Advanced Topics in Geometry, Autumn 2024 - Kieran O'Grady

Problem set 2

In these exercises $\mathbb{K}$ is an algebraically closed field, and algebraic varieties are defined over $\mathbb{K}$.

Exercise 1. Let X be an irreducible affine variety.

(1) Let N be a $\mathbb{K}(X)$ vector space. Note that N is a $\mathbb{K}[X]$-module (because $\mathbb{K}[X] \subset \mathbb{K}(X)$), and hence there is an associated sheaf of $\mathscr{O}_X$-modules $\widetilde{N}$. Show that if $\dim X > 0$ and $N \neq 0$ then $\widetilde{N}$ is not coherent (of course it is quasi-coherent by definition).

(2) Let M be a $\mathbb{K}[X]$-module, and let $N := M \otimes_{\mathbb{K}[X]} \mathbb{K}(X)$. Thus N is a $\mathbb{K}(X)$ vector space. We have a morphism of sheaves of $\mathscr{O}_X$-modules $\varphi^M \colon \widetilde{M} \to \widetilde{N}$ defined by

$$\begin{array}{ccc} \widetilde{M}(U) & \xrightarrow{\varphi^M_U} & \widetilde{N}(U) \\ (s_x)_{x\in U} & \mapsto & (s_x \otimes 1)_{x \in U} \end{array}$$

(Note that if $p \in X$ then $N_{\mathfrak{m}_p} \cong M_{\mathfrak{m}_p} \otimes \mathbb{K}(X)$.) Let $X = \mathbb{A}^1(\mathbb{K})$ with afffine coordinate t, and let $M = \mathbb{K}[t] \oplus \mathbb{K}[t]/(t)$. Describe $\ker(\varphi^M)$.

Exercise 2. Let X be a topological space with a sheaf of rings $\mathscr{R}_X$. We assume that for each non empty open subset $U \subset X$ the ring $\mathscr{R}_X(U)$ is an integral domain. Let $\mathscr{F}$ be a sheaf of $\mathscr{R}_X$-modules. For $U \subset X$ open let

$$T_{\mathrm{pre}}(\mathscr{F})(U) := \mathrm{Tors}\,\mathscr{F}(U) := \{s \in \mathscr{F}(U) \mid \exists\, 0 \neq f \in \mathscr{R}_X(U) \text{ such that } fs = 0\}.$$

Note that $T_{\mathrm{pre}}(\mathscr{F})(U)$ is a sub $\mathscr{R}_X(U)$-module of $\mathscr{F}(U)$ (the *torsion submodule* of the $\mathscr{R}_X(U)$-module $\mathscr{F}(U)$), and that $\rho^{\mathscr{F}}_{U,V}(T_{\mathrm{pre}}(\mathscr{F})(U)) \subset T_{\mathrm{pre}}(\mathscr{F})(V)$ for all open $V \subset U$ in X. Thus $T_{\mathrm{pre}}(\mathscr{F})(U)$ is a presheaf of $\mathscr{R}_X$-modules. Let

$$T(\mathscr{F}) := T_{\mathrm{pre}}(\mathscr{F})^+$$

be the sheafification of $T_{\mathrm{pre}}(\mathscr{F})$. (This is the *torsion subsheaf* of the sheaf of $\mathscr{R}_X$-modules $\mathscr{F}$.)

(1) Give examples in which $T_{\mathrm{pre}}(\mathscr{F})$ is not a sheaf.

(2) Let X be an irreducible affine variety, and let M be a $\mathbb{K}[X]$-module. Define an isomorphism

$$\widetilde{(\mathrm{Tors}\,M)} \cong T(\widetilde{M}).$$

(3) Let X be an irreducible affine variety. Let M be a $\mathbb{K}[X]$-module and let φ^M be as in Exercise 1. Prove that $\ker(\varphi^M) = T(\widetilde{M})$.

Let X be an affine variety, and let M be a $\mathbb{K}[X]$-module. Recall (see (Quasi)Coherent-sheaves, p.8) that the fiber at p of $\widetilde{M}$ is the $\mathbb{K}$ vector space

$$\widetilde{M}(p) := \widetilde{M}_p/\mathfrak{m}_p\widetilde{M} = M/\mathfrak{m}_p M.$$

For $\underline{a} = (a_1, \ldots, a_d) \in M^d$ let

$$\phi^{\underline{a}} \colon \mathscr{O}_X^{\oplus d} \longrightarrow \widetilde{M}$$

be the morphism defined by setting (for $V \subset X$ open)

$$\phi^{\underline{a}}_V(\lambda_1, \ldots, \lambda_d) := \lambda_1(a_{1|V}) + \ldots + \lambda_d(a_{d|V}).$$

Exercise 3. Let X and M be as above, with M a finitely generated $\mathbb{K}[X]$-module. Let $p \in X$.

(a) Suppose that the fiber $\widetilde{M}(p)$ is generated by $\overline{a}_1, \ldots, \overline{a}_d \in M/\mathfrak{m}_p M$. Prove that there exists an open $U \subset X$ containing p such that $\phi^{\underline{a}}_{|U} \colon \mathscr{O}_U^{\oplus d} \to \widetilde{M}_{|U}$ is surjective.

(b) Suppose that $\widetilde{M}(p)$ is a free $\mathscr{O}_{X,p}$-module. Prove that there exists an open $U \subset X$ containing p such that $\widetilde{M}_{|U}$ is free.

Let X be an algebraic variety. A quasi-coherent sheaf $\mathscr{F}$ on X is *torsion free* if $T(\mathscr{F}) = 0$.

Exercise 4. Let $\mathscr{F}$ be a torsion free coherent sheaf on X.

(a) Prove that if X is a smooth curve then $\mathscr{F}$ is locally free.
(b) Give an example of a singular curve X and an $\mathscr{F}$ as above with $\mathscr{F}$ not locally free.
(c) Give an example of a smooth surface X and an $\mathscr{F}$ as above with $\mathscr{F}$ not locally free.