Advanced Topics in Geometry, Autumn 2024 - Kieran O'Grady

PROBLEM SET 3

In these exercises $\mathbb{K}$ is an algebraically closed field, and algebraic varieties are defined over $\mathbb{K}$.

Exercise 1. An algebraic variety X is *locally factorial* if for every $p \in X$ the local ring $\mathscr{O}_{X,p}$ is a unique factorization domain[1] (note: in particular X is locally irreducible).
Prove that X is locally factorial if and only if the following holds: Given a closed subset $A \subset$X of pure codimension 1 there exists an open affine covering $\mathscr{U} = \{U_i\}_{i \in I}$ of X and $f_i \in \mathbb{K}[U_i]$ for each $i \in I$ such that $I(A \cap U_i) = (f_i)$.

As you know, a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ on an algebraic variety X is (quasi)coherent if there exists an affine open covering $\mathscr{U} = \{U_i\}_{i \in I}$ of X such that for all $i \in I$ we have $\mathscr{F}_{|U_i} \cong \widetilde{M}_i$ for a suitable $\mathbb{K}[U_i]$-module M_i (finitely generated if $\mathscr{F}$ is coherent). Similarly, a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is locally-free of rank r if there exists an open affine covering $\mathscr{U} = \{U_i\}_{i \in I}$ of X such that for all $i \in I$ we have $\mathscr{F}_{|U_i} \cong \widetilde{\mathbb{K}[U_i]^{\oplus r}}$. Now suppose that X is an affine variety. As you know, any (quasi)coherent sheaf on X is isomorphic to $\widetilde{M}$ for a suitable $\mathbb{K}[X]$-module M (finitely generated if $\mathscr{F}$ is coherent). On the other hand it is not true that any locally-free sheaf of rank r on X is isomorphic to $\widetilde{\mathbb{K}[U_i]^{\oplus r}}$ (i.e. to $\mathscr{O}_X^{\oplus r}$). The next two exercise illustrates this fact.

Exercise 2. Let X be an irreducible locally factorial affine variety.

(a) Suppose that $\mathrm{Pic}(X)$ is trivial. Show that if $A \subset X$ is irreducible of codimension 1 then $I(A)$ is a principal prime ideal (the coherent sheaf sheaf $\widetilde{I(D)}$ is locally-free of rank 1, hence...). From this deduce that if $f \in \mathbb{K}[X]^*$ is an irreducible non unit then $I(V(f)) = (f)$, and that $V(f)$ is irreducible. This implies that f is prime.

(b) Suppose that $\mathbb{K}[X]$ is a UFD. Show that if $A \subset X$ is irreducible then $I(A)$ is a principal prime ideal.

Exercise 3. Let $D \subset \mathbb{P}^n$ be an irreducible hypersurface of degree d, and let $X := \mathbb{P}^n \setminus D$. Note that X is a locally factorial irreducible affine variety.

(a) Prove that $\mathrm{Pic}(X) \cong \mathbb{Z}/(d)$, in particular if $d \geq 2$ then $\mathrm{Pic}(X)$ is non trivial.

(b) Suppose that $d \geq 2$. Exhibit a non unit $f \in \mathbb{K}[X]^*$ which is irreducible but not prime.

Exercise 4. Let $0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$ be an exact sequence of coherent sheaves on an algebraic variety X. Let $\mathscr{H}$ be a coherent sheaf on X, and consider the complex of coherent sheaves on X

$$0 \longrightarrow \mathscr{E} \otimes \mathscr{H} \longrightarrow \mathscr{F} \otimes \mathscr{H} \longrightarrow \mathscr{G} \otimes \mathscr{H} \longrightarrow 0. \tag{1}$$

(a) Prove that if $\mathscr{H}$ is locally-free then the complex in (1) is exact.

(b) Give examples for which the complex in (1) is not exact (of course $\mathscr{H}$ must be non locally-free).

An exact sequence

$$0 \longrightarrow \mathscr{E} \xrightarrow{\alpha} \mathscr{F} \xrightarrow{\beta} \mathscr{G} \longrightarrow 0 \tag{2}$$

of coherent sheaves on an algebraic variety X *splits* if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{E} & \xrightarrow{\alpha} & \mathscr{F} & \xrightarrow{\beta} & \mathscr{G} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{E} & \xrightarrow{\iota} & \mathscr{E} \oplus \mathscr{G} & \xrightarrow{\pi} & \mathscr{G} & \longrightarrow & 0 \end{array}$$

where the vertical arrows are isomorphisms and ι, π are the standard inclusion and projection maps.

[1] Recall the following non trivial result: a smooth variety is locally factorial.

Exercise 5. Suppose that we have an exact sequence as in (2), with $\mathscr{G}$ locally-free. The exact sequence (see Exercise 4) $0 \longrightarrow \mathscr{G}^\vee \otimes \mathscr{E} \longrightarrow \mathscr{G}^\vee \otimes \mathscr{F} \longrightarrow \mathscr{G}^\vee \otimes \mathscr{G} \longrightarrow 0$ gives rise to a long exact sequence of cohomology groups

$$0 \longrightarrow H^0(X, \mathscr{G}^\vee \otimes \mathscr{E}) \longrightarrow H^0(X, \mathscr{G}^\vee \otimes \mathscr{F}) \longrightarrow H^0(X, \mathscr{G}^\vee \otimes \mathscr{G}) \xrightarrow{\partial} H^1(X, \mathscr{G}^\vee \otimes \mathscr{E}) \longrightarrow \cdots \tag{3}$$

The element $e := \partial(\mathrm{Id}_{\mathscr{G}}) \in H^1(X, \mathscr{G}^\vee \otimes \mathscr{E})$ is the *extension class* of the exact sequence in (2).

(a) Prove that the exact sequence in (2) splits if and only if its extension class vanishes.
(b) Give examples for which the extension class is non zero. Use this to prove that $H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-2)) \neq 0$.