**Advanced Topics in Geometry, Autumn 2024 - Kieran O'Grady**

Problem set 5

In the following "topology" means Euclidean topology unless we specify to the contrary. Let $U \subset \mathbb{C}$ be an open subset, and let $C^\infty(U)$ be the ring of $C^\infty$ functions from $U$ to $\mathbb{C}$. Recall that we have derivations $\frac{\partial}{\partial z}\colon C^\infty(U) \to C^\infty(U)$ and $\frac{\partial}{\partial \overline{z}}\colon C^\infty(U) \to C^\infty(U)$ defined by

$$\frac{\partial}{\partial z}(f) = \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial}{\partial \overline{z}}(f) = \frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

where $z = x + iy$. Note that $f$ is holomorphic if and only if $\frac{\partial}{\partial \overline{z}}(f) = 0$, and it is anti-holomorphic (i.e. its conjugate is holomorphic) if and only if $\frac{\partial}{\partial z}(f) = 0$. If $U \subset \mathbb{C}^n$ is an open subset we let $C^\infty(U)$ be the ring of $C^\infty$ functions from $U$ to $\mathbb{C}$. We have derivations $\frac{\partial}{\partial z_\alpha}\colon C^\infty(U) \to C^\infty(U)$ and $\frac{\partial}{\partial \overline{z}_\alpha}\colon C^\infty(U) \to C^\infty(U)$ for $\alpha \in \{1, \ldots, n\}$ by "fixing" the variables different from $z_\alpha$.

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**Exercise 1.** Compute $\frac{\partial f}{\partial z_\alpha}$ and $\frac{\partial f}{\partial \overline{z}_\alpha}$ for the $C^\infty$ function on $\mathbb{C}^n$ defined by

$$f(z_1, \ldots, z_n) := z_1^{d_1} \cdot \ldots \cdot z_n^{d_n} \cdot \overline{z}_1^{e_1} \cdot \ldots \cdot \overline{z}_n^{e_n}.$$

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Let $X$ be a complex manifold of dimension $n$, i.e. the local charts map to open subsets of $\mathbb{C}^n$, and the transition functions between local charts are holomorphic functions (between open subsets of $\mathbb{C}^n$). Let $\mathscr{A}^m(X)$ be the complex vector space of smooth (i.e. $C^\infty$) $m$-forms on $X$ with complex coefficients. Let $\varphi \in \mathscr{A}^m(X)$. Then there is a unique decomposition

$$\varphi = \varphi^{m,0} + \varphi^{m-1,1} + \ldots + \varphi^{0,m},$$

such that in local holomorphic coordinates $(z_1, \ldots, z_n)$ on an open subset $U \subset \mathbb{C}^n$ we have

$$\varphi^{p,q}_{|U} = \sum_{\substack{|A|=p\\|B|=q}} f_{A,\overline{B}} dz_A \wedge d\overline{z}_B \tag{1}$$

where $A, B \subset \{1, \ldots, n\}$, $f_{A,\overline{B}} \in C^\infty(U)$, $dz_A = dz_{\alpha_1} \wedge \ldots \wedge dz_{\alpha_p}$ with $A = \{\alpha_1, \ldots, \alpha_p\}$ (and $\alpha_1 < \ldots < \alpha_p$) and $d\overline{z}_B = d\overline{z}_{\beta_1} \wedge \ldots \wedge d\overline{z}_{\beta_q}$ with $B = \{\beta_1, \ldots, \beta_q\}$ (and $\beta_1 < \ldots < \beta_q$). It follows that we have a direct sum decomposition

$$\mathscr{A}^m(X) = \bigoplus_{p+q=m} \mathscr{A}^{p,q}(X)$$

where $\mathscr{A}^{p,q}(X)$ is the subspace of smooth $m$-forms which can be written locally as in (1). Next recall that the differential $d\colon \mathscr{A}^m(X) \to \mathscr{A}^{m+1}(X)$ decomposes as $d = \partial + \overline{\partial}$ where, in local holomorphic coordinates $(z_1, \ldots, z_n)$ we have

$$\partial\left(\sum_{A,B} f_{A,\overline{B}} dz_A \wedge d\overline{z}_B\right) := \sum_{\substack{A,B\\ \gamma\in\{1,\ldots,n\}}} \frac{\partial f_{A,\overline{B}}}{\partial z_\gamma} dz_\gamma \wedge dz_A \wedge d\overline{z}_B$$

and

$$\overline{\partial}\left(\sum_{A,B} f_{A,\overline{B}} dz_A \wedge d\overline{z}_B\right) := \sum_{\substack{A,B\\ \gamma\in\{1,\ldots,n\}}} \frac{\partial f_{A,\overline{B}}}{\partial \overline{z}_\gamma} d\overline{z}_\gamma \wedge dz_A \wedge d\overline{z}_B$$

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**Exercise 2.** Prove that

$$\partial \circ \partial = 0, \qquad \partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0, \qquad \overline{\partial} \circ \overline{\partial} = 0. \tag{2}$$

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Let $X$ be a complex manifold and let $\pi\colon L \to X$ be a holomorphic line bundle on $X$, i.e. such that the transition functions are holomorphic. A *Hermitian metric $h$ on $L$* consists of a Hermitian metric $h(p)$ on the fiber $L(p) = \pi^{-1}(p)$ for each each $p \in X$ which varies smoothly, i.e. such that for each trivialization $\phi\colon U \times \mathbb{C} \xrightarrow{\sim} \pi^{-1}(U)$ over an open subset $U \subset X$ we have

$$||\phi(z,t)||_h := h(\phi(z,t), \phi(z,t)) = \rho(z)t\overline{t}$$

where $\rho \in C^\infty(U)$. Hermitian metrics on $L$ exist by a partition of unity argument.

**Exercise 3.** Let $X$ be a complex manifold and let $\mathrm{Pic}(X)$ be the group of isomorphism classes of holomorphic line bundles on $X$, with multiplication defined by tensor product of line-bundles. The goal of this exercise is to define a homomorphism of groups

$$\mathrm{Pic}(X) \xrightarrow{c_1^{\mathrm{DR}}} H^2_{\mathrm{DR}}(X;\mathbb{C}) \tag{3}$$

where $H^2_{\mathrm{DR}}(X;\mathbb{C})$ is the complex De Rham cohomology group of degree 2. Of course the trivial homomorphism is one such homomorphism, but uninteresting. Later we will see that $c_1$ is far from trivial.

(a) Let $h$ be a Hermitian metric on $L$. Let $U \subset X$ be open, and let $s_1, s_2 \colon U \to \pi^{-1}(U)$ be nowhere vanishing holomorphic sections of $L$ over $U$ (nowhere vanishing means that $s_i(p) \neq 0$ for all $p \in U$). Prove that

$$\partial \circ \overline{\partial} \log ||s_1||_h^2 = \partial \circ \overline{\partial} \log ||s_2||_h^2$$

and conclude that there is a well-defined smooth 2-form $\omega_h \in \mathscr{A}^{1,1}(X)$ such that on each open subset $U \subset X$ with a nowhere vanishing holomorphic section $s \colon U \to \pi^{-1}(U)$ of $L$ over $U$ one has

$$\omega_{h|U} = \frac{1}{2\pi i} \partial \circ \overline{\partial} \log ||s||_h^2.$$

(b) Prove that $\omega_h$ is closed, and hence it defines a class $[\omega_h] \in H^2_{\mathrm{DR}}(X;\mathbb{C})$ (actually it is a real class, see (6)). Let $h_1, h_2$ be Hermitian metrics on $L$. Prove that $[\omega_{h_1}] = [\omega_{h_2}]$. We let

$$c_1^{\mathrm{DR}}(L) := [\omega_h] \in H^2_{\mathrm{DR}}(X;\mathbb{C}) \tag{4}$$

where $h$ is any Hermitian metric on $L$.

(c) Prove that $c_1^{\mathrm{DR}}$ is a homomorphism of groups.

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Let $X$ be a complex manifold. Let $\pi \colon L \to X$ be a holomorphic line bundle on $X$, with a Hermitian metric $h$. Then $h$ induces a Hermitian metric $h^{\otimes m}$ on the tensors power $L^{\otimes m}$ for all $m \in \mathbb{Z}$ as follows. If $m \geq 0$ we declare that

$$||s^{\otimes m}||_{h^{\otimes m}} := ||s||^m$$

for any local section $s$ of $L$ (and hence $s^{\otimes m}$ is a local section of $L^{\otimes m}$). The induced Hermitian metric on $L^{-1} = L^{\vee}$ is defined by stipulating that for local sections $s$ of $L$ and $t$ of $L^{\vee}$ (on the same open subset $U \subset X$) we have

$$|\langle s, t\rangle| = ||s||_h \cdot ||t||_{h^{-1}},$$

where $\langle s, t\rangle$ is the (complex) function on $U$ whose value at $p$ is the evaluation of $t(p)$ on $s(p)$. If $m > 0$ the Hermitian metric on $L^{\otimes(-m)} = (L^{\vee})^{\otimes m}$ is defined to be $h_{L^{\vee}}^{\otimes m}$.

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**Exercise 4.** Let $L \subset \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1}$ be the tautological line bundle on $\mathbb{P}^n(\mathbb{C})$. Here we regard $\mathbb{P}^n(\mathbb{C})$ as a complex manifold, i.e. we should write $\mathbb{P}^n(\mathbb{C})^{\mathrm{an}}$, but we avoid this cumbersome notation. If $p \in \mathbb{P}^n(\mathbb{C})$ then the fiber $L(p)$ is a subspace of $\mathbb{C}^{n+1}$, and hence it inherits a Hermitian metric from the standard Hermitian metric on $\mathbb{C}^{n+1}$. This is the *Fubini-Study* Hermitian metric on $L$, we denote it by FS. It induces Hermitian metrics on $L^{\otimes m}$ for any $m \in \mathbb{Z}$ - we call such metric the *Fubini-Study* Hermitian metric on $L^{\otimes m}$.

(a) Prove that

$$\int_{\mathbb{P}^1(\mathbb{C})} \omega_{\mathrm{FS}} = -1.$$

(b) From (a) deduce that there is a unique class $c_1(L) \in H^2(\mathbb{P}^n(\mathbb{C});\mathbb{Z})$ such that $c_1^{\mathrm{DR}}(L)$ is in the image of $c_1(L)$ via the homomorphism $H^2(\mathbb{P}^n(\mathbb{C});\mathbb{Z}) \to H^2(\mathbb{P}^n(\mathbb{C});\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^2_{\mathrm{DR}}(\mathbb{P}^n(\mathbb{C});\mathbb{C})$.

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**Exercise 5.** Let $X$ be a complex manifold. Let $\pi \colon L \to X$ be a holomorphic line bundle on $X$, with a Hermitian metric $h$.

(a) Let $U \subset X$ be open, with local holomorphic coordinates $(z_1, \ldots, z_n)$. Show that

$$\omega_{h|U} = \frac{i}{2\pi} \sum_{\substack{1 \leq \alpha \leq n \\ 1 \leq \beta \leq n}} f_{\alpha,\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta \tag{5}$$

where the matrix $(f_{\alpha,\overline{\beta}})_{\substack{1 \leq \alpha \leq n \\ 1 \leq \beta \leq n}}$ is Hermitian, i.e. equal to the conjugate of its transpose. Conclude that the class $c_1^{\mathrm{DR}}(L)$ is a real cohomology class, i.e.

$$c_1^{\mathrm{DR}}(L) \in H^2_{\mathrm{DR}}(X;\mathbb{R}). \tag{6}$$

(b) Show that if one changes local holomorphic coordinates $(z_1, \dots, z_n)$, then the Hermitian matrix $f := (f_{\alpha,\overline{\beta}})_{\substack{1\leq\alpha\leq n\\ 1\leq\beta\leq n}}$ changes as the matrix of a Hermitian form under a pointwise linear change of coordinates. Conclude that we get a well defined Hermitian product $H_h(p)$ on the holomorphic tangent space $T'_pX$ at each point $p \in X$.

(c) The metric $h$ is *positive* if the associated Hermitian product $H_h(p)$ is positive definite at each point $p \in X$. The line bundle $L$ is *positive* if there exists a positive Hermitian metric on $L$. Let $L$ be the dual of the tautological line bundle on $\mathbb{P}^n(\mathbb{C})$. Show that $L$ is positive by checking that the Fubini-Study Hermitian metric on $L$ is positive.

(d) Suppose that $X$ is compact of (pure) dimension $n$. Since $X$ is a complex manifold it is orientable, and it has a preferred orientaion, the *complex orientation* $[X]$. Show that if $L$ is a positive line bundle on $X$ then

$$\int_{[X]} \underbrace{c_1^{\mathrm{DR}}(L) \wedge \ldots \wedge c_1^{\mathrm{DR}}(L)}_{n} > 0. \tag{7}$$

(e) Let $Y \subset \mathbb{P}^N(\mathbb{C})$ be a smooth Zariski closed subset of pure dimension $n$. Hence $Y$ is a complex projective variety, and the associated analytic space $Y^{\mathrm{an}}$ is a complex manifold. Let $L$ be the dual of the tautological line bundle on $\mathbb{P}^n(\mathbb{C})$. Prove that $c_1^{\mathrm{DR}}(L_{|Y^{\mathrm{an}}}) \neq 0$ by showing that $L_{|Y^{\mathrm{an}}}$ is positive.

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**Exercise 6.** Choose $r \in (1, +\infty)$, and let $\mathbb{Z}$ act on $X := \mathbb{C}^2 \setminus \{0\}$ via $az := r^a \cdot z$.

(1) Let

$$Y := X/\mathbb{Z}$$

(topological quotient). Show that the quotient map $p\colon X \to Y$ is a topological covering, and hence $Y$ is a topological manifold, and it inherits a structure of (2 dimensional) complex manifold from that of $X$. This is an example of a *Hopf surface*.

(2) Show that $Y$ is compact and that $H^2(Y;\mathbb{R}) = 0$. Conclude that $Y$ is a complex manifold which is not isomorphic to the analytification $W^{\mathrm{an}}$ of any complex algebraic variety $W$.