Advanced Topics in Geometry, Autumn 2024 - Kieran O'Grady

Problem set 4

In the exercises below $\mathbb{K}$ is an algebraically closed field, and algebraic varieties are defined over $\mathbb{K}$.

Exercise 1. Let V be a finite dimensional $\mathbb{K}$ vector space. Let L be the tautological (sub) line bundle on $\mathbb{P}(V)$, and let Q be the tautological quotient vector bundle on $\mathbb{P}(V)$ fitting into the exact sequence (of vector bundles, or locally free sheaves, on $\mathbb{P}(V)$)

$$0 \longrightarrow L \longrightarrow \mathscr{O}_{\mathbb{P}(V)} \otimes_{\mathbb{K}} V \longrightarrow Q \longrightarrow 0.$$

Prove that there is an isomorphism of vector bundles

$$L^{\vee} \otimes Q \cong \Theta_{\mathbb{P}(V)}$$

proceeding as follows.

(a) Given a point $\xi_0 := ([v_0], \phi_0) \in L$, where $[v_0] \in \mathbb{P}(V)$ and $\phi_0 \colon L([v]) \to Q([v_0])$ is a linear map (i.e. ϕ_0 is a linear map $\langle v_0 \rangle \to V/\langle v_0 \rangle$) let $\phi \colon \langle v_0 \rangle \to V$ be a linear lift of ϕ_0 (i.e. $[\phi(v)] \in V/\langle v_0 \rangle$ equals $\phi_0(v)$ for every $v \in \langle v_0 \rangle$) and let $D_{v_0,\phi}$ be the derivation in $[v_0]$ defined by

$$\begin{array}{ccc} \mathscr{O}_{\mathbb{P}(V),[v_0]} & \xrightarrow{D_{v_0,\phi}} & \mathbb{K} \\ f & \mapsto & \frac{d}{dt}\left(f(v_0 + t\phi(v_0))\right) \end{array}$$

Prove that $D_{v_0,\phi}$ does not change if we change representative of $[v_0]$ or if we change lift of ϕ_0, and hence we have a well-defined map (of sets)

$$\begin{array}{ccc} L^{\vee} \otimes Q & \longrightarrow & \Theta_{\mathbb{P}(V)} \\ \xi & \mapsto & D_{\xi} \end{array} \tag{1}$$

by associating to ξ the derivation D_{ξ}.

(b) Prove that the map in (1) is an isomorphism of algebraic varieties.

Exercise 2. (a) Deduce from Exercise 1 that we have exact sequences of (locally free) sheaves on $\mathbb{P}^n$:

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}^{\oplus(n+1)}(1) \longrightarrow \Theta_{\mathbb{P}^n} \longrightarrow 0, \tag{2}$$

and

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}^{\oplus(n+1)}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow 0. \tag{3}$$

(b) From the exact sequence in (2) deduce that $h^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}) = (n+1)^2 - 1$. Give a basis of $H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n})$ and identify $H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n})$ with the Lie algebra $\mathfrak{sl}(n+1)$.

Exercise 3. (a) Suppose that

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of locally free sheaves of rank respectively a, b, c on an algebraic variety X. Prove that

$$\bigwedge^b B \cong \bigwedge^a A \otimes \bigwedge^c C.$$

(b) Let F be a locally free sheaf of rank r on an algebraic variety X. Prove that if L is a line bundle on X then

$$\bigwedge^r (F \otimes L) \cong \bigwedge^r F \otimes L^{\otimes r}.$$

(c) From the exact sequence in (3) and Item (a) (or Item (b)) above deduce that

$$K_{\mathbb{P}^n} = \Omega^n_{\mathbb{P}^n} = \bigwedge^n \Omega_{\mathbb{P}^n} \cong \mathscr{O}_{\mathbb{P}^n}(-n-1),$$

and hence

$$h^p(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}) = \begin{cases} 1 & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Exercise 4. (a) Prove that

$$h^p(\mathbb{P}^n, \Omega_{\mathbb{P}^n}) = \begin{cases} 1 & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) From the exact sequence in (3) deduce that we have an exact sequence

$$0 \longrightarrow \Omega^2_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}^{\oplus\binom{n+1}{2}}(-2) \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow 0, \tag{5}$$

and from this get that

$$h^p(\mathbb{P}^n, \Omega^2_{\mathbb{P}^n}) = \begin{cases} 1 & \text{if } p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 5. Let Y be an irreducible smooth algebraic variety. Let $X \subset Y$ be an irreducible smooth subvariety of pure codimension 1, and let $i\colon X \hookrightarrow Y$ be the inclusion map. Then $\mathscr{O}_Y(-X) = \mathscr{I}_X \subset \mathscr{O}_Y$ is an invertible sheaf on Y (if $U \subset Y$ is an open affine subset such that the ideal $I(X \cap U) \subset \mathbb{K}[U]$ is generated by f, then the restriction of $\mathscr{I}_X$ to U is the sheafification of the $\mathbb{K}[U]$-module (f)). We let $\mathscr{O}_X(-X)$ be the restriction of $\mathscr{O}_Y(-X)$ to X, i.e. $i^*\mathscr{O}_Y(-X)$. Then we have an exact sequence[1] of locally free sheaves on X:

$$0 \longrightarrow \mathscr{O}_X(-X) \xrightarrow{\alpha} i^*\Omega_Y \xrightarrow{\beta} \Omega_X \longrightarrow 0, \tag{6}$$

where $\alpha([g]) := (dg)_{|X}$ (check that this definition makes sense[2]), and β is given by the differential.

(a) Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree d. Use the exact sequence in (6) to prove that

$$K_X \cong \mathscr{O}_{\mathbb{P}^n}(d-n-2), \tag{7}$$

and show that

$$h^p(X, \Omega^n_X) = \begin{cases} \binom{d-1}{n+1} & \text{if } p = 0, \\ 1 & \text{if } p = n, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

(b) Let $X \subset \mathbb{P}^3$ be a smooth surface of degree d. Compute $h^p(X, \Omega_X)$.

Exercise 6. If X is a smooth projective variety over the complex numbers, and X^{an} is X viewed as complex manifold (in particular topological manifold), then

$$b_m(X^{\text{an}}) = \sum_{p+q=m} h^q(X, \Omega^p_X).$$

Test the above equality in some of the cases for which you have computed the numbers $h^q(X, \Omega^p_X)$.

[1]For exactness look at the notes of A. Gathmann or R. Vakil.

[2]Attention: $(dg)_{|X}$ and $d(g_{|X})$ are different, if $g \in \mathscr{O}_Y(-X)$ the second one is 0, the first one in general is not 0.