Advanced Topics in Geometry, Autumn 2024 - Kieran O'Grady

Problem set 1

Exercise 1. Let X be a topological space. For $U \subset X$ open let

$$\mathscr{F}(U) := \{f : U \to \mathbb{R} \mid f \text{ is cosntant}\},$$

and for $V \subset U$ with V open let $\rho^{\mathscr{F}}_{U,V} : \mathscr{F}(U) \to \mathscr{F}(V)$ be the restriction map.

(1) Check that $\mathscr{F}$ is a presheaf of abelian groups.
(2) Determine under which hypotheses $\mathscr{F}$ is a sheaf.
(3) Describe explicitly the sheafification $\mathscr{F}^+$ of $\mathscr{F}$.

Exercise 2. Let $d \in \mathbb{N}$, and let $P \in \mathbb{K}[Z_0, \ldots, Z_n]_d$. The section $s_P \in \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$ is defines as follows. Given $\ell \in \mathbb{P}^n$, i.e. a 1-dimensional vector subspace $\ell \subset \mathbb{K}^{n+1}$ the value $s_P(\ell)$ is the restriction to ℓ of P. (Recall that the fiber of $\mathscr{O}_{\mathbb{P}^n}(d)$ at ℓ is the dual of $\ell^{\otimes d}$ i.e. the vector space of homogeneous degree-d maps $\ell \to \mathbb{K}$.) Prove that the map

$$\begin{array}{ccc} \mathbb{K}[Z_0, \ldots, Z_n]_d & \longrightarrow & \Gamma(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)) \\ P & \mapsto & s_P \end{array}$$

is an isomorphism of $\mathbb{K}$-vector spaces.

Let $\mathscr{R}$ be a sheaf of rings on a topological space X, and let $\mathscr{F}, \mathscr{G}$ be sheaves of $\mathscr{R}$-modules. The *direct sum* $\mathscr{F} \oplus \mathscr{G}$ is the sheaf of $\mathscr{R}$ modules defined by

$$(\mathscr{F} \oplus \mathscr{G})(U) := \mathscr{F}(U) \oplus \mathscr{G}(U)$$

for $U \subset X$ open, with restriction maps given by $\rho^{\mathscr{F}}_{U,V} \oplus \rho^{\mathscr{G}}_{U,V}$ for $V \subset U$ open.

Exercise 3. Let $d \in \mathbb{N}$. The sections $Z_i^d \in \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$ for $i \in \{0, 1\}$ define morphisms of sheaves

$$\mathscr{O}_{\mathbb{P}^1}(-d) \xrightarrow{Z_i^d} \mathscr{O}_{\mathbb{P}^1},$$

hence letting $\alpha := (Z_0^d, Z_1^d)$ we have a morphism

$$\mathscr{O}_{\mathbb{P}^1}(-d) \xrightarrow{\alpha} \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}.$$

(1) Show that φ is injective, and hence it defines an inclusion of sheaves $\alpha : \mathscr{O}_{\mathbb{P}^1}(-d) \hookrightarrow \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}$.
(2) Show that we have an isomorphism

$$\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} / \operatorname{im}(\alpha) \cong \mathscr{O}_{\mathbb{P}^1}(d).$$

(3) Let

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(-d) \xrightarrow{\alpha} \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \xrightarrow{\beta} \mathscr{O}_{\mathbb{P}^1}(d) \longrightarrow 0 \qquad (1)$$

be the short exact sequence one gets from (1)-(2). Show that if $d \geq 2$ then the map

$$\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}) \xrightarrow{\Gamma(\beta)} \Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(d))$$

is not surjective (and hence β is not surjective as morphism of preasheaves).