\mathbb{K}))$ belongs to $\Sigma_{m,n}$ if and only if the determinants of all 2×2 minors of T vanish. This is the *Segre variety* in $\mathbb{P}(M_{m+1,n+1}(\mathbb{K}))$.

If $[W] \in \mathbb{P}^m$ and $[Z] \in \mathbb{P}^n$, viewed as column matrices, then $W \cdot Z^t \in M_{m+1,n+1}(\mathbb{K})$ and the rank of $W \cdot Z^t$ is 1. If we rescale W or Z then $W \cdot Z^t$ gets rescaled. Thus we have a well defined *Segre map*

$$\begin{array}{ccc} \mathbb{P}^m \times \mathbb{P}^n & \xrightarrow{\sigma_{m,n}} & \Sigma_{m,n} \\ ([W],[Z]) & \mapsto & [W \cdot Z^t] \end{array} \tag{2.3.22}$$

Explicitly

$$\sigma_{m,n}([W],[Z]) = \left[\begin{pmatrix} W_0 \cdot Z_0 & W_0 \cdot Z_1 & \dots & W_0 \cdot Z_0 \\ W_1 \cdot Z_0 & W_1 \cdot Z_1 & \dots & W_1 \cdot Z_n \\ \dots & \dots & \dots & \dots \\ W_m \cdot Z_0 & W_m \cdot Z_1 & \dots & W_m \cdot Z_n \end{pmatrix}\right] \tag{2.3.23}$$

Proposition 2.3.29. *The Segre map in* (2.3.22) *is an isomorphism of algebraic varieties.*

Proof. First we prove that the Segre map is bijective. Let $[T] \in \Sigma_{m,n}$. Then T has rank 1 because $T \neq 0$. Hence the associated linear map $L_T \colon \mathbb{K}^{n+1} \to \mathbb{K}^{m+1}$ (given by $L_T(U) := T \cdot U$, where U is a column matrix) can be factored as $L_T = L_W \circ L_{Z^t}$ where $L_{Z^t} \colon \mathbb{K}^{n+1} \to \mathbb{K}$ is surjective and $L_W \colon \mathbb{K} \to \mathbb{K}^{m+1}$ is injective. This gives that $T = W \cdot Z^t$. We also get that $\ker(L_T) = \ker(L_{Z^t})$ and $\operatorname{im}(L_T) = \operatorname{im}(L_W)$. Thus W and Z are determined by $[T]$ up to a scalar factor, and hence $\sigma_{m,n}$ is injective.

Next we prove that the Segre map is a homeomorphism. Let $C \subset \Sigma_{m,n}$ be a closed subset, i.e. $C = \Sigma_{m,n} \cap V(P_1, \dots, P_r)$ where $P_i \in \mathbb{K}[T_{00}, T_{01}, \dots, T_{mn}]_{d_i}$. Then

$$\sigma_{m,n}^{-1}(C) = V(P_1(W_0 \cdot Z_0, W_0 \cdot Z_1, \dots, W_m \cdot Z_n), \dots, P_r(W_0 \cdot Z_0, W_0 \cdot Z_1, \dots, W_m \cdot Z_n)). \tag{2.3.24}$$

Since $P_i(W_0 \cdot Z_0, W_0 \cdot Z_1, \dots, W_m \cdot Z_n)$ for $i \in \{1, \dots, r\}$ is a bihomogeneous polynomial (of degree (d_i, d_i)), it follows that $\sigma_{m,n}^{-1}(C)$ is closed in $\mathbb{P}^m \times \mathbb{P}^n$, se Claim 2.3.27. This shows that $\sigma_{m,n}$ is continuous. Now suppose that $D \subset \mathbb{P}^m \times \mathbb{P}^n$ is closed. By Claim 2.3.27 there exist bihomogeneous polynomials $F_1, \dots, F_r \in \mathbb{K}[W_0, \dots, W_m, Z_0, \dots, Z_n]$ such that $X = V(F_1, \dots, F_r)$. As noticed in Remark 2.3.28 we may assume that F_i is bihomogeneous polynomial of degree (d_i, d_i) for each $i \in \{1, \dots, r\}$, and hence there exists $P_i \in \mathbb{K}[T_{00}, T_{01}, \dots, T_{mn}]_{d_i}$ such that

$$P_i(W_0 \cdot Z_0, W_0 \cdot Z_1, \dots, W_m \cdot Z_n) = F_i(W_0, \dots, W_m; Z_0, \dots, Z_n). \tag{2.3.25}$$

This implies that $\sigma_{m,n}(D) = \Sigma_{m,n} \cap V(P_1, \dots, P_r)$ and hence $\sigma_{m,n}(D)$ is closed. Thus also the inverse of the Segre map is continuous and hence $\sigma_{m,n}$ is a homemomorphism.

It remains to show that the Segre map is an isomorphism of algebraic varieties. Recall that we have the open covering in (2.3.19). Now $\sigma_{m,n}$ maps the affine space $\mathbb{P}^n_{W_i} \times \mathbb{P}^m_{Z_j}$ to the open set $\Sigma_{m,n} \backslash V(T_{ij})$ and, as is easily checked the map

$$\mathbb{P}^n_{W_i} \times \mathbb{P}^m_{Z_j} \longrightarrow \Sigma_{m,n} \backslash V(T_{ij}) \tag{2.3.26}$$

is an isomorphism (of affine spaces). It follows that $\sigma_{m,n}$ is an isomorphism of algebraic varieties. □

Proof of Proposition 2.3.26. We may assume that $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ are locally closed. Then $X \times Y \subset \mathbb{P}^m \times \mathbb{P}^n$ is locally closed, and it is the product of X and Y, see Remark 2.3.9. Since the Segre map $\sigma_{m,n} \colon \mathbb{P}^m \times \mathbb{P}^n \to \Sigma_{m,n}$ is an isomorphism, it restricts to an isomorphism between $X \times Y$ and the locally closed subset $\sigma_{m,n}(X \times Y) \subset \Sigma_{m,n}$. Since $\Sigma_{m,n}$ is a projective variety, $\sigma_{m,n}(X \times Y)$ is a quasi projective variety.

□

2.4 Complete varieties

In the present section we introduce the notion of complete varieties, which are the analogues of compact topological space in the category of prevarieties. The prime example of complete varieties are projective varieties. We note that a complex quasi projective variety is complete if and only if, equipped with the Euclidean topology, it is compact. Since every quasi projective variety is quasi compact (and also every prevariety of finite type), one defines "compactness" for algebraic varieties by relying on a different characterization of compact topological spaces.

Let M be a topological space. Then M is quasi compact, i.e. every open covering has a finite subcovering, if and only if M is universally closed, i.e. for any topological space T, the projection map $T \times M \to T$ is closed, i.e. it maps closed sets to closed sets. (See tag/005M in [?].) A quasi projective variety X is quasi compact, but it is not generally true that, for a variety T, the projection $T \times X \to T$ is closed. In fact, let $X \subset \mathbb{P}^n$ be locally closed; then Δ_X, the diagonal of X, is closed in $X \times \mathbb{P}^n$, because it is the intersection of $X \times X \subset \mathbb{P}^n \times \mathbb{P}^n$ with the diagonal $\Delta_{\mathbb{P}^n} \subset \mathbb{P}^n \times \mathbb{P}^n$, which is closed. The projection $X \times \mathbb{P}^n \to \mathbb{P}^n$ maps X to X, hence if X is not closed in $\mathbb{P}^n$, then X is not universally closed. This does not contradict the result in topology quoted above, because the Zariski topology of the product of quasi projective varieties is not the product topology.

Definition 2.4.1. An algebraic prevariety X is *complete* (or *proper over* $\mathbb{K}$) if X is an algebraic variety and it is universally closed, i.e. for any prevariety T, the projection map $T \times X \to T$ is closed.

As noticed above, if $X \subset \mathbb{P}^n$ is not closed (e.g. $\mathbb{P}^n_{Z_i}$ if $n > 0$), then it is not universally closed, and hence X is not complete.

The following is a key result.

Theorem 2.4.2 (Main Theorem of Elimination Theory)**.** *Projective varieties are complete.*

Proof. Let X be projective variety. Since X is an algebraic variety we must prove that X is universally closed.

By hypothesis we may assume that $X \subset \mathbb{P}^n$ is closed. We claim that it suffices to prove that $\mathbb{P}^n$ is universally closed. In fact assume that that $\mathbb{P}^n$ is universally closed. Let T be a prevariety and let $\pi^X_T \colon T \times X \to T$ be the projection. Let $C \subset T \times X$ be closed. Since $T \times X \subset T \times \mathbb{P}^n$ is closed, C is closed also in $T \times \mathbb{P}^n$. Let $\pi^{\mathbb{P}^n}_T \colon T \times \mathbb{P}^n \to T$ be the projection. Then $\pi^X_T(C) = \pi^{\mathbb{P}^n}_T(C)$, and hence it is closed because by assumption $\mathbb{P}^n$ is universally closed. Since T is covered by open affine subsets, we may assume that T is affine, i.e. T is (isomorphic to) a closed subset of $\mathbb{A}^m$ for some m. Lastly, we may as well assume that $T = \mathbb{A}^m$.

To sum up: it suffices to prove that if $C \subset \mathbb{A}^m \times \mathbb{P}^n$ is closed, then $\pi(C)$ is closed in $\mathbb{A}^m$, where $\pi \colon \mathbb{A}^m \times \mathbb{P}^n \to \mathbb{A}^m$ is the projection. We will show that $(\mathbb{A}^m \backslash \pi(C))$ is open. By Claim 2.3.27 there exist $F_i \in \mathbb{K}[t_1, \ldots, t_m, Z_0, \ldots, Z_n]$ for $i = 1, \ldots, r$, *homogeneous* as polynomial in $Z_0, \ldots, Z_n$ such that

$$C = \{(t, [Z]) \mid 0 = F_1(t, Z) = \ldots = F_r(t, Z)\}.$$

Suppose that $F_i \in \mathbb{K}[t_1, \ldots, t_m][Z_0, \ldots, Z_n]_{d_i}$ i.e. F_i is homogeneous of degree d_i in $Z_0, \ldots, Z_n$. Let $\bar{t} \in (\mathbb{A}^m \backslash \pi(C))$. By Hilbert's Nullstellensatz, there exists $N \geqslant 0$ such that

$$(F_1(\bar{t}, Z), \ldots, F_r(\bar{t}, Z)) \supset \mathbb{K}[Z_0, \ldots, Z_n]_N. \tag{2.4.1}$$

We may assume that $N \geqslant d_i$ for $1 \leqslant i \leqslant r$. For $t \in \mathbb{A}^m$ let

$$\begin{array}{ccc} \mathbb{K}[Z_0, \ldots, Z_n]_{N-d_1} \times \ldots \times [Z_0, \ldots, Z_n]_{N-d_r} & \xrightarrow{\Phi(t)} & \mathbb{K}[Z_0, \ldots, Z_n]_N \\ (G_1, \ldots, G_r) & \mapsto & \sum_{i=1}^r G_i \cdot F_i \end{array}$$

Thus $\Phi(t)$ is a linear map: choose bases of domain and codomain and let $M(t)$ be the matrix associated to $\Phi(t)$. Clearly the entries of $M(t)$ are elements of $\mathbb{K}[t_1, \ldots, t_m]$. By hypothesis $\Phi(\bar{t})$ is surjective and hence there exists a maximal minor of $M(t)$, say $M_{I,J}(t)$, such that $\det M_{I,J}(\bar{t}) \neq 0$. The open $(\mathbb{A}^m \backslash V(\det M_{I,J}))$ is contained in $(T \backslash \pi(C))$. This finishes the proof of Theorem 2.4.2. □

Next give a few general results on complete algebraic varieties.

Proposition 2.4.3. *Let X, Y be complete (algebraic) varieties.*

1. *If $W \subset X$ is closed then (with its canonical structure of variety, see Proposition 2.3.24) W is complete.*

2. *The product $X \times Y$ is complete.*

Proof. (1): We must check that W is universally closed. One argues as in the second paragraph of the proof of Theorem 2.4.2. (2): By Proposition 2.3.25 $X \times Y$ is an algebraic variety. Hence it remains to check that $X \times Y$ is universally closed. Let T be a prevariety, and let $C \subset T \times (X \times Y)$ be closed. Factoring the projection $\pi \colon T \times (X \times Y) \to T$ as the composition of $f \colon T \times (X \times Y) \to T \times Y$ followed by $g \colon T \times Y \to T$, we get that $f(C) \subset T \times Y$ is closed because X is universally closed, and $g(f(C)$, i.e. $\pi(C)$, is closed because Y is universally closed. □

If $f \colon X \to Y$ is a regular map between prevarieties, the *graph of f* is the subset Γ_f of $X \times Y$ defined by

$$\Gamma_f := \{(x, f(x)) \mid x \in X\}. \tag{2.4.2}$$

Lemma 2.4.4. *Let $f \colon X \to Y$ be a regular map between algebraic prevarieties, and suppose that Y is separated. Then the graph of f is closed in $X \times Y$.*

Proof. The map

$$\begin{array}{ccc} X \times Y & \xrightarrow{f \times \mathrm{Id}_Y} & Y \times Y \\ (x, y) & \mapsto & (f(x), y) \end{array} \tag{2.4.3}$$

is regular, and hence continuous. Since $\Gamma_f = (f \times \mathrm{Id}_X)^{-1}(\Delta_Y)$ and Δ_Y is closed in $Y \times Y$ (because Y is separated) it follows that Γ_f is closed. □

Proposition 2.4.5. *Let X, Y be algebraic varieties, with X complete and Y separated. If $f \colon X \to Y$ is a regular map then it is closed.*

Proof. Since, by Proposition 2.4.3, closed subsets of X are complete varieties, it suffices to prove that $f(X)$ is closed in Y. Let $\pi \colon X \times Y \to Y$ be the projection map. By Lemma 2.4.4 Γ_f is closed in $X \times Y$, and hence $\pi(\Gamma_f)$ is closed in Y because X is complete. Since $f(X) = \pi(\Gamma_f)$ we are done. □

Corollary 2.4.6. *Let X be a complete algebraic variety and let $Y \subset X$ be a locally closed subset (with its canonical structure of algebraic variety, see Proposition 2.3.24). Then Y is complete if and only if Y is closed.*

Proof. If Y is closed then it is complete by Proposition 2.4.3. Conversely, suppose that Y is complete. Since the inclusion map $i\colon Y \hookrightarrow X$ is regular and X is separated, $Y = i(Y)$ is closed in X by Proposition 2.4.5. □

Remark 2.4.7. In particular a locally closed of a projective space is projective only if it is closed. By way of contrast, notice that it is *not* true that a locally-closed subset of $\mathbb{A}^n$ is affine if and only if it is closed. In fact the complement of a hypersurface $V(f) \subset \mathbb{A}^n$ is affine but not closed.

Corollary 2.4.8. *Let X be a complete algebraic variety. A regular map $f\colon X \to \mathbb{K}$ is locally constant. If X is irreducible (recall Proposition 2.2.11) then f is constant.*

Proof. Composing f with the inclusion $j\colon \mathbb{K} \hookrightarrow \mathbb{P}^1$

$$\begin{array}{ccc} \mathbb{K} & \overset{j}{\hookrightarrow} & \mathbb{P}^1 \\ z & \mapsto & [1, z] \end{array} \tag{2.4.4}$$

we get the regular map $j \circ f\colon X \to \mathbb{P}^1$. By Proposition 2.4.5 $j \circ f(X)$ is closed, i.e. $j \circ f(X) = V(I)$ for some homogeneous ideal $I \subset \mathbb{K}[Z_0, Z_1]$. Since $[0, 1] \notin j \circ f(X)$ there exists a non zero polynomial $P \in I$ and hence $j \circ f(X) = f(X)$ is contained in the finite set $V(P)$. The second statement follows at once from the first. □

2.5 Algebraic vector bundles

Definitions and first examples

A very important notion in Topology and in Differential Geometry is that of continuous and C^∞ vector bundle respectively. One defines an analogous notion in the context of algebraic varieties.

Definition 2.5.1. Let X be an algebraic variety defined over $\mathbb{K}$. A rank r algebraic vector bundle over X (we call it a *line bundle* if $r = 1$)consists of the following data:

1. A regular map $\pi\colon E \to X$ of algebraic varieties.
2. For each $x \in X$ a structure of $\mathbb{K}$ vector space of dimension r on the fiber $E(x) := \pi^{-1}(x)$.

These data are subject to the following conditions.

(a) There exist an open cover $X = \bigcup_{\alpha\in A} U_\alpha$ and for each $\alpha \in A$ an isomorphism of varieties $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{K}^r$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times \mathbb{K}^r \\ & {\scriptstyle \pi_{|U_\alpha}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_\alpha} & \\ & U_\alpha & \end{array} \tag{2.5.1}$$

where pr_α is the projection. (This is a *trivialization of E over U_α*.)

(b) For each $\alpha \in A$ and $x \in U_\alpha$ the restriction of φ_α to $E(x)$, which is an isomorphism

$$\varphi_\alpha(x)\colon E(x) \xrightarrow{\sim} \{x\} \times \mathbb{K}^r = \mathbb{K}^r \tag{2.5.2}$$

by Item (a), is a $\mathbb{K}$-linear map (and hence an isomorphism of $\mathbb{K}$ vector spaces)

Example 2.5.2. Let X be an algebraic variety. Then $E := X \times \mathbb{K}^r$ with $\pi\colon E \to X$ given by the projection map is clearly a rank r algebraic vector bundle on X.

An algebraic vector bundle is always of a fixed rank, even if we do not mention explicitly the value of the rank. From now on by vector bundle on an algebraic variety X we mean an algebraic vector bundle on X.

Definition 2.5.3. Let X be an algebraic variety, and let $\pi\colon E \to X$ be a vector bundle of rank r on X. If $Y \subset X$ is a locally closed subset with its canonical structure of algebraic variety, then $\pi^{-1}(Y) \to Y$ is a vector bundle of rank r on Y. We denote it by $E_{|Y}$.

Definition 2.5.4. Let X be an algebraic variety (defined over $\mathbb{K}$), and let $\pi\colon E \to X$, $\rho\colon F \to X$ be vector bundles on X. A *morphism of vector bundles* $E \to F$ consists of a regular map of algebraic varieties $g\colon E \to F$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\quad g \quad} & F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & X & \end{array} \tag{2.5.3}$$

is commutative, and such that for every $x \in X$ the restriction of g to $E(x)$ is a linear map $g(x)\colon E(x) \to F(x)$ of $\mathbb{K}$ vector spaces.

The identity map $\mathrm{Id}_E\colon E \to E$ of a vector bundle is a morphism of vector bundles, and the composition of morphisms of vector bundles on X is a morphism of vector bundles on X. Hence vector bundles on X form a category. In particular we have the notion of *isomorphic vector bundles on* X.

Definition 2.5.5. Let X be an algebraic variety. A vector bundle E of rank r on X is *trivial* if it is isomorphic to the vector bundle in Example 2.5.2.

Next we define a fundamental line bundle on projective space. Let V be a finite dimensional $\mathbb{K}$ vector space. We view points of $\mathbb{P}(V)$ as 1 dimensional (vector) subspaces $\ell \subset V$. Let $L \subset \mathbb{P}(V) \times V$ be defined by

$$L := \{(\ell, v) \in \mathbb{P}(V) \times V \mid v \in \ell\}. \tag{2.5.4}$$

We claim that L is a closed subset of $\mathbb{P}(V) \times V$, and hence an algebraic variety. In fact choose a basis of V so that V and $\mathbb{P}(V)$ are identified with $\mathbb{K}^{n+1}$ and $\mathbb{P}^n$ respectively. Then

$$L = \{([Z_0, \ldots, Z_n], (W_0, \ldots, W_n)) \in \mathbb{P}^n \times \mathbb{K}^{n+1} \mid \mathrm{rk}\begin{pmatrix} Z_0 & \ldots & Z_n \\ W_0 & \ldots & W_n \end{pmatrix} \leqslant 1\}. \tag{2.5.5}$$

This shows that L is closed in $\mathbb{P}(V) \times V$. Let $\pi\colon L \to \mathbb{P}(V)$ be the restriction of the projection $\mathbb{P}(V) \times V \to \mathbb{P}(V)$. If $\ell \in \mathbb{P}(V)$ then $L(\ell) = \pi^{-1}(\ell) = \ell$ and this gives the structure of 1 dimensional $\mathbb{K}$ vector space to $L(\ell)$. Let $i \in \{0, \ldots, n\}$. We define $\varphi_i\colon \pi^{-1}(\mathbb{P}^n_{Z_i}) \to \mathbb{P}^n_{Z_i} \times \mathbb{K}$ as follows:

$$\begin{array}{ccc} \pi^{-1}(\mathbb{P}^n_{Z_i}) & \xrightarrow{\varphi_i} & \mathbb{P}^n_{Z_i} \times \mathbb{K} \\ ([Z], W) & \mapsto & ([Z], W_i) \end{array} \tag{2.5.6}$$

This shows that $L \to \mathbb{P}(V)$ is a line bundle. It is called the *tautological line bundle on* $\mathbb{P}(V)$. If $n > 0$ then L is not trivial. Before showing this we introduce sections of a vector bundle.

Definition 2.5.6. Let X be an algebraic variety, and let $\pi\colon E \to X$ be a vector bundle on X. A *section of* E is a map $\sigma\colon X \to E$ such that $\pi \circ \sigma = \mathrm{Id}_X$, i.e. such that $\sigma(x) \in E(x)$ for every $x \in X$. The section σ is *regular* if it is regular as map of algebraic varieties.

Example 2.5.7. Let X be an algebraic variety, and let $E = X \times \mathbb{K}^r$ with $\pi\colon E \to X$ the projection. A regular section $\sigma\colon X \to E$ is equivalent to the r-tuple of regular maps $f_i\colon X \to \mathbb{K}$ that one gets by projecting to factors of $\mathbb{K}^r$. Let σ_i for $i \in \{1, \ldots, r\}$ be the section corresponding to the r-tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 is in the i-th place. Then for every $x \in X$ the vectors $\sigma_1(x), \ldots, \sigma_r(x) \in E(x)$ form a basis of $E(x)$.

Example 2.5.8. Let X be an algebraic variety, and let $\pi\colon E \to X$ be an algebraic vector bundle. The *zero section* of E is defined by setting $\sigma(x) := 0 \in E(x)$ for every $x \in X$. This is a regular section.

We let

$$\Gamma(X, E) := \{\sigma\colon X \to E \mid \sigma \text{ is a section of } E\}. \tag{2.5.7}$$

The sum of regular sections is a regular section, and the product of an element of $\mathbb{K}$ by a regular section is a regular section. With these operations $\Gamma(X, E)$ acquires the structure of a $\mathbb{K}$ vector space. The zero of the vector space is the zero section.

Remark 2.5.9. Let X be an algebraic variety, and let $\pi\colon E \to X$ be an algebraic vector bundle. Let σ be a section of E. If $x \in X$ then it makes sense to state that $\sigma(x) \in E(x)$ is zero or not. The notation $\sigma(x) = 0$ means that $\sigma(x)$ is zero. The zero-set of σ is the subset of X defined by

$$Z(\sigma) := \{x \in X \mid \sigma(x) = 0\}. \tag{2.5.8}$$

As is easily checked $Z(\sigma)$ is a closed subset of X.

Let $E \to X$ and $F \to X$ be vector bundles on X, and suppose that $\varphi\colon E \to F$ is a morphism of vector bundles. If σ is a regular section of $E \to X$ then $\varphi \circ \sigma\colon X \to F$ is a regular section of F. Usually one denotes $\varphi \circ \sigma$ by $\varphi(\sigma)$. As is easily checked the map

$$\begin{array}{ccc} \Gamma(X, E) & \longrightarrow & \Gamma(X, F) \\ \sigma & \mapsto & \varphi(\sigma) \end{array} \tag{2.5.9}$$

is $\mathbb{K}$ linear. In particular if E and F are isomorphic, then their spaces of global sections are isomorphic.

Remark 2.5.10. Let $E \to X$ be a vector bundle of rank r on X. Then E is trivial if and only if there exist $\sigma_i \in \Gamma(X, E)$ for $i \in \{1, \ldots, r\}$ such that for every $x \in X$ the vectors $\sigma_1(x), \ldots, \sigma_r(x) \in E(x)$ form a basis of $E(x)$. In fact if E is trivial then $\sigma_1, \ldots, \sigma_r$ exist by Example 2.5.7. Conversely, suppose that there exist such $\sigma_1, \ldots, \sigma_r$. Then the map

$$\begin{array}{ccc} X \times \mathbb{K}^r & \longrightarrow & E \\ (x, t) & \mapsto & \sum_{i=1}^r t_i \sigma_i(x) \end{array} \tag{2.5.10}$$

is an isomorphism of vector bundles.

Proposition 2.5.11. *Let V be a finitely generated $\mathbb{K}$ vector space of dimension at least 2, and let L be the tautological line bundle on $\mathbb{P}(V)$. Then $\Gamma(\mathbb{P}(V), L) = \{0\}$ and L is non trivial.*

Proof. Let $\sigma \in \Gamma(\mathbb{P}(V), L)$. The composition

$$\mathbb{P}(V) \xrightarrow{\sigma} L \hookrightarrow \mathbb{P}(V) \times V \longrightarrow V \cong \mathbb{K}^{r+1} \tag{2.5.11}$$

is regular and hence constant by Corollary 2.4.8. The unique element in the image is a vector which belongs to every 1 dimensional (vector) subspace of V. Since $\dim V \geqslant 2$ it follows that it is the zero vector. Since there are no nonzero sections of L it follows that L is non trivial, see Example 2.5.7. □

Vector bundles and 1-cocycles

Let $\pi\colon E \to X$ be a rank r vector bundle. We assume that it is trivial on each open set of a covering $\{U_\alpha\}_{\alpha \in A}$ as in Definition 2.5.1. For $\alpha, \beta \in A$ we define the corresponding *transition map* as follows:

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & \mathrm{GL}_r(\mathbb{K}) \\ x & \mapsto & \varphi_\alpha(x) \circ \varphi_\beta^{-1}(x) \end{array} \tag{2.5.12}$$

Note that $g_{\alpha\beta}$ is a regular map between algebraic varieties ($\mathrm{GL}_r(\mathbb{K}) = M_{r,r}(\mathbb{K}) \backslash V(\mathrm{Det}_r)$ where $\mathrm{Det}_r(g) := \mathrm{Det}(g)$ is the determinant of $g \in M_{r,r}(\mathbb{K})$, and hence $\mathrm{GL}_r(\mathbb{K})$ is an affine variety).

Remark 2.5.12. Let $\{g_{\alpha\beta}\}$ be as above. Then the following hold:

1. For $\alpha \in A$ we have $g_{\alpha\alpha}(x) = 1_r$ for all $x \in U_\alpha$ (1_r is the unit $r \times r$ matrix).
2. For $\alpha, \beta \in A$ we have $g_{\beta\alpha}(x) = g_{\alpha\beta}(x)^{-1}$.
3. For $\alpha, \beta, \gamma \in A$ we have $g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$ for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Example 2.5.13. Let $\pi \colon L \to \mathbb{P}^n$ be the tautological line bundle. Then we have the trivialization φ_i of L over $\mathbb{P}^n_{Z_i}$ given by (2.5.6). Thus we have

$$\begin{array}{ccc} \mathbb{P}^n_{Z_i} \times \mathbb{K} & \xrightarrow{\varphi_j^{-1}} & \pi^{-1}(\mathbb{P}^n_{Z_i}) \\ ([Z], t) & \mapsto & ([Z], (t \cdot Z_j^{-1}) \cdot Z) \end{array} \tag{2.5.13}$$

It follows that $g_{ij} = \varphi_i \circ \varphi_j^{-1}$ is given by

$$g_{ij}([Z]) = \frac{Z_i}{Z_j}, \qquad [Z] \in \mathbb{P}^n_{Z_i} \cap \mathbb{P}^n_{Z_j}. \tag{2.5.14}$$

Here we identify $\mathrm{GL}_1(\mathbb{K})$ with $\mathbb{K}^\times$.

Definition 2.5.14. Let X be an algebraic variety and $\{U_\alpha\}_{\alpha \in A}$ an open covering of X. A 1-*cochain with values in* $\mathrm{GL}_r(\mathbb{K})$ (relative to the given open covering) consists of the assignment of a regular function $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{K})$ to each couple $(\alpha, \beta) \in A^2$. We denote it by $g = \{g_{\alpha\beta}\}$. The 1-cochain g is a 1-*cocycle* if Items (1), (2) and (3) hold.

Thus we have assigned a 1-cocycle with values in $\mathrm{GL}_r(\mathbb{K})$ to every rank r vector bundle $\pi \colon E \to X$ with local trivializations.

Remark 2.5.15. Let $\pi \colon E \to X$ be a rank r vector bundle with local trivializations as in Definition 2.5.1. For $\alpha \in A$ let $h_\alpha \colon U_\alpha \to \mathrm{GL}_r(\mathbb{K})$ be a regular map. Then also $h_\alpha \cdot \varphi_\alpha \colon U_\alpha \to \mathrm{GL}_r(\mathbb{K})$ is a trivialization of $\pi^{-1}(U_\alpha) \to U_\alpha$ and conversely, every trivialization of $\pi^{-1}(U_\alpha) \to U_\alpha$ is obtained in this way. The 1-cocycle $\widetilde{g}_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{K})$ corresponding to this new local trivialization is given by $\widetilde{g}_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}$. In particular a moment's thought shows that E is trivial if and only if there exists $\{h_\alpha\}_{\alpha \in A}$ as above such that $g_{\alpha\beta} = h_\alpha^{-1} \cdot h_\beta$, or equivalently such that $g_{\alpha\beta} = h_\alpha \cdot h_\beta^{-1}$. Beware that the last formula is formally the same as the formula in (2.5.22) defining the 1-cocycle $g_{\alpha\beta}$, but in (2.5.22) we compose two linear maps with inverted domains and codomains, while $h_\alpha \cdot h_\beta^{-1}$ is the composition (or product) of two automorphisms of $\mathbb{K}^r$.

Remark 2.5.16. Let $\pi \colon E \to X$ be a rank r vector bundle with trivializations as in Definition 2.5.1 with corresponding 1-cocycle $g \coloneqq \{g_{\alpha\beta}\}_{(\alpha,\beta) \in A^2}$. Let $\{V_\lambda\}_{\lambda \in \Lambda}$ be an open covering of X with a *refinement map* $\rho \colon \Lambda \to A$, i.e. such that $V_\lambda \subset U_{\rho(\lambda)}$ for each $\lambda \in \Lambda$. Then we get an induced 1-cocycle $\{h_{\lambda\mu}\}_{(\lambda,\mu) \in \Lambda^2}$ by setting $h_{\lambda\mu} \coloneqq (g_{\rho(\lambda)\rho(\mu)})_{|V_\lambda \cap V_\mu}$. Let us denote $\{h_{\lambda\mu}\}_{(\lambda,\mu) \in \Lambda^2}$ by $\rho(g)$.

Now let $\pi \colon E \to X$ and $\nu \colon F \to X$ be rank r vector bundles on X with local trivializations over the sets of open coverings $\{U_\alpha\}_{\alpha \in A}$ and $\{V_\lambda\}_{\lambda \in \Lambda}$. Let $g \coloneqq \{g_{\alpha\beta}\}_{(\alpha,\beta) \in A^2}$ and $h \coloneqq \{h_{\lambda\mu}\}_{(\lambda,\mu) \in \Lambda^2}$ be the corresponding 1-cocycles. There exists a common refinement, i.e. an open covering $\{W_\xi\}_{\xi \in \Xi}$ and maps $\rho \colon \Xi \to A$, $\omega \colon \Xi \to \Lambda$ such that $W_\xi \subset U_{\rho(\xi)} \cap V_{\omega(\xi)}$ for each $\xi \in \Xi$ (e.g. consider the open sets given by intersections of an open set U_α and an open set V_λ). Thus $\pi \colon E \to X$ and $\nu \colon F \to X$ have also associated 1-cocycles $\{\rho(g))_{\xi\zeta}\}_{(\xi,\zeta) \in \Xi^2}$ and $\{\omega(h))_{\xi\zeta}\}_{(\xi,\zeta) \in \Xi^2}$ relative to the same open covering $\{W_\xi\}_{\xi \in \Xi}$. It follows from Remark 2.5.15 that the vector bundles $\pi \colon E \to X$ and $\nu \colon F \to X$ are isomorphic if and only if there exists a collection of regular maps $m_\xi \colon W_\xi \to \mathrm{GL}_r(\mathbb{K})$ for $\xi \in \Xi$ such that $\rho(g)_{\xi\zeta} = m_\xi \cdot \omega(h)_{\xi\zeta} \cdot m_\zeta^{-1}$ for all $(\xi, \zeta) \in \Xi^2$.

Above we have associated to a vector bundle with local trivializations a 1-cocycle. One can invert this construction. Let X be an algebraic variety and $\{U_\lambda\}_{\lambda \in \Lambda}$ an open covering of X, and let $g = \{g_{\lambda\mu}\}$ be a 1-cocycle with values in $\mathrm{GL}_r(\mathbb{K})$ relative to the given open covering. Then we can define a vector bundle $E \to X$ as follows. First, since open affine subsets of an algebraic are a basis of the Zariski topology, we may refine the open covering, see Remark 2.5.16, and get an induced 1-cocycle relative to an open covering by affine sets. The construction that we give does not depend (up to isomorphism) on the refinement, so we assume from the beginning that the open subsets U_λ are affine. For $\lambda, \mu \in A$ let

$$\begin{array}{ccc} (U_\lambda \cap U_\mu) \times \mathbb{K}^r & \xrightarrow{\varphi_{\lambda\mu}} & (U_\lambda \cap U_\mu) \times \mathbb{K}^r \\ (x, \xi) & \mapsto & (x, g_{\lambda\mu} \cdot \xi) \end{array} \tag{2.5.15}$$

If we let $A_{\lambda\mu} := (U_\lambda \cap U_\mu) \times \mathbb{K}^r \subset U_\lambda \times \mathbb{K}^r$ and $B_{\lambda\mu} := (U_\lambda \cap U_\mu) \times \mathbb{K}^r \subset U_\mu \times \mathbb{K}^r$ then Hypothesis 2.2.13 are satisfied, and hence we may glue the affine varieties $U_\lambda \times \mathbb{K}^r$ via the $\varphi_{\lambda\mu}$ see the Gluing Construction 2.2.14. Let E be the prevariety that we get. The regular maps $U_\lambda \times \mathbb{K}^r \to U_\lambda \hookrightarrow X$ glue to give a regular map $\pi\colon E \to X$.

Claim 2.5.17. *The prevariety E is an algebraic variety.*

Proof. We must check that E is of finite type and separated. Since X is of finite type it has a finite cover $X = V_1 \cup \ldots \cup V_m$ by open affine sets. Since V_i is quasi compact the covering $V_i = \bigcup_{\lambda \in \Lambda}(V_i \cap U_\lambda)$ has a finite subcover. Each $V_i \cap U_\lambda$ is an open affine set by Proposition 2.3.20. We have $\pi^{-1}(V_i \cap U_\lambda) \cong V_i \cap U_\lambda \times \mathbb{K}^r$, and hence E is a finite union of open affine subsets. This proves that E is of finite type.

In order to prove that E is separated we notice that if $\lambda, \mu \in \Lambda$ then

$$(U_\lambda \times \mathbb{K}^r) \times (U_\mu \times \mathbb{K}^r) \cap \Delta_E = ((U_\lambda \times U_\mu) \cap \Delta_X) \cap (U_\lambda \times U_\mu) \times \Delta_{\mathbb{K}^r}. \tag{2.5.16}$$

This shows that $(U_\lambda \times \mathbb{K}^r) \times (U_\mu \times \mathbb{K}^r) \cap \Delta_E$ is closed. Since $E \times E$ is the union of the open subsets $(U_\lambda \times \mathbb{K}^r) \times (U_\mu \times \mathbb{K}^r)$, it follows that Δ_E is closed in $E \times E$, i.e. E is separated. □

Linear algebra constructions on vector bundles

One can produce vector bundles from given vector bundles by lifting linear algebra constructions.

Direct sum of vector bundles

Let $\pi\colon E \to X$ and $\rho\colon F \to X$ be algebraic vector bundles. Let

$$E \oplus F := E \times_X F = \{(e, f) \in E \times F \mid \pi(e) = \rho(f)\}. \tag{2.5.17}$$

The map $\mu\colon E \oplus F \to X$ defined by setting $\mu(e, f) := \pi(e) = \rho(f)$ is regular. If $x \in X$ the fiber $\mu^{-1}(x)$ is identified with $E(x) \oplus F(x)$ and hence it has a structure of $\mathbb{K}$ vector space of dimension $\mathrm{rk}(E) + \mathrm{rk}(F)$. Lastly, by choosing an open cover of X which trivializes both E and F we get that $\mu\colon E \oplus F \to X$ has a local trivialization. Thus $E \oplus F$ is an algebraic vector bundle over X. This is the *direct sum of E and F*.

Functorial constructions

Let $E \to X$ be an algebraic vector bundles. Then one constructs a vector bundle $E^\vee \to X$ whose fiber over $x \in X$ is identified with the dual vector space $E(x)^\vee$. Analogously one constructs a vector bundle $E \otimes E \to X$ whose fiber over $x \in X$ is identified with the tensor square $E(x) \otimes E(x)$. More generally let Λ be a functor (possibly contravariant) from the category of $\mathbb{K}$ vector spaces to itself. Then one can construct a vector bundle $\Lambda(E) \to X$ whose fiber over $x \in X$ is identified with the vector space $\Lambda(E(x))$. In fact let $g := \{g_{\alpha\beta}\}$ be the 1-cocycle corresponding to local trivializations of E as in Definition 2.5.1. Then $\Lambda(g) := \{\Lambda(g_{\alpha\beta})\}$ is a 1-cocycle which defines by gluing a vector bundle $F \to X$. If we change local trivializations of E the vector bundle obtained from the new 1-cocycle is isomorphic to F by Remark 2.5.16. Thus we have produced a vector bundle well determined up to isomorphism, that we denote by $\Lambda(E)$. In order to define an isomorphism between $\Lambda(E)(x)$ and $\Lambda(E(x))$ one proceeds as follows. The vector bundle $\Lambda(E)$ is obtained by gluing the affine varieties $U_\alpha \times \Lambda(\mathbb{K}^r)$ (by refining the open covering $\{U_\alpha\}$ we may assume that U_α is affine for every $\alpha \in A$) via the maps $(\mathrm{Id}_{U_\alpha \cap U_\beta}, \Lambda(g_{\alpha\beta}))$. Thus for $x \in U_\alpha$ we have the isomorphism of vector spaces $\psi_\alpha(x)\colon \Lambda(E)(x) \xrightarrow{\sim} \Lambda(\mathbb{K}^r)$. If $x \in U_\alpha$ then we also have the isomorphism $\Lambda(\varphi_\alpha(x))\colon \Lambda(E(x)) \xrightarrow{\sim} \Lambda(\mathbb{K}^r)$. The composition gives the isomorphism of vector spaces

$$\Lambda(\varphi_\alpha(x))^{-1} \circ \psi_\alpha(x)\colon \Lambda(E)(x) \xrightarrow{\sim} \Lambda(E(x)). \tag{2.5.18}$$

By functoriality the above isomorphism is independent of the open set U_α containing x.

Example 2.5.18. If $\{g_{\alpha\beta}\}$ is a 1-cocycle representing $E \to X$, then $E^\vee \to X$ is represented by the 1-cocycle $\{(g^t_{\alpha\beta})^{-1}\}$.

Example 2.5.19. Let $L \to \mathbb{P}^n$ be the tautological line bundle. In Example 2.5.13 we have show that L is represented by the 1-cocycle $g = \{g_{ij}\}$ relative to the open cover $\{\mathbb{P}^n_{Z_i}\}_{i=0}^n$ given by $g_{ij}([Z]) = Z_i/Z_j$ for $[Z] \in \mathbb{P}^n_{Z_i} \cap \mathbb{P}^n_{Z_j}$. It follows that the dual $L^\vee \to \mathbb{P}^n$ is represented by the 1-cocycle $h = \{h_{ij}\}$ relative to the open cover $\{\mathbb{P}^n_{Z_i}\}_{i=0}^n$ given by $h_{ij}([Z]) = Z_j/Z_i$ for $[Z] \in \mathbb{P}^n_{Z_i} \cap \mathbb{P}^n_{Z_j}$.

Remark 2.5.20. Let $L \to \mathbb{P}^n$ be the tautological line bundle and let $L^\vee \to \mathbb{P}^n$ be its dual. Let $f\colon V \to \mathbb{K}$ be a linear map. We associate a section $\sigma_f\colon \mathbb{P}(V) \to L^\vee$ by mapping $[Z] \in \mathbb{P}^n$ to the linear function on $L([Z]) = \operatorname{span}(Z)$ given by the restriction of f to $\operatorname{span}(Z)$. The section σ_f is regular. In fact the trivialization of $L^\vee$ considered in Example 2.5.19 gives a generator ψ_i of $L^\vee_{|\mathbb{P}^n_{Z_i}}$ characterized by the fact that ψ_i takes the value W_i on $([Z], W)$. We have the equality

$$\sigma_{f|\mathbb{P}^n_{Z_i}} = \frac{f(Z)}{Z_i}\psi_i. \tag{2.5.19}$$

This show that σ_f is regular. As an exercise one should check that the local sections on the right hand side of the above equation do indeed glue to give a global section of $L^\vee$.

Example 2.5.21. Let $L \to X$ be a line bundle, represented by the 1-cocycle $g = \{g_{\alpha\beta}\}$ relative to an open cover $\{U_\alpha\}$. The tensor power $L^{\otimes m} \to X$ is represented by the 1-cocycle $h = \{h_{\alpha\beta}\}$ where $h_{\alpha\beta} := g^m_{\alpha\beta}$. Note that if we set $m = -1$ we get a 1-cocycle representing L^{-1}. This is one reason for using L^{-1} as alternative notation for the dual $L^\vee$. Of course L^{-m} is used to denote $(L^{\otimes m})^{-1}$. Note also that the 1-cocycle $g^0_{\alpha\beta}$ represents the trivial line bundle. This justifies setting $L^{\otimes 0}$ equal to the trivial line bundle.

Tensor product of vector bundles

Let $\pi\colon E \to X$ and $\rho\colon F \to X$ be algebraic vector bundles. One constructs a *tensor product vector bundle* $E \otimes F \to X$ whose fiber over $x \in X$ is identified with the tensor square $E(x) \otimes F(x)$ by a procedure which is analogous to what was done in the previous subsubsection. We leave details to the reader. Of course if $E = F$ then the tensor product vector bundle is the square tensor vector bundle of the previous subsubsection.

Quotient of a vector bundle by a subbundle

Let $\pi\colon E \to X$ and $\rho\colon G \to X$ be algebraic vector bundles. A morphism $\theta\colon G \to E$ of vector bundles, see Definition 2.5.4, is an *injection of vector bundles* if for every $x \in X$ the linear map $\theta(x)\colon G(x) \to E(x)$ is injective. If this is the case then the image $\operatorname{im}(\theta)$ is a closed subset of E.

Definition 2.5.22. Let $\pi\colon E \to X$ be an algebraic vector bundle. A closed subset $F \subset E$ is a *subvector bundle of rank* s if there exists an injection of vector bundles $\theta\colon G \to E$, where G has rank s, such that $F = \operatorname{im}(\theta)$.

Note that, by definition, a subvector bundle of rank s of $\pi\colon E \to X$ is a vector bundle of rank s on X.

Let $\pi\colon E \to X$ be an algebraic vector bundle r and let $F \subset E$ be a subvector bundle of rank s. One defines a vector bundle with fiber over $x \in X$ identified with $E(x)/F(x)$ proceeding as follows. For $x \in X$ let $\mu(x)\colon E(x) \to E(x)/F(x)$ be the quotient map. Let $\{U_\alpha\}_{\alpha\in A}$ be an open covering of X which trivializes E, as in Definition 2.5.1. Let $\alpha \in A$. Let $\psi_\alpha \colon \mathbb{K}^{r-s} \hookrightarrow \mathbb{K}^r$ be an injection of vector spaces, and for $x \in U_\alpha$ let

$$\mathbb{K}^{r-s} \overset{\mu_\alpha(x)}{\longrightarrow} E(x)/F(x) \tag{2.5.20}$$

be the composition

$$\mathbb{K}^{r-s} \overset{\psi_\alpha(x)}{\longrightarrow} \mathbb{K}^r \overset{\varphi_\alpha(x)^{-1}}{\longrightarrow} E(x) \overset{\mu(x)}{\longrightarrow} E(x)/F(x). \tag{2.5.21}$$

By refining the covering $\{U_\alpha\}$ and choosing appropriately the injections ψ_α we may assume that $\mu_\alpha(x)$ is an isomorphism for all $\alpha \in A$ and all $x \in U_\alpha$. For $\alpha, \beta \in A$ we define the map $U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{K})$ as follows:

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \overset{g_{\alpha\beta}}{\longrightarrow} & \mathrm{GL}_r(\mathbb{K}) \\ x & \mapsto & \mu_\alpha(x) \circ \mu_\beta^{-1}(x) \end{array} \tag{2.5.22}$$

Then $\{g_{\alpha\beta}\}_{\alpha,\beta \in A}$ is a 1-cocycle with values in $\mathrm{GL}_{r-s}(\mathbb{K})$, and hence there is an associated vector bundle. Up to isomorphism the vector bundle does not depend on the choices that we made: this the *quotient vector bundle* E/F. Let $x \in X$: proceeding as has been done for previous constructions one defines an isomorphism between the fiber $(E/F)(x)$ and the quotient $E(x)/F(x)$.

Note that the map $\mu \colon E \to E/F$ defined by setting $\mu_{|E(x)} := \mu(x)$ for all $x \in X$ is a (regular) map of vector bundles. Suppose that $G \hookrightarrow E$ is a subvector bundle such that for all $x \in X$ the restriction of $\mu(x)$ to $G(x)$ is an isomorphism. Then the composition $G \hookrightarrow E \overset{\mu}{\longrightarrow} E/F$ is an isomorphism of vector bundles. One could hope to define the quotient vector bundle as being isomorphic to any subvector bundle $G \subset E$ with the above property. This would not be an acceptable definition because in general there is no such subvector bundle, see Exercise 2.6.7.

Sheaves

There is a different way of viewing a vector bundle, namely as a particular kind of sheaf. First we introduce sheaves.

Definition 2.5.23. Let X be a topological space. A *sheaf of sets* $\mathscr{F}$ on X consists of the following data:

1. for each open $U \subset X$ a set $\mathscr{F}(U)$, and
2. for each inclusion $U \subset V$ of open subsets of X a *restriction map* $\rho_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$,

such that the following hold:

(a) $\rho_{U,U} = \mathrm{Id}_{\mathscr{F}(U)}$.

(b) If $U \subset V \subset W$ are inclusions of open subset of X then $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$.

(c) Let $V \subset X$ be open and suppose that $V = \bigcup_{i \in I} V_i$ where each V_i is open.

  (c1) If $\sigma, \tau \in \mathscr{F}(U)$ and $\rho_{V,V_i}(\sigma) = \rho_{V,V_i}(\tau)$ for all $i \in I$ then $\sigma = \tau$.

  (c2) If there exists a collection of $\sigma_i \in \mathscr{F}(V_i)$ for every $i \in I$ such that $\rho_{V_i, V_i \cap V_j}(\sigma_i) = \rho_{V_j, V_i \cap V_j}(\sigma_j)$ for all $i, j \in I$ then there exists $\sigma \in \mathscr{F}(V)$ such that $\rho_{V,V_i}(\sigma) = \rho_{V,V_i}(\tau)$ for all $i \in I$.

If each of the sets $\mathscr{F}(U)$ has a structure of group, and $\rho_{V,U}$ is a homomorphism of groups, then we say that $\mathscr{F}$ is a *sheaf of groups*. If each of the sets $\mathscr{F}(U)$ has a structure of ring, and $\rho_{V,U}$ is a homomorphism of groups, then we say that $\mathscr{F}$ is a *sheaf of groups*.

Example 2.5.24. Let X, Y be topological space. For $U \subset X$ open let $\mathscr{F}(U)$ be the set whose elements are the continuous maps $f \colon U \to Y$. If $U \subset V$ is an inclusion of open subsets of X let $\rho_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$ be defined by setting $\rho_{V,U}(f) := f_{|U}$. Then $\mathscr{F}$ is a sheaf of sets on X. Suppose that Y is a topological group, i.e. that multiplication and inverse are continuos maps. Then pointwise multiplication defines a structure of group on $\mathscr{F}(U)$, and we get a sheaf of groups.

Example 2.5.25. Let X be a prevariety. For $U \subset X$ open let $\mathscr{F}(U)$ be the ring whose elements are the regular maps $f\colon U \to \mathbb{K}$ with addition and multiplication defined pointwise. If $U \subset V$ is an inclusion of open subsets of X let $\rho_{V,U}\colon \mathscr{F}(V) \to \mathscr{F}(U)$ be the homomorphism of rings defined by setting $\rho_{V,U}(f) \coloneqq f_{|U}$. Then $\mathscr{F}$ is a sheaf of rings on X. This is the *structure sheaf of* X and is denoted by $\mathscr{O}_X$.

Definition 2.5.26. Let X be a topological space, and let $\mathscr{R}$ be a sheaf of rings on X. A *sheaf of $\mathscr{R}$-modules* is a sheaf of sets $\mathscr{F}$ on X with the extra datum of a structure of $\mathscr{R}(U)$-module on $\mathscr{F}(U)$ for every opne $U \subset X$. We require that for every inclusion $U \subset V$ of open subsets of X and $\sigma \in \mathscr{F}(V)$, $f \in \mathscr{R}(V)$ we have

$$\rho^{\mathscr{F}}_{V,U}(f \cdot \sigma) = \rho^{\mathscr{R}}_{V,U}(f) \cdot \rho^{\mathscr{F}}_{V,U}(\sigma), \tag{2.5.23}$$

where $\rho^{\mathscr{F}}_{V,U}$ and $\rho^{\mathscr{R}}_{V,U}$ are the restriction maps of $\mathscr{F}$ and $\mathscr{R}$ respectively.

Definition 2.5.27. Let X be a topological space, and let $\mathscr{F}$ be a sheaf of sets on X. If $V \subset X$ is open then we get a sheaf of sets on V by assigning to $U \subset V$ open the set $\mathscr{F}(U)$, and by defining, for $U \subset W \subset V$ open, the restriction map equal to the restriction map $\rho_{W,U}$ of $\mathscr{F}$. This sheaf of sets on V is the *restriction of $\mathscr{F}$ to* V and is denoted by $\mathscr{F}_{|V}$. If $\mathscr{F}$ is a sheaf of (groups)/(rings)/(modules over a sheaf of rings) then $\mathscr{F}_{|V}$ is a sheaf of (groups)/(rings)/(modules over a sheaf of rings) in a natural way.

Remark 2.5.28. Let X be a topological space, and let $\mathscr{F}$ be a sheaf of (sets)/(groups)/(rings)/(modules over a sheaf of rings) over X. If $U \subset X$ is open then $\mathscr{F}(U)$ is the set/group/ring/module of *sections of $\mathscr{F}$ over* U, and is denoted also by $\Gamma(U, \mathscr{F}_{|U})$.

Example 2.5.29. Let X be an algebraic prevariety. If $U \subset X$ is open it has a canonical structure of algebraic prevariety. Restriction of regular functions defines an identification between $\mathscr{O}_{X|U}$ and $\mathscr{O}_U$. The ring of sections $\mathscr{O}_X(U) = \Gamma(U, \mathscr{F}_{|U})$ is equal to $\mathbb{K}[U]$.

Definition 2.5.30. Let $\mathscr{F}, \mathscr{G}$ be sheaves of (sets)/(groups)/(rings)/(modules over a sheaf of rings) on a topological space X. A *morphism* $\varphi\colon \mathscr{F} \to \mathscr{G}$ consists of the assignment of a morphism $\varphi_U \in \mathscr{F}(U) \to \mathscr{G}(U)$ (i.e. respectively a (map of sets)/(homomorphism of groups)/(homomorphism of rings)/(homomorphism of modules)) such that the following holds. If $U \subset V$ are open subsets of X then the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\varphi_V} & \mathscr{G}(V) \\ {\scriptstyle \rho^{\mathscr{F}}_{V,U}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathscr{G}}_{V,U}} \\ \mathscr{F}(U) & \xrightarrow{\varphi_U} & \mathscr{G}(U) \end{array} \tag{2.5.24}$$

If $\mathscr{F}$ is a sheaf as above on X the identity map $\mathrm{Id}_U \in \mathscr{F}(U) \to \mathscr{F}(U)$ defines a morphism of $\mathscr{F}$: this is the *Identity morphism.* If $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are sheaves as above on X, and $\varphi\colon \mathscr{F} \to \mathscr{G}$, $\psi\colon \mathscr{G} \to \mathscr{H}$ are morphisms of sheaves, one gets a morphism of sheaves $\psi \circ \varphi\colon \mathscr{F} \to \mathscr{H}$ by setting $(\psi \circ \varphi)_U \coloneqq \psi_U \circ \varphi_U$. Thus we have the *category of sheaves of (sets)/(groups)/(rings)/(modules over a sheaf of rings) on* X. In particular we have the notion of *isomorphism of sheaves.*

Vector bundles and locally free sheaves

Let $\pi\colon E \to X$ be a vector bundle on an algebraic variety X. For $U \subset X$ open (in the Zariski topology) let

$$\mathscr{S}(E)(U) \coloneqq \{\sigma\colon U \to E \mid \sigma \text{ is regular and } \pi \circ \sigma = \mathrm{Id}_U\} = \Gamma(U, E_{|U}). \tag{2.5.25}$$

If $\sigma, \tau \in \mathscr{S}(E)(U)$ then the map $(\sigma + \tau)\colon U \to E_{|U}$ mapping x to $\sigma(x) + \tau(x)$ is a regular section of $E_{|U}$. If $f \in \mathscr{O}_X(U)$ then the map $f\sigma\colon U \to E_{|U}$ mapping x to $f(x) \cdot \sigma(x)$ is a regular section of $E_{|U}$. With these operations $\mathscr{S}(E)(U)$ is an $\mathscr{O}_X(U)$-module. Let $U \subset V$ be open subsets of X and let $\sigma \in \mathscr{S}(E)(V)$. Then the restriction of σ to U is a regular section of $E_{|U}$. Thus we have a map $\rho^E_{V,U}\colon \mathscr{S}(E)(V) \to \mathscr{S}(E)(U)$. One easily checks that this gives a sheaf of $\mathscr{O}_X$-modules.

Definition 2.5.31. Let $\pi\colon E \to X$ be a vector bundle on an algebraic variety X. The *sheaf of germs of sections of* E is the sheaf $\mathscr{S}(E)$ of $\mathscr{O}_X$-modules defined above.

Example 2.5.32. Let $\pi\colon L \to X$ be the trivial line bundle, i.e. $L = X \times \mathbb{K}$ and π is the projection. Then $\mathscr{S}(L)$ is isomorphic to $\mathscr{O}_X$. In fact if $U \subset X$ is open then

$$\mathscr{S}(L)(U) = \{f\colon U \to \mathbb{K} \mid f \text{ is regular}\} = \mathscr{O}_X(U). \tag{2.5.26}$$

Definition 2.5.33. Let $L \to \mathbb{P}^n$ be the tautological line bundle. Then

1. $\mathscr{O}_{\mathbb{P}^n}(-1)$ is the sheaf of germs of sections of L, i.e. $\mathscr{O}_{\mathbb{P}^n}(-1) := \mathscr{S}(L)$.

2. $\mathscr{O}_{\mathbb{P}^n}(1)$ is the sheaf of germs of sections of $L^\vee$, i.e. $\mathscr{O}_{\mathbb{P}^n}(1) := \mathscr{S}(L^\vee)$.

3. If $d \in \mathbb{N}$ then $\mathscr{O}_{\mathbb{P}^n}(d)$ is the sheaf of germs of sections of $(L^\vee)^{\otimes d}$ and $\mathscr{O}_{\mathbb{P}^n}(-d)$ is the sheaf of germs of sections of $(L)^{\otimes d}$. Note that $\mathscr{O}_{\mathbb{P}^n}(0) \cong \mathscr{O}_X$, see Example 2.5.21.

Suppose that $\pi\colon E \to X$, $\rho\colon F \to X$ are vector bundles on an algebraic variety X, and that $f\colon E \to F$ is a morphism of vector bundles. If $U \subset X$ is open let $\mathscr{S}(f)_U\colon \mathscr{S}(E)(U) \to \mathscr{S}(F)(U)$ be defined by $\mathscr{S}(f)(\sigma) := f \circ \sigma$. The collection of the maps $\mathscr{S}(f)_U$ defines a morphism of $\mathscr{O}_X$-modules $\mathscr{S}(E) \to \mathscr{S}(F)$. Thus we have defines a functor from the category of vector bundles over X to the category of $\mathscr{O}_X$-modules.

The sheaf of germs of sections of a vector bundle is a very particular kind of sheaf of $\mathscr{O}_X$-modules, i.e. the image of the functor that we have defined is far from being the whole category of $\mathscr{O}_X$-modules (unless X is a finite set). In order to give a precise characterization of the image we need to go through a few (more) definitions. Let X be a topological space, let $\mathscr{R}$ be a sheaf of rings on X, and let $\mathscr{F}, \mathscr{G}$ be sheaf of $\mathscr{R}$-modules on X. By associating to $U \subset X$ open the direct sum $\mathscr{F}(U) \oplus \mathscr{G}(U)$ we get an $\mathscr{R}(U)$-module. If $U \subset V$ are open the restriction maps $\rho^{\mathscr{F}}_{V,U}$ and $\rho^{\mathscr{G}}_{V,U}$ define maps $\rho^{\mathscr{F}}_{V,U}\colon (\mathscr{F}(V) \oplus \mathscr{G}(V)) \to (\mathscr{F}(U) \oplus \mathscr{G}(U))$. As is easily checked this defines a sheaf of $\mathscr{R}$-modules on X.

Definition 2.5.34. Let X be a topological space, let $\mathscr{R}$ be a sheaf of rings on X, and let $\mathscr{F}, \mathscr{G}$ be sheaf of $\mathscr{R}$-modules on X. The *direct sum of* $\mathscr{F}$ *and* $\mathscr{G}$ is the sheaf of $\mathscr{R}$-modules on X defined above, and is denoted by $\mathscr{F} \oplus \mathscr{G}$.

Example 2.5.35. Let $\pi\colon E \to X$ and $\rho\colon F \to X$ be vector bundles on the algebraic variety X. Then $\mathscr{S}(E \oplus F)$, i.e. the sheaf of germs of sections of $E \oplus F$, is isomorphic to the direct sum $\mathscr{S}(E) \oplus \mathscr{S}(F)$.

Definition 2.5.36. Let X be a topological space, and let $\mathscr{R}$ be a sheaf of rings on X. A sheaf $\mathscr{F}$ of $\mathscr{R}$-modules is *locally free of rank* r if there exists an open covering $\{U_\alpha\}_{\alpha \in A}$ of X such that for every $\alpha \in A$ the restriction $\mathscr{F}_{|U_\alpha}$ is isomorphic to $\mathscr{R}^{\oplus r}_{|U_\alpha}$, i.e. the direct sum of r copies of $\mathscr{R}_{|U_\alpha}$.

Claim 2.5.37. *Let $\pi\colon E \to X$ be a vector bundle of rank r on an algebraic variety X. Then $\mathscr{S}(E)$, i.e. the sheaf of germs of sections of E, is locally free of rank r.*

Proof. By definition there exists an open covering $\{U_\alpha\}_{\alpha \in A}$ of X such that $E_{|U_\alpha}$ is trivial of rank r. By Examples 2.5.32 and 2.5.35 it follows that $E_{|U_\alpha}$ is isomorphic to $\mathscr{O}^{\oplus r}_{X|U_\alpha}$. □

The following result gives that vector bundles and locally free sheaves are equivalent notions.

Proposition 2.5.38. *Let X be an algebraic variety. By assigning to a vector bundle E on X its sheaf of germs of sections $\mathscr{S}(E)$ and to a morphism $f\colon E \to F$ of vector bundles on X the morphism of $\mathscr{O}_X$-modules $\mathscr{S}(f)\colon \mathscr{S}(E) \to \mathscr{S}(F)$ we get an equivalence between the functor of vector bundles (of constant rank) on X and the functor of locally free sheaves of $\mathscr{O}_X$-modules (of constant rank).*

Proof. Let $\mathscr{F}$ be a locally free sheaf of $\mathscr{O}_X$-modules of rank r. By hypothesis there exists an open covering $\{U_\alpha\}_{\alpha\in A}$ and for each $\alpha \in A$ an isomorphism

$$\varphi_\alpha\colon \mathscr{F}_{|U_\alpha} \xrightarrow{\sim} \mathscr{O}^{\oplus r}_{X|U_\alpha}. \tag{2.5.27}$$

From this we produce a 1-cocycle with values in $\mathrm{GL}_r(\mathbb{K})$ as follows. For $\alpha, \beta \in A$ the composition

$$\mathscr{O}^{\oplus r}_{X|U_\alpha\cap U_\beta} \overset{\varphi^{-1}_{\beta|\ldots}}{\longrightarrow} \mathscr{F}_{|U_\alpha\cap U_\beta} \overset{\varphi_{\alpha|\ldots}}{\longrightarrow} \mathscr{O}^{\oplus r}_{X|U_\alpha\cap U_\beta} \tag{2.5.28}$$

is an isomorphism

$$\psi_{\alpha\beta}\colon \mathscr{O}^{\oplus r}_{X|U_\alpha\cap U_\beta} \xrightarrow{\sim} \mathscr{O}^{\oplus r}_{X|U_\alpha\cap U_\beta}. \tag{2.5.29}$$

There exist $g^{ij}_{\alpha\beta} \in \mathscr{O}_X(U_\alpha \cap U_\beta)$ for $i, j \in \{1, \ldots, r\}$ such that

$$\psi_{\alpha\beta}(e^{\alpha\beta}_j) = \sum_{i=1}^{r} g^{ij}_{\alpha\beta}(x) e^{\alpha\beta}_i. \tag{2.5.30}$$

The $r \times r$ matrix $g_{\alpha\beta} := (g^{ij}_{\alpha\beta})$ with values in $\mathscr{O}_X(U_\alpha \cap U_\beta)$ is invertible because $\psi_{\alpha\beta} \circ \psi_{\beta\alpha}$ is the identity. Thus we have the 1-cochain $g = \{g_{\alpha\beta}\}$ with values in $\mathrm{GL}_r(\mathbb{K})$. One checks that g is a 1-cocycle. Let $\pi\colon E \to X$ be the associated rank r vector bundle. The sheaf of germs of sections $\mathscr{S}(E)$ is isomorphic to $\mathscr{F}$. Moreover if $\varphi\colon \mathscr{E} \to \mathscr{F}$ is a morphism of locally free sheaves, and E, F are vector bundles such that $\mathscr{S}(E) \cong \mathscr{E}$, $\mathscr{S}(F) \cong \mathscr{F}$, then there exists a morphism of vector bundles $f\colon E \to F$ such that $\mathscr{S}(f) = \varphi$. We leave details of the proofs to the reader. □

Because of the above result one does not distinguish between vector bundles and locally free sheaves. For example $\mathscr{O}_{\mathbb{P}^n}(d)$ (see Definition 2.5.33), which strictly speaking is a locally free sheaf of rank 1, denotes also the corresponding line bundle on $\mathbb{P}^n$, e.g. the dual of the tautological line bundle if $d = 1$.

Line bundles and regular maps to projective spaces

2.6 Exercises

Exercise 2.6.1. Let R be an integral domain, and let $(m, n) \in (\mathbb{N}^2\backslash\{0\})$. Let $F \in R[X, Y]_m$ and $G \in R[X, Y]_n$. The *resultant* $\mathscr{R}(F, G)$ is the element of R defined as follows. Consider the map of free R-modules

$$\begin{array}{ccc} R[X,Y]_{n-1} \oplus R[X,Y]_{m-1} & \overset{L(F,G)}{\longrightarrow} & R[X,Y]_{m+n-1} \\ (\Phi, \Psi) & \mapsto & \Phi \cdot F + \Psi \cdot G \end{array} \tag{2.6.31}$$

and let $S(F, G)$ be the matrix of $L(F, G)$ relative to the basis

$$(X^{n-1}, 0),\ (X^{n-2}Y, 0),\ \ldots\ ,\ (Y^{n-1}, 0),\ (0, X^{m-1}),\ (0, X^{m-2}Y),\ \ldots\ ,\ (0, Y^{m-1}) \tag{2.6.32}$$

of the domain and the basis

$$X^{m+n-1},\ X^{m+n-2}Y,\ \ldots\ ,\ XY^{m+n-2},\ Y^{m+n-1} \tag{2.6.33}$$

of the codomain. Then $\mathscr{R}(F, G)$ is defined by

$$\mathscr{R}(F, G) := \det S(F, G). \tag{2.6.34}$$

Explicitly: if

$$F = \sum_{i=0}^{m} a_i X^{m-i} Y^i, \quad G = \sum_{j=0}^{n} b_j X^{n-j} Y^j \tag{2.6.35}$$

then

$$\mathscr{R}(F,G) = \det \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 & b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & a_0 & \vdots & \vdots & \cdots & b_0 \\ a_m & a_{m-1} & \cdots & \vdots & b_n & b_{n-1} & \cdots & \vdots \\ 0 & a_m & \cdots & \vdots & 0 & b_n & \cdots & \vdots \\ 0 & 0 & \cdots & \vdots & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_m & 0 & 0 & \cdots & b_n \end{pmatrix}. \tag{2.6.36}$$

Now let K be a field and $K \subset \mathbb{K}$ be an algebraic closure of K. Let $F \in K[X,Y]_m$ and $G \in K[X,Y]_n$.

(a) Prove that $\mathscr{R}(F,G) = 0$ if and only if there exists $H \in K[X,Y]_d$ with $d > 0$ which divides both F and G (in $K[X,Y]$).

(b) Prove that $\mathscr{R}_{m,n}(F,G) = 0$ if and only if there exists a common non-trivial root of F and G in $\mathbb{K}^2$, i.e. $[X_0, Y_0] \in \mathbb{P}^1(\mathbb{K})$ such that $F(X_0,Y_0) = G(X_0,Y_0) = 0$.

(c) Let $f(t,x) \in K[t_1,\ldots,t_m][x]$ and $g(t,x) \in K[t_1,\ldots,t_m][x]$ (here $t = t_1,\ldots,t_m$) be polynomials of degrees m and n in x respectively, i.e.

$$f(t,x) = \sum_{i=1}^{m} a_i(t)x^{m-i}, \quad g(t,x) = \sum_{j=1}^{n} b_j(t)x^{n-j} \quad a_i(t), b_j(t) \in K[t_1,\ldots,t_m], \quad a_0(t) \neq 0 \neq b_0(t).$$

We let

$$D(f,g) := \{\bar{t} \in \mathbb{A}^m(\mathbb{K}) \mid \exists x \in \mathbb{K} \text{ such that } f(\bar{t},x) = g(\bar{t};x) = 0\}.$$

Using the properties of the resultant proved above show that if f,g are both monic, i.e. $a_0(t) = b_0(t) = 1$, then there exists $\varphi \in K[t_1,\ldots,t_m]$ such that $D(f,g) = V(\varphi)$.

(d) Give examples of $f(t,x) \in K[t_1,\ldots,t_m][x]$ and $g(t,x) \in K[t_1,\ldots,t_m][x]$ for which there exists no $\varphi \in K[t_1,\ldots,t_m]$ such that $D(f,g) = V(\varphi)$.

Exercise 2.6.2. The goal of the exercise is to prove the Main Theorem of Elimination Theory, i.e. Theorem 2.4.2, without invoking the Nullstellensatz.

(a) Let $\pi\colon \mathbb{A}^m \times \mathbb{P}^1 \to \mathbb{A}^m$ be the projection. Prove that if $X \subset \mathbb{A}^m \times \mathbb{P}^1$ is closed then $\pi(X)$ is closed in $\mathbb{A}^m$ by using Item (b) of Exercise 2.6.1.

(b) Let $\mu_n\colon (\mathbb{P}^1)^n \to \mathbb{P}^n$ be the map defined by

$$\begin{array}{ccc} (\mathbb{P}^1)^n & \xrightarrow{\mu_n} & \mathbb{P}(\mathbb{K}[X,Y]_n) \cong \mathbb{P}^n \\ ([a_0,b_0],[a_1,b_1],\ldots,[a_n,b_n]) & \mapsto & [(a_0X - b_0Y)\cdots(a_1X-b_1Y)\cdot\ldots\cdot(a_nX-b_nY) \end{array} \tag{2.6.37}$$

Prove that μ_n is regular.

(c) Let $\pi\colon \mathbb{A}^m \times \mathbb{P}^n \to \mathbb{A}^m$ be the projection. Let $X \subset \mathbb{A}^m \times \mathbb{P}^1$ be closed. Prove that $\pi(X)$ is closed in $\mathbb{A}^m$ by considering the closed subset $\mu_n^{-1}(X) \subset (\mathbb{P}^1)^n$ (see Item (b)), and applying Item (a) to the projections $\mathbb{A}^m \times (\mathbb{P}^1)^n \to \mathbb{A}^m \times (\mathbb{P}^1)^{n-1}$, $\mathbb{A}^m \times (\mathbb{P}^1)^{n-1} \to \mathbb{A}^m \times (\mathbb{P}^1)^{n-2}$ etc.

Let V be a $\mathbb{K}$ vector space of finite dimension, and let $0 \leqslant h \leqslant \dim V$. The *Grassmannian*

$$\mathrm{Gr}\,(h,V) := \{W \subset V \mid \dim W = h\}\,.$$

is the set of subvector spaces of V of dimension h. The Zariski topology on $\mathrm{Gr}\,(h,V)$ is defined as follows. Let $\mathrm{Fr}(h,V)$ be the set of ordered lists of linearly independent vectors $v_1,\ldots,v_h \in V$. We define the left action

$$\begin{array}{ccc} \mathrm{GL}_h(\mathbb{K}) \times \mathrm{Fr}(h,V) & \longrightarrow & \mathrm{Fr}(h,V) \\ ((a_{ij}),\{v_1,\ldots,v_h\}) & \mapsto & \{\sum_{i=1}^h a_{1i}v_i,\, \sum_{i=1}^h a_{2i}v_i,\, \ldots,\, \sum_{i=1}^h a_{hi}v_i\} \end{array} \tag{2.6.38}$$

The quotient for the equivalence relation defined by the above action is the map

$$\begin{array}{ccc} \mathrm{Fr}(h,V) & \xrightarrow{\pi} & \mathrm{Gr}\,(h,V) \\ v_1,\ldots,v_h & \mapsto & \mathrm{span}(v_1,\ldots,v_h) \end{array} \tag{2.6.39}$$

Since $\mathrm{Fr}(h,V) \subset V^h$ (as an open subset) it inherits a Zariski topology from $V^h \cong \mathbb{A}^{h\cdot\dim V}$. The Zariski topology on $\mathrm{Gr}\,(h,V)$ is the quotient topology.

Exercise 2.6.3. The goal of the exercise is to provide $\mathrm{Gr}\,(h,V)$ with the structure of an algebraic variety. Let $U \subset V$ be a vector subspace of dimension $\dim V - h$, i.e. an element of $\mathrm{Gr}\,(\dim V - h, V)$. Let $\mathrm{Gr}\,(h,V)_U \subset \mathrm{Gr}\,(h,V)$ be the subset of W which are transverse to U.

(a) Show that $\mathrm{Gr}\,(h,V)_U$ is open.

(b) Show that the action of $\mathrm{Hom}(V/U,U)$ on $\mathrm{Gr}\,(h,V)_U$ defined by

$$\begin{array}{ccc} \mathrm{Hom}(V/U,U) & \longrightarrow & \mathrm{Gr}\,(h,V)_U \\ (f,W) & \mapsto & \{w+\varphi(\overline{w}) \mid w \in W\} \end{array} \tag{2.6.40}$$

is simply transitive ($\overline{w}$ is the equivalence class of w in V/U), and hence it gives a bijection

$$\varphi_U\colon\ \mathrm{Hom}(V/U,U) \to \mathrm{Gr}\,(h,V)_U\,. \tag{2.6.41}$$

To be precise there is one such bijection for each choice of $W \in \mathrm{Gr}\,(h,V)_U$, but they are all equivalent for what follows. Show that φ_U is a homemomorphism, and that the collection of $\mathrm{Gr}\,(h,V)_U$'s and homemomorphisms φ_U is an algebraic atlas of $\mathrm{Gr}\,(h,V)$. Thus we have given $\mathrm{Gr}\,(h,V)$ the structure of an algebraic prevariety.

(c) Prove that $\mathrm{Gr}\,(h,V)$ is an algebraic variety, i.e. that it is of finite type and separated. (It might help to unwind the definitions above for $V = \mathbb{K}^n$, replacing $\{v_1,\ldots,v_h\} \in \mathrm{Fr}(h,V)$ by the $h \times n$ matrix whose rows are the v_i's.)

(d) Prove that $\mathrm{Gr}\,(h,V)$ is irreducible. (Recall that prevarieties of finite type have an irreducible decomposition.)

Exercise 2.6.4. The goal of the exercise is to show that $\mathrm{Gr}\,(h,V)$, with the structure of algebraic variety provided by Exercise 2.6.3, is a projective variety.

1. Let $v_1,\ldots,v_a \in V$ be linearly independent, and let $\alpha \in \bigwedge^h V$. Prove that $v_i \wedge \alpha = 0$ for all $i \in \{1,\ldots,a\}$ if and only if there exists $\beta \in \bigwedge^{h-a} V$ such that $\alpha = v_1 \wedge \ldots \wedge v_a \wedge \beta$.
2. For $\alpha \in \bigwedge^h V$, let m_α be the linear map

$$\begin{array}{ccc} V & \xrightarrow{m_\alpha} & \bigwedge^{h+1} V \\ v & \mapsto & v \wedge \alpha \end{array}$$

Show that if $\alpha \neq 0$, then the kernel of m_α has dimension at most h, and that $\dim\ker(m_\alpha) = h$ if and only if α is *decomposable*, i.e. $\alpha = w_1 \wedge \ldots \wedge w_h$, where $w_1 \wedge \ldots \wedge w_h \in V$ are linearly independent.

3. If $W \in \mathrm{Gr}\,(h,V)$ then $\bigwedge^h W$ is a 1-dimensional subspace of $\bigwedge^h V$, i.e. a point of $\mathbb{P}(\bigwedge^h V)$. Hence we have a well defined *Plücker map*

$$\begin{array}{ccc} \mathrm{Gr}\,(h,V) & \xrightarrow{\mathscr{P}} & \mathbb{P}\left(\bigwedge^h V\right) \\ W & \mapsto & \bigwedge^h W. \end{array}$$

Show that

$$\mathrm{im}(\mathscr{P}) = \left\{[\alpha] \in \mathbb{P}\left(\bigwedge^h V\right) \mid \dim(\ker m_\alpha) \geqslant h\right\}, \tag{2.6.42}$$

and if $[\alpha] \in \mathrm{im}(\mathscr{P})$, then $[\alpha] = \bigwedge^h \ker(m_\alpha)$. Conclude that $\mathscr{P}$ is injective and that $\mathrm{im}(\mathscr{P})$ is closed in $\mathbb{P}(\bigwedge^h V)$.

4. Prove that the Plücker map defines an isomorphism $\mathrm{Gr}\,(h,V) \xrightarrow{\sim} \mathrm{im}(\mathscr{P})$ between algebraic varieties, and hence $\mathrm{Gr}\,(h,V)$ is a projective variety.

Let

$$\mathbb{G}\mathrm{r}(k, \mathbb{P}(V)) := \{L \subset \mathbb{P}(V) \mid \quad L \text{ is a linear subspace, } \dim L = k\}. \tag{2.6.43}$$

We have natural identification

$$\begin{array}{ccc} \mathrm{Gr}(k+1, V) & \longrightarrow & \mathbb{G}\mathrm{r}(k, \mathbb{P}(V)) \\ W & \mapsto & \mathbb{P}(W) \end{array} \tag{2.6.44}$$

Thus $\mathbb{G}\mathrm{r}(k, \mathbb{P}(V))$ is a projective variety.

Exercise 2.6.5. Let V be a 4-dimensional $\mathbb{K}$ vector space and

$$\mathbb{G}\mathrm{r}(1, \mathbb{P}(V)) \xrightarrow{\mathscr{P}} \mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^5$$

the Plücker map.

1. Prove that the image of $\mathscr{P}$ is a non degenerate quadric hypersurface, i.e. that the ideal of $\operatorname{im} \mathscr{P}$ is generated by a non degenerate quadratic polynomial F.
2. Let $X \subset \mathbb{G}\mathrm{r}(1, \mathbb{P}(V))$. Prove that $\mathscr{P}(X)$ is a line if and only if X is a pencil of lines, i.e. the set of lines containing point p and belonging to a plane Λ containing p.
3. Let $X \subset \mathbb{G}\mathrm{r}(1, \mathbb{P}(V))$. Prove that $\mathscr{P}(X)$ is a plane if and only if one of the following holds:
   a) X is the set of lines containing a point p.
   b) X is the set of lines contained in a plane Λ.

Exercise 2.6.6.

1. Let $X \subset \mathbb{P}^n$ be closed. Given $0 \leqslant k \leqslant n$ let

$$F_k(X) := \{\Lambda \in \mathbb{G}\mathrm{r}(k, \mathbb{P}^n) \mid \Lambda \subset X\}. \tag{2.6.45}$$

   Prove that $F_k(X)$ is a closed subset of $\mathbb{G}\mathrm{r}(k, \mathbb{P}^n)$.
2. Let $X = V(Z_0 Z_3 - Z_1 Z_2) \subset \mathbb{P}^3$ be a non degenerate quadric surface. Describe $F_1(X) \subset \mathbb{G}\mathrm{r}(1, \mathbb{P}^3) \subset \mathbb{P}^5$.

Exercise 2.6.7. Let $L \to \mathbb{P}^1$ be the tautological line bundle. Let $E \to \mathbb{P}^1$ be the trivial vector bundle of rank 2, i.e. $E = \mathbb{P}^1 \times \mathbb{K}^2$ with map the projection. We have an obvious injection of vector bundles $L \hookrightarrow E$, and therefore we may consider L as a subbundle of E. Prove that there is no algebraic subvector bundle $G \subset E$ such that for all $[Z] \in \mathbb{P}^1$ the map $G([Z]) \to E([Z])/L([Z])$ is an isomorphism. You may find the following observations useful:

1. The quotient line bundle E/L has sections whose zero set is a point (see Remark 2.5.9 for zero sets of sections of vector bundles).
2. Any section of G is constant (viewed as a section of V).

Chapter 3

Rational maps, dimension and degree

3.1 Introduction

3.2 Rational maps

Let X, Y be algebraic varieties. We define a relation on the set of couples (U, φ) where $U \subset X$ is open dense and $\varphi\colon U \to Y$ is a regular map as follows: $(U, \varphi) \sim (V, \psi)$ if the restrictions of φ and ψ to $U \cap V$ are equal. Then $\sim$ is an equivalence relation. In fact reflexivity and symmetry are trivially true. To prove transitivity suppose that $(U, \varphi) \sim (V, \psi)$ and $(V, \psi) \sim (W, \mu)$. Then the restrictions of φ and μ to $U \cap V \cap W$ are equal. Since V is open dense in X, the intersection $U \cap V \cap W$ is (open) dense in $U \cap W$. Since X is separable it follows that the restrictions of φ and μ to $U \cap W$ are equal, i.e. $(U, \varphi) \sim (W, \mu)$.

Definition 3.2.1. A *rational map* $f\colon X \dashrightarrow Y$ is a $\sim$-equivalence class of couples (U, φ) where $U \subset X$ is open dense and $\varphi\colon U \to Y$ is a regular map.

1. The map f is *regular* at $x \in X$ (equivalently x is a *regular point* of f), if there exists (U, φ) in the equivalence class of f such that $x \in U$. We let $\operatorname{Reg}(f) \subset X$ be the set of regular points of f. By definition $\operatorname{Reg}(f)$ is an open subset of X.

2. The *indeterminancy set* of f is $\operatorname{Ind}(f) := X \setminus \operatorname{Reg}(f)$ (notice that $\operatorname{Ind}(f)$ is closed). A point $x \in X$ is a *point of indeterminancy* if it belongs to $\operatorname{Ind}(f)$.

Example 3.2.2. If $f\colon X \to Y$ is a regular map, we may consider f as a rational map represented by (X, f).

Example 3.2.3. Let X be an algebraic variety, and let $U \subset X$ be open. Let $\iota\colon U \hookrightarrow X$ be the inclusion map. Then (U, ι) represents a rational map $f\colon X \dashrightarrow U$ (note that f goes in the "wrong" direction). Clearly $\operatorname{Reg}(f) = U$.

Example 3.2.4. Let V be a finitely generated vector space and let $[v_0] \in \mathbb{P}(V)$. Let $U := (\mathbb{P}(V) \setminus \{[v_0]\})$. We assume that $\dim V \geqslant 2$, and hence U is open dense in $\mathbb{P}(V)$. The map

$$\begin{array}{ccc} U & \overset{\varphi}{\dashrightarrow} & \mathbb{P}(V/\langle v_0 \rangle) \\ [w] & \mapsto & [\overline{w}] \end{array}$$

where $\overline{w}$ is the equivalence class of w, is regular. Hence (U, φ) represents a rational map $f\colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V/\langle v_0 \rangle)$, which is called the *projection from* $[v_0]$. If $\dim V = 2$ then φ is constant and hence φ is regular. If $\dim V > 2$ then the regular locus of φ is equal to U.

From now on we will consider only rational maps between *irreducible* algebraic varieties. Let $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow W$ be rational maps between (irreducible) algebraic varieties. It might happen that for all $x \in \operatorname{Reg}(f)$ the image $f(x)$ does not belong to $\operatorname{Reg}(g)$, and hence the composition $g \circ f$ makes no sense. In order to deal with compositions of rational maps, we give the following definition.

Definition 3.2.5. A rational map $f\colon X \dashrightarrow Y$ between irreducible algebraic varieties is *dominant* if it is represented by a couple (U,φ) such that $\varphi(U)$ is dense in Y.

Remark 3.2.6. Let $f\colon X \dashrightarrow Y$ be a dominant rational map between irreducible algebraic varieties. If (V,ψ) is an arbitrary representative of f then $\psi(V)$ is dense in Y. In fact by definition f is represented by a couple (U,φ) such that $\varphi(U)$ is dense in Y. Replacing V by $V\cap U$ (which is open dense in X) we may assume that $V\subset U$, and hence $\psi=\varphi_{|V}$. Suppose that $\psi(V)$ is not dense in Y, i.e. there exists a proper closed $W\subsetneq Y$ containing $\psi(V)$. Since $\varphi^{-1}(W)\subset U$ is closed and it contains the dense subset $V\subset U$, it is equal to U. Thus $\varphi(U)\subset W$, and this is a contradiction.

Let X,Y,W be irreducible algebraic varieties. Let

$$X \overset{g}{\dashrightarrow} Y \overset{f}{\dashrightarrow} W \tag{3.2.1}$$

be dominant rational maps, represented by (U,φ) and (V,ψ) respectively. Since $\varphi(U)$ is dense in Y, $\varphi(U)\cap V$ is non empty and hence $\varphi^{-1}(V)$ is non empty. Since $\varphi^{-1}(V)$ is open and X is irreducible, it follows that $\varphi^{-1}(V)$ is dense in X.

Definition 3.2.7. Keeping notation as above, the *composition* $f\circ g$ is the rational map $X \dashrightarrow W$ represented by $(\varphi^{-1}(V),\psi\circ\varphi)$. (The equivalence class of $(\varphi^{-1}(V),\psi\circ\varphi)$ is independent of the representatives (U,φ) and (V,ψ).)

Definition 3.2.8. A dominant rational map $f\colon X \dashrightarrow Y$ between irreducible algebraic varieties is *birational* if there exists a dominant rational map $g\colon Y \dashrightarrow X$ such that $g\circ f=\mathrm{Id}_X$ and $f\circ g=\mathrm{Id}_Y$. An irreducible algebraic variety X is *rational* if it is birational to $\mathbb{P}^n$ for some n, it is *unirational* if there exists a dominant rational map $f\colon \mathbb{P}^n \dashrightarrow X$.

Example 3.2.9. Of course isomorphic irreducible quasi projective varieties are birational. Example 3.2.3 is a slightly less trivial instance of birational map. The inclusion map $\iota\colon U\hookrightarrow X$ has rational inverse the map $f\colon X\dashrightarrow U$ of Example 3.2.3.

Example 3.2.10. Let V be a $\mathbb{K}$ vector space of dimension $n+1$. Suppose that $P\colon V\to\mathbb{K}$ is a quadratic form of rank at least 3, i.e. $\ker P$ has codimension at least 3 (recall that $\ker P\subset V$ is the subspace of vectors u such that $P(u+v)=P(v)$ for all $v\in V$). Then P is not the product of linear functions and hence $Q := V(P)\subset\mathbb{P}(V)$ is an irreducible quadric. Let $[v_0]\in(Q\backslash\mathbb{P}(\ker P))$. The restriction of the projection from $[v_0]$ (see Example 3.2.4) is a rational map

$$Q \overset{f}{\dashrightarrow} \mathbb{P}(V/\langle v_0\rangle). \tag{3.2.2}$$

We claim that f is birational, and hence Q is rational. The reason is the following. First note that by associating to a line $\mathbb{P}(W)\subset\mathbb{P}(V)$ containing $[v_0]$ the element $W/\langle v_0\rangle$ of $\mathbb{P}(V/\langle v_0\rangle)$ we get a bijection between the set of lines containing $[v_0]$ and $\mathbb{P}(V/\langle v_0\rangle)$. Thus we view the latter as parametrizing lines through $[v_0]$. An open dense subset of lines through $[v_0]$ intersect Q in $[v_0]$ and another point (because P has degree 2). Thus for an open dense $U\subset\mathbb{P}(V/\langle v_0\rangle)$ we may define a map $U\to Q$ by associating to the line $\Lambda\in U$ the unique point in $\Lambda\cap Q$ other than $[v_0]$. This is a regular map $U\to Q$ defining a rational map $g\colon\mathbb{P}(V/\langle v_0\rangle)\dashrightarrow Q$ which is the rational inverse of f. More explicitly: in suitable coordinates $Z_0,\ldots,Z_n$ we have $v_0=(0,0,\ldots,0,1)$ and $F=Z_0Z_n-G$, where $0\neq G\in\mathbb{K}[Z_0,\ldots,Z_{n-1}]_2$. Then

$$\begin{array}{ccc} Q & \overset{f}{\dashrightarrow} & \mathbb{P}^{n-1} \\ [Z_0,\ldots,Z_n] & \mapsto & [Z_0,\ldots,Z_{n-1}] \end{array}$$

and

$$\begin{array}{ccc} \mathbb{P}^{n-1} & \overset{g}{\dashrightarrow} & Q^{n-1} \\ [T_0,\ldots,T_{n-1}] & \mapsto & [T_0^2,T_0T_1,\ldots,T_0T_{n-1},G(T_0,\ldots,T_{n-1})] \end{array}$$

Notice that if $n=2$, then f and g are regular (see Example 1.5.9). If $n\geqslant 3$ then neither f nor g is regular. Moreover the quadric Q is not isomorphic to $\mathbb{P}^{n-1}$. We cannot prove this now in general. For $\mathbb{K}=\mathbb{C}$ and $n=3$ you may show that $Q\subset\mathbb{P}^3(\mathbb{C})$ with the Euclidean topology is not homeomorphic to $\mathbb{P}^2(\mathbb{C})$ with the Euclidean topology, and hence they are not isomorphic as algebraic varieties.

Proposition 3.2.11. *Irreducible algebraic varieties X, Y are birational if and only if there exist open dense subsets $U \subset X$ and $V \subset Y$ that are isomorphic.*

Proof. An isomorphism $\varphi\colon U \xrightarrow{\sim} V$ clearly defines a birational map $f\colon X \dashrightarrow Y$. To prove the converse, let

$$X \overset{g}{\dashrightarrow} Y \overset{f}{\dashrightarrow} X \tag{3.2.3}$$

be birational inverse maps. Let (U, φ) represent g and (V, ψ) represent f. Then $\varphi^{-1}(V)$ and $\psi^{-1}(U)$ are open dense subsets of U and V respectively. By hypothesis the composition

$$\psi \circ \left(\varphi_{|\varphi^{-1}(V)}\right) : \varphi^{-1}(V) \to U$$

is equal to the identity on an open non-empty subset of $\varphi^{-1}(V)$. By separability of X we get that $\psi \circ \left(\varphi_{|\varphi^{-1}(V)}\right) = \mathrm{Id}_{\varphi^{-1}(V)}$. In particular $\psi \circ \varphi\left(\varphi^{-1}(V)\right) \subset U$, i.e. $\varphi\left(\varphi^{-1}(V)\right) \subset \psi^{-1}(U)$. Similarly

$$\varphi \circ \left(\psi_{|\psi^{-1}(U)}\right) = \mathrm{Id}_{\psi^{-1}(U)}, \quad \psi\left(\psi^{-1}(U)\right) \subset \varphi^{-1}(V).$$

Thus the restrictions of φ and ψ define regular maps $\varphi^{-1}(V) \xrightarrow{\sim} \psi^{-1}(U)$ and $\psi^{-1}(U) \xrightarrow{\sim} \varphi^{-1}(V)$ which are inverse of each other. □

Example 3.2.12. Let f, g be the birational maps in Example 3.2.10. Assume that $n \geqslant 3$, so that both non regular. Then

$$\mathrm{Reg}(f) = Q\backslash\{[0, 0, \ldots, 0, 1]\}, \qquad \mathrm{Reg}(g) = \mathbb{P}^{n-1}\backslash V(T_0, G(T_0, \ldots, T_{n-1})). \tag{3.2.4}$$

On the other hand open dense subsets which are isomorphic are strictly smaller than the regular loci. In fact f defines an isomorphism

$$Q\backslash V(Z_0) \xrightarrow{\sim} \mathbb{P}^{n-1}\backslash V(T_0). \tag{3.2.5}$$

If X, Y are algebraic varieties defined over a subfield $F \subset \mathbb{K}$, then one defines the notion of *rational map $f\colon X \dashrightarrow Y$ defined over F* by considering equivalence classes of couples (U, φ) where $U \subset X$ is an open subset defined over F and $\varphi\colon U \to Y$ is defined over F. As a consequence we have the notion of algebraic varieties defined over F which are *birational over F*. In particular we have the notion of an algebraic varieties defined over F which is *rational over F*.

Example 3.2.13. Let V_0 be an F vector space of dimension $n+1$, and let $P_0\colon V_0 \to F$ be a quadratic form of rank at least 3. Let $V := V_0 \otimes_F \mathbb{K}$ and let $P\colon V \to \mathbb{K}$ be the quadratic form obtained from P_0 by extending scalars. Then $Q := V(P)$ is a quadric defined over F. We claim that Q is rational over F if and only if $Q(F)\backslash\mathbb{P}(\ker P_0)$ is not empty. In fact suppose that there exists a birational map from a projective $\mathbb{P}^m$ (for some m) space to Q, and hence a regular dominant map $\varphi\colon U \to Q$ where $U \subset \mathbb{P}^m$ is open dense. There are plenty of points in U defined over F and their images are points in $Q(F)$. Moreover not all of these rational points are contained in $\mathbb{P}(\ker P_0)$ because φ is dominant. Hence $Q(F)\backslash\mathbb{P}(\ker P_0)$ is non empty. On the other hand, if there exists a point $[v_0]$ in $(Q(F)\backslash\mathbb{P}(\ker P_0))$, then the procedure described in Example 3.2.10 gives a birational map $f\colon Q \dashrightarrow \mathbb{P}(V/\langle v_0\rangle)$ defined over F. In fact this holds because we can choose coordinates $Z_0, \ldots, Z_n$ for V_0 such that $v_0 = (0, 0, \ldots, 0, 1)$ and $F = Z_0 Z_n - G$, where $0 \neq G \in F[Z_0, \ldots, Z_{n-1}]_2$.

Many natural invariants of complete algebraic varieties do not separate between birational varieties. This fact gives practical criteria that allow to establish that couples of complete varieties are not birational. On the other hand, it leads one to approach the classification of isomorphism classes of complete (or projective) varieties in two steps: first one classifies equivalence classes for birational equivalence, then one distinguishes isomorphim classes within each birational equivalence class.

3.3 The field of rational functions

If X is an affine variety then one can reconstruct X from the ring $\mathbb{K}[X]$ of regular functions on X. Actually there is a contravariant equivalence between the category of affine varieties and the category of finitely generated $\mathbb{K}$ algebras with no non zero nilpotents, see Section 1.8. On the other hand if X is proper then, since every regular function is locally constant, the ring $\mathbb{K}[X]$ gives very little information about X (unless X is a finite set, i.e. affine). One gets a rich algebraic object by associating to an irreducible algebraic variety the field of rational functions. From this field one reconstructs the algebraic variety modulo birational maps.

Let X be an irreducible algebraic variety. A *rational function on* X is a rational map $X \dashrightarrow \mathbb{K}(= \mathbb{A}^1)$. We define addition and multiplication of rational functions on X by adding and multiplying regular representatives. Let $f, g\colon X \dashrightarrow \mathbb{K}$ be represented by (U, φ) and (V, ψ) respectively. Then

$$\begin{aligned} f + g &:= [(U \cap V, \varphi_{|U\cap V} + \psi_{|U\cap V})], \\ f \cdot g &:= [(U \cap V, \varphi_{|U\cap V} \cdot \psi_{|U\cap V})]. \end{aligned}$$

The definition makes sense because changing representatives of f and g we get equivalent couples. We claim that with the above operations the set of rational functions on X is a field. It is obvious that it is a ring. To check that every non zero element has a multiplicative inverse let $f\colon X \dashrightarrow \mathbb{K}$ be a non zero rational function. Then $f = [(U, \varphi)]$ where $\varphi \neq 0$. Thus $V(\varphi) \subset U$ is a proper closed subset and therefore $U^0 := (U \backslash V(\varphi))$ is open dense in X. Then $g := [(U^0, \varphi^{-1})]$ is the multiplicative inverse of f.

Definition 3.3.1. Let X be an irreducible algebraic variety. The *field of rational functions on* X is the set of rational functions on X with the above operations. It is denoted by $\mathbb{K}(X)$.

Remark 3.3.2. Let X be an irreducible algebraic variety. We have a canonical embedding $\mathbb{K} \hookrightarrow \mathbb{K}(X)$ as the subfield of constant functions.

Remark 3.3.3. Let X be an irreducible algebraic variety. Let $U \subset X$ be a dense open subset. The map

$$\begin{array}{ccc} \mathbb{K}(U) & \xrightarrow{\alpha} & \mathbb{K}(X) \\ [(V, \varphi)] & \mapsto & [(V, \varphi)] \end{array} \tag{3.3.6}$$

is an isomorphism of extensions of $\mathbb{K}$, i.e. it is an isomorphism of fields and the composition $\mathbb{K} \hookrightarrow \mathbb{K}(U) \xrightarrow{\alpha} \mathbb{K}(X)$, where the first map is the the canonical embedding, equals the canonical embedding $\mathbb{K} \hookrightarrow \mathbb{K}(X)$. In particular $\mathbb{K}(X)$ is isomorphic (as extension of $\mathbb{K}$) to the field of rational functions of any of its dense open affine subsets.

The field of rational functions of an irreducible affine variety is isomorphic to the field of fractions of its ring of regular functions. To see this, first note that if X is an irreducible algebraic variety we have an inclusion of $\mathbb{K}$ extensions:

$$\begin{array}{ccc} (\text{field of fractions of } \mathbb{K}[X]) & \hookrightarrow & \mathbb{K}(X) \\ \frac{\alpha}{\beta} & \mapsto & [(X \backslash V(\beta), \frac{\alpha}{\beta})] \end{array} \tag{3.3.7}$$

Claim 3.3.4. *Let X be an affine irreducible variety. Then* (3.3.7) *is an isomorphism.*

.

Proof. We must prove that the map in (3.3.7) is surjective. Let $f \in \mathbb{K}(X)$, and let (U, φ) represent f. By Example 1.6.5, there exists $0 \neq \gamma \in \mathbb{K}[X]$ such that the dense principal open subset X_γ is contained in U. Moreover, by Example 1.6.5 and Theorem 1.6.2, $\mathbb{K}[X_f]$ is generated as $\mathbb{K}$-algebra by $\mathbb{K}[X]$ and γ^{-1}, hence ϕ is represented by $(X_\gamma, \frac{\alpha}{\gamma^m})$ where $\alpha \in \mathbb{K}[X]$. Let $\beta := \gamma^m$. Since $X_\gamma = X_\beta$, we have proved that f belongs to the image of (3.3.7). □

Example 3.3.5. By Claim 3.3.4 the field $\mathbb{K}(\mathbb{A}^n)$ is the field of fractions of $\mathbb{K}[z_1, \ldots, z_n]$, i.e. $\mathbb{K}(z_1, \ldots, z_n)$. By Remark 3.3.3 we also have $\mathbb{K}(\mathbb{P}^n) \cong \mathbb{K}(z_1, \ldots, z_n)$.

Remark 3.3.6. If X is an irreducible algebraic variety then $\mathbb{K}(X)$ is finitely generated over $\mathbb{K}$. In fact by Remark 3.3.3 we may replace X by a dense open affine $Y \subset X$. Then $\mathbb{K}(Y)$ is the field of quotients of $\mathbb{K}[Y]$ by Claim 3.3.4. Let $Y \subset \mathbb{A}^n$ as a closed subset. By Theorem 1.6.2 the restriction of coordinate functions $z_{1|X}, \ldots, z_{n|X}$ generate $\mathbb{K}[Y]$ as $\mathbb{K}$-algebra and hence they generate $\mathbb{K}(Y)$ as extension of $\mathbb{K}$. In particular we can extract a transendence basis of $\mathbb{K}(Y)$ from $z_{1|X}, \ldots, z_{n|X}$.

Let $f\colon X \dashrightarrow Y$ be a dominant rational map of irreducible algebraic varieties. Since f is dominant the *pull-back* map

$$\begin{array}{ccc} \mathbb{K}(Y) & \xrightarrow{f^*} & \mathbb{K}(X) \\ \varphi & \mapsto & \varphi \circ f \end{array}$$

is well defined. The map f^* is an inclusion of fields and if $\mathbb{K} \hookrightarrow \mathbb{K}(Y)$ is the canonical inclusion then the composition $\mathbb{K} \hookrightarrow \mathbb{K}(Y) \xrightarrow{\varphi^*} \mathbb{K}(X)$ is the canonical inclusion. Thus f^* is a morphism of extensions of $\mathbb{K}$. Suppose that $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow W$ are dominant rational maps of irreducible algebraic varieties. Then $g \circ f\colon X \dashrightarrow W$ is dominant and

$$f^* \circ g^* = (g \circ f)^* . \tag{3.3.8}$$

Of course $\mathrm{Id}_X^*\colon \mathbb{K}(X) \to \mathbb{K}(X)$ is the identity map. This gives a contravariant functor

$$\begin{array}{ccc} \mathrm{RAT}/\mathbb{K} & \longrightarrow & \mathrm{FGF}/\mathbb{K} \\ X & \mapsto & \mathbb{K}(X) \\ X \overset{f}{\dashrightarrow} Y & \mapsto & f^* \end{array} \tag{3.3.9}$$

where $\mathrm{RAT}/\mathbb{K}$ is the category whose objects are irreducible algebraic varieties and $\mathrm{FGF}/\mathbb{K}$ is the category of finitely generated field extensions of $\mathbb{K}$ (with morphisms the morphisms as extensions of $\mathbb{K}$).

Proposition 3.3.7. *The functor in* (3.3.9) *is an equivalence between the category of irreducible algebraic varieties with homomorphisms dominant rational maps and the category of finitely generated field extensions of* $\mathbb{K}$.

Proposition 3.3.7 follows from Proposition 3.3.8, which proves that the functor in (3.3.9) is essentially surjective, and Proposition 3.3.10, which proves that the functor in (3.3.9) is fully faithful.

Proposition 3.3.8. *Let E be a finitely generated field extension of $\mathbb{K}$. There exist an irreducible algebraic variety X and an isomorphisms of $E \xrightarrow{\sim} \mathbb{K}(X)$ of extensions of $\mathbb{K}$.*

Proof. Let m be the transcendenece degree of E over $\mathbb{K}$. By Corollary A.4.8, there exist a prime polynomial $P \in \mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]$ and an isomorphism of extensions of $\mathbb{K}$

$$E \xrightarrow{\sim} \mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]/(P). \tag{3.3.10}$$

Write

$$P = z_{m+1}^d + c_1 z_{m+1}^{d-1} + \cdots + c_d, \quad c_i \in \mathbb{K}\left(z_1, \ldots, z_m\right).$$

Then, for $i \in \{1, \ldots, d\}$, we have $c_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{K}[z_1, \ldots, z_m]$ and $b_i \neq 0$. Let $\widetilde{P} \in \mathbb{K}[z_1, \ldots, z_{m+1}]$ be obtained from P by clearing denominators, i.e. $\widetilde{P} = (b_1 \cdot \ldots \cdot b_d)P$. Lastly, let $Q \in \mathbb{K}[z_1, \ldots, z_{m+1}]$ be obtained from $\widetilde{P}$ by factoring out the maximum common divisor of the coefficients of $\widetilde{P}$ as polynomial in z_{m+1} (recall that $\mathbb{K}[z_1, \ldots, z_m]$ is a UFD). Notice that Q is irreducible and hence prime. Write

$$Q = e_0 z_{m+1}^d + e_1 z_{m+1}^{d-1} + \cdots + e_d, \qquad e_i \in \mathbb{K}[z_1, \ldots, z_m], \quad e_0 \neq 0.$$

Then $X := V(Q) \subset \mathbb{A}^{m+1}$ is an irreducible hypersurface because Q is prime. Because of the isomorphism in (3.3.10) it suffices to prove that there is an isomorphism of extensions of $\mathbb{K}$

$$\mathbb{K}(z_1, \ldots, z_m)[z_{m+1}]/(P) \xrightarrow{\sim} \mathbb{K}(X). \tag{3.3.11}$$

Let $\overline{z}_i := z_{i|X}$. We claim that the rational functions on X represented by $\{\overline{z}_1, \ldots, \overline{z}_m\}$ are algebraically independent over $\mathbb{K}$. In fact, suppose that $R \in \mathbb{K}[t_1, \ldots, t_m]$ and $R(\overline{z}_1, \ldots, \overline{z}_n) = 0$. By the fundamental Theorem of Algebra, for any $(\xi_1, \ldots, \xi_m) \in (\mathbb{A}^m \backslash V(e_0))$ there exists $\xi_{m+1} \in \mathbb{K}$ such that $(\xi_1, \ldots, \xi_m, \xi_{m+1}) \in X$. It follows that $R(\xi_1, \ldots, \xi_m) = 0$ for all $(\xi_1, \ldots, \xi_m) \in (\mathbb{A}^n \backslash V(e_0))$, and hence $R \cdot e_0$ vanishes identically on $\mathbb{A}^m$. Thus $R \cdot e_0 = 0$, and since $e_0 \neq 0$ it follows that $R = 0$. This proves that $\{\overline{z}_1, \ldots, \overline{z}_m\}$ are algebraically independent over $\mathbb{K}$. On the other hand $\overline{z}_{m+1}$ is algebraic over $\mathbb{K}(\overline{z}_1, \ldots, \overline{z}_m)$ and its minimal polynomial equals P. Thus by mapping z_i to $\overline{z}_i$ for $i \in \{1, \ldots, n+1\}$ (and mapping $\mathbb{K}$ to $\mathbb{K}$ by the identity map) we get an isomorphism of extensions of $\mathbb{K}$ as in (3.3.11). □

Proposition 3.3.9. *Let X and Y be irreducible algebraic varieties, and let $\alpha\colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$ is an inclusion of extensions of $\mathbb{K}$. There exists a unique dominant rational map $f\colon X \dashrightarrow Y$ such that $f^* = \alpha$.*

Proof. By remark 3.3.3 we may assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are closed. Hence by Claim 3.3.4 $\mathbb{K}(X)$, $\mathbb{K}(Y)$ are the fields of fractions of $\mathbb{K}[X]$ and $\mathbb{K}[Y]$ respectively. By Theorem 1.6.2, $\mathbb{K}[X] = \mathbb{K}[z_1, \ldots, z_n]/I(X)$ and $\mathbb{K}[Y] = \mathbb{K}[w_1, \ldots, w_m]/I(Y)$. Given $p \in \mathbb{K}[z_1, \ldots, z_n]$ and $q \in \mathbb{K}[w_1, \ldots, w_m]$ we let $\overline{p} := p|_X$ and $\overline{q} := q|_Y$. We have

$$\alpha\left(\overline{w}_i\right) = \frac{\overline{f}_i}{\overline{g}_i}, \quad f_i, g_i \in \mathbb{K}[z_1, \ldots, z_n], \quad \overline{g}_i \neq 0.$$

Let $U := X \backslash (V(g_1) \cup \ldots \cup V(g_m))$. Then U is open and dense in X. Let

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{\varphi}} & \mathbb{A}^m \\ a & \mapsto & \left(\frac{f_1(a)}{g_1(a)}, \ldots, \frac{f_m(a)}{g_m(a)}\right) \end{array}$$

We claim that $\widetilde{\varphi}(U) \subset Y$. In fact let $h \in I(Y)$. Since α is an inclusion of extensions of $\mathbb{K}$,

$$h(\overline{f}_1/\overline{g}_1, \ldots, \overline{f}_m/\overline{g}_m) = h(\alpha(\overline{w}_1), \ldots, \alpha(\overline{w}_m)) = \alpha(h(\overline{w}_1, \ldots, \overline{w}_m)) = \alpha(0) = 0.$$

This proves that if $h \in I(Y)$ then h vanishes on $\widetilde{\varphi}(U)$, i.e. $\widetilde{\varphi}(U) \subset Y$. Thus $\widetilde{\phi}$ induces a regular map $\varphi\colon U \to Y$. If $b \in \mathbb{K}[Y] \subset \mathbb{K}(Y)$ then

$$\varphi^*(b) \in \mathbb{K}[U] \subset \mathbb{K}(U) = \mathbb{K}(X)$$

is equal to $\alpha(b)$. It follows that if $b \neq 0$ then $\varphi^*(b) \neq 0$. Thus φ is dominant. Let $f\colon X \dashrightarrow Y$ be the equivalence class of (U, ϕ). Then $f^* = \alpha$.

Moreover it is clear from the above construction that f is the unique rational (dominant) map such that $f^* = \alpha$. □

The result below follows at once from what has been proved above.

Corollary 3.3.10. *Irreducible algebraic varieties are birational if and only if their fields of rational functions are isomorphic as extensions of $\mathbb{K}$.*

Example 3.3.11. Let $p \in \mathbb{K}[z]$ be free of square factors (and $\deg p \geqslant 1$). Then $t^2 - p(z)$ is prime and hence $X := V\left(t^2 - p(z)\right) \subset \mathbb{A}^2$ is irreducible. Thus we have the extensions of fields $\mathbb{K}(X) \supset \mathbb{K}(z) \supset \mathbb{K}$ where the top extension is algebraic of degree 2. Then X is rational if and only if $\mathbb{K}(X)$ is a purely trascendental extension of $\mathbb{K}$. If $\deg p = 1$ then $\mathbb{K}(X)$ is a purely trascendental extension of $\mathbb{K}$ because it is generated (over $\mathbb{K}$) by t. Similarly it is a purely trascendental extension of $\mathbb{K}$ if $\deg p = 2$ by Example 1.5.9. If $\deg p \geqslant 3$ then X is not rational (the proof of this fact this requires new ideas) and hence $\mathbb{K}(X)$ is *not* a purely trascendental extension of $\mathbb{K}$.

The result below follows from the above corollary and the proof of Proposition **??**.

Proposition 3.3.12. *Let X be an irreducible algebraic variety and let $m := \mathrm{Tr.deg}_{\mathbb{K}}\, \mathbb{K}(X)$. Then X is birational to an irreducible hypersurface in $\mathbb{A}^{m+1}$.*

3.4 Dimension

Definition 3.4.1. 1. The *dimension* of an irreducible algebraic variety X is the transcendence degree of $\mathbb{K}(X)$ over $\mathbb{K}$.

2. Let X be an arbitrary quasi projective variety, and let $X = X_1 \cup \cdots \cup X_r$ be its irreducible decomposition. The *dimension* of X is the maximum of the dimensions of its irreducible components. We say that X has *pure dimension* n if every irreducible component of X has dimension n.

3. Let $p \in X$. The *dimension of* X *at* p is the maximum of the dimensions of the irreducible components of X containing p.

Remark 3.4.2. The dimension of an irreducible algebraic variety X is equal to the dimension of any open dense subset $U \subset X$. In fact, by definition it suffices to prove it for irreducible X, and in that case it holds because the fields of rational functions $\mathbb{K}(X)$ and $\mathbb{K}(U)$ are isomorphic extensions of $\mathbb{K}$.

Example 3.4.3. The dimension of $\mathbb{A}^n$ and of $\mathbb{P}^n$ is equal to n. In fact $\mathbb{K}(\mathbb{A}^n) = \mathbb{K}(\mathbb{P}^n) = \mathbb{K}(z_1, \ldots, z_n)$, and $\{z_1, \ldots, z_n\}$ is a transcendence basis of $\mathbb{K}(z_1, \ldots, z_n)$ over $\mathbb{K}$.

Example 3.4.4. The dimension of $\mathrm{Gr}(h, V)$ is equal to $h \cdot (\dim V - h)$, because it is irreducible and it contains an open dense subset isomorphic to an affine space of dimension $h \cdot (\dim V - h)$ (actually many such subsets), see Exercise 2.6.3.

Example 3.4.5. Let $X \subset \mathbb{A}^{n+1}$ be a hypersurface. We claim that X has pure dimension n. Since the irreducible components of X are hypersurfaces, in fact the zero loci of the prime factors of f, it suffices to show that if X is an irreducible hypersurface then it has dimensjon n. Let $I(X) = (f)$. Reordering the coordinates $(z_1, \ldots, z_n, z_{n+1})$ we may assume that

$$f = c_0 z_{n+1}^d + c_1 z_{n+1}^{d-1} + \cdots + c_d, \quad c_i \in \mathbb{K}[z_1, \ldots, z_n], \quad c_0 \neq 0, \quad d > 0. \tag{3.4.1}$$

For $i \in \{1, \ldots, n+1\}$ let $\overline{z}_i := z_{i|X}$. In the proof of Proposition 3.3.8 we showed that $\overline{z}_1, \ldots, \overline{z}_n$ are algebraically independent in $\mathbb{K}(X)$. Since $\mathbb{K}(X)$ is generated over $\mathbb{K}$ by $\overline{z}_1, \ldots, \overline{z}_n, \overline{z}_{n+1}$ and since $\overline{z}_{n+1}$ is algebraic over the subfield generated by $\overline{z}_1, \ldots, \overline{z}_n$ it follows that $\overline{z}_1, \ldots, \overline{z}_n$ is transcendence basis of $\mathbb{K}(X)$ over $\mathbb{K}$. Similarly, a hypersurface in $\mathbb{P}^{n+1}$ has pure dimension n. (Intersect with $\mathbb{P}^n_{Z_i}$ for $i \in \{0, 1, \ldots, n+1\}$.)

Remark 3.4.6. An algebraic variety has dimension 0 if and only if it is a finite set.

Remark 3.4.7. If $f\colon X \dashrightarrow Y$ is a dominant map of irreducible algebraic varieties then $\dim X \geqslant \dim X$ because we have the inclusion $f^*\colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$ of field extensions of $\mathbb{K}$.

Proposition 3.4.8. *Let X be an irreducible algebraic variety and let $Y \subset X$ be a proper closed subset. Then* $\dim Y < \dim X$.

Proof. We may assume that Y is irreducible. Since X is covered by open affine varieties, we may assume that X is affine. Thus we may assume that $X \subset \mathbb{A}^n$. Thus Y is also closed in $\mathbb{A}^n$. We may choose a transcendence basis $\{f_1, \ldots, f_d\}$ of $\mathbb{K}(Y)$, where each f_i is a regular function on Y, see Remark 3.3.6.

Let $\tilde{f}_1, \ldots, \tilde{f}_d \in \mathbb{K}[X]$ such that $\tilde{f}_{i|}W = f_i$. Since Y is a proper closed subset of X, there exists a non zero $g \in \mathbb{K}[X]$ such that $g_{|Y} = 0$. It suffices to prove that $\tilde{f}_1, \ldots, \tilde{f}_d, g$ are algebraically independent over. We argue by contradiction. Suppose that there exists $0 \neq P \in \mathbb{K}[S_1, \ldots, S_d, T]$ such that $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$. Since X is irreducible we may assume that P is irreducible. Restricting to Y the equality $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$, we get that $P(f_1, \ldots, f_d, 0) = 0$. Thus $P(S_1, \ldots, S_d, 0) = 0$, because $f_1, \ldots, f_d$ are algebraically independent. This means that T divides P. Since P is irreducible $P = cT$, $c \in \mathbb{K}^*$. Thus $P(\tilde{f}_1, \ldots, \tilde{f}_d, g) = 0$ reads $g = 0$, and that is a contradiction. □

Corollary 3.4.9. *A (non empty) closed subset $X \subset \mathbb{A}^{n+1}$ has pure dimension n if and only if it is a hypersurface. Similarly, a closed subset $X \subset \mathbb{P}^{n+1}$ has pure dimension n if and only if it is a hypersurface.*

Proof. If $X \subset \mathbb{A}^{n+1}$ is a hypersurface then it has pure dimension n, see Eaxmple 3.4.1.

In order to prove the converse, suppose that $X \subset \mathbb{A}^{n+1}$ is a closed subset of pure dimension n. Thus every irreducible component of X is a closed subset of $\mathbb{A}^{n+1}$ of dimension n. Since the union of hypersurfaces in $\mathbb{A}^{n+1}$ is a hypersurface in $\mathbb{A}^{n+1}$, it suffices to prove that each irreducible component of X is a hypersurface. Thus we may assume that X is irreducible. Since $\dim X = n < \dim \mathbb{A}^{n+1}$, there exists a non zero $f \in I(X) \subset \mathbb{K}[z_1, \ldots, z_{n+1}]$. Since X is irreducible, the ideal $I(X)$ is prime, and hence there exists a prime factor g of f which vanishes on X. Thus $X \subset V(g)$ and $V(g)$ is irreducible. By Example 3.4.1 we have $\dim V(g) = n$, and hence $\dim X = \dim V(g)$. Since X is closed it follows from Proposition 3.4.8 that $X = V(g)$. This finishes the proof for closed subsets of $\mathbb{A}^{n+1}$.

The result for closed subsets of $\mathbb{P}^{n+1}$ follows by a smilar proof, or by intersecting with the standard open affine subsets $\mathbb{P}^n_{Z_i}$ for $i \in \{0, \ldots, n+1\}$. □

Proposition 3.4.10. *Let X, Y be algebraic varieties. Then* $\dim(X \times Y) = \dim X + \dim Y$.

Proof. We may assume that X and Y are irreducible affine varieties. There exist transcendence bases $\{f_1, \ldots, f_d\}$, $\{g_1, \ldots, g_e\}$ of $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ respectively given by regular functions. Let $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ be the projections. We claim that $\{\pi_X^*(f_1), \ldots, \pi_X^*(f_d), \pi_Y^*(g_1), \ldots, \pi_Y^*(g_e)\}$ is a transcendence basis of $\mathbb{K}(X \times Y)$.

First, by Proposition 2.3.6 $\mathbb{K}[X \times Y]$ is algebraic over the subring generated (over $\mathbb{K}$) by $\pi_X^*(f_1), \ldots, \pi_Y^*(g_e)$.

Secondly, let us show that $\pi_X^*(f_1), \ldots, \pi_Y^*(g_e)$ are algebraically independent. Suppose that there is a polynomial relation

$$\sum_{0 \leqslant m_1, \ldots, m_e \leqslant N} P_{m_1, \ldots, m_e}(\pi_X^*(f_1), \ldots, \pi_X^*(f_d)) \cdot \pi_Y^*(g_1)^{m_1} \cdot \ldots \cdot \pi_Y^*(g_e)^{m_e} = 0,$$

where each $P_{m_1, \ldots, m_e}$ is a polynomial. Since $g_1, \ldots, g_e$ are algebraically independent we get that $P_{m_1, \ldots, m_e}(f_1(a), \ldots, f_d(a)) = 0$ for every $a \in X$. Since $f_1, \ldots, f_d$ are algebraically independent, it follows that $P_{m_1, \ldots, m_e} = 0$ for every $0 \leqslant m_1, \ldots, m_e \leqslant N$, and hence $P = 0$. This proves that $\pi_X^*(f_1), \ldots, \pi_Y^*(g_e)$ are algebraically independent. □

3.5 Dimension and intersection

Closed subsets of $\mathbb{P}^n$: dimension and intersection with linear subspaces

Let $X \subset \mathbb{P}^n$ be a hypersurface. Thus $X = V(F)$ where $F \in \mathbb{K}[Z_0, \ldots, Z_n]_d$ with $d > 0$ and $F \neq 0$. Let $\Lambda = \mathbb{P}(U)$ be a linear subspace of $\mathbb{P}^n$, i.e. $U \subset \mathbb{K}^{n+1}$ is a $\mathbb{K}$ vector subspace. Then $\Lambda \cap X = V(F_{|U})$. It follows that if $\dim \Lambda \geqslant 1$ then Λ has non empty intersection with X. If, on the other hand, $\dim \Lambda = 0$ i.e. Λ is a point, then $\Lambda \cap X$ is empty for all points in the dense open subset $\mathbb{P}^n \backslash X$. An analogous characterization of the dimension of a closed subset of $\mathbb{P}^n$ holds in general. In order to formulate the relevant result we introduce a definition and a classical piece of terminology.

Definition 3.5.1. Let X be an irreducible algebraic variety, and let $Y \subset X$ be a closed subset. The *codimension of* Y *in* X is equal to $\dim X - \dim Y$, and is denoted by $\mathrm{cod}(Y, X)$.

Terminology 3.5.2. Let X be an algebraic variety, and let $\mathscr{P}$ be a property that each point of X might or might not have (formally "the subset of points of X having the property $\mathscr{P}$"). Then *a general point of X has property $\mathscr{P}$* if there is a dense open subset of X of points having property $\mathscr{P}$.

Proposition 3.5.3. *Let $X \subset \mathbb{P}^n$ be closed.*

(a) *Let $k < \mathrm{cod}(X, \mathbb{P}^n)$. Then for a general $\Lambda \in \mathrm{Gr}(k, \mathbb{P}^n)$ we have $\Lambda \cap X = \varnothing$ (i.e. there exists a dense open $U \subset \mathrm{Gr}(k, \mathbb{P}^n)$ such that $\Lambda \cap X = \varnothing$ for all $\Lambda \in U$).*

(b) *Let $\Lambda \subset \mathbb{P}^n$ be a linear subspace such that* $\dim \Lambda \geqslant \mathrm{cod}(X, \mathbb{P}^n_{\mathbb{C}})$. *Then* $\Lambda \cap X \neq \varnothing$.

The proof of Proposition 3.5.3 is given after a few preliminary results.

Definition 3.5.4. Let $X \subset \mathbb{P}^n$ be closed. For $k \in \{0, \ldots, n\}$ let $\Gamma_X(k) \subset X \times \mathbb{Gr}(k, \mathbb{P}^n)$ be given by

$$\Gamma_X(k) = \{(p, \Lambda) \in X \times \mathbb{Gr}(k, \mathbb{P}^n) \mid p \in \Lambda\}.$$

Proposition 3.5.5. *Let $X \subset \mathbb{P}^n$ be closed and irreducible. Then $\Gamma_X(k)$ is closed in $X \times \mathbb{Gr}(k, \mathbb{P}^n)$, irreducible, and*

$$\dim \Gamma_X(k) = \dim X + k(n-k). \tag{3.5.1}$$

Proof. Let us show that $\Gamma_X(k)$ is closed if $X = \mathbb{P}^n$. Let $A := (a_{i,j}) \in M_{k+1,n+1}(\mathbb{K})$ be a matrix of maximal rank, i.e. of rank $k+1$. Thus the rows of A span a subspace $U_A \subset \mathbb{K}^{n+1}$ of dimension $k+1$, and hence $\mathbb{P}(U_A) \in \mathbb{Gr}(k, \mathbb{P}^n)$. Let $[Z] \in \mathbb{P}^n$. Then $([Z], \mathbb{P}(U_A)) \in \Gamma_{\mathbb{P}^n}(k)$ if and only if the $(k+2) \times (n+1)$ matrix obtained by adding the row Z to A has rank less than $k+2$, i.e. if and only if for all $0 \leqslant j_0 < j_1 < \ldots < j_{k+1} \leqslant (n+1)$ we have

$$\operatorname{Det} \begin{bmatrix} X_{j_0} & X_{j_1} & \cdots & \cdots & X_{j_{k+1}} \\ a_{0,j_0} & a_{0,j_1} & \cdots & \cdots & a_{0,j_{k+1}} \\ a_{1,j_0} & a_{1,j_1} & \cdots & \cdots & a_{1,j_{k+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k,j_0} & a_{k,j_1} & \cdots & \cdots & a_{k,j_{k+1}} \end{bmatrix} = 0$$

Expanding the determinant on the left hand side we get that $([Z], \mathbb{P}(U_A)) \in \Gamma_{\mathbb{P}^n}(k)$ if and only if

$$\sum_{s=0}^{k+1} p_{j_0, j_1, \ldots, j_{k+1}} X_{j_s} = 0 \tag{3.5.2}$$

for all $0 \leqslant j_0 < j_1 < \ldots < j_{k+1} \leqslant (n+1)$, where $[\ldots, p_{j_0,j_1,\ldots,j_{k+1}}, \ldots]$ are the Plücker coordinates of $\mathscr{P}(U_A)$ (see Exercise 2.6.4) with respect to the basis of $\bigwedge^{k+1} \mathbb{K}^{n+1}$ associated to the standard basis of $\mathbb{K}^{n+1}$. This shows that $\Gamma_{\mathbb{P}^n}(k)$ is closed.

Now we show that $\Gamma_X(k)$ is closed for $X \subset \mathbb{P}^n$ closed. Let $\pi \colon \mathbb{P}^n \times \mathbb{Gr}(k, \mathbb{P}^n) \to \mathbb{P}^n$ be the projection. Then $\Gamma_X(k) = \pi^{-1}(X) \cap \Gamma_{\mathbb{P}^n}(k)$. Since X is closed in $\mathbb{P}^n$ and π is regular $\pi^{-1}(X)$ is closed in $\mathbb{P}^n \times \mathbb{Gr}(k, \mathbb{P}^n)$ and hence $\Gamma_X(k)$ is closed in $\mathbb{P}^n \times \mathbb{Gr}(k, \mathbb{P}^n)$ because $\Gamma_{\mathbb{P}^n}(k)$ is closed. Of course this gives that $\Gamma_X(k)$ is closed in $X \times \mathbb{Gr}(k, \mathbb{P}^n)$.

Next we prove that $\Gamma_X(k)$ is irreducible of dimension as claimed. For $i \in \{0, \ldots, n\}$ we have the isomorphism

$$\begin{array}{ccc} X_{Z_i} \times \operatorname{Gr}(k,n) & \xrightarrow{\alpha_i} & \Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathbb{Gr}(k, \mathbb{P}^n)) \\ (p, W) & \mapsto & (p, \overline{p+W}) \end{array} \tag{3.5.3}$$

where $W \in \operatorname{Gr}(k,n)$, i.e. W is a k-dimensional vector subspace of $\mathbb{K}^n$ viewed as the vector space acting on the affine space $\mathbb{P}^n_{Z_i} \simeq \mathbb{A}^n$ and $\overline{p+W}$ denotes the closure in $\mathbb{P}^n$ of the affine subspace $p + W \subset \mathbb{P}^n_{Z_i} \simeq \mathbb{A}^n$. Suppose that $\Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathbb{Gr}(k, \mathbb{P}^n))$ is non empty. Then by the isomorphism in (3.5.3) it is irreducible, and

$$\dim \left(\Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathbb{Gr}(k, \mathbb{P}^n))\right) = \dim X_{Z_i} \times \operatorname{Gr}(k,n) = \dim X + \dim \operatorname{Gr}(k,n) = \dim X + k(n-k).$$

Since $\Gamma_X(k)$ is covered by the open non empty subsets $\Gamma_X(k) \cap (\mathbb{P}^n_{Z_i} \times \mathbb{Gr}(k, \mathbb{P}^n))$, any such open subset is irreducible, and any two (non empty) such subsets have non empty intersection (because X is irreducible), it follows that $\Gamma_X(k)$ is is irreducible of dimension given by (3.5.1). □

Corollary 3.5.6. *Let $X \subset \mathbb{P}^n$ be closed. Then $\Gamma_X(k)$ is closed of dimension given by*

$$\dim \Gamma_X(k) = \dim X + k(n-k). \tag{3.5.4}$$

If $k \leqslant \operatorname{cod}(X, \mathbb{P}^n)$ then

$$\dim \Gamma_X(k) \leqslant \dim \mathbb{Gr}(k, \mathbb{P}^n) \tag{3.5.5}$$

with equality if and only if $k = \operatorname{cod}(X, \mathbb{P}^n)$.

Proof. Let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. Then

$$\Gamma_X(k) = \Gamma_{X_1}(k) \cup \cdots \cup \Gamma_{X_r}(k).$$

Thus the equality in (3.5.4) follows from Proposition 3.5.5. Let's prove (3.5.5). Let $c := \operatorname{cod}(X, \mathbb{P}^n)$ and let $i \in \{1, \ldots, r\}$ be such that $c = n - \dim X_i$. Then

$$\dim \Gamma_{X_i}(c) = n - c + c(n-c) = (c+1)(n-c) = \dim \mathbb{G}r(c, \mathbb{P}^n).$$

This proves that the inequality in (3.5.5) holds and also the last statement. □

Proposition 3.5.7. *Let $X \subset \mathbb{P}^n$ be closed. Suppose that $p \in \mathbb{P}^n \backslash X$ and that $H \subset \mathbb{P}^n$ is a hyperplane not containing p. Let*

$$\begin{array}{ccc} (\mathbb{P}^n \backslash \{p\}) & \xrightarrow{\pi_p} & H \\ q & \mapsto & \langle p, q \rangle \cap H \end{array}$$

be projection from p. Then $\pi_p(X)$ is a closed subset of H and $\dim \pi_p(X) = \dim X$.

Proof. We may assume that X is irreducible. Since $\pi_{p|X}$ is regular and X is projective $\pi_p(X)$ is closed. It remains to prove that $\dim \pi_p(X) = \dim X$. We may assume that $p = [0, \ldots, 0, 1]$ and $H = V(X_n)$. We have

$$\pi_p([Z_0, \ldots, Z_n]) = [Z_0, \ldots, Z_{n-1}].$$

Let $Y := \pi_p(X)$. The map π_p defines a regular surjective map $\rho \colon X \to Y$ between irreducible (projective) varieties. We have the injection of fields $\rho^* \colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$. It suffices to prove that $\mathbb{K}(X)$ is algebraic over $\rho^* \mathbb{K}(Y)$.

One of $V(Z_0), \ldots, V(Z_{n-1})$ does *not* contain Y, say $V(Z_0)$, and hence $\mathbb{K}(Y)$ is generated over $\mathbb{K}$ by

$$(Z_1/Z_0)|_Y, \ldots, (Z_{n-1}/Z_0)|_Y.$$

On the other hand $\mathbb{K}(X)$ is generated by

$$(Z_1/Z_0)|_X = \rho^*\left((Z_1/Z_0)|_Y\right), \ldots, (Z_{n-1}/Z_0)|_X = \rho^*\left((Z_{n-1}/Z_0)|_Y\right)$$

and $(Z_n/Z_0)|_X$. There exists $F \in I(X)$ such that $F(p) \neq 0$ because $p \notin X$. Since $p = [0, \ldots, 0, 1]$ we get that

$$F = a_0 Z_n^d + a_1 Z_n^{d-1} + \cdots + a_d, \quad a_i \in \mathbb{K}[Z_0, \ldots, Z_{n-1}]_i, \quad a_0 \neq 0. \tag{3.5.6}$$

Dividing by Z_0^d and restricting to X we get that

$$\overline{a}_0 \cdot ((Z_n/Z_0)_{|X})^d + \overline{a}_1 \cdot ((Z_n/Z_0)_{|X})^{d-1} + \cdots + \overline{a}_d = 0$$

where for $0 \leqslant j \leqslant d$

$$\overline{a}_j := (a_j/Z_0^j)_{|X} \in \mathbb{K}\left(\rho^*\left((Z_1/Z_0)|_Y\right), \ldots, \rho^*\left((Z_{n-1}/Z_0)|_Y\right)\right). \tag{3.5.7}$$

Since $\overline{a}_0 \neq 0$ this proves that $(Z_n/Z_0)|_X$ is algebraic over $\rho^* \mathbb{K}(Y)$.

□

Proof of Proposition 3.5.3. By considering an irreducible component of X of maximum dimension we may assume that X is irreducible. Let $\rho \colon \Gamma_X(k) \to \mathbb{G}r(k, \mathbb{P}^n)$ be the restriction of the projection map $\mathbb{P}^n \times \mathbb{G}r(k, \mathbb{P}^n) \to \mathbb{G}r(k, \mathbb{P}^n)$. Then $\Lambda \in \mathbb{G}r(k, \mathbb{P}^n)$ has non empty intersection with X if and only if it belongs to $\operatorname{im}(\rho)$. The map ρ is closed because $\Gamma_X(k)$ is projective, hence $\operatorname{im}(\rho)$ is closed. Moreover $\operatorname{im}(\rho)$ is irreducible because X is irreducible. Thus ρ defines a dominant map $\Gamma_X(k) \to \operatorname{im}(\rho)$ of irreducible varieties. It follows that $\dim(\operatorname{im}(\rho)) \leqslant \Gamma_X(k)$. Now suppose that $k < \operatorname{cod}(X, \mathbb{P}^n)$. By Corollary 3.5.6 we get that $\dim(\operatorname{im}(\rho)) < \dim \mathbb{G}r(k, \mathbb{P}^n)$ and hence $\mathbb{G}r(k, \mathbb{P}^n) \backslash \operatorname{im}(\rho)$ is an open dense subset of $\dim \mathbb{G}r(k, \mathbb{P}^n)$. Item (a) follows because any $\Lambda \in (\mathbb{G}r(k, \mathbb{P}^n) \backslash \operatorname{im}(\rho))$ does not intersect X.

Next we prove (b). The proof is by induction on $\mathrm{cod}(X,\mathbb{P}^n)$. If $\mathrm{cod}(X,\mathbb{P}^n) = 0$ the result is trivial (if you don't like to start from $\mathrm{cod}(X,\mathbb{P}^n) = 0$ you may begin from $\mathrm{cod}(X,\mathbb{P}^n) = 1$, i.e. X is a hypersurface). Let's prove the inductive step. Let $p \in \Lambda$. If $p \in X$ there is nothing to prove; thus we may assume that $p \notin X$. Choose a hyperplane $H \subset \mathbb{P}^n$ not containing p and let π be projection from p to H as in (3.6.8). Then $\pi(X) \subset H \simeq \mathbb{P}^{n-1}$ is closed because X is projective, and $\dim \pi(X) = \dim X$ by Proposition 3.5.7. Thus

$$\mathrm{cod}(\pi(X),\mathbb{P}^{n-1}) = \mathrm{cod}(X,\mathbb{P}^n) - 1. \tag{3.5.8}$$

Let $\Omega := \pi(\Lambda\backslash\{p\})$. Then $\Omega \subset H$ is a linear subspace and $\dim\Omega = (\dim\Lambda - 1)$. By the equality in (3.5.8) it follows that $\dim\Omega \geqslant \mathrm{cod}(\pi(X),\mathbb{P}^{n-1})$. Hence $\Omega \cap \pi(X)$ is non empty by the inductive hypothesis. Let $q \in \Omega \cap \pi(X)$. Since $q \in \pi(X)$ there exists $\tilde{q} \in X$ such that $\pi(\tilde{q}) = q$. But $\tilde{q} \in \Lambda$ because $q \in \Omega$. Thus $\tilde{q} \in X \cap \Lambda$. □

Dimension of intersections

The result below is a remarkable generalization of the well-known result in linear algebra stating that the set of solutions of a system of m homogeneous linear equations in $n \geqslant m$ unknowns has dimension at least $n - m$.

Proposition 3.5.8. *Let $X, Y \subset \mathbb{P}^n$ be closed and suppose that $(\dim X + \dim Y) \geqslant n$. Then $X \cap Y$ is non empty and each of its irreducible components has dimension at least $\dim X + \dim Y - n$.*

Remark 3.5.9. It is clear that one needs the hypothesis that X, Y be closed for the thesis of Proposition 3.5.8 to hold. The hypothesis that the ambient algebraic variety is $\mathbb{P}^n$ is also a key hypothesis. As soon as one replaces $\mathbb{P}^n$ by other complete algebraic varieties the thesis fails to hold. As a test consider replacing $\mathbb{P}^n$ by a product of projective spaces, or by a Grassmannian.

We prove Proposition 3.5.8 after going through a series of preliminary results.

Let $X, Y \subset \mathbb{P}^N$ be two closed subsets. Let $\langle X\rangle \subset \mathbb{P}^N$ and $\langle Y\rangle \subset \mathbb{P}^N$ be the linear subspaces generated by X and Y respectively.

Definition 3.5.10. Suppose that

$$\langle X\rangle \cap \langle Y\rangle = \varnothing. \tag{3.5.9}$$

The *join $J(X,Y)$ of X and Y* is the subset of $\mathbb{P}^N$ swept out by the lines joining a point of X to a point of Y, i.e.

$$J(X,Y) := \bigcup_{p\in X, q\in Y} \langle p, q\rangle. \tag{3.5.10}$$

Claim 3.5.11. *Let $X, Y \subset \mathbb{P}^N$ be closed and assume that* (3.5.9) *holds.*

1. *$J(X,Y)$ is closed.*
2. *If X and Y are irreducible then $J(X,Y)$ is irreducible.*
3. $\dim J(X,Y) = \dim X + \dim Y + 1$.

Proof. Let $m := \dim\langle X\rangle$ and $n := \dim\langle Y\rangle$. There exist homogeneous coordinates

$$[S_0,\ldots,S_m,T_0,\ldots,T_n,U_0,\ldots,U_p]$$

on $\mathbb{P}^N$ such that $\langle X\rangle = \{[S_0,\ldots,S_m,0,\ldots,0]\}$ and $\langle Y\rangle = \{[0,\ldots,0,T_0,\ldots,T_n,0,\ldots,0]\}$. Then

$$J(X,Y) = \{[S_0,\ldots,S_m,T_0,\ldots,T_n,0,\ldots,0] \mid [S_0,\ldots,S_m] \in X, \quad [T_0,\ldots,T_n] \in Y\}. \tag{3.5.11}$$

Item (1) follows at once.

Let $r \in (J(X,Y)\backslash X\backslash Y)$. By (3.5.9) there is *unique* couple $(\varphi_1(r), \varphi_2(r)) \in X \times Y$ such that $r \in \langle\varphi_1(r), \varphi_2(r)\rangle$. Thus we have a map

$$\begin{array}{ccc} (J(X,Y)\backslash X\backslash Y) & \xrightarrow{\varphi} & X \times Y \\ r & \mapsto & (\varphi_1(r), \varphi_2(r)) \end{array} \tag{3.5.12}$$

As is easily checked φ is regular. The fibers of φ are isomorphic to $\mathbb{K}^\times$. Moreover for any $i \in \{0,\ldots,m\}$ and $j \in \{0,\ldots,n\}$ we have

$$\varphi^{-1}(X_{S_i} \times Y_{T_j}) \cong X_{S_i} \times Y_{T_j} \times \mathbb{K}^\times. \tag{3.5.13}$$

Items (2) and (3) follow from this. □

The result below is the special case of Proposition 3.5.8 one gets by letting Y be a hyperplane.

Proposition 3.5.12. *Let $X \subset \mathbb{P}^n$ be closed, irreducible of strictly positive dimension. Let $H \subset \mathbb{P}^n$ a hyperplane not containing X. Then $X \cap H$ is non empty and it has pure dimension equal to* $\dim X - 1$.

Proof. Since $X \cap H \subsetneq X$ we have $\dim X \cap H < \dim X$ by Proposition 3.4.8. Let $c := \mathrm{cod}(X, \mathbb{P}^n)$. Let $\Lambda \subset H$ be a linear subspace such that $\dim \Lambda = c$. Note that such subspaces exist because by hypothesis $c \leqslant (n-1) = \dim H$. By Proposition 3.5.3 applied to $X \subset \mathbb{P}^n$ we have $\Lambda \cap X \neq \varnothing$, and since $\Lambda \subset H$ we have $\Lambda \cap X \subset \Lambda \cap (X \cap H)$. This proves that $X \cap H$ is non empty and also, by Proposition 3.5.3, that $\mathrm{cod}(X \cap H, H) \leqslant c$. The latter inequality gives that

$$\dim(X \cap H) \geqslant \dim H - c = n - 1 - c = \dim X - 1. \tag{3.5.14}$$

This proves that $X \cap H$ is non empty and $\dim(X \cap H) = \dim X - 1$. It does not suffice because the proposition states a stronger result namely that $X \cap H$ has pure dimension equal to $\dim X - 1$.

The proof of the stronger statement is by induction on $\mathrm{cod}(X, \mathbb{P}^n)$. If $\mathrm{cod}(X, \mathbb{P}^n) = 0$ then $X = \mathbb{P}^n$ and the statement of the proposition is trivially true. If $\mathrm{cod}(X, \mathbb{P}^n) = 1$ then X is a hypersurface by Corollary 3.4.9, hence $X \cap H$ is a hypersurface in H and hence every irreducible component of $X \cap H$ has codimension one in H by Corollary 3.4.9. This proves the validity of the proposition if $\mathrm{cod}(X, \mathbb{P}^n) = 1$. Now we prove the inductive step. Assume that $\mathrm{cod}(X, \mathbb{P}^n) = c \geqslant 2$. Let Y be an irreducible component of $X \cap H$. Pick a point $p \in H\backslash X$ and a hyperplane L not containing p and different from H. Let

$$\begin{array}{ccc} \mathbb{P}^n\backslash\{p\} & \xrightarrow{\pi_p} & L \\ q & \mapsto & \langle p, q\rangle \cap L \end{array}$$

be the projection from p. Let $H_0 := \pi_p(H\backslash\{p\})$. Note that $H_0 \subset L$ is a hyperplane. We consider $\pi_p(X) \cap H_0$. Let $X \cap H = Y \cup Y_1 \cup \cdots \cup Y_r$ be the irreducible decomposition of $X \cap H$. We have

$$\pi_p(X) \cap H_0 = \pi_p(Y) \cup \pi_p(Y_1) \cup \ldots \cup \pi_p(Y_r),$$

and, since $p \notin X$, each of $\pi_p(Y), \pi_p(Y_1), \ldots, \pi_p(Y_r)$ is closed by Proposition 3.5.7. We claim that there exists p such that

$$\pi_p(Y) \not\subset \pi_p(Y_i) \quad \forall i \in \{1,\ldots,r\}. \tag{3.5.15}$$

In fact let $q \in Y\backslash\bigcup_{i=1}^r Y_i$. By Claim 3.5.11 $J(q, Y_i)$ is closed, irreducible, and

$$\dim J(q, Y_i) = \dim Y_i + 1. \tag{3.5.16}$$

Since $\dim Y_i \leqslant \dim X - 1$ and since $\mathrm{cod}(X, \mathbb{P}^n) \geqslant 2$ we have $\dim Y_i \leqslant \dim H - 2$. Thus (3.5.16) gives that $J(q, Y_i) \neq H$. Hence there exists

$$p \in H\backslash\bigcup_{i=1}^{r} J(q, Y_i). \tag{3.5.17}$$

For such a p the statement in (3.5.15) holds, and hence $\pi_p(Y)$ is an irreducible component of $\pi_p(X) \cap H_0$.

By the inductive hypothesis we get that $\dim \pi_p(Y) = \dim \pi_p(X) - 1$. Since $\dim \pi_p(Y) = \dim Y$ and $\dim \pi_p(X) = \dim X$ (by Proposition 3.5.7) we are done. □

Proof of Proposition 3.5.8. Let $[S_0, \ldots, S_n, T_0, \ldots, T_n]$ be homogeneous coordinates on $\mathbb{P}^{2n+1}$. We have the two embeddings

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{i} & \mathbb{P}^{2n+1} \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0, \ldots, Z_n, 0, \ldots, 0] \end{array} \qquad \begin{array}{ccc} \mathbb{P}^n & \xrightarrow{j} & \mathbb{P}^{2n+1} \\ [Z_0, \ldots, Z_n] & \mapsto & [0, \ldots, 0, Z_0, \ldots, Z_n] \end{array} \tag{3.5.18}$$

Since the images of i and j are disjoint linear subspaces of $\mathbb{P}^{2n+1}$ the join $J(i(Y), j(W))$ is defined. Let $\Lambda \subset \mathbb{P}^{2n+1}$ be the linear subspace given by

$$\Lambda := V(S_0 - T_0, \ldots, S_n - T_n). \tag{3.5.19}$$

We have the isomorphism

$$\begin{array}{ccc} X \cap Y & \xrightarrow{\sim} & \Lambda \cap J(i(X), j(Y)) \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0, \ldots, Z_n, Z_0, \ldots, Z_n] \end{array} \tag{3.5.20}$$

By Claim 3.5.11 the closed subset $J(i(X), j(Y)) \subset \mathbb{P}^{2n+1}$ has dimension equal to $\dim X + \dim Y + 1$. On the other hand Λ is a codimension-$(n+1)$ linear subspace of $\mathbb{P}^{2n+1}$, hence by repeated application of Proposition 3.5.12 we get that $\Lambda \cap J(i(X), j((Y))$ is non empty and each of its irreducible components has dimension at least equal to $(\dim X + \dim Y - n)$. By the isomorphism in (3.5.20) the proposition follows. □

Dimension of fibers

Theorem 3.5.13. *Let X be an irreducible algebraic variety. Let $f\colon X \to \mathbb{K}$ be a non zero regular function, and let $V(f) := f^{-1}(0)$. Every irreducible component of $V(f)$ has dimension equal to* $\dim X - 1$.

Proof. Since X is a (finite) union of open affine subsets we may assume that X is affine. Thus $X \subset \mathbb{A}^n$ is a closed subset. By Theorem 1.6.2 there exists $\tilde{f} \in \mathbb{K}[z_1, \ldots, z_n]$ such that $f = \tilde{f}_{|X}$. Let $Y := V(\tilde{f})$, and let W be an irreducible component of $X \cap Y$. We must prove that $\dim W = \dim X - 1$. We have $\mathbb{A}^n = \mathbb{P}^n_{Z_0} \subset \mathbb{P}^n$ as open dense subset. Let $\overline{X}, \overline{Y}, \overline{W} \subset \mathbb{P}^n$ be the closures of X, Y and W respectively. Then $\overline{Y} \subset \mathbb{P}^n$ is a hypersurface. Let $P \in \mathbb{K}[Z_0, \ldots, Z_n]$ be a homogeneous polynomial such that $\overline{Y} = V(P)$, and let d be its degree. Let $N := \binom{d+n}{n} - 1$, and let

$$\begin{array}{ccc} \mathbb{P}^n & \xrightarrow{\nu_d^n} & \mathbb{P}^N \\ [Z_0, \ldots, Z_n] & \mapsto & [Z_0^d, Z_0^{d-1} X_1, \ldots, Z_n^d] \end{array}$$

be the Veronese map. Since $\overline{Y} = V(P)$ and P has degree d, there exists a hyperplane $H \subset \mathbb{P}^N$ such that $(\nu_d^n)^{-1}(H) = \overline{Y}$. Thus ν_d^n defines an isomorphism $\overline{X} \cap \overline{Y} \xrightarrow{\sim} \nu_d^n(\overline{X}) \cap H$, and $\nu_d^n(\overline{W})$ is an irreducible component of $\nu_d^n(\overline{X}) \cap H$. By Proposition 3.5.12 we have

$$\dim W = \dim \overline{W} = \dim \nu_d^n(\overline{W}) = \dim \nu_d^n(\overline{X}) - 1 = \dim \overline{X} - 1 = \dim X - 1.$$

□

Corollary 3.5.14. *Let $f\colon X \to Y$ be a regular map of algebraic varieties. Let $p \in X$. Every irreducible component of $f^{-1}(f(p))$ has dimension at least equal to* $\dim X - \dim_{f(p)} Y$.

Proof. Since X and Y are covered by open affine subsets, we may assume that X and Y are affine. Let $q := f(p)$ and let $m := \dim_q Y$. We claim that there exist $\varphi_1, \ldots, \varphi_m \in \mathbb{K}[Y]$ such that q is an irreducible component of $V(\varphi_1, \ldots, \varphi_m)$. In fact one may argue by induction on m. If $m = 0$ the statement is trivially true. Let $m > 0$ and assume that the claim holds for lower values of m. Since $\dim_q Y > 0$ there exists $\varphi_m \in \mathbb{K}[Y]$ vanishing at q and not vanishing on any irreducible component of Y containing q. Then $V(\varphi_m)$ contains q, and by Theorem 3.5.13 its dimension at q is equal to $m - 1$. By the inductive hypothesis there exist $\psi_1, \ldots, \psi_{m-1} \in \mathbb{K}[V(\varphi_1)]$ such that q is an irreducible component of $V(\psi_1, \ldots, \psi_{m-1}) \subset V(\varphi_1)$. Since $V(\varphi_1)$ is a closed affine subset of the affine variety Y, there exist $\varphi_1, \ldots, \varphi_{m-1} \in \mathbb{K}[Y]$ whose restrictions to $V(\varphi_1)$ are equal to $\psi_1, \ldots, \psi_{m-1}$ respectively. Then q is an irreducible component of $V(\varphi_1, \ldots, \varphi_m)$. Thus we have

$$V(f^*(\varphi_1), \ldots, f^*(\varphi_m)) = f^{-1}(q) \sqcup W,$$

where W is closed in X, i.e. $f^{-1}(q)$ is a union of irreducible components of $V(f^*(\varphi_1), \ldots, f^*(\varphi_m))$. By repeated application of Theorem 3.5.13 every irreducible component of $V(f^*(\varphi_1), \ldots, f^*(\varphi_m))$ has dimension at least equal to $\dim X - m = \dim X - \dim Y$. □

3.6 Degree

Degree of a map

Definition 3.6.1. Let $f\colon X \to Y$ be a regular map of irreducible algebraic varieties. The *degree of f*, denoted by $\deg f$, is given by

$$\deg f := \begin{cases} 0 & \text{if } f \text{ is not dominant,} \\ [\mathbb{K}(X) : f^*\mathbb{K}(Y)] & \text{if } f \text{ is dominant.} \end{cases}$$

The *separable degree of f*, denoted by $\deg_s f$, is given by

$$\deg_s f := \begin{cases} 0 & \text{if } f \text{ is not dominant,} \\ [\mathbb{K}(X)^s : f^*\mathbb{K}(Y)] & \text{if } f \text{ is dominant,} \end{cases}$$

where $\mathbb{K}(X)^s \subset \mathbb{K}(X)$ is the maximal separable extension of $f^*\mathbb{K}(Y)$.

Thus $0 < \deg f < \infty$ if and only if f is dominant and $\dim W = \dim Z$. Note that $\deg_s f$ divides $\deg f$, and that if $\mathbb{K}$ has characteristic 0 then $\deg_s f = \deg f$.

Example 3.6.2. Let $(z_1, \ldots, z_n, w)$ be affine coordinates on $\mathbb{A}^{n+1}$. Let $X \subset \mathbb{A}^{n+1}$ be an irreducible hypersurface and let $I(X) = P$. Write

$$P = a_0 w^d + a_1 w^{d-1} + \cdots + a_d, \qquad a_i \in \mathbb{K}[z_1, \ldots, z_n], \quad a_0 \neq 0$$

Let $Y = \mathbb{A}^n$ and let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ (z_1, \ldots, z_n, w) & \mapsto & (z_1, \ldots, z_n) \end{array}$$

Then $\deg f = d$. In fact if $d = 0$ then $\operatorname{im} f = V(a_0) \subsetneq \mathbb{A}^n$ and hence f is not dominant. If $d > 0$ then

$$\mathbb{K}(X) = \mathbb{K}(z_1, \ldots, z_n)[w]/(P)$$

and hence $[\mathbb{K}(X) : \mathbb{K}(z_1, \ldots, z_n)] = d$.

If $\mathbb{K}$ has characteristic 0 then $\deg_s f = \deg f$. Suppose that $\operatorname{char} \mathbb{K} = p > 0$. Let m be the maximum integer such that $p^m \mid (d - i)$ for all $i \in \{0, \ldots, d\}$ such that $a_i \neq 0$. Then $\deg_s f = d/p^m$.

Below is the main result of the present section.

Proposition 3.6.3. *Let $f\colon X \to Y$ be a regular map of irreducible algebraic varieties such that* $\deg f < \infty$*. Then there exists an open dense $Y^0 \subset Y$ such that*

$$|f^{-1}(q)| = \deg_s f \qquad \forall q \in Y^0. \tag{3.6.1}$$

Example 3.6.4. Let us check the statement of Proposition 3.6.3 for the map $f\colon X \to \mathbb{A}^n$ of Example 3.6.2. Let $P \in \mathbb{K}[z_1, \ldots, z_n, w]$ be as in that example. Let $Q \in \mathbb{K}[z_1, \ldots, z_n, w]$ be defined as follows: $Q = P$ if $\operatorname{char}\mathbb{K} = 0$, and

$$Q(z_1, \ldots, z_n, w^{p^m}) = P(z_1, \ldots, z_n, w), \qquad \frac{\partial Q}{\partial w} \neq 0.$$

In particular m is the maximum integer such that $p^m \mid (d-i)$ for all $i \in \{0, \ldots, d\}$ such that $a_i \neq 0$, and hence $\deg_s f$ is the degree in w of Q. Let $Y := V(Q) \subset \mathbb{A}^{n+1}$. Let $g\colon X \to Y$ be defined by $g(z,w) := w^{p^m})$, and let $h\colon Y \to \mathbb{A}^n$ be defined by $h(z,w) := z$. The regular map $f\colon X \to \mathbb{A}^n$ factorizes as the composition

$$X \xrightarrow{g} Y \xrightarrow{h} \mathbb{A}^n. \tag{3.6.2}$$

Clearly the map g is bijective, hence it suffices to check that $|h^{-1}(\overline{z})| = d$ for a general $\overline{z} \in \mathbb{A}^n$. Since $\frac{\partial Q}{\partial w} \neq 0$, closed subset of Y defined by $V(Q, \partial Q/\partial w)$ is a proper subset and hence it has dimension strictly smaller than $\dim Y = n$. Thus $\Delta := \overline{h(V(Q, \partial Q/\partial w))}$ is contained in a proper closed subset of $\mathbb{A}^n$ and hence $(\mathbb{A}^n \backslash \Delta \backslash V(a_0))$ contains an open dense subset $U \subset \mathbb{A}^n$. Let $\overline{z} \in U$. Then $Q(\overline{z}, w) \in \mathbb{K}[w]$ is a polynomial with simple roots of degree $\deg_s f$ and hence $|h^{-1}(\overline{z})| = \deg_s f$.

Example 3.6.5. We consider a more general version of Example 3.6.2. Let Y be an affine variety. Let $P \in \mathbb{K}(Y)[t]$ be an *irreducible* polynomial:

$$P = t^d + a_1 t^{d-1} + \cdots + a_d, \quad a_i \in f^*(\mathbb{K}(Y)).$$

Since Y is affine $\mathbb{K}(Y)$ is the field of fractions of $\mathbb{K}[Y]$. Thus there exists $0 \neq b \in \mathbb{K}[Y]$ such that $b \cdot a_i \in f^*(\mathbb{K}[Y])$ for all $1 \leqslant i \leqslant d$. Let $c_0 := b$, $c_i := b \cdot a_i$, $1 \leqslant i \leqslant d$ and

$$Q := c_0 y^d + c_1 y^{d-1} + \cdots + c_d \in \mathbb{K}[Y][w]. \tag{3.6.3}$$

If $\mathbb{K}[Y]$ is a UFD we may factor out the gcd $\{c_0, \ldots, c_d\}$ and hence by renaming the c_i's we may assume that $\gcd\{c_0, \ldots, c_d\} = 1$. It follows that $V(Q)$ is irreducible (the proof is the same as the one for hypersurfaces in $\mathbb{A}^n$). In general $\mathbb{K}[Y]$ is not a UFD and hence there might be no way of "reducing" the polynomial of (3.6.5) in order to get that $V(Q)$ is irreducible. An example of this phenomenon is the following: $Y := V(z_1 z_2 - z_3 z_4)$ and $V := V(z_1 y - z_3)$.

Let hypotheses and notation be as in Example 3.6.5, and let $\pi\colon X \times \mathbb{A}^1 \to X$ be the projection map. An irreducible component V_i of $V(Q)$ *dominates* X if $\overline{\pi(V_i)} = X$.

Claim 3.6.6. *Keep hypotheses and notation as in Example 3.6.5. There is one and only one irreducible component of $V(Q)$ which dominates Y, call it V_*. Let $\pi_*\colon V_* \to Y$ be the restriction of π. There is an open dense $U \subset Y$ such that $|\pi_*^{-1}(q)| = \deg_s \pi_*$ for every $q \in U$.*

Proof. We have $\pi(V(Q)) \supset Y \backslash V(c_0)$. Then $Y \backslash V(c_0)$ is dense in Y because $c_0 \neq 0$. It follows that there exists at least one irreducible component V_* of V such that $\pi(\bar{V}_*) = Y$. Let V_* be such an irreducible component. Let $g \in I(V_*)$. We claim that

$$Q | g \text{ in } \mathbb{K}(Y)[w]. \tag{3.6.4}$$

(Notice: we do *not* claim that $Q|g$ in $\mathbb{K}[Y][w]$.) In fact suppose that $Q \nmid g$. Then Q and g are coprime (in $\mathbb{K}(Y)[w]$) because Q is prime, and hence there exist $\alpha, \beta \in \mathbb{K}(Y)[w]$ such that

$$\alpha \cdot Q + \beta \cdot g = 1.$$

Multiplying by $0 \neq \gamma \in \mathbb{K}[Y][w]$ such that $\alpha \cdot \gamma, \beta \cdot \gamma \in \mathbb{K}[Y][w]$ we get that

$$(\alpha \cdot \gamma) Q + (\beta \cdot \gamma) g = \gamma.$$

It follows that if $q \in V_*$ then $\gamma(q) = 0$. Since $\gamma \neq 0$ we get that $\pi(\bar{V}_*) \neq Y$: that is a contradiction. This proves (3.6.4). Let $I(V_*) = (g_1, \ldots, g_r)$. From (3.6.4) we get that there exist $h_1, \ldots, h_r \in \mathbb{K}[Y][w]$ and $m_1, \ldots, m_r \in \mathbb{K}[Y]$ such that

$$m_i \cdot g_i = Q \cdot h_i, \quad m_i \neq 0, \quad i = 1, \ldots, r.$$

Set $m = m_1 \cdot \cdots \cdot m_r$. Then $V_*\backslash V(m) = V\backslash V(m)$ and it follows that V_* is the unique irreducible component of $V(Q)$ dominating Y. Now let
****

$$Q' := dc_0 y^{d-1} + (d-1)c_1 y^{d-2} + \cdots + c_{d-1} \in \mathbb{K}[Y][w]. \tag{3.6.5}$$

be the derivetive of Q with respect to y. Then $Q' \neq 0$ and $\deg Q' < \deg Q$. Thus Q and Q' are coprime in $\mathbb{K}(Y)[w]$ and hence there exist $\mu, \nu \in \mathbb{K}(Y)[w]$ such that

$$\mu \cdot Q + \nu \cdot Q' = 1.$$

Arguing as above we get that there exists a *proper* closed $W \subset Y$ such that

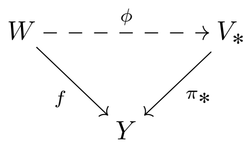
$$\pi^{-1}(Y\backslash W) \cap V(Q) \cap V(Q') = \varnothing. \tag{3.6.6}$$

Now let $U := (Y\backslash W\backslash V(c_0)\backslash V(m))$: then $|\pi_*^{-1}(q)| = d$ for every $q \in U$. □

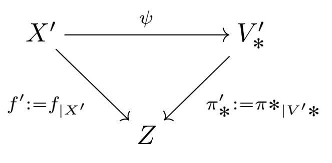
Proof of Proposition 3.6.3. Suppose that $\deg f = 0$. Then $\overline{f(X)} \neq Y$ and $Y^0 := Y\backslash f(\bar{X})$ does the job. Now suppose that $d := \deg f > 0$. Since Y is covered by open affine sets we may assume that Y itself is affine. By definition we have an inclusion $f^*\colon \mathbb{K}(Y) \hookrightarrow \mathbb{K}(X)$ and $\mathbb{K}(X)$ as vector space over $\mathbb{K}(Y)$ has dimension d. Since we are in characteristic zero there exists $\xi \in \mathbb{K}(X)$ primitive over $f^*(\mathbb{K}(Y))$. Let

$$P = t^d + a_1 t^{d-1} + \cdots + a_d, \quad a_i \in f^*(\mathbb{K}(Y))$$

be the minimal polynomial of ξ. Let $V(\widetilde{P}) \subset Y \times \mathbb{A}^1$ - notation as in Claim 3.6.6. Let $V_* \subset V(\widetilde{P})$ be the unique irreducible component dominating Y. We have a commutative diagram



with ϕ birational. By Proposition 3.2.11 there exist open dense subsets $X' \subset X$ and $V_*' \subset V_*$ fitting into a commutative diagram



(3.6.7)

with ψ an isomorphism. Since $X\backslash X' \neq X$ and $\dim X = \dim Y$ we have

$$\overline{f(X\backslash X')} \neq Y.$$

On the other hand

$$f^{-1}\{q\} = (f')^{-1}\{q\} \quad \text{if} \quad q \in Y\backslash\overline{f(X\backslash X')}.$$

By commutativity of (3.6.7) and the fact that ψ is an isomorphism we get that

$$|(f')^{-1}\{q\}| = |(\pi_*')^{-1}\{q\}|, \quad q \in Y.$$

Hence the proposition follows from ******. □

Degree of a closed subset of a projective space

Let $X \subset \mathbb{P}^n$ be closed, and let c be its codimension. Suppose that X is irreducible and let π be the forgetful regular map

$$\begin{array}{ccc} \Gamma_X(c) & \stackrel{\pi}{\longrightarrow} & \mathbb{G}\mathrm{r}(c, \mathbb{P}^n) \\ (p, \Lambda) & \mapsto & \Lambda \end{array} \tag{3.6.8}$$

Since $\Gamma_X(c)$ and $\mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ are irreducible we have a well-defined $\deg \pi$. By Corollary 3.5.6 we have $\dim \Gamma_X(c) = \dim \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$. Thus $\deg \pi < \infty$. The *degree of* X is defined to be the separable degree

$$\deg X \coloneqq \deg_s(\Gamma_X(c) \stackrel{\pi}{\longrightarrow} \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)). \tag{3.6.9}$$

In general let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. The *degree of* X is defined to be the sum of the degrees of irreducible components of X which realize the dimension of X:

$$\deg X := \sum_{\dim X_i = \dim X} \deg X_i. \tag{3.6.10}$$

Proposition 3.6.7. *Let $X \subset \mathbb{P}^n$ be closed of codimension c. There exists an open dense $U \subset \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ with the following property: if $\Lambda \in U$ then $X \cap \Lambda$ is finite of cardinality equal to* $\deg X$. *Moreover* $\deg X$ *is positive.*

Proof. If X is irreducible the first statement follows from Proposition 3.6.3 applied to the map π in (3.6.8), and the positivity of $\deg X$ follows from Proposition 3.5.3. In general let $X = X_1 \cup \cdots \cup X_r$ be the irreducible decomposition of X. If $\Lambda \in \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ is general then by Proposition 3.5.3

$$\Lambda \cap X_i = \emptyset \text{ if } \dim X_i < \dim X, \quad \Lambda \cap (X_i \cap X_j) = \emptyset \text{ if } i \neq j. \tag{3.6.11}$$

It follows that if $\Lambda \in \mathbb{G}\mathrm{r}(c, \mathbb{P}^n)$ is general then

$$\Lambda \cap X = \bigsqcup_{\dim X_i = \dim X} \Lambda \cap X_i, \tag{3.6.12}$$

and hence the claim follows from the case when X is irreducible. □

Example 3.6.8. Let $X \subset \mathbb{P}^n$ be a hypersurface and let $I(Z) = (F)$. Then $\deg X = \deg F$. In fact *******

Example 3.6.9. Let $\mathcal{C}_d \subset \mathbb{P}^d$ be the *rational normal curve*, i.e. the image of the Veronese map

$$\begin{array}{ccc} \mathbb{P}^1 & \stackrel{\nu_d^1}{\longrightarrow} & \to \mathbb{P}^d \\ [S, T] & \mapsto & [S^d, S^{d-1}T, \ldots, T^d] \end{array} \tag{3.6.13}$$

Then $\deg \mathcal{C}_d = d$. ********

3.7 Exercises

Exercise 3.7.1. The Veronese map is

$$\begin{array}{ccc} \mathbb{P}^2 & \stackrel{f}{\dashrightarrow} & \mathbb{P}^2 \\ [Z_0, Z_1, Z_2] & \mapsto & [Z_1 Z_2, Z_0 Z_2, Z_0 Z_1] \end{array} \tag{3.7.14}$$

1. Prove that f is a birational map.
2. Determine $\mathrm{Reg}(f)$.
3. Describe maximal open sets $U, V \subset \mathbb{P}^2$ such that f induecs an isomorphism $U \xrightarrow{\sim} V$.

Exercise 3.7.2. An *algebraic group* is an algebraic variety G equipped with a group structure such that the map

$$\begin{array}{ccc} G \times G & \longrightarrow & G. \\ (x,y) & \mapsto & xy^{-1} \end{array} \tag{3.7.15}$$

is regular. For example $\mathrm{GL}_n(\mathbb{K})$ with matrix multiplication is an algebraic group. Prove that the irreducible components of an algebraic groups are pairwise disjoint and they all have the same dimension.

Exercise 3.7.3. Let $M_{n,n}(\mathbb{K})$ be the vector-space of $n \times n$ matrices with entries in $\mathbb{K}$. If $\operatorname{char}\mathbb{K} \neq 2$ define $\mathrm{O}_n(\mathbb{K})$ and $\mathrm{SO}_n(\mathbb{K})$ as usual:

$$\mathrm{O}_n(\mathbb{K}) := \{A \in M_{n,n}(\mathbb{K}) \mid A^t \cdot A = 1_n\}, \qquad \mathrm{SO}_n(\mathbb{K}) := \{A \in \mathrm{O}_n(\mathbb{K}) \mid \operatorname{Det} A = 1\}, \tag{3.7.16}$$

where $1_n \in M_{n,n}(\mathbb{K})$ is the unit matrix.

1. Let $Q := V(z_1^2 + z_2^2 + \ldots + z_n^2 - 1) \subset \mathbb{A}^n$, and let $f\colon \mathrm{SO}_n(\mathbb{K}) \to Q$ be the map associating to $A \in \mathrm{SO}_n(\mathbb{K})$ its first column. Prove that $f^{-1}(z)$ is isomorphic to $\mathrm{SO}_{n-1}(\mathbb{K})$ for every $z \in Q$.
2. Let X be an irreducible component of $\mathrm{SO}_n(\mathbb{K})$. Prove that $f(X)$ is dense in Q. Prove that if X is the irreducible component containing 1_n then $f(X)$ contains an open dense subset of Q.
3. Prove by induction on n that $\mathrm{SO}_n(\mathbb{K})$ is irreducible.
4. Prove that $\mathrm{O}_n(\mathbb{K})$ has two irreducible components.

Exercise 3.7.4. Let , and

$$U_n(\mathbb{K}) := \{Z \in M_{n,n}(\mathbb{K}) \mid \operatorname{Det}(1_n - Z) \neq 0\}.$$

The *Cayley map* is given by

$$\begin{array}{ccc} U_n(\mathbb{K}) & \overset{\varphi}{\longrightarrow} & M_{n,n}(\mathbb{K}) \\ Z & \mapsto & (1_n + Z)\cdot(1_n - Z)^{-1} \end{array} \tag{3.7.17}$$

1. Prove that φ defines a birational map $f\colon M_{n,n}(\mathbb{K}) \dashrightarrow M_{n,n}(\mathbb{K})$. Determine the rational inverse $f^{-1}\colon M_{n,n}(\mathbb{K}) \dashrightarrow M_{n,n}(\mathbb{K})$
2. Assume that $\operatorname{char}\mathbb{K} \neq 2$. Let $\mathfrak{o}_n(\mathbb{K}) \subset M_{n,n}(\mathbb{K})$ be the subspace of anti-symmetric matrices and let $\mathrm{SO}_n(\mathbb{K}) \subset M_{n,n}(\mathbb{K})$ be the group of special orthogonal matrices. Prove that if $Z \in \mathfrak{o}_n(\mathbb{K}) \cap U_n(\mathbb{K})$ then $\varphi(Z) \in \mathrm{SO}_n(\mathbb{K})$. Let $\psi\colon \mathfrak{o}_n(\mathbb{K}) \cap U_n(\mathbb{K}) \to \mathrm{SO}_n(\mathbb{K})$ be the restriction of φ.
3. Prove that the image of ψ is dense in $\mathrm{SO}_n(\mathbb{K})$, and hence ψ defines a dominant rational map $g\colon \mathfrak{o}_n(\mathbb{K}) \dashrightarrow \mathrm{SO}_n(\mathbb{K})$.
4. Prove that $\operatorname{Reg}(f^{-1})$ contains an open dense subset of $\mathrm{SO}_n(\mathbb{K})$ and hence g is a birational map.
5. Notice that g is defined over the prime field. Produce many matrices in $\mathrm{SO}_3(\mathbb{Q})$.

Exercise 3.7.5. Let $U_d^n \subset \mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d)$ be the set of points $[F]$ such that F is a prime polynomial.

1. Prove that if $n \geqslant 2$ then U_d^n is a dense open subset of $\mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d)$.
2. Prove that if $d \geqslant 2$ then the codimension of the complement of U_d^n in $\mathbb{P}(\mathbb{K}[Z_0,\ldots,Z_n]_d)$ is equal to

$$\binom{d+n-1}{n-1} - n. \tag{3.7.18}$$

Let $\operatorname{Div}(\mathbb{P}^n)$ be the abelian group with generators the irreducible hypersurfaces in $\mathbb{P}^n$. Thus an element of $\operatorname{Div}(\mathbb{P}^n)$ is a *formal* finite sum $D = \sum_{i\in I} m_i D_i$, where each m_i is an integer, and the D_i's are pairwise distinct irreducible hypersurface in $\mathbb{P}^n$. The *degree of* D is defined to be $\sum_{i\in I} m_i \deg D_i$. The divisor $\sum_{i\in I} m_i D_i$ is *effective* if $m_i > 0$ for all $i \in I$.

Let $F \in \mathbb{K}[Z_0,\ldots,Z_n]_d$ be non zero, and let $F = \prod_{i=1}^{r} F_i^{m_i}$ be the decomposition into prime factors, where for $i \neq j$ the factors F_i and F_j are not associated. The *divisor of* F is the element of $\operatorname{Div}(\mathbb{P}^n)$ defined by

$$\operatorname{div}(F) := \sum_{i=1}^{r} m_i V(F_i). \tag{3.7.19}$$

Let $\mathrm{Div}^d_+(\mathbb{P}^n) \subset \mathrm{Div}(\mathbb{P}^n)$ be the subset of effective divisors of degree d. The map

$$\begin{array}{ccc} \mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d) & \stackrel{\mathrm{div}}{\longrightarrow} & \mathrm{Div}^d_+(\mathbb{P}^n) \\ [F] & \mapsto & \mathrm{div}(F) \end{array} \tag{3.7.20}$$

is a bijection. This gives a geometric interpretation of $\mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d)$. From now on we identify $\mathrm{Div}^d_+(\mathbb{P}^n)$ with $\mathbb{P}(\mathbb{K}[Z_0, \ldots, Z_n]_d)$ via the bijection in (3.7.20). If $D = \sum_{i \in I} m_i D_i$ is an effective divisor, i.e. $m_i > 0$ for each $i \in I$, the *support of* D is the union of the D_i's and is denoted by $\mathrm{supp}\, D$.

Exercise 3.7.6. Let $R_d \subset \mathrm{Div}^d_+(\mathbb{P}^n)$ be the subset defined as follows:

$$R_d := \{D \in \mathrm{Div}^d_+(\mathbb{P}^n) \mid \text{there exists a line } \Lambda \subset \mathrm{supp}\, D \}. \tag{3.7.21}$$

1. Prove that R_d is closed in $\mathrm{Div}^d_+(\mathbb{P}^n)$.
2. Prove that if $d \geqslant 4$ then $R_d \neq \mathrm{Div}^d_+(\mathbb{P}^3)$.