Ad-nilpotent ideals of Borel subalgebras: combinatorics and representation theory

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Preliminary version

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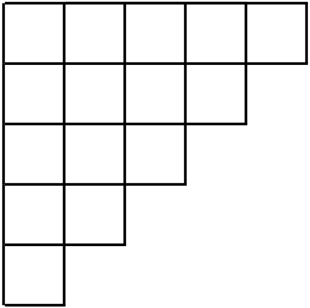
General Plan

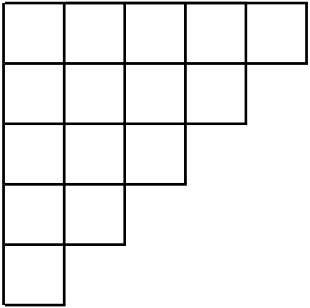
1. Lecture 1
   - Two combinatorial Problems
   - Outline of the course: statement of results
2. Lecture 2
   - Background
3. Lecture 3
   - Affine root systems and affine Weyl groups
   - *ad*-nilpotent ideals of Borel subalgebras
4. Lecture 4
   - Panyushev's theory of rootlets
   - Abelian ideals and $\bigwedge \mathfrak{g}$
5. Lecture 5
   - Kostant-Macdonald formulas
   - Applications to number theory
   - Application to affine Lie algebras
   - Developments and open problems

Introduction: two combinatorial problems

Consider the staircase shape $\mathcal{T}_n = (n, n-1, \dots, 1)$



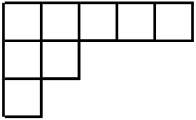
Introduction: two combinatorial problems



Label the boxes as matrix entries with row (resp. column) indices increasing from left to right (resp. from top to bottom).

Two combinatorial problems

A subdiagram D of $\mathcal{T}_n$, is a shape like



let h_D denote the hook length of box $(1,1)$, i.e. the Upper-Left corner box. In the example at hand, $h_D = 7$.

Two combinatorial problems

Problems

We want to count

1. the number N_{adnilp} of subdiagrams of $\mathcal{T}_n$;
2. the number N_{ab} of subdiagrams D of $\mathcal{T}_n$ such that $h_D \leq n$.

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Answers

If $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the *Catalan number*, then

$$N_{adnilp} = \mathcal{C}_{n+1}.$$

Moreover,

$$N_{ab} = 2^n.$$

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3. Dyck paths of semilength $n+1$;
4. (weak) Dyck words of length $2(n+1)$

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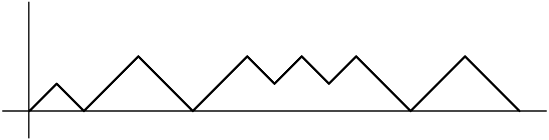

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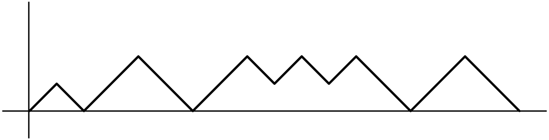

abaabbaababbaabb

Counting Dyck words

Notation

$\mathcal{W}_{a,b}$: set of words w in the alphabet $\{A, B\}$ with $N_A(w) = a$, $N_B(w) = b$.
$\mathcal{W}_{a,b}(A)$: set of words in $\mathcal{W}_{a,b}$ starting with A
$[w]_k$: k-th prefix of w
$\Delta_{A,B}(w) = N_A(w) - N_B(w)$.

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Example

$$\mathcal{W}_{2,2} = \{aabb, abab, abba, bbaa, baba, baab\}$$
$$\mathcal{W}_{2,2}(A) = \{aabb, abab, abba\}$$
$$w = abbba, [w]_1 = a, [w]_2 = ab, [w]_3 = abb, \ldots$$
$$\Delta_{A,B}(w) = -1, \Delta_{A,B}([w]_4) = -2$$

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Definition

A weak (strong) Dyck word is a word $w \in \mathcal{W}_{a,b}$ such that $\Delta_{A,B}([w]_k) \geq 0$ (resp. $\Delta_{A,B}([w]_k) > 0$) for all $1 \leq k \leq a + b$.

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Note that such a word belongs to $\mathcal{W}_{a,b}(A)$, and that

$$|\mathcal{W}_{a,b}(A)| = \binom{a+b-1}{b}.$$

Counting Dyck words

Now remark that

$$T^c = \{w \in \mathcal{W}_{a,b}(A) \mid \Delta_{A,B}([w]_k) = 0 \text{ for some } k\}.$$

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Let $w \in T^c$ and $k = 2m$ be the rightmost position such that $\Delta_{A,B}([w]_k) = 0$. Then w has the form

$$w = [w]_k\alpha, \qquad N_A(\alpha) = a - m,\ N_B(\alpha) = b - m.$$

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We have

$$N_A(w') = m + (b - m) = b, \quad N_B(w') = m + (a - m) = a,$$

$$\text{so that } w' \in \mathcal{W}_{b,a}, \text{ indeed } w' \in \mathcal{W}_{b,a}(A).$$

Counting Dyck words

Claim

The map $\Phi : T^c \to \mathcal{W}_{b,a}(A), \Phi(w) = w'$ is a bijection.

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Example

$\mathcal{W}_{2,3}(A)$

w	w'
$abba\|a$	$abbab$
$abab\|a$	$ababb$
$ab\|aab$	$abbba$
$aabb\|a$	$aabbb$
$aabab$	
$aaabb$	

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The map $\Phi : T^c \to \mathcal{W}_{b,a}(A), \Phi(w) = w'$ is a bijection.

Corollary

$$|T| = |\mathcal{W}_{a,b}(A)| - |T^c| = \binom{a+b-1}{b} - \binom{a+b-1}{a}$$
$$= \frac{a-b}{a+b}\binom{a+b}{b}.$$

Counting Dyck words

Now remark that

- $w \in \mathcal{W}_{a,b}$ is weak Dyck iff $Aw \in \mathcal{W}_{a+1,b}$ is strong Dyck.

Hence

Proposition

The number of weak Dyck words in $\mathcal{W}_{a,b}$ is

$$\frac{a-b+1}{a+1}\binom{a+b}{a}.$$

In particular, for $a = b = n+1$,

$$N_{adnilp} = \frac{1}{n+2}\binom{2n+2}{n+1} = \mathcal{C}_{n+1}.$$

Elementary combinatorial calculation of N_{ab}

We claim that the number N_k of subdiagrams of $\mathcal{T}_n$ with hook lenght k is $2^{k-1}, k \geq 1$.

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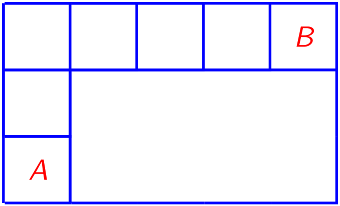

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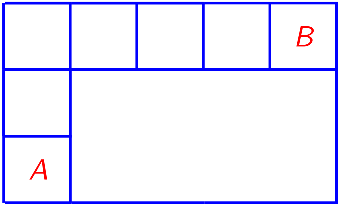

Fix a hook, and let A, B denote the end boxes in the leg and arm of cell $(1,1)$. We have to count all lattice paths from A to B inside the small rectangle. If the arm and leg length of the hook are $h+1$, $k-h$, they are $\binom{k-1}{h}$. So

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Now simply remark that

$$N_{ab} = 1 + \sum_{k=1}^{n} N_k = 1 + \sum_{k=1}^{n} 2^{k-1} = 2^n$$

Relationships between combinatorial and algebraic objects

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- $\mathfrak{u}$-cohomology;

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The role of *ad*-nilpotent and abelian ideal of a Borel subalgebras, which we have seem embodied in one special case, will naturally emerge. In particular, the two combinatorial results we have proved will be part of a more general and intrinsic theory.

Macdonald-Kostant Theorem

Notation

$\mathfrak{g}$ complex simple finite dimensional Lie algebra; $\mathfrak{b}$ Borel subalgebra, with Cartan component $\mathfrak{h}$ and nilradical $\mathfrak{n}$

Δ root system of $(\mathfrak{g}, \mathfrak{h})$, Δ^+ set of positive roots with basis Π and fundamental chamber C

$\rho = 1/2 \sum_{\alpha \in \Delta^+} \alpha$ Weyl vector, P^+ dominant integral weights

W Weyl group of $\mathfrak{g}$, $(\cdot, \cdot)$ Killing form of $\mathfrak{g}$

V_λ irreducible $\mathfrak{g}$-module of highest weight λ, χ_λ character of V_λ

$Cas(\lambda) = (\lambda, \lambda + 2\rho)$, eigenvalue of the Casimir operator $\Omega_{\mathfrak{g}}$ on V_λ

Standard Lie algebra notation

Euler product

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$$\frac{1}{\phi(x)} = \sum_{n \geq 0} p(n)x^n$$

where $p(n)$ is the classical partition function.

Standard Lie algebra notation

Theorem

Let $\phi(x) = \prod_{n=1}^{\infty}(1 - x^n) \in \mathbb{C}[[x]]$ *denote the Euler product. Then*

$$\phi(x)^{\dim \mathfrak{g}} = \sum_{\lambda \in P^+} \chi_\lambda(e^{2\pi\sqrt{-1}\,2\rho}) \dim V_\lambda \; x^{Cas(\lambda)}.$$

Moreover $\chi_\lambda(e^{2\pi\sqrt{-1}\,2\rho}) \in \{-1, 0, 1\}$ *for* $\lambda \in P^+$.

Problem 1

Single out the subset of P^+ consisting of weights giving nonzero contribution to the sum. Find the coefficients b_k in

$$\phi(x)^{\dim \mathfrak{g}} = \sum_{k=0}^{\infty} b_k x^k.$$

First answer to problem 1

$Q^\vee$ coroot lattice, $\widehat{W} \cong W \ltimes Q^\vee \leq Aff(\mathfrak{h}^*_\mathbb{R})$

A_1 fundamental alcove, $A_w = wA_1$, $w \in \widehat{W}$.

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$$\begin{aligned}
&\widehat{W}^+ = \{w \in \widehat{W} \mid A_w \subset C\},\\
&\lambda^w = w(2\rho)/2 - \rho,\\
&D^+ = \{\lambda^w\}_{w \in \widehat{W}^+}.
\end{aligned}$$

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Theorem (Kostant, 2004)

$$\chi_\lambda(e^{2\pi\sqrt{-1}\,2\rho}) = \begin{cases} (-1)^{\ell(w)} & \lambda = \lambda^w,\ w \in \widehat{W}^+, \\ 0 & \textit{otherwise.} \end{cases}$$

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$$b_k = \sum_{w \in \widehat{W}^+, Cas(\lambda^w) = k} (-1)^{\ell(w)} \dim V_{\lambda^w}.$$

Nilradical homology for affine algebras

Notation

$\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$ affine Kac-Moody algebra attached to $\mathfrak{g}$

$\mathfrak{u} = t\mathfrak{g}[t],\ \mathfrak{u}^- = t^{-1}\mathfrak{g}[t^{-1}]$ opposite niradicals in $\widehat{\mathfrak{g}}$.

Bigrading on $\bigwedge \mathfrak{u}^-$

$$\bigwedge \mathfrak{u}^- = \bigoplus_{(n,k)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}} \left(\bigwedge^n \mathfrak{u}^-\right)_k .$$

where the subscript k denotes the subspace of t-degree $-k$. This grading descends to homology.

Nilradical homology for affine algebras

Theorem

As a $\mathfrak{g}$-module

$$H^*(\mathfrak{u}) \cong H_*(\mathfrak{u}^-) = \bigoplus_{w \in \widehat{W}^+} V_{\lambda^w}.$$

Moreover

$$H_n(\mathfrak{u}^-)_k = \bigoplus_{w \in \widehat{W}^+, \ell(w)=n, Cas(\lambda^w)=k} V_{\lambda^w}.$$

Interlude: *ad*-nilpotent and abelian ideals of Borel subalgebras

Let $\mathfrak{i}$ be an ideal of $\mathfrak{b}$ contained in $\mathfrak{n}$. It consists of *ad*-nilpotent elements, so we'll call it an *ad-nilpotent ideal* and we'll denote by $\mathcal{I}$ the set of *ad*-nilpotent ideals.

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$$\mathfrak{i} = \bigoplus_{\alpha \in \Phi_{\mathfrak{i}}} \mathfrak{g}_{\alpha}$$

where $\Phi_{\mathfrak{i}} \subset \Delta^{+}$ is dual order ideal of the *root poset*.

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where $\Phi_{\mathfrak{i}} \subset \Delta^+$ is dual order ideal of the *root poset*.
$\mathcal{I}$ contains the remarkable subset of *abelian ideals* of $\mathfrak{b}$:

$$\mathcal{I}_{ab} = \{\mathfrak{i} \in \mathcal{I} \mid [x, y] = 0 \, \forall \, x, y \in \mathfrak{i}\}.$$

ad-nilpotent and abelian ideals of Borel subalgebras

Theorem

1. $|\mathcal{I}_{ab}| = 2^{\mathrm{rk}\ (\mathfrak{g})}$.
2. *If h denotes the Coxeter number and m_i are the exponents of $\mathfrak{g}$ then*

$$|\mathcal{I}| = \frac{\prod_{i=1}^{\mathrm{rk}\ \mathfrak{g}}(h + m_i + 1)}{|W|}.$$

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Example

If $\mathfrak{g} = sl(n+1)$, then

$$\mathrm{rk}\ \mathfrak{g} = n, \quad |W| = |S_{n+1}| = (n+1)!, \quad h = n+1, \quad m_i = i,$$

so that

$$N_{ab} = 2^n, \qquad N_{adnilp} = \mathcal{C}_{n+1}.$$

Relationships between $\mathcal{I}$, $\mathcal{I}_{ab}$, $\widehat{W}$

The above theorem has a more significant formulation.

Relationships between $\mathcal{I}$, $\mathcal{I}_{ab}$, $\widehat{W}$

Let θ be the highest root of Δ and set, for $\mathfrak{i} \in \mathcal{I}$

$$\langle \mathfrak{i} \rangle = \sum_{\alpha \in \Phi_{\mathfrak{i}}} \alpha, \quad \widehat{W}_2^+ = \{ w \in \widehat{W} \mid A_w \subset 2A_1 \}.$$

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Theorem

There are natural bijections

$$\eta : \mathcal{I} \to Q^\vee/(h+1)Q^\vee, \quad \zeta : \mathcal{I}_{ab} \to \widehat{W}_2^+.$$

Moreover, for $\mathfrak{i} \in \mathcal{I}_{ab}$

$$\langle \mathfrak{i} \rangle = \lambda^{\zeta(\mathfrak{i})}.$$

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Next we'll see some representation theoretic applications.

The structure of $\bigwedge \mathfrak{g}$ as a $\mathfrak{g}$-module

Notation

If $\mathfrak{a} = \oplus_{i=1}^{k} \mathbb{C} v_i$ is an abelian subalgebra of $\mathfrak{g}$, set

$$v_{\mathfrak{a}} = v_1 \wedge \ldots \wedge v_k \in \bigwedge^{k} \mathfrak{g}.$$

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- m_k is the maximum eigenvalue of $\Omega_{\mathfrak{g}}$ on $\bigwedge^k \mathfrak{g}$
- M_k eigenspace of $\Omega_{\mathfrak{g}}$ on $\bigwedge^k \mathfrak{g}$ of eigenvalue k
- $\mathfrak{C}_k = Span(v_{\mathfrak{a}} \mid \mathfrak{a} \text{ abelian}, \dim(\mathfrak{a}) = k)$

Kostant Theorems

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M_k eigenspace of $\Omega_{\mathfrak{g}}$ on $\bigwedge^k \mathfrak{g}$ of eigenvalue k
$\mathfrak{C}_k = Span(v_{\mathfrak{a}} \mid \mathfrak{a} \text{ abelian}, \dim(\mathfrak{a}) = k), \;\; \mathfrak{C} = \bigoplus_k \mathfrak{C}_k$

Theorem

1. *$m_k \leq k$, and $m_k = k$ iff $\mathfrak{C}_k \neq \emptyset$. In such a case $M_k = \mathfrak{C}_k$.*
2. *$\mathfrak{C}$ is a multiplicity-free $\mathfrak{g}$-module. Moreover*

$$\mathfrak{C}_k = \bigoplus_{\mathfrak{i} \in \mathcal{I}_{ab}, \dim \mathfrak{i} = k} V_{\langle \mathfrak{i} \rangle} = \bigoplus_{w \in \widehat{W}_2^+, \ell(w) = k} V_{\lambda \zeta(\mathfrak{i})}.$$

3. *If $\mathbf{d}$ is the Chevalley-Eilenberg differential affording Lie algebra cohomology, then*

$$\bigwedge \mathfrak{g} = \mathfrak{C} \oplus \langle \mathbf{d}\mathfrak{g} \rangle,$$

where $\langle \mathbf{d}\mathfrak{g} \rangle$ denotes the ideal generated by $\mathbf{d}\mathfrak{g}$ under wedge multiplication.

Final results

Theorem

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If moreover $k \leq h^\vee$*, the dual Coxeter number of* $\mathfrak{g}$*, they are also equal to*

- $(-1)^k b_k$

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A crash course in semisimple Lie algebras 1

Examples: the classical Lie algebras

- $sl(n,\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid tr(A) = 0\}$
- $so(2n+1, C) = \{\begin{pmatrix} A & B & v \\ C & -A^t & u \\ -v^t & -u^t & 0 \end{pmatrix} \mid B = -B^t, C = -C^t, u, v \in \mathbb{C}^n\}$
- $sp(2n, C) = \{\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid B = B^t, C = C^t\}$
- $so(2n, C) = \{\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid B = -B^t, C = -C^t\}$

A crash course in semisimple Lie algebras 2

General Definitions

A complex finite-dimensional Lie algebra $\mathfrak{g}$ is said to be

1. *simple* if it is not abelian and has no nontrivial ideals
2. *semisimple* if has no solvable ideals

A crash course in semisimple Lie algebras 2

General Definitions

A complex finite-dimensional Lie algebra $\mathfrak{g}$ is said to be

1. *simple* if it is not abelian and has no nontrivial ideals
2. *semisimple* if has no solvable ideals

Characterization of semisimple Lie algebras

$\mathfrak{g}$ is semisimple if and only if one of the following conditions is verified

1. $\mathfrak{g}$ is a direct sum of simple ideals.
2. The Killing form of $\mathfrak{g}$, defined as

$$(x, y) = tr(ad(x)\, ad(y)),$$

is nondegenerate.

Interlude

Why are semisimple Lie algebras important ?

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Let G a Lie group (for instance a closed subgroup of $GL(n)$). Then

$$\mathfrak{g} = \{c'(0) \mid c : \mathbb{R} \to G, C^\infty \text{ curve with } c(0) = I\}.$$

has a natural Lie algebra structure, which makes $\mathfrak{g}$ a first-order approximation of G.

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has a natural Lie algebra structure, which makes $\mathfrak{g}$ a first-order approximation of G.

Theorem

If G is compact then $\mathfrak{g}$ is reductive, i.e. is a direct sum as Lie algebras of a semisimple Lie algebra and an abelian one.

A crash course in semisimple Lie algebras 3

Structure Theory

A crash course in semisimple Lie algebras 3

Structure Theory

- Recall that and element x is said to be semisimple if $ad(x)$ is diagonalizable as an endomorphism of $\mathfrak{g}$.

A crash course in semisimple Lie algebras 3

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- By Engel's theorem, if $\mathfrak{g}$ is semisimple then there exist semisimple elements, so we can consider subalgebras formed by semisimple elements, and in turn subalgebras maximal w.r.t. this property, which are called **Cartan subalgebras**.

A crash course in semisimple Lie algebras 3

Structure Theory

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- By Engel's theorem, if $\mathfrak{g}$ is semisimple then there exist semisimple elements, so we can consider subalgebras formed by semisimple elements, and in turn subalgebras maximal w.r.t. this property, which are called **Cartan subalgebras**.
- A Cartan subalgebra $\mathfrak{h}$ turns out to be abelian, hence is a set of commuting diagonalizable operators on $\mathfrak{g}$. We can therefore consider the corresponding eigenspace decomposition.

A crash course in semisimple Lie algebras 4

Root Space decomposition

A crash course in semisimple Lie algebras 4

Root Space decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \,\forall h \in \mathfrak{h}\}.$$

A crash course in semisimple Lie algebras 4

Root Space decomposition

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Since $\mathfrak{h}$ is self-centralizing, we can rewrite the previous decomposition as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta \subset \mathfrak{h}^* \setminus \{0\}$ is a certain finite set, called the *root system* of $\mathfrak{g}$ w.r.t. $\mathfrak{h}$.

A crash course in semisimple Lie algebras 5

Basic Theorems

A crash course in semisimple Lie algebras 5

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1. Root systems, as we shall see, can be studied and classified combinatorially.

A crash course in semisimple Lie algebras 5

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2. One proves that the classification of roots systems induces the classification of semisimple Lie algebras, meaning that there is no dependence, up to isomorphism, on the choice of the Cartan subalgebra and other choices which should be done in classifying root systems.

A crash course in semisimple Lie algebras 5

Basic Theorems

1. Root systems, as we shall see, can be studied and classified combinatorially.
2. One proves that the classification of roots systems induces the classification of semisimple Lie algebras, meaning that there is no dependence, up to isomorphism, on the choice of the Cartan subalgebra and other choices which should be done in classifying root systems.
3. The final outcome is that there are the four infinite series we have seen in a previous slide (named A_n, B_n, C_n, D_n) plus five exceptional Lie algebras (named E_6, E_7, E_8, G_2, F_4).

Finite root systems

Reflections

Let E be an Euclidean space. If $0 \neq \alpha \in E$, the reflection in α is the orthogonal transformation defined by

$$s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha.$$

Finite root systems

Reflections

Let E be an Euclidean space. If $0 \neq \alpha \in E$, the reflection in α is the orthogonal transformation defined by

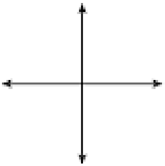
$$s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha.$$

Definition

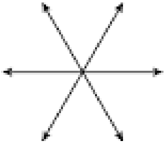
A finite set $\Delta \subset E$ of nonzero vectors is a root system in E if

1. $E = Span_{\mathbb{R}}\Delta$;
2. if $\alpha \in \Delta$ then $c\alpha \in \Delta \iff c = \pm 1$;
3. $s_\alpha(\Delta) \subset \Delta \,\forall\, \alpha \in \Delta$;
4. $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \,\forall\, \alpha, \beta \in \Delta$

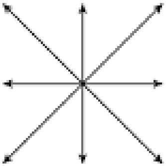
Finite root systems: examples in rank 2



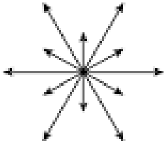
$A_1 \times A_1$



A_2



$B_2 \cong C_2$



G_2

Finite root systems: structure

Notice that to a root system we can associate

- the central hyperplane arrangement in E given by the equations $(\alpha, x) = 0$, $\alpha \in \Delta$;
- a *reflection group*, i.e. the subgroup W of $O(E)$ generated by s_α, $\alpha \in \Delta$.

Finite root systems: structure

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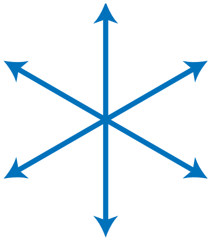
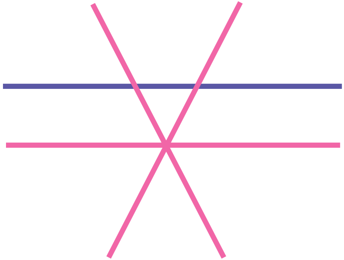
- the central hyperplane arrangement in E given by the equations $(\alpha, x) = 0$, $\alpha \in \Delta$;
- a *reflection group*, i.e. the subgroup W of $O(E)$ generated by s_α, $\alpha \in \Delta$.

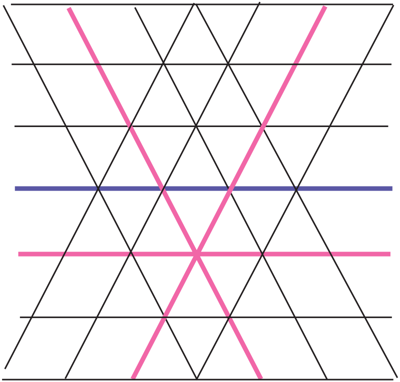
The complement $\mathfrak{E} = E \setminus \bigcup_{\alpha\in\Delta} \alpha^\perp$ is a union of convex cones acted on by W. We say that a vector $v \in \mathfrak{E}$ is *regular*.

Example: central arrangement





Example: affine arrangement



Finite root systems: structure

Fix a regular vector γ and set $\Delta^+ = \{\alpha \in \Delta \mid (\gamma, \alpha) > 0\}$, so that $\Delta = \Delta^+ \cup -\Delta^+$.

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Proposition

Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the set of roots Δ^+ which are not sum of two roots from Δ^+. Then

1. *Π is a linear basis of E;*
2. *$\Delta^+ = \{\sum_{i=1}^r a_i\alpha_i \in \Delta \mid a_i \geq 0\}$;*
3. *W acts simply transitively on chambers and bases.*

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Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be the set of roots Δ^+ which are not sum of two roots from Δ^+. Then

1. Π *is a linear basis of* E*;*
2. $\Delta^+ = \{\sum_{i=1}^r a_i\alpha_i \in \Delta \mid a_i \geq 0\}$*;*
3. W *acts simply transitively on chambers and bases.*

Corollary

One associates to Π a graph with some combinatorial data, the Dynkin diagram. The classification of the possible Dynkin diagrams, affords the classification of root systems.

Recollections from the theory of semisimple Lie algebras

Triangular decomposition

$\mathfrak{g}$ semisimple, $\mathfrak{h}$ Cartan, Δ roots, Δ^+ positive roots, Π simple roots.

Recollections from the theory of semisimple Lie algebras

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$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \quad \mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \pm\Delta^+} \mathfrak{g}_\alpha$$

Recollections from the theory of semisimple Lie algebras

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$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \quad \mathfrak{n}^\pm = \bigoplus_{\alpha \in \pm\Delta^+} \mathfrak{g}_\alpha$$

- $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \,\forall\, h \in \mathfrak{h}\}$ is one-dimensional.
- $\mathfrak{g}_\alpha \oplus g_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a copy of $sl(2, \mathbb{C})$.
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, a *Borel subalgebra*, is a maximal solvable subalgebra of $\mathfrak{g}$.
- $\mathfrak{n}^+$ is the nilradical of $\mathfrak{b}$.

Weyl group action

Notation

Let $V = \mathfrak{h}_{\mathbb{R}}$.

- A vector in $E = V^* \setminus \bigcup_{\alpha \in \Delta} \alpha^{\perp}$ is said to be *regular*
- The connected components of E are called *chambers*.

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Proposition

W acts simply transitively on chambers and bases.

Corollary

Fixing a chamber C and labelling it by $1 \in W$, the map $w \mapsto C_w := w\,C_1$ is a bijection between W and the set of chambers.

Weyl groups as Coxeter groups

Recall that W is generated by the reflections s_α, $\alpha \in \Delta$.

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Coxeter relations

$$(s_\alpha s_\beta)^{m_{\alpha,\beta}} = 1$$

where $\alpha, \beta \in \Pi$ $m_{\alpha,\beta} \in \mathbb{N} \cup \{\infty\}$, $m_{\alpha,\alpha} = 1$.

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Remark

One has a natural length function ℓ on W.

Example: $sl(n)$

Take $\mathfrak{g} = sl(n)$. Then one can choose

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Denote by e_{ij} the standard matrix unit. The basic computation is as follows; if $h = diag(h_1, \ldots, h_n)$, and ϵ_i is the i-th coodinate function of $\mathfrak{h}$, then

$$\begin{aligned}[h, e_{ij}] &= (h_i - h_j)e_{ij} \\ &= (\epsilon_i - \epsilon_j)(h)e_{ij}.\end{aligned}$$

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Hence

$$\Delta = \{\epsilon_i - \epsilon_j \mid i \neq j\}.$$

Example: $sl(n)$

One can choose

$$\Delta^+ = \{\epsilon_i - \epsilon_j \mid i < j\}.$$

so that

$$\Pi = \{\epsilon_i - \epsilon_{i+1} \mid 1 \leq i \leq n-1\}.$$

since $\epsilon_i - \epsilon_j = (\epsilon_i - \epsilon_{i+1}) + (\epsilon_{i+1} - \epsilon_{i+2}) + \dots ((\epsilon_{j-1} - \epsilon_j))$.

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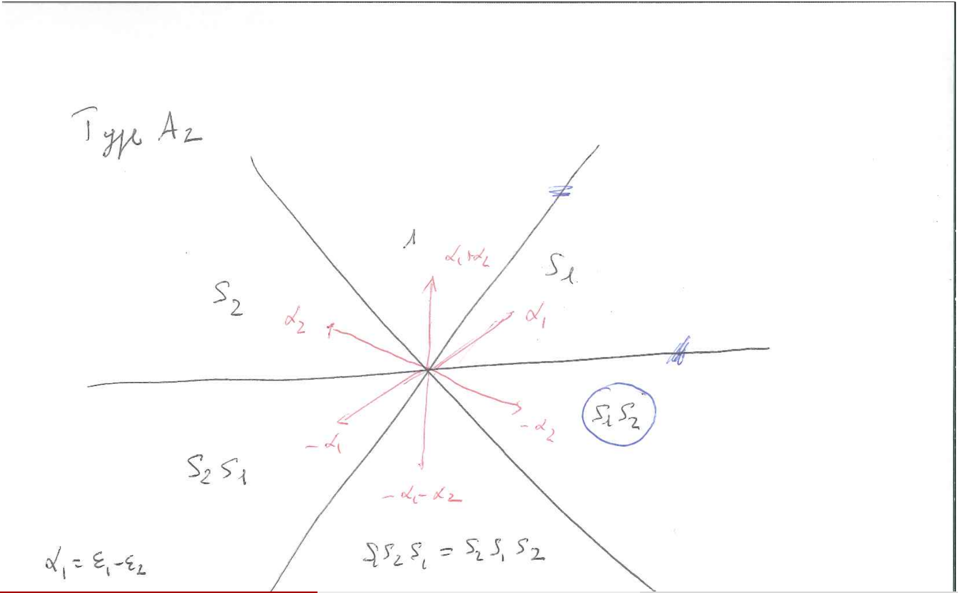
Note that the Coxeter relations read

$$(i, i+1)^2 = 1, \quad (i, i+1)(h, h+1) = (h, h+1)(i, i+1) \text{ if } i+1 < h,$$

$$(i+1, i+2)(i, i+1)(i+1, i+2) = (i, i+1)(i+1, i+2)(i, i+1)$$

Example


Type A_2
1
$\alpha_1+\alpha_2$
s_1
s_2
α_2
α_1
$s_1 s_2$
$-\alpha_2$
$-\alpha_1$
$s_2 s_1$
$-\alpha_1-\alpha_2$
$s_1 s_2 s_1 = s_2 s_1 s_2$
$\alpha_1 = \varepsilon_1 - \varepsilon_2$

Interpretation of the length function

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$w \in W$

$$N(w) = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) \in -\Delta^+\}.$$

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Proposition

1. *$\ell(w) = |N(w)| = \#$ hyperplanes separating $C_1,\ C_w$. More precisely, $\alpha \in N(w)$ iff $(\alpha, x) = 0$ separates C_1, C_w.*
2. *If $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression, then*

$$N(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\}.$$

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$$N(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\}.$$

Example

If $\sigma \in S_n$, the $N(\sigma)$ is the set of its *inversions*:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 3 & 1 & 4 & 5 \end{pmatrix} = s_1 s_2 s_5 s_4 s_3 s_2$$

$$N(\sigma) = \{\epsilon_1 - \epsilon_2, \epsilon_1 - \epsilon_3, \epsilon_5 - \epsilon_6, \epsilon_4 - \epsilon_6, \epsilon_1 - \epsilon_6, \epsilon_3 - \epsilon_6\}$$

An important technical Lemma

Definition

Say that $L \subset \Delta^+$ is *root-closed* if

$$\alpha, \beta \in L, \alpha + \beta \in \Delta \implies \alpha + \beta \in L$$

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Remark

The fact that $\Delta^+ \setminus L$ is root-closed means that

$$\alpha \in L, \alpha = \beta + \gamma, \beta, \gamma \in \Delta^+ \implies \beta \in L \text{ or } \gamma \in L.$$

An important technical Lemma

Example-Algorithm

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Given L biclosed, choose a simple root $\alpha \in L$ and iterate starting from $s_\alpha(L \setminus \{\alpha\})$.

An important technical Lemma

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Given L biclosed, choose a simple root $\alpha \in L$ and iterate starting from $s_\alpha(L \setminus \{\alpha\})$.For instance, in type A_5

$$L = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\}.$$

$$\begin{array}{lr}
s_1 & \{\alpha_2, \alpha_2 + \alpha_3, \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5\} \\
s_1 s_2 & \{\alpha_3, \alpha_5, \alpha_3 + \alpha_4 + \alpha_5\} \\
s_1 s_2 s_3 & \{\alpha_3, \alpha_3 + \alpha_4\} \\
s_1 s_2 s_3 s_5 & \{\alpha_4\} \\
s_1 s_2 s_3 s_5 s_4 = w & \emptyset
\end{array}$$

Another technical Lemma

Definition – Weyl vector

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$$\rho - w(\rho) = \sum_{\alpha \in N(w)} \alpha.$$

Proof.

By induction on $\ell(w)$. If $w = s_\alpha, \alpha \in \Pi$, it is known that $s_\alpha(\Delta^+ \setminus \{\alpha\}) \subset \Delta^+ \setminus \{\alpha\}$, hence

$$\rho - s_\alpha(\rho) = \rho - (\rho - \alpha) = \alpha = \sum_{\beta \in N(w)} \beta.$$

Another technical Lemma

Proposition

For $w \in W$

$$\rho - w(\rho) = \sum_{\alpha \in N(w)} \alpha.$$

Proof.

Now assume $w = s_\alpha w', \alpha \in \Pi,\ \ell(w') = \ell(w) - 1$

$$\begin{aligned} \rho - s_\alpha w'(\rho) &= s_\alpha(\rho) + \alpha - s_\alpha w'(\rho) = s_\alpha(\rho - w'(\rho)) + \alpha \\ &= s_\alpha(\sum_{\beta \in N(w')} \beta) + \alpha = \sum_{\beta \in N(w)} \beta \end{aligned}$$

□

Representations

Recall that a representation V of $\mathfrak{g}$ is a Lie algebra homomorphism $\mathfrak{g} \to gl(V)$.

Representations

Basic theorems

For semisimple Lie algebras:

1. finite dimensional representations are completely reducible.

Representations

Basic theorems

For semisimple Lie algebras:

1. finite dimensional representations are completely reducible.
2. finite dimensional representations are in bijection with the set of dominant weights

$$P^+ = \left\{ \lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_{\geq 0} \text{ for any simple root } \alpha \right\}.$$

Representations

Abstract construction

For $\lambda \in P^+$, the attached irreducible representation V_λ is the unique irreducible quotient of

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the $\mathfrak{b}$-module with basis v_λ and action $x.v_\lambda = 0$, $x \in \mathfrak{b}$, $h.v_\lambda = \lambda(h)v_\lambda$ and $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

Representations

Abstract construction

For $\lambda \in P^+$, the attached irreducible representation V_λ is the unique irreducible quotient of

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the $\mathfrak{b}$-module with basis v_λ and action $x.v_\lambda = 0$, $x \in \mathfrak{b}$, $h.v_\lambda = \lambda(h)v_\lambda$ and $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

Any quotient of M_λ, in particular V_λ is a *highest weight module*, i.e. it is generated under $U(\mathfrak{g})$ by a vector v such that

$$\mathfrak{n}^+.v = 0, \qquad h.v = \lambda(h)v \;\forall\, h \in \mathfrak{h}.$$

Representations

Theorem (Weyl dimension formula)

If $\nu \in P^+$, then

$$\dim V_\nu = \frac{\prod_{\beta \in \Delta^+} (\nu + \rho, \beta)}{\prod_{\beta \in \Delta^+} (\rho, \beta)}$$

Cohomology of Lie algebras

Definition

Let $\mathfrak{g}$ be (any) Lie algebra and V be a representation of $\mathfrak{g}$. The Lie algebra cohomology $H^*(\mathfrak{g}, V)$ is the cohomology of the complex

$$0 \to C^0 \xrightarrow{\mathbf{d}_0} C^1 \xrightarrow{\mathbf{d}_1} C^2 \dots C^p \xrightarrow{\mathbf{d}_p} C^{p+1} \to \dots$$

where $C^p = Hom(\bigwedge^p \mathfrak{g}, V)$ and

$$(\mathbf{d}_p\omega)(x_1 \wedge \dots \wedge x_{p+1}) =$$
$$\sum_{i<j}(-1)^{i+j}\omega([x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \dots \wedge \hat{x}_j \dots \wedge x_{p+1})$$
$$+ \sum_i (-1)^{i+1} x_i.\omega(x_1 \wedge \dots \wedge \hat{x}_i \dots \wedge x_{p+1})$$

Cohomology of Lie algebras

General Facts

- $H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$
- $H^1(\mathfrak{g}, V) = Der(\mathfrak{g}, V)/InnDer(\mathfrak{g}, V)$
- $H^2(\mathfrak{g}, V) =$ iso-classes of abelian extension of $\mathfrak{g}$ by V

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Proposition

Write $H^\bullet(\mathfrak{g})$ for cohomology with trivial coefficients.

- *If $\mathfrak{g}$ is semisimple then $H^1(\mathfrak{g}, V) = 0$ (implies complete reducibility of reps).*
- *If $\mathfrak{g}$ is semisimple then $H^2(\mathfrak{g}, V) = 0$ (implies Levi decomposiiton).*
- *If G is compact $H^\bullet_{DR}(G) = H^\bullet(\mathfrak{g}) = (\bigwedge \mathfrak{g})^{\mathfrak{g}}$*

Dual version, homology

Complex

$$\to \Lambda_p \xrightarrow{\partial_p} \Lambda_{p-1} \xrightarrow{\partial_{p-1}} \ldots \Lambda_1 \xrightarrow{\partial_1} \Lambda_0 \to 0$$

where

$$\Lambda_p = \Lambda_p(\mathfrak{g}, V) = \bigwedge^p \mathfrak{g} \otimes V$$

and

$$\partial_p(x_1 \wedge \ldots \wedge x_p \otimes v) = \sum_{i<j} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \ldots \wedge \hat{x}_j \ldots \wedge x_p \otimes v$$
$$+ \sum_i (-1)^i x_1 \wedge \ldots \wedge \hat{x}_i \ldots \wedge x_p \otimes x_i.v$$

Computing homology

Kostant's approach

One can put on $C = \bigwedge \mathfrak{g} \otimes V$ a Hilbert space structure, and then one defines a positive semidefinite operator L_V on C by putting $L_V = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d}$ where $\mathbf{d}^*$ is the Hermitian adjoint of $\mathbf{d}$. One then has a natural isomorphism

$$KerL_V = H_*(\mathfrak{g}, V)$$

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$$KerL_V = H_*(\mathfrak{g}, V)$$

Kostant has a nice spectral resolution for L_V for a class of subalgebras which includes the parabolic subalgebras of semisimple Lie algebras (i.e., the subalgebras containing a Borel subalgebra).

Kostant theorem on $\mathfrak{u}$-cohomology

Theorem

Let $\mathfrak{p}$ be a parabolic subalgebra of a semisimple Lie algebra $\mathfrak{g}$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. Then, as $\mathfrak{l}$-modules,

$$H_p(\mathfrak{u}^-, V_\lambda) = \bigoplus_{w \in W', \ell(w) = p} V(w(\lambda + \rho) - \rho)$$

where ρ is the Weyl vector and W' is the set of minimal length right coset representatives for $W_{\mathfrak{l}} \backslash W$.
Moreover, a representative for the highest weight vector is given by the decomposable vector $x_{\beta_1} \wedge \ldots \wedge x_{\beta_p} \otimes v_{w\lambda}$, where $N(w) = \{\beta_1, \ldots, \beta_p\}$ and $v_{w(\lambda)}$ is a nonzero weight vector of weight $w(\lambda)$.

Affine root systems

Affine root system

Let F be the space of affine-linear functions on $V = \mathfrak{h}_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} Q^\vee$, where $Q^\vee = \sum_{\alpha \in \Pi} \mathbb{Z}\alpha^\vee$ is the coroot lattice.

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$$\widehat{\Delta} = \{a_{\alpha,j} \mid \alpha \in \Delta, j \in \mathbb{Z}\}$$

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$$\widehat{\Delta} = \{a_{\alpha,j} \mid \alpha \in \Delta, j \in \mathbb{Z}\}$$

Affine Weyl group

For $\alpha \in \Delta$, $j \in \mathbb{Z}$ let $s_{\alpha,j}$ be the affine reflection around $\alpha(x) = j$:

$$s_{\alpha,j}(v) = v - a_{\alpha,-j}(v)\alpha^{\vee}.$$

Let $\widehat{W}$ be the subgroup of $Isom(V)$ generated by $\{s_{\alpha,j} \mid a_{\alpha,j} \in \widehat{\Delta}\}$

Affine Weyl groups

Proposition

Let t_v be the translation by v.

1. $$\widehat{W} = W \ltimes Q^\vee$$
   where $Q^\vee$ is viewed inside $\widehat{W}$ via $\alpha^\vee \mapsto t_{\alpha^\vee}$
2. *$\widehat{W}$ is a Coxeter group with generating set*
   $$s_0 = s_{\theta,1} = t_{\theta^\vee} s_{\theta,0}, s_i = s_{\alpha_i,0}, \; i = 1, \dots, n.$$
   Here $\theta = \sum_{i=1}^n c_i\alpha_i$ is the highest root of Δ.
3. *A fundamental domain for the action of $\widehat{W}$ on V is given by*
   $$\{v \in V \mid \alpha(v) \geq 0 \, \forall \, \alpha \in \Delta^+, \, \theta(v) \leq 1\}.$$

Alcoves

Identifying V and V^* by means of $(\cdot, \cdot)$, we can also define an action of $\widehat{W}$ on V^*;

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Identifying V and V^* by means of $(\cdot,\cdot)$, we can also define an action of $\widehat{W}$ on V^*; then

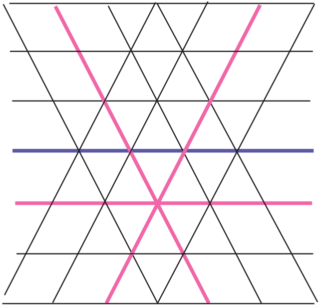
$$\bar{A}_1 = \{\lambda \in V^* \mid (\alpha,\lambda) \geq 0 \,\forall\, \alpha \in \Delta^+,\, (\theta,\lambda) \leq 1\}$$

is a fundamental domain for this action, called the *fundamental alcove*. We will refer to the alcoves as the $\widehat{W}$-translates of A_1 (i.e., $A_w = wA_1$).

Example

Disclaimer

Although I kept the Killing form since the beginning (and this will be important in the sequel), in many of the following pictures I use the more usual normalization $(\theta, \theta) = 2$.



Positive systems

The set

$$\widehat{\Delta}^+ = \{a_{\alpha,j} \mid \alpha \in \Delta, j > 0\} \cup \{a_{\alpha,0} \mid \alpha \in \Delta^+\}$$

can be shown to be a set of positive roots in $\widehat{\Delta}$ and the corresponding set of simple roots is $\widehat{\Pi} = \{\alpha_0, \dots, \alpha_n\}$, where $\alpha_0 = a_{-\theta,1}$ and we identify α_i with $a_{\alpha_i,0}$, $i = 1, \dots, n$.

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If we set $c_0 = 1$ we have $\delta = \sum_{i=0}^{n} c_i \alpha_i$, so that we might write

$$\widehat{\Delta}^+ = \{\alpha + n\delta \mid \alpha \in \Delta, n > 0\} \cup \Delta^+$$

Algebraic intepretation

The elements of $\widehat{\Delta}$ can be regarded as (part of the) roots of an infinite dimensional Lie algebra. Here is a sketch of its construction.

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Start with a fd-simple Lie algebra $\mathfrak{g}$ and form the loop algebra

$$\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}], \quad [x \otimes p(t), y \otimes q(t)] = [x, y] \otimes p(t)q(t).$$

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One shows that

$$H^2(\widetilde{\mathfrak{g}}) = \mathbb{C}\psi, \quad \psi(x \otimes p(t), y \otimes q(t)) = (x, y)Res_t(\frac{d\,p(t)}{dt}\,q(t)).$$

One can therefore form an infinite dimensional Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

where the (canonical) central extension is determined by ψ, K is central and d acts as the Euler operator $t\frac{d}{dt}$.

Algebraic intepretation

Facts

- If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, the subalgebra of $\widehat{\mathfrak{g}}$ is $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d \subset \widehat{\mathfrak{g}}$ is maximal commutative $ad_{\widehat{\mathfrak{g}}}$-diagonalizable.
- $\widehat{\mathfrak{g}}$ has an invariant nondegenerate bilinear form
- If $\delta \in \widehat{\mathfrak{h}}^*$ is defined by $\delta(\mathfrak{h}) = \delta(d) = 0$, $\delta(K) = 1$, then δ generates the kernel of the restriction of the bilinear form to $[\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]$.
- One has a root space decomposition w.r.t. $\widehat{\mathfrak{h}}$; the root system is

$$\widehat{\Delta} = \widehat{\Delta}_{re} \cup \pm\mathbb{N}\delta, \quad \widehat{\Delta}_{re} = \{\alpha + n\delta \mid \alpha \in \Delta\}.$$

  Note that $\widehat{\Delta}_{re}$ is our previous $\widehat{\Delta}$.
- the simple systems $\widehat{\Pi}$ give rise to the extended Dynkin diagrams of $\mathfrak{g}$, i.e. ordinary Dynkin diagrams to which $-\theta$ is added as an independent simple root.

ad-nilpotent of Borel subalgebras

Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{b}$ be a Borel subalgebra. Let $\mathfrak{h}$ be the Cartan component and Δ^+ the positive system.

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Definition

Let $\mathfrak{i}$ be an ideal of $\mathfrak{b}$ contained in $\mathfrak{n}$. It consists of *ad*-nilpotent elements, so we'll call it an *ad-nilpotent ideal* and we denote by $\mathcal{I}$ the set of *ad*-nilpotent ideals.

ad-nilpotent of Borel subalgebras

If $\mathfrak{i} \in \mathcal{I}$, then $\mathfrak{i}$ is $\mathfrak{h}$-stable, hence it admits a decomposition

$$\mathfrak{i} = \bigoplus_{\alpha \in \Phi_{\mathfrak{i}}} \mathfrak{g}_\alpha$$

where as usual $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \,\forall\, h \in \mathfrak{h}\}$ and $\Phi_{\mathfrak{i}} \subset \Delta^+$ is dual order ideal of the *root poset*.
More precisely, recall the partial order on Q defined by

$$\alpha \leq \beta \iff = \beta - \alpha \in \sum_{\gamma \in \Delta^+} \mathbb{Z}_{\geq 0}\gamma$$

Then it is clear that

$$\mathfrak{i} \in \mathcal{I} \iff \alpha \in \Phi_{\mathfrak{i}}, \beta \in \Delta^+, \alpha + \beta \in \Delta^+ \implies \alpha + \beta \in \Phi_{\mathfrak{i}}.$$

Encoding *ad*-nilpotent ideals

Definition

For $\mathfrak{i} \in \mathcal{I}$ set $\Phi_{\mathfrak{i}}^1 = \Phi_{\mathfrak{i}}$, $\Phi_{\mathfrak{i}}^i = (\Phi_{\mathfrak{i}}^{i-1} + \Phi_i) \cap \Delta^+$ and

$$L_{\mathfrak{i}} = \bigcup_{k \geq 1} (-\Phi_{\mathfrak{i}}^k + k\delta) \subset \widehat{\Delta}^+.$$

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Theorem

1. *$L_{\mathfrak{i}}$ is biclosed in $\widehat{\Delta}^+$, hence there exists a unique $w_{\mathfrak{i}} \in \widehat{W}$ such that $L_{\mathfrak{i}} = N(w_{\mathfrak{i}})$.*
2. *Given $w \in \widehat{W}$, there exists $\mathfrak{i} \in \mathcal{I}$ such that $w = w_{\mathfrak{i}}$ if and only if*
   - $w^{-1}(\alpha) > 0 \, \forall, \alpha \in \Pi$ *(i.e., $w_{\mathfrak{i}} \in \widehat{W}^+$);*
   - *If $w(\alpha) < 0$ for $\alpha \in \widehat{\Pi}$, then there exists $\beta \in \Delta^+$ such that $w(\alpha) = \beta - \delta$.*

Abelian ideals

Definition

We denote by $\mathcal{I}_{ab}$ the set of abelian ideals of $\mathfrak{b}$. Clearly $\mathcal{I}_{ab} \subset \mathcal{I}$.

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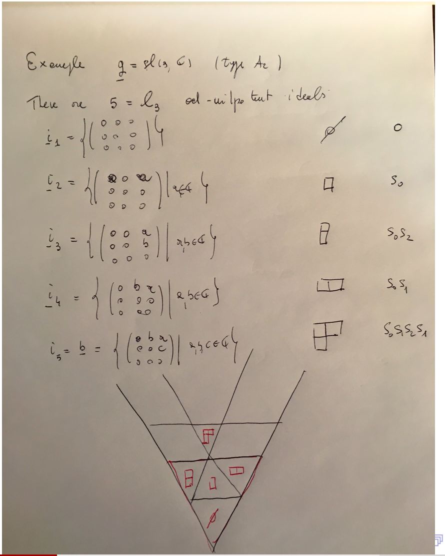
Theorem

The following statements are equivalent

1. $\mathfrak{i} \in \mathcal{I}_{ab}$
2. $L_\mathfrak{i} = -\Phi_\mathfrak{i} + \delta$ *is biclosed, hence there exists a unique* $w_\mathfrak{i} \in \widehat{W}$ *such that* $L_\mathfrak{i} = N(w_\mathfrak{i})$.
3. $w_\mathfrak{i}(A_1) \subset 2A_1$ *(i.e.* $w_\mathfrak{i} \in \widehat{W}_2^+$*).*

In particular, $|\mathcal{I}_{ab}| = 2^{\mathrm{rk}\ \mathfrak{g}}$ *(Peterson's abelian ideal Theorem)*

Example



Abelian ideals

Proof

(1) $\iff$ (2). If $\mathfrak{i}$ is abelian, it is clear that if $\alpha, \beta \in \Phi_{\mathfrak{i}}$, then $-(\alpha+\beta)+2\delta \notin \widehat{\Delta}^+$; now assume $-\alpha+\delta = \xi+\eta$, $\alpha \in \Phi_{\mathfrak{i}}, \xi, \eta \in \widehat{\Delta}^+$. Then $\xi = \xi_0 + \delta, \eta \in \Delta^+$, so that $-\alpha = \xi_0 + \eta$; in particular $\xi_0 \in \Delta^-$ and since $\Phi_{\mathfrak{i}}$ is a dual order ideal, $-\xi_0 = \alpha + \eta \in \Phi_{\mathfrak{i}}$, as required. The converse is easy.

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(2) $\iff$ (3). It is obvious that $A_w \subset 2A_1$, otherwise the hyperplane $\theta = 2$ separates A_1 and A_w, and $-\theta + 2\delta \in N(w)$, against the assumption. Conversely, if $A_w \subset 2A_1$, then each hyperplane which separates A_1 and A_w intersects $2A_1$. Now $-\alpha + k\delta \in N(w)$ iff $\alpha = k$ separates A_1, A_w But for each $x \in 2A_1$ and for each $\alpha \in \Delta^+$ we have $0 < (x,\alpha) < (x,\theta) < 2$. Therefore if a bounding hyperplane $\alpha = k$ intersects $2A_1$, we have $0 < k < 2$.

Another encoding of *ad*-nilpotent ideals

Recall that $\widehat{W} \cong Q^\vee \rtimes W$.

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Recall that $\widehat{W} \cong Q^\vee \rtimes W$.

Proposition

1. *The map* $\phi : \mathfrak{i} \mapsto v_{\mathfrak{i}}^{-1}(\tau_{\mathfrak{i}})$, *where* $w_{\mathfrak{i}} = t_{\tau_{\mathfrak{i}}} v_{\mathfrak{i}}$ *is a bijection*

$$\mathcal{I} \to D = \{\tau \in Q^\vee \mid (\tau, \alpha) \leq 1 \ \forall \alpha \in \Pi, \ (\tau, \theta) \geq -2\}.$$

2. *The map* ϕ *restricts to a bijection*

$$\mathcal{I}_{ab} \to D_{ab} = \{\tau \in Q^\vee \mid (\tau, \alpha) \in \{1, 0, -1, -2\} \ \forall\, \alpha \in \Delta^+\}.$$

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Corollary

If h *denotes the Coxeter number, and* m_i *the exponents, then*
$$|\mathcal{I}| = \frac{\prod_{i=1}^{\mathrm{rk}\,\mathfrak{g}} (m_i + 1 + h)}{|W|}.$$

Proof of the proposition

Let $t_{\tau_i} v_i = w_i$, $t_{\tau_j} v_j = w_j$ for some i and j in $\mathcal{I}$. Assume $v_i^{-1}(\tau_i) = v_j^{-1}(\tau_j)$. Since $\tau_i, \tau_j \in \overline{C}_1$, which is a fundamental domain for W, we have $\tau_i = \tau_j$ and $v_i v_j^{-1}(\tau_i) = \tau_i$. Hence $t_{\tau_i} v_i(A_1) = t_{\tau_i} v_i v_j^{-1} v_j(A_1) =$ $v_i v_j^{-1}(t_{\tau_i} v_j(A_1)) = v_i v_j^{-1}(t_{\tau_j} v_j(A_1)) \subset v_i v_j^{-1}(C_1)$. But $t_{\tau_i} v_i(A_1) \subset C_1$, hence $v_i v_j^{-1} = 1$. Thus F is injective. Next let $\sigma \in D$. We first see that there exists $v \in W$ such that $t_{v(\sigma)} v(A_1) \subset C_1$: simply take the unique $v \in W$ such that $v(\sigma + A_1) \subset C_1$. Now it is immediate that, since $\sigma \in D$, $t_{v(\sigma)} v$ also satisfies the second condition of part 2 in our characterization Theorem, hence $t_{v(\sigma)} v = w_i$ for some i in $\mathcal{I}$. It is obvious that F maps $t_{v(\sigma)} v$ to σ, thus F is surjective.

Proof of the Corollary

Let

$$X = \{x \in V \mid (x, \alpha_i) \leq 1 \text{ for each } i \in \{1, \dots, n\} \text{ and } (x, \theta) \geq -2\}$$
$$= t_{\rho^\vee} w_0(\overline{A}_{h+1}).$$

where $\rho^\vee = \omega_1^\vee + \cdots + \omega_n^\vee$. One can show that there exists $w \in \widehat{W}$ such that $X = w(\overline{A}_{h+1})$. Such a w gives a bijection from

$$\overline{A}_{h+1} \cap Q^\vee \to D = X \cap Q^\vee.$$

If $\mathfrak{i} \in \mathcal{I}$ and $w_\mathfrak{i} = t_{\tau_\mathfrak{i}} v_\mathfrak{i}$, with $\tau_\mathfrak{i} \in Q^\vee$ and $v_\mathfrak{i} \in W$, then we obtain that $w^{-1} v_\mathfrak{i}^{-1}(\tau_\mathfrak{i})$ belongs to $\overline{A}_{h+1} \cap Q^\vee$ and

$$\mathfrak{i} \mapsto w^{-1} v_\mathfrak{i}^{-1}(\tau_\mathfrak{i}), \qquad \mathcal{I} \to \overline{A}_{h+1} \cap Q^\vee$$

is a bijection. Since elements in $\overline{A}_{h+1} \cap Q^\vee$ are a natural set of representatives of the W-orbits of $Q^\vee/(h+1)Q^\vee$, we are done. The combinatorial enumeration is due to Haiman.

ρ-points

Definitions

Take as invariant form on $\mathfrak{h}$ the Killing form

1. The ρ-points are the $\widehat{W}$-orbit of 2ρ.
2. The *weight* of $\mathfrak{i} \in \mathcal{I}_{ab}$ is $\langle \mathfrak{i} \rangle = \sum_{\mathfrak{g}_\alpha \subset \mathfrak{i}} \alpha$.

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Then $2\rho \in A_1$, and

$$2(\lambda^w + \rho) = w(2\rho).$$

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Introducing a linear version of $\widehat{W}$ as a subgroup of $O(\widehat{\mathfrak{h}}^*)$, one can prove that

$$\lambda^{w_\mathfrak{i}} = \langle \mathfrak{i} \rangle.$$

ρ-points

Recall that an antichain $\mathcal{A}$ in a poset $\mathcal{P}$ is a set of mutually non-comparable elements

Proposition

The following sets are in bijection with $\mathcal{I}_{ab}$:

1. *the set of abelian dual order ideals in Δ^+;*
2. *the set $\widehat{W}_2^+$ in $\widehat{W}$ (the minuscule elements);*
3. *the set of alcoves in $2A_1$;*
4. *the set of ρ-points in $2A_1$;*
5. *the set of weights of abelian ideals.*
6. *the set $D_{ab} = \{\eta \in Q^\vee \mid \eta(\alpha) \in \{-2-1, 0, 1\}\ \forall\, \alpha \in \Delta^+\}$;*
7. *the set of antichains $\mathcal{A} \subset \Delta^+$ such that for any $\alpha, \beta \in \mathcal{A}$ we have $\alpha + \beta \not\leq \theta$.*

Afterwords

Lemma (Kostant)

Let $\mathfrak{i}_1, \mathfrak{i}_2 \in I$ be such that $\langle \mathfrak{i}_1 \rangle = \langle \mathfrak{i}_2 \rangle$. Then $\mathfrak{i}_1 = \mathfrak{i}_2$.

Proof.

Set $\Phi_i = \Phi_{\mathfrak{i}_i}$, $\Phi := \Phi_1 \cap \Phi_2$. Assume by contradiction that $\Phi_1 \neq \Phi_2$. Then since $\langle \Phi_1 \rangle = \langle \Phi_2 \rangle$ both $\Phi_1 - \Phi$ and $\Phi_2 - \Phi$ are nonempty. Pick $\varphi_i \in \Phi_i - \Phi$ $(i = 1, 2)$. We must have $(\varphi_1 | \varphi_2) \leqslant 0$. Otherwise $\varphi_1 - \varphi_2$ would be a root which can be assumed positive by possibly interchanging the indices 1 and 2. By the ideal property $\Phi_i + \Delta^+ \subseteq \Phi_i$ we then have $\varphi_1 = \varphi_2 + (\varphi_1 - \varphi_2) \in \Phi_2$, a contradiction. Thus $(\varphi_1 | \varphi_2) \leqslant 0$. Hence since $\langle \Phi_1 - \Phi \rangle = \langle \Phi_2 - \Phi \rangle$ we obtain

$$0 \leqslant \left\| \langle \Phi_i - \Phi \rangle \right\|^2 = \big(\langle \Phi_1 - \Phi \rangle \mid \langle \Phi_2 - \Phi \rangle \big) \leqslant 0$$

and so $\Phi = \Phi_1 = \Phi_2$. $\square$

Panyushev's theory of rootlets

For $\alpha \in \Delta_\ell^+$ define

$$\mathcal{I}_{ab}(\alpha) = \{\mathfrak{i} \in \mathcal{I} \mid w_{\mathfrak{i}}^{-1}(-\theta + 2\delta) = \alpha\},$$
$$\widehat{W}_\alpha = \langle s_\beta \mid \beta \in \widehat{\Pi},\ \beta \perp \alpha \rangle \leq \widehat{W},$$
$$W_\alpha = \langle s_\beta \mid \beta \in \Pi,\ \beta \perp \alpha \rangle \leq \widehat{W}.$$

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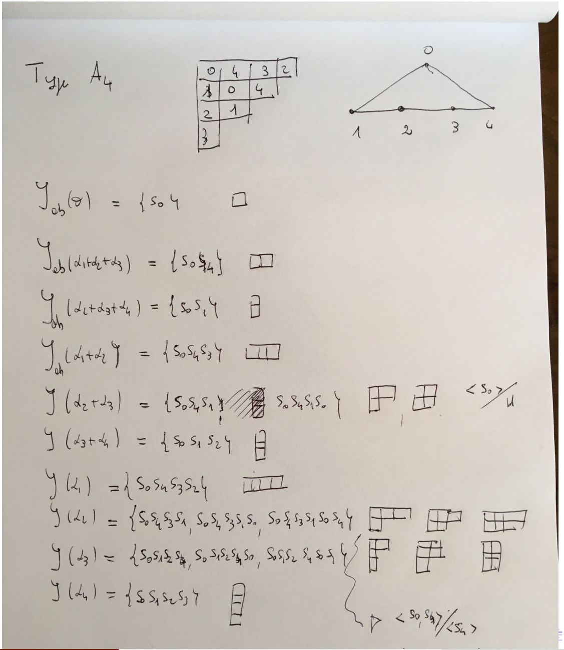
Theorem

1. *The set $\mathcal{I}'_{ab}$ of nonzero abelian ideals of $\mathfrak{b}$ decomposes as*
$$\mathcal{I}'_{ab} = \bigsqcup_{\alpha \in \Delta_\ell^+} \mathcal{I}_{ab}(\alpha)$$
2. *There an explicit poset isomorphism $\mathcal{M} : \mathcal{I}_{ab}(\alpha) \to \widehat{W}_\alpha / W_\alpha$. In particular $\mathcal{I}_{ab}(\alpha)$ has minimum and maximum.*
3. *$\mathcal{M}$ gives rise to a natural bijection maximal abelian ideals of $\mathfrak{b}$ and Π_ℓ.*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Maximal dimension

Proposition (Suter)

For $\alpha \in \Pi_\ell$

$$\dim \max \mathcal{I}_{ab}(\alpha) = h^\vee - 1 + |\Delta^+(\widehat{W}_\alpha)| - |\Delta^+(W_\alpha)|.$$

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Remark

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$$M = \max \dim\{\mathfrak{a} \mid \mathfrak{a} \text{ abelian subalgebra of } \mathfrak{g}\ \} = \dim \max I_{ab}(\beta)$$

where β is a simple roots having maximum distance from α_0 in $\widehat{\Pi}$.

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Example

$$M(E_6) = \dim \max \mathcal{I}_{ab}(\alpha_1) = 11 + |\Delta^+(A_5)| - |\Delta^+(A_4)| = 11 + 15 - 10 = 16.$$

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Comments of the proof - (1)

We show that if w is a non-trivial minuscule element, then $w^{-1}(-\theta+2\delta)\in\Delta_\ell^+$.
Since $w^{-1}(-\theta+\delta)$ is negative, we can write $w^{-1}(-\theta+\delta)=-k\delta-\gamma_0$, where $k\in\{0,1,2,\dots\}$ and $\gamma_0\in\Delta$.
a) Assume $k\geq 2$. Then $w^{-1}(2\delta-\theta)=-(k-1)\delta-\gamma_0<0$. This contradicts the fact that w is minuscule.
b) Assume $k=0$. Then $w^{-1}(\delta-\theta)=-\gamma_0$ and $\gamma_0\in\Delta^+$. It is clear that $w\in\widehat{W}\setminus W$. Write the expression of θ through the simple roots: $\theta=\sum_{i=1}^p c_i\alpha_i$ and set $\gamma_i=w^{-1}(\alpha_i)$.
Then $\sum_{i=1}^p c_i\gamma_i=\gamma_0+\delta$. Since γ_i's are positive and $\gamma_0\in\Delta$, there exists a unique $i_0\in\{1,\dots,p\}$ such that $c_{i_0}=1$, $\gamma_{i_0}\in\delta+\Delta$ and $\gamma_i\in\Delta$ for $i\neq i_0$.
It follows that the elements $-\gamma_0$, γ_j $(j\geq 1,\ j\neq i_0)$ form a basis for Δ.
Hence there is $w'\in W$ which takes $-\gamma_0$, γ_j $(j\neq i_0)$ to $\alpha_1,\dots,\alpha_p$.

Sketch of proof - (1)

Because $w'(\gamma_{i_0}) \in \delta + \Delta$ and the elements $w'(\gamma_i)$ $(i = 0, 1, \ldots, p)$ form a basis for $\widehat{\Delta}$, we see that $w'(\gamma_{i_0}) = -\theta + \delta$.

Thus, $w'w^{-1}$ takes $\widehat{\Pi}$ to itself and hence $w' = w$.

This is however impossible, since $w \notin W$.

Thus, $k = 1$ and $\mu := w^{-1}(\delta - \theta) + \delta = w^{-1}(2\delta - \theta) \in \Delta$.

Since δ is isotropic and θ is long, μ is long as well.

Finally, since w is minuscule, $2\delta - \theta \notin \widehat{N}(w)$. Hence μ is positive.

Sketch of proof - (2)

One shows that, if $\mathfrak{i} \in \mathcal{I}(\alpha), \alpha \in \Delta_\ell^+$

$$w_\mathfrak{i} = s_0 v_\alpha \tilde{v}_{\mathfrak{i},\alpha}$$

where

$$v_\alpha = \text{ element of minimal length in } W \text{ s.t. } v_\alpha(\alpha) = \theta$$
$$v_{\mathfrak{i},\alpha} \in \widehat{W}_\alpha / W_\alpha.$$

Proposition

$w_\mathfrak{i} \mapsto \tilde{v}_{\mathfrak{i},\alpha}$ is a bijection $\mathcal{I}_{ab}(\alpha) \to \widehat{W}_\alpha / W_\alpha$

Examples

$\mathcal{I}_{ab}(\theta) = \{s_0\}$; if $\bar{\alpha} \in \Pi$ is such that $(\bar{\alpha}, \theta) \neq 0$, then

$$\mathcal{I}_{ab}(\bar{\alpha}) = \{s_0 v_{\bar{\alpha}}\}.$$

Sketch of proof - (3)

Proposition

$\mathfrak{i} \in \mathcal{I}_{ab}$ *is maximal if and only if* $w_{\mathfrak{i}}(\widehat{\Pi}) \cap (-\Delta^+ + \delta) = \emptyset$. *In this case* $-\theta + 2\delta \in w_{\mathfrak{i}}(\Pi_\ell)$

Sketch of proof - (3)

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Proof.

The abelian ideal $\mathfrak{i}$ is maximal in $\mathcal{I}_{ab}$ if and only if, for all $w \in \widehat{W}$ such that $w_{\mathfrak{i}} \leq w$ in the weak order, $w \notin \widehat{W}_2^+$.
But $u \in \widehat{W}_2^+, u' \leq u \implies u' \in \widehat{W}_2^+$. This implies that $\mathfrak{i}$ is maximal in $\mathcal{I}_{ab}$ if and only if, for all $\alpha \in \widehat{\Pi}$ such that $w_{\mathfrak{i}}(\alpha) > 0$, we have that $w_{\mathfrak{i}} s_\alpha \notin \widehat{W}_2^+$.
Since for $w_{\mathfrak{i}}(\alpha) > 0$ we have $N(w_{\mathfrak{i}} s_\alpha) = N(w_{\mathfrak{i}}) \cup \{w_{\mathfrak{i}}(\alpha)\}$, this happens if and only if $w_{\mathfrak{i}}(\alpha) \notin -\Delta^+ + \delta$. This proves the first statement.

Sketch of proof - (3)

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$\mathfrak{i} \in \mathcal{I}_{ab}$ is maximal if and only if $w_{\mathfrak{i}}(\widehat{\Pi}) \cap (-\Delta^+ + \delta) = \emptyset$. In this case $-\theta + 2\delta \in w_{\mathfrak{i}}(\Pi_\ell)$

Proof.

Now if $w_{\mathfrak{i}}(\widehat{\Pi}) \cap (-\Delta^+ + \delta) = \emptyset$, we have in particular that $w_{\mathfrak{i}}$ is not the identity of $\widehat{W}$. Since no translation may correspond to some non zero abelian ideal, if $w_{\mathfrak{i}} = t_\tau v$, then v is not the identity; therefore $v(\alpha) < 0$ for at least one $\alpha \in \Pi$. $t_\tau v(\widehat{\Pi}) \subseteq \Delta^+ - \delta \cup \Pi \cup \{-\theta + 2\delta\}$, hence for such an α, $v(\alpha) = -\theta$ and $w_i(\alpha) = -\theta + 2\delta$. Moreover, since θ is long, α is long too. □

A result on dominant elements in $\widehat{W}$

Theorem

If $w \in \widehat{W}^+ \setminus \widehat{W}_2^+$, then $\ell(w) \geq h^\vee$. Here $h^\vee = 1 + \sum_i d_i$ if $\theta^\vee = \sum_i d_i \alpha_i^\vee$

Proof.

Recall that there are exactly $h^\vee - 2$ decompositions of θ as a sum of two positive roots $\theta = \alpha_1 + \beta_1 = \ldots = \alpha_{h^\vee-2} + \beta_{h^\vee-2}$. By assuption $(\theta, x) = 2$ separates A_w and A_1. This implies that $-\theta + 2\delta \in N(w)$. If we consider

$$-\theta + 2\delta = (-\alpha_1 + \delta) + (-\beta_1 + \delta) = \ldots = (-\theta + \delta) + \delta$$

we deduce that $|N(w)| \geq h^\vee - 1$. Since $w \notin \widehat{W}_2^+$ there exist $-\xi + \delta, -\eta + \delta \in N(w)$ such that $\gamma = \xi + \eta \in \Delta^+$. Hence the decomposition $-\gamma + 2\delta = (-\xi + \delta) + (-\eta + \delta)$ afford at least one more element in $N(w)$, which has therefore at least $h^\vee$ elements, as desired. □

Preliminaries on affine algebras

Denote by $\mathfrak{h}_{\mathbb{R}}$ the real span of $\alpha_0^\vee, \dots, \alpha_n^\vee$ and let $\widehat{\mathfrak{g}}_{\mathbb{R}}$ be the real algebra generated by $\mathfrak{h}_{\mathbb{R}} \oplus \mathbb{R}d$ together with the Chevalley generators $e_0 = t^{-1} \otimes e_\theta, f_0 = t \otimes f_\theta, e_i, f_i, 1 \leq i \leq n$ for $\widehat{\mathfrak{g}}$.
Let *conj* be the conjugation of $\widehat{\mathfrak{g}}$ corresponding to the real form $\widehat{\mathfrak{g}}_{\mathbb{R}}$ and define the conjugate linear antiautomorphism σ_o of $\widehat{\mathfrak{g}}$ by setting $\sigma_o(h) = conj(h)$, $\sigma_o(e_i) = f_i$, and $\sigma_o(f_i) = e_i$. We extend the form $(\cdot, \cdot)$ to $\widehat{\mathfrak{g}}$ by setting ($x_r = t^r \otimes x$)

$$(x_r, y_s) = \delta_{r,-s}(x, y),\ (\tilde{\mathfrak{g}}, d) = (\tilde{\mathfrak{g}}, K) = (d, d) = 0,\ (K, d) = 1.$$

Preliminaries on affine algebras

Since $(\cdot,\cdot)$ is real on $\mathfrak{h}_{\mathbb{R}}$, we have that $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}) \subset \mathbb{R}$. Following Kumar, we can define a contravariant (i.e. $\{[a,x],y\} = -\{x,[\sigma_o(a),y]\}$) Hermitian form $\{\cdot,\cdot\}$ on $\widehat{\mathfrak{g}}$ by setting

$$\{x,y\} = (x,\sigma_o(y)).$$

$\mathfrak{u}$-homology

If $\alpha = \alpha_0 + k\delta$ set $\widehat{\mathfrak{g}}_\alpha = t^k \otimes \mathfrak{g}_{\alpha_0}$; if $\alpha = k\delta$ set $\widehat{\mathfrak{g}}_\alpha = t^k \otimes \mathfrak{h}$. We also set

$$\mathfrak{m} = \mathfrak{g} + \widehat{\mathfrak{h}},$$
$$\mathfrak{u} = \sum_{\alpha(d)>0} \widehat{\mathfrak{g}}_\alpha = t\mathfrak{g}[t],$$
$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{u}.$$

We also set $\mathfrak{u}^- = \sum_{\alpha(d)<0} \widehat{\mathfrak{g}}_\alpha = t^{-1}\mathfrak{g}[t^{-1}]$, $\mathfrak{q}^- = \mathfrak{m} \oplus \mathfrak{u}^-$; note that $\sigma_o(\mathfrak{u}) = \mathfrak{u}^-$. Since $(\mathfrak{u}, \mathfrak{q}) = 0$ and the form $(\cdot, \cdot)$ is nondegenerate on $\widehat{\mathfrak{g}}$, it follows that $\{\cdot, \cdot\}$ defines a nondegenerate hermitian form on $\mathfrak{u}^-$, which is known to be positive definite.

$\mathfrak{u}$-homology

Extend $\{\cdot,\cdot\}$ to $\wedge\mathfrak{u}^-$ in the usual way: elements in $\wedge^r\mathfrak{u}^-$ are orthogonal to elements of $\wedge^s\mathfrak{u}^-$ if $r \neq s$ whereas

$$\{X_1 \wedge \cdots \wedge X_r, Y_1 \wedge \cdots \wedge Y_r\} = \det\left(\{X_i, Y_j\}\right).$$

Similarly, we can extend $(\cdot,\cdot)$ to define a symmetric bilinear form on $\bigwedge \widehat{\mathfrak{g}}$. If we extend σ_o to $\bigwedge^k \widehat{\mathfrak{g}}$ by setting

$$\sigma_o(x^1 \wedge \cdots \wedge x^k) = \sigma_o(x^1) \wedge \cdots \wedge \sigma_o(x^k),$$

then obviously relation $\{x,y\} = (x, \sigma_o(y))$ still holds with $x, y \in \wedge\mathfrak{u}^-$.

$\mathfrak{u}$-homology

Set $\partial_p : \wedge^p \mathfrak{u}^- \to \wedge^{p-1}\mathfrak{u}^-$ to be the usual Chevalley-Eilenberg boundary operator defined by

$$\partial_p(X_1 \wedge \ldots \wedge X_p) = \sum_{i<j} (-1)^{i+j} [X_i, X_j] \wedge X_1 \ldots \widehat{X_i} \ldots \widehat{X_j} \cdots \wedge X_p$$

if $p > 1$ and $\partial_1 = \partial_0 = 0$ and let $H_p(\mathfrak{u}^-, \mathbb{C})$ be its homology.

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if $p > 1$ and $\partial_1 = \partial_0 = 0$ and let $H_p(\mathfrak{u}^-, \mathbb{C})$ be its homology.

Let $L_p : \wedge^p\mathfrak{u}^- \to \wedge^p\mathfrak{u}^-$ be the corresponding Laplacian:

$$L_p = \partial_{p+1}\partial^*_{p+1} + \partial^*_p\partial_p.$$

where ∂^*_p denotes the adjoint of ∂_p with respect to $\{\cdot, \cdot\}$.
We shall use the following two basic properties of L_p

$$Ker\ L_p \cong H_p(\mathfrak{u}^-),$$
$$(Ker\ L_p)^\perp = Im\,\partial^*_p + Im\,\partial_{p+1}.$$

$\mathfrak{u}$-homology

Since $\mathfrak{u}^-$ is stable under $ad(\mathfrak{m})$ we have an action of $\mathfrak{m}$ on $\mathfrak{u}^-$. Restricting this action to $\mathfrak{g}$ we get an action of $\mathfrak{g}$ on $\mathfrak{u}^-$. Notice also that, since K is a central element, the action of K on $\mathfrak{u}^-$ is trivial.

Recall that the Casimir operator $\Omega_\mathfrak{g}$ of $\mathfrak{g}$ is the element of the universal enveloping algebra of $\mathfrak{g}$ defined by setting $\Omega_\mathfrak{g} = \sum_{i=1}^N b_i b_i'$, where $\{b_1, \dots, b_N\}$, $\{b_1', \dots, b_N'\}$ are dual bases of $\mathfrak{g}$ with respect to $(\cdot, \cdot)$.

Set $\{u_1, \dots, u_n\}$ and $\{u^1, \dots, u^n\}$ to be bases of $\mathfrak{h}$ dual to each other with respect to $(\cdot, \cdot)$. If ρ_0 denotes the Weyl vector of $\mathfrak{g}$, it is well known that $\Omega_\mathfrak{g}$ can be rewritten as

$$\Omega_\mathfrak{g} = \sum_{i=1}^n u_i u^i + 2\rho_0 + \sum_{\alpha \in \Delta^+} x_{-\alpha} x_\alpha.$$

Define $\Lambda_0 \in \widehat{\mathfrak{h}}^*$ by setting $\Lambda_0(\mathfrak{h}_0) = \Lambda_0(d) = 0$ and $\Lambda_0(K) = 1$. Set

$$\rho = \frac{1}{2}\Lambda_0 + \rho_0$$

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$$\rho = \frac{1}{2}\Lambda_0 + \rho_0$$

Fact

$$\rho(\alpha_i^\vee) = 1, i = 0, \dots, n, \qquad \rho(d) = 0$$

Casimir element vs Laplacian

Proposition

$$L_p(x) = -(d + \Omega_{\mathfrak{g}})(x) \quad (x \in \wedge \mathfrak{u}^-).$$

Proof.

Note that $\{u_1, \dots, u_n, c, d\}$ and $\{u^1, \dots, u^n, d, c\}$ are bases of $\widehat{\mathfrak{h}}$ dual to each other with respect to $(\cdot, \cdot)$. Then, following Kumar, we set

$$\Omega = \sum_{i=1}^{n} u_i u^i + 2K\, d + 2\rho + \sum_{\alpha \in \Delta^+} x_{-\alpha} x_{\alpha}.$$

By what observed before about ρ, we have that

$$\Omega = \Omega_{\mathfrak{g}} + d + 2d\, K.$$

Casimir element vs Laplacian

Proposition

$$L_p(x) = -(d + \Omega_{\mathfrak{g}})(x) \quad (x \in \wedge \mathfrak{u}^-).$$

Proof.

$$\Omega = \Omega_{\mathfrak{g}} + d + 2d\,K.$$

By a Laplacian calculation due to Kumar applied to $\wedge \mathfrak{u}^- \simeq \mathbb{C} \otimes \wedge \mathfrak{u}^-$ we have that if $x \in \wedge \mathfrak{u}^-$, then $L_p(x) = -\Omega(x)$. Hence, by observing that K acts trivially on $\wedge \mathfrak{u}^-$, the result follows. □

Garland-Lepowsky generalization of Kostant's theorem

Notation

If $\lambda \in \widehat{\mathfrak{h}}^*$ is such that $\overline{\lambda} = \pi(\lambda)$ ($\pi : \widehat{\mathfrak{h}} \to \mathfrak{h}$ projection) is dominant integral for Δ^+, let $V(\lambda)$ denote the irreducible $\mathfrak{m}$-module of highest weight λ.

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Theorem

$$H_p\left(\mathfrak{u}^-\right) = \bigoplus_{\substack{w \in \widehat{W}^+ \\ \ell(w)=p}} V\left(w(\rho) - \rho\right).$$

Moreover a representative of the highest weight vector of $V\left(w(\rho) - \rho\right)$ is given by

$$X_{-\beta_1} \wedge \cdots \wedge X_{-\beta_p}$$

where $N(w) = \{\beta_1, \ldots, \beta_p\}$ and the $X_{-\beta_i}$ are root vectors in $\widehat{\mathfrak{g}}$.

A natural bigrading

For $x^i \in \mathfrak{g}$, set $x^i_j = t^j \otimes x_i$ and define

$$\wedge^{(r,s)}\mathfrak{u}^- = Span\left\{ x^1_{i_1} \wedge x^2_{i_2} \wedge \cdots \wedge x^r_{i_r} \mid -\sum_{i=1}^{r} i_j = s \right\}.$$

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Note that the map $x^1_{-1} \wedge \ldots \wedge x^r_{-1} \mapsto x^1 \wedge \ldots \wedge x^r$ affords a canonical identification

$$Z : \wedge^{(r,r)}\mathfrak{u}^- \xrightarrow{\cong} \wedge^r \mathfrak{g}$$

that intertwines the adjoint action of $\mathfrak{g}$.

Abelian subspaces as cycles

Lemma

Given linearly independent elements $x^1, \dots, x^p$ of $\mathfrak{g}$, set $v = x^1_{-1} \wedge \dots \wedge x^p_{-1}$. Then $\partial_p(v) = 0$ if and only if $[x^i, x^j] = 0$ for all i, j.

Proof.

This follows readily from the definition of ∂_p:

$$\partial_p(v) = \sum (-1)^{i+j} [x^i, x^j]_{-2} \wedge x^1_{-1} \dots \widehat{x^i_{-1}} \dots \wedge \widehat{x^j_{-1}} \dots \wedge x^p_{-1}.$$

□

Notation

For a p-dimensional subspace $\mathfrak{a} = \bigoplus_{i=1}^{p} \mathbb{C} v^i$ of $\mathfrak{g}$ define

$$v_{\mathfrak{a}} = v^1 \wedge \ldots \wedge v^p \in \wedge^p \mathfrak{g},$$
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Theorem

The maximal eigenvalue for the action of $\Omega_{\mathfrak{g}}$ on $\wedge^p \mathfrak{g}$ is at most p. Equality holds if and only if there exists a commutative subspace $\mathfrak{a}$ of $\mathfrak{g}$ of dimension p. In such a case, $v_{\mathfrak{a}}$ is an eigevector for $\Omega_{\mathfrak{g}}$ relative to the eigenvalue p.

Proof.

To prove that the maximal eigenvalue is at most p, remark that L_p is self-adjoint and positive semidefinite on $\wedge\mathfrak{u}^-$ with respect to $\{\,,\,\}$. Since $\Omega_{\mathfrak{g}} = -d - L_p$, the claim follows.

Proof.

To prove that the maximal eigenvalue is at most p, remark that L_p is self-adjoint and positive semidefinite on $\wedge\mathfrak{u}^-$ with respect to $\{\,,\,\}$. Since $\Omega_\mathfrak{g} = -d - L_p$, the claim follows.
Suppose that $\mathfrak{a}$ is an abelian subspace of $\mathfrak{g}$ of dimension p. Then, by the Lemma $\partial_p(\widehat{v}_\mathfrak{a}) = 0$. Since $\widehat{v}_\mathfrak{a} \in \wedge^{(p,p)}\mathfrak{u}^-$, we have that $\partial^*_{p+1}(\widehat{v}_\mathfrak{a}) = 0$, hence $L_p(\widehat{v}_\mathfrak{a}) = (\partial_{p+1}\partial^*_{p+1} + \partial^*_p\partial_p)(\widehat{v}_\mathfrak{a}) = 0$. We have then $\Omega_\mathfrak{g}(v_\mathfrak{a}) = p\, v_\mathfrak{a}$.

Proof.

To prove that the maximal eigenvalue is at most p, remark that L_p is self-adjoint and positive semidefinite on $\wedge\mathfrak{u}^-$ with respect to $\{\,,\,\}$. Since $\Omega_{\mathfrak{g}} = -d - L_p$, the claim follows.
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Conversely, if $\Omega_{\mathfrak{g}}$ has eigenvalue p on $\wedge^p\mathfrak{g}$, then $Ker\, L_p \cap \wedge^{(p,p)}\mathfrak{u}^- \neq 0$. By GL-theorem we know that $Ker\, L_p$ decomposes with multiplicity one. Since $\wedge^{(p,p)}\mathfrak{u}^-$ is $\mathfrak{m}$-stable, we deduce that one of the highest weight vectors, say $x^1_{-1} \wedge \cdots \wedge x^p_{-1}$, must belong to $Ker\, L_p \cap \wedge^{(p,p)}\mathfrak{u}^-$. Since $\partial^*_p\partial_p = 0$ implies that $\partial_p = 0$, the above Lemma gives that $Span(x^1, \ldots, x^p)$ is the required abelian subspace. □

$\bigwedge \mathfrak{g}$ and $\widehat{W}$.

We now relate the vectors $v_{\mathfrak{a}}$ to distinguished elements of $\widehat{W}$. Suppose that $\mathfrak{i}$ is a $\mathfrak{h}$-stable subspace of $\mathfrak{g}$. Set

$$\Phi_{\mathfrak{i}} = \{\alpha \in \Delta^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{i}\}, \qquad \widehat{\Phi}_{\mathfrak{i}} = \{\delta - \alpha \mid \alpha \in \Phi_{\mathfrak{i}}\}.$$

$\bigwedge \mathfrak{g}$ and $\widehat{W}$.

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Theorem (*)

The following statements are equivalent

1. $\mathfrak{i}$ *is an abelian* $\mathfrak{b}$*-stable subspace of* $\mathfrak{g}$.
2. *There is an element* $w_{\mathfrak{i}} \in \widehat{W}$ *such that* $N(w_{\mathfrak{i}}) = \widehat{\Phi}_{\mathfrak{i}}$.
3. $\mathfrak{i}$ *is a* $\mathfrak{b}$*-stable subspace of* $\mathfrak{g}$ *and* $\Omega_{\mathfrak{g}} v_{\mathfrak{i}} = (\dim \mathfrak{i}) v_{\mathfrak{i}}$.

Proof of the Theorem

Proof.

1) $\implies$ 2). Set $p = \dim \mathfrak{i}$. Then, since $\mathfrak{i}$ is abelian, $\partial_p(\widehat{v_{\mathfrak{i}}}) = 0$. Notice that $\widehat{v_{\mathfrak{i}}} \in \wedge^{(p,p)}\mathfrak{u}^-$, so $\partial_p^*(\widehat{v_{\mathfrak{i}}}) = 0$. It follows that $L_p(\widehat{v_{\mathfrak{i}}}) = 0$. Since $\mathfrak{i}$ is $\mathfrak{b}$-stable, $\widehat{v_{\mathfrak{i}}}$ is a maximal vector for $\mathfrak{m}$ in $\wedge\mathfrak{u}^-$. By GL-Theorem, there is an element $w_{\mathfrak{i}} \in \widehat{W}$ such that $\wedge_{\alpha \in N(w_{\mathfrak{i}})} X_{-\alpha} = \widehat{v_{\mathfrak{i}}}$ and this implies that $N(w_{\mathfrak{i}}) = \widehat{\Phi}_{\mathfrak{i}}$.
2) $\implies$ 3). By GL-Theorem, we have that $\widehat{v_{\mathfrak{i}}}$ is a maximal vector for the action of $\mathfrak{m}$ on $\wedge\mathfrak{u}^-$, hence $\mathfrak{i}$ is a $\mathfrak{b}$-stable subspace of $\mathfrak{g}$. Moreover $L_p(\widehat{v_{\mathfrak{i}}}) = 0$ therefore

$$\Omega_{\mathfrak{g}}(\widehat{v_{\mathfrak{i}}}) = -(L_p + d)(\widehat{v_{\mathfrak{i}}}) = (dim\, \mathfrak{i})\widehat{v_{\mathfrak{i}}},$$

and this implies that $\Omega_{\mathfrak{g}} v_{\mathfrak{i}} = (dim\, \mathfrak{i}) v_{\mathfrak{i}}$.
3) $\implies$ 1). This follows from what we have seen about eigenvectors for $\Omega_{\mathfrak{g}}$. □

On the structure of $\bigwedge \mathfrak{g}$

Notation

$$\widehat{\mathfrak{C}}_p = Span(\widehat{v}_{\mathfrak{a}} \mid \mathfrak{a} \text{ abelian subalgebra of dimension } p)$$
$$\widehat{M}_p = \text{ eigenspace of eigenvalue } p \text{ for the action of } \Omega_{\mathfrak{g}} \text{ on } \wedge^{(p,p)}\mathfrak{u}^-$$
$$\mathfrak{a}_1, \dots, \mathfrak{a}_r : \text{ abelian ideals of } \mathfrak{b} \text{ of dimension } p$$
$$\mu_i = \langle N(w_{\mathfrak{a}_i}) \rangle = -\dim(\mathfrak{a}_i)\delta + \langle \Phi_{\mathfrak{a}_i} \rangle,$$
$$\widehat{J} = \text{ ideal (for exterior multiplication) in } \wedge\mathfrak{u}^- \text{ generated by } \partial_2^*(\mathfrak{u}^-),$$
$$\widehat{J}_p = \widehat{J} \cap \wedge^{(p,p)}\mathfrak{u}^-.$$

On the structure of $\bigwedge \mathfrak{g}$

Proposition

1. $\widehat{\mathfrak{C}}_p = \widehat{M}_p = \bigoplus_{i=1}^{r} V(\mu_i) = Ker(L_{p|\wedge^{(p,p)}}) = H_p(\mathfrak{u}^-)_p.$
2. *The following orthogonal decomposition with respect to* $\{\cdot,\cdot\}$ *holds:*

$$\wedge^{(p,p)}\mathfrak{u}^- = \widehat{\mathfrak{C}}_p \oplus \widehat{J}_p$$

In particular, letting $\mathcal{A}$ *be the subalgebra of* $\bigoplus_{p\geq 0} \wedge^{(p,p)}\mathfrak{u}^-$ *generated by* 1 *and* $\partial_2^*(\mathfrak{u}^-)$ *then*

$$\bigoplus_{p\geq 0} \wedge^{(p,p)}\mathfrak{u}^- = \mathcal{A} \wedge \sum_{p\geq 0} \widehat{\mathfrak{C}}_p$$

On the structure of $\bigwedge \mathfrak{g}$ – Proofs

Proof.

1). We know that the linear generators of $\widehat{\mathfrak{C}}_p$ are eigenvectors for $\Omega_\mathfrak{g}$ of eigenvalue p, hence $\widehat{\mathfrak{C}}_p \subseteq \widehat{M}_p$. Clearly, $\widehat{M}_p \subseteq Ker\, L_p$. We know from the first lecture that $w(\rho) - \rho = -\langle N(w) \rangle$. Combining this observation with GL-Theorem and Theorem $(*)$, we have that $Ker\, L_p = \bigoplus_{i=1}^{r} V(\mu_i)$.

Finally, by Theorem $(*)$, $V(\mu_i)$ is linearly generated by elements in $\widehat{\mathfrak{C}}_p$, hence $\bigoplus_{i=1}^{r} V(\mu_i) \subseteq \widehat{\mathfrak{C}}_p$.

On the structure of $\bigwedge \mathfrak{g}$ – Proofs

Proof.

2). We have

$$\widehat{\mathfrak{C}}_p^{\perp} = (KerL_p)^{\perp} = \partial_p^*(\wedge^{(p-1,p)}\mathfrak{u}^-).$$

The first equality is clear from part 1), whereas the second follows combining relation $H_p(\mathfrak{u}^-) = Ker\ L_p$ with the fact that $\wedge^{(p+1,p)}\mathfrak{u}^- = 0$.

On the structure of $\bigwedge \mathfrak{g}$ – Proofs

Proof.

It remains to prove that $\partial^*_p(\wedge^{(p-1,p)}\mathfrak{u}^-) = \widehat{J}_p$. Observe that, if $v \in \wedge^{(p-1,p)}\mathfrak{u}^-$, then necessarily v is a sum of decomposable elements of type $x^1_{-1} \wedge x^2_{-1} \wedge \cdots \wedge x^{p-1}_{-1}$. Assume that $v = x^1_{-1} \wedge x^2_{-1} \wedge \cdots \wedge x^{p-1}_{-1}$. Since ∂^* is a skew-derivation and $\partial^*_{p-1}(x^2_{-1} \wedge \cdots \wedge x^{p-1}_{-1}) \in \wedge^{(p-1,p-2)}\mathfrak{u}^- = 0$, we have

$$\partial^*_p(v) = \partial^*_2(x^1_{-1}) \wedge x^2_{-1} \wedge \cdots \wedge x^{p-1}_{-1},$$

so that $\partial^*_p(v) \in \widehat{J}_p$.

On the structure of $\bigwedge \mathfrak{g}$ – Proofs

Proof.

Conversely, if $w \in \widehat{J}_p$, then $w = \partial_2^*(x) \wedge y$ with $x \in \wedge^{(1,s)}\mathfrak{u}^-$, $y \in \wedge^{(p-2,r)}\mathfrak{u}^-$. Since $s + r = p$, $r \geq p-2$, $s \geq 2$, we have necessarily $s = 2, r = p-2$. Therefore $\partial_{p-1}^*(y) = 0$, hence $w = \partial_p^*(x \wedge y) \in \partial_p^*(\wedge^{(p-1,p)}\mathfrak{u}^-)$.
Finally, if $x \in \bigoplus_{p\geq 0} \wedge^{(p,p)}\mathfrak{u}^-$, then $x = a_1 + \partial_2^*(j_1) \wedge b_1$ with $a_1 \in \widehat{\mathfrak{C}}_p$, $j_1 \in \mathfrak{u}^-$, $b_1 \in \wedge^{(p-2,p-2)}\mathfrak{u}^-$. In turn, we can write $b_1 = a_2 + \partial_2^*(j_2) \wedge b_2$ as above, and so on. The last claim now follows. □

On the structure of $\bigwedge \mathfrak{g}$

Using the map $Z : \wedge^{(r,r)}\mathfrak{u}^- \to \wedge^r \mathfrak{g}$, the previous Proposition can be restated as a result on the algebra $\wedge\mathfrak{g}$. We set $\mathfrak{C}_p$ to be the linear span of the vectors $v_\mathfrak{a}$ when $\mathfrak{a}$ ranges over the set of commutative subalgebras of $\mathfrak{g}$ of dimension p, M_p to denote the eigenspace corresponding to the eigenvalue p for the action of $\Omega_\mathfrak{g}$ on $\wedge^p\mathfrak{p}$. Let J be the ideal (for exterior multiplication) in $\wedge\mathfrak{g}$ generated by $\mathbf{d}(\mathfrak{g})$ and set $J_p = J \cap \wedge^p\mathfrak{g}$.

Theorem

1. $\mathfrak{C}_p = M_p = \bigoplus_{i=1}^{r} L(\langle \mathfrak{a}_i \rangle).$
2. *The following orthogonal decomposition holds*

$$\wedge^p \mathfrak{g} = \mathfrak{C}_p \oplus J_p$$

Moreover, letting A be the subalgebra of $\bigwedge \mathfrak{g}$ generated by 1 *and* $\mathbf{d}(\mathfrak{g})$ *then* $\bigwedge \mathfrak{g} = A \wedge \sum_{p\geq 0} \mathfrak{C}_p$.

Euler product

Recall that we have denoted by

$$\phi(x) = \prod_{n=1}^{\infty}(1 - x^n) \in \mathbb{C}[[x]]$$

the Euler product. Its importance is due to the expansion

$$\frac{1}{\phi(x)} = \sum_{n\geq 0} p(n)x^n$$

where $p(n)$ is the classical partition function.

Euler product

Known facts

$$\phi(x) = \sum_{n\in\mathbb{Z}} (-1)^n x^{\frac{3n^2+n}{2}} = 1 - x - x^2 + x^5 + \dots \qquad \text{(Euler)}$$

$$\phi(x)^3 = \sum_{n\in\mathbb{Z}_{\geq 0}} (-1)^n (2n+1) x^{\frac{n^2+n}{2}} \qquad \text{(Jacobi, 1828)}$$

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$$\phi(x)^{10} = -----$$

Winquist (1969) has shown that he expression of $\phi(x)^{10}$ affords an elementary proof of Ramanujan's third congruence for p

$$p(11m+6) \equiv 0 \mod 11.$$

Macdonald formula

Let

$$\eta(x) = x^{1/24}\phi(x)$$

be the Dedekind η-function.

Theorem

If $\mathfrak{g}$ is a simple Lie algebra, Q its root lattice and $h^\vee$ is the dual Coxeter number

$$\eta(x)^{\dim \mathfrak{g}} = \sum_{\nu \in h^\vee Q} \frac{\prod_{\beta\in\Delta^+}(\nu+\rho,\beta)}{\prod_{\beta\in\Delta^+}(\rho,\beta)} x^{(\nu+\rho,\nu+\rho)}.$$

Kostant 1976

Theorem

1. *Macdonald formula implies*

$$\phi(x)^{\dim \mathfrak{g}} = \sum_{\nu \in P^+} \mathrm{tr}\,(\theta_\lambda(\tau)) \dim V_\lambda x^{Cas(\lambda)}.$$

where $\theta_\lambda : W \to GL(V_\lambda^0)$ *and* τ *is a Coxeter element.*

2. *If* $a = e^{2\pi\sqrt{-1}\ 2\rho}$ *then*

$$\mathrm{tr}\ \theta_\lambda(\tau) = \chi_\lambda(a) \in \{0, \pm 1\}.$$

Comment on the proof

Recall Freudenthal-de Vries strange formula

$$\dim \mathfrak{g}/24 = (\rho, \rho).$$

Then

$$\begin{aligned}\phi(x)^{\dim \mathfrak{g}} &= \eta(x)^{\dim \mathfrak{g}} x^{-\dim \mathfrak{g}/24} = \eta(x)^{\dim \mathfrak{g}} x^{-(\rho,\rho)} \\ &= \left(\sum_{\lambda \in P^+} \dim V_\lambda \epsilon(\lambda) x^{(\lambda+\rho,\lambda+\rho)}\right) x^{-(\rho,\rho)} = \sum_{\lambda \in P^+} \epsilon(\lambda) \dim V_\lambda x^{Cas(\lambda)}\end{aligned}$$

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The change of summation range is not completely obvious.
The theory of regular elements in the connected simply connected Lie group with Lie algebra $\mathfrak{g}$ allows to evaluate $\epsilon(\lambda)$ as $\chi_\lambda(a)$.

Kostant 2004

Proposition

$$\chi_\lambda(a) = \begin{cases} 0 & \text{if } \lambda \notin D^+, \\ (-1)^{\ell(w)} & \text{if } \lambda = \lambda^w. \end{cases}$$

Kostant 2004

Corollary

Recall that we set $\phi^{\dim \mathfrak{g}} = \sum_k b_k x^k$. *If* $k \leq h^\vee$, *then* $(-1)^k b_k = \dim M_k$.

Proof.

By the proposition $b_k = \sum_{w \in \widehat{W}^+, Cas(\lambda^w)=k} (-1)^{\ell(w)} \dim V_{\lambda^w}$. It can be shown that $Cas(\lambda^w) \geq \ell(w)$, hence $\ell(w) \leq k \leq h^\vee$, so that $w \in \widehat{W}_2^+$ and $\ell(w) = Cas(\lambda^w) = k$, so the formula reduces to

$$b_k = \sum_{w \in \widehat{W}_2^+, \ell(w)=k} (-1)^k \dim V_{\lambda^w} = \sum_{\mathfrak{i} \in \mathcal{I}_{ab}, \dim \mathfrak{i}=k} (-1)^k \dim V_{\langle \mathfrak{i} \rangle}$$

$$= (-1)^k \dim M_k = (-1)^k \dim \mathfrak{C}_k$$

□

Number theoretical applications

For any complex number s one can define s power of Euler product $\prod_{n=1}^{\infty}(1-x^n)$ by taking the logarithm of the Euler product, multiplying by s and then exponentiating.

$$\left(\prod_{n=1}^{\infty}(1-x^n)\right)^s = \sum_{k\geq 0} f_k(s)x^k, \tag{5.1}$$

Number theoretical applications

$$\left(\prod_{n=1}^{\infty}(1-x^n)\right)^s = \sum_{k\geq 0} f_k(s)x^k, \tag{5.1}$$

$f_k(s)$ is a polynomial of degree k defined as follows: Let $\mu : \mathbb{N} \to \mathbb{Q}$ be defined by putting $\mu(m) = \sum_{d|m} 1/d$. For $k, n \in \mathbb{N}$, $n \leq k$, let

$$Q_{k,n} = \{q \in \mathbb{N}^n \mid q = (m_1, \ldots, m_n), \sum_{i=1}^{n} m_i = k\}$$

and using this notation let

$$q_{k,n} = \sum_{q \in Q_{k,n}} \mu(m_1) \cdots \mu(m_n)$$

Put

$$f_0 = 1, \qquad f_k(s) = \sum_{n=1}^{k} q_{k,n} (-s)^n / n!$$

Number theoretical applications

In the following ressult Kostant gives a representation theoretical interpretation of some linear factors of f_2, f_3, f_4.

Proposition

We have $f_1(s) = -s$ *and*

$$\begin{aligned} f_2(s) &= 1/2!\, s(s-3) \\ -f_3(s) &= 1/3!\, s(s-1)(s-8) \\ f_4(s) &= 1/4!\, s(s-1)(s-3)(s-14) \end{aligned}$$

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Theorem

Let k be a positive integer. Then $f_k(s)$ is determined by the numbers $\dim \mathfrak{C}_k(sl_m)$ *for k different values of $m \in \mathbb{Z}_{\geq 0}$ where $m \geq k$ and $m > 1$.*

Proof of the Proposition

$$\left(\prod_{n=1}^{\infty}(1-x^n)\right)^s = \sum_{k\geq 0} f_k(s)x^k, \qquad (5.2)$$

$f_2(s) = 1/2!\, s(s-3) \quad -f_3(s) = 1/3!\, s(s-1)(s-8) \quad f_4(s) = 1/4!\, s(s-1)(s-3)(s-14)$

Euler has determined the right side of (5.2) when $s = 1$. The only nonzero coefficients on the right side of (5.2) are the coefficients of the pentagonal powers $x^{(3n^2-n)/2}$ where $n \in \mathbb{Z}$. Since 3 and 4 are not pentagonal numbers it follows that 1 must be a root of $f_3(s)$ and $f_4(s)$. Now Jacobi has determined the right side of (5.2) when $s = 3$. Here the only nonzero coefficients on the right side of (5.2) are the coefficients of the triangular powers $x^{n(n+1)/2}$ for $n \in \mathbb{Z}_+$. Since 4 is not a triangular number, 3 must be a root of $f_4(s)$.

Proof of the Proposition

$$\left(\prod_{n=1}^{\infty}(1-x^n)\right)^s = \sum_{k\geq 0} f_k(s)x^k, \tag{5.2}$$

$f_2(s) = 1/2!\, s(s-3) \quad -f_3(s) = 1/3!\, s(s-1)(s-8) \quad f_4(s) = 1/4!\, s(s-1)(s-3)(s-14)$

If M denotes the maximum dimension of an abelian subalgebra, we have $M < h^\vee$, just when $\mathfrak{g}$ is of A_1, A_2 and G_2. More precisely $M = 1, 2, 3$, $h^\vee = 2, 3, 4$, respectively. In these cases we have

$$(-1)^k \dim M_k = b_k = f_k(\dim \mathfrak{g}) \tag{5.3}$$

But then, $M_4 = \mathfrak{C}_4 = 0$ if $\mathfrak{g}$ is of type G_2, $M_3 = \mathfrak{C}_3 = 0$ if $\mathfrak{g}$ is of type A_2 and $M_2 = \mathfrak{C}_2 = 0$ is of type A_1 and by (5.3)

$$b_{h^\vee} = f_{h^\vee}(\dim \mathfrak{g}) = 0.$$

Hence we have proved that the missing roots are the complex dimensions of G_2, A_2 and A_1, namely $14, 8, 3$

Application to affine Lie algebras

Map $\mathfrak{g} \to so(\mathfrak{g})$ via the adjoint reps. This map can be lifted to $\widehat{\mathfrak{g}} \to \widehat{so(\mathfrak{g})}$

Application to affine Lie algebras

Map $\mathfrak{g} \to so(\mathfrak{g})$ via the adjoint reps. This map can be lifted to $\widehat{\mathfrak{g}} \to \widehat{so(\mathfrak{g})}$

Theorem (Cellini-Kac-Möseneder-P.)

Let $\epsilon = 0$ or 1. Then one has the following decomposition of the basic and vector $\widehat{so(\mathfrak{g})}$-modules with respect to $\widehat{\mathfrak{g}}$.

$$L(\tilde{\Lambda}_\epsilon) = \bigoplus_{\substack{\mathfrak{i} \in \mathcal{I}_{ab} \\ |\mathfrak{i}| \equiv \epsilon \ mod\ 2}} L(h^\vee \Lambda_0^{\mathfrak{g}} + \langle \mathfrak{i} \rangle - \frac{1}{2}(|\mathfrak{i}| - \epsilon)\delta)$$

Moreover, the highest weight vector $\mathbf{v}_\mathfrak{i}$ of the submodule $L(h^\vee \Lambda_0^{\mathfrak{g}} + \langle \mathfrak{i} \rangle - \frac{1}{2}(|\mathfrak{i}| - \epsilon)\delta)$ is, up to a constant factor, the following pure spinor (of the spin representation of $Cl_0(\tilde{\mathfrak{g}})$):

$$\mathbf{v}_\mathfrak{i} = \prod_{\alpha \in \Phi_\mathfrak{i}} (t^{-1} x_\alpha).$$

Combinatorics

ad-nilpotent ideals offer many nice examples in bijective combinatorics. An instance is the following

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Proposition (Andrews-Krattenthaler-Orsina.P.)

There is an explicit bijection between ad-nilpotent ideals in $sl(n)$ and Dyck paths of semilength n mapping ideals of class of nilpotence k to paths of height $k+1$.

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Proposition (Andrews-Krattenthaler-Orsina.P.)

There is an explicit bijection between ad-nilpotent ideals in $sl(n)$ and Dyck paths of semilength n mapping ideals of class of nilpotence k to paths of height $k+1$.

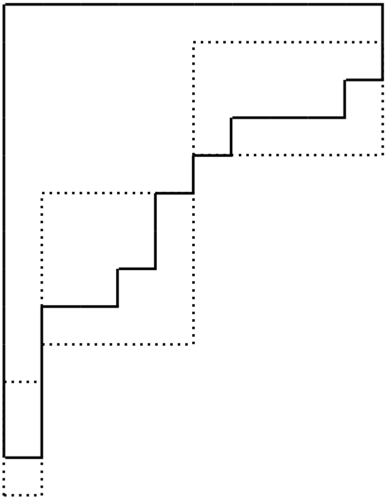
Panyushev has written several papers on questions having the same flavour: for instance, for sl_n, he proves that

$$\#\{\mathfrak{i} \in \mathcal{I} \mid \mathfrak{i} \text{ is generated by } k \text{ elements as a } \mathfrak{b}\text{-module}\} = \frac{1}{n}\binom{n}{k}\binom{n}{k+1},$$

the so-called *Narayana numbers*.

AKOP bijection

$D = (10, 10, 9, 6, 5, 4, 4, 3, 1, 1, 1, 1, 0) \subset \mathcal{T}_{13}$. $i_3 = 10$, $i_2 = 5$, $i_1 = 1$



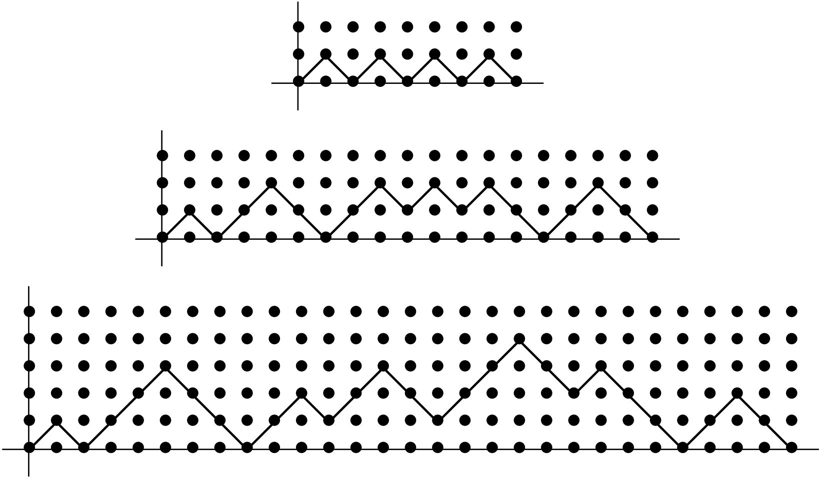
AKOP bijection

Procedure

1. Start with $n+1-i_k$ up-down pieces
2. Write the word corresponding to the path inside the k-th rectangle $l^{a_0} d\, l^{a_1} d \dots d\, l^{a_{i_k-i_{k-1}}}$
3. Insert $a_0, a_1, \dots$ picks
4. Iterate the proedure on the next rectangle

AKOP bijection

k-th rectangle: $dldllldl \rightarrow a_0 = 0, a_1 = 1, a_2 = 3, a_4 = 1$



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Combinatorics

Another seemingly unrelated topic relies on the following definition, coming from operational research

Definition

A subset $Y \subset S_n$ is *inversion complete* if $\bigcup_{x \in Y} N(x) = \Delta^+$, and is *minimal inversion complete* if it is inversion complete and minimal wrt this property.

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Clearly, the same definition extends to any Weyl group (indeed finite reflection group). Malvenuto-Möseneder-Orsina-P. started the investigation of MICS of maximum cardinality, and subsequent work of Panyushev made there approach more transparent.The most relevant unsolved problem is dealt with in the following

Conjecture

Let W be the Weyl group of $\mathfrak{g}$, simply laced. The maximum cardinality of a MICS equals the maximal dimension of an abelian subalgebra of $\mathfrak{g}$.

Representation theory

There is a nontrivial theory parallel to that of abelian ideals in the graded case. More precisely, let $\mathfrak{g}$ be a simple Lie algebra endowed with an indecomposable involution σ.

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There is a nontrivial theory parallel to that of abelian ideals in the graded case. More precisely, let $\mathfrak{g}$ be a simple Lie algebra endowed with an indecomposable involution σ.

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding Cartan decomposition. The analog of $\mathcal{I}_{ab}$ is

$$\mathcal{I}_{ab}^{\sigma} = \text{ abelian } \mathfrak{b}_0\text{-stable subspaces of } \mathfrak{g}_1.$$

Representation theory

$$\mathcal{I}_{ab}^{\sigma} = \text{ abelian } \mathfrak{b}_0\text{-stable subspaces of } \mathfrak{g}_1.$$

Results

1. There is an encoding of $\mathcal{I}_{ab}^{\sigma}$ via a subset $\mathcal{W}_{\sigma}^{ab} \in \widehat{W}$. The alcoves in $\mathcal{W}_{\sigma}^{ab} \in \widehat{W}$ pave a polytope with explicit defining inequalities. This description yields a uniform formula for $|\mathcal{I}_{ab}^{\sigma}|$ [Cellini-Möseneder-P., IMRN]
2. It is possible to describe the maximal elements in $\mathcal{I}_{ab}^{\sigma}$ and to obtain uniform enumerative formulas for their dimensions [Cellini-Möseneder-P.-Pasquali, Selecta]

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2. It is possible to describe the maximal elements in $\mathcal{I}^{\sigma}_{ab}$ and to obtain uniform enumerative formulas for their dimensions [Cellini-Möseneder-P.-Pasquali, Selecta]
3. There are many applications to affine and vertex algebras, especially in the theory of *conformal embeddings* [Adamovic-Kac-Möseneder-P.-Perse]

Connections with spherical varieties

Let G be a connected simply connected semisimple complex algebraic group with Lie algebra $\mathfrak{g}$. Let B be a Borel subgroup, and set $\mathfrak{b} = \mathrm{Lie} B$.

Panyushev

If $\mathfrak{a} \in \mathcal{I}_{ab}$ then

1. any nilpotent orbit meeting $\mathfrak{a}$ is a G-spherical variety
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Subsequently, Panyushev dealt with the $\mathbb{Z}_2$-graded case, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

Definition

We say that $\mathfrak{a} \in \mathcal{I}^{\sigma}_{ab}$ is *G-spherical* (resp. *G_0-spherical*) if all orbits Gx, $x \in \mathfrak{a}$ are G-spherical (resp. if all orbits G_0x, $x \in \mathfrak{a}$ are G_0-spherical).

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i) We clarify the connections between G_0-orbits of nilpotent elements in $\mathfrak{g}_1$, spherical G-orbits of nilpotent elements in $\mathfrak{g}_1$ and G_0-orbits of abelian subalgebras in $\mathfrak{g}_1$ which are stable under some Borel subalgebra of $\mathfrak{g}_0$.

ii) We prove that B_0 acts on $\mathfrak{a}$ with finitely many orbits, independently of its sphericity. Moreover, we parametrize orbits via orthogonal set of weights of $\mathfrak{a}$.

iii) Assume that there exist non-spherical subalgebras. We give a construction of a canonical non-spherical subalgebra $\mathfrak{a}_p$.

iv) We give a simple criterion to decide whether $\mathfrak{a}$ is spherical or not: in the main theorem we show that there exists $\overline{\mathfrak{a}} \in \mathcal{I}^{\sigma}_{ab}$ such that $\mathfrak{a}$ is non-spherical if and only if $\mathfrak{a} \supset \overline{\mathfrak{a}}$.