POSITIVE AND NEGATIVE CORRELATIONS FOR CONDITIONAL ISING DISTRIBUTIONS

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ABSTRACT. In the Ising model at zero external field with ferromagnetic first neighbors interaction the Gibbs measure is investigated using the group properties of the contours configurations. Correlation inequalities expressing positive dependence among groups and comparison among groups and cosets are used. An improved version of the Griffiths' inequalities is proved for the Gibbs measure conditioned to a subgroup. Examples of positive and negative correlations among the spin variables are proved under conditioning to a contour or to a separation line.

1. INTRODUCTION

We consider the Ising model on a finite set V, which to fix the ideas is chosen to be a cube of the d-dimensional integer lattice. The configuration space is

$$S_V = \{-1, +1\}^V$$

and an element $s \in S_V$ is a finite sequence

$$s = (s_i, \quad i \in V)$$

Two sites i and j are called first neighbors if their distance is 1; we denote by B the set of the first neighbors pairs of V. The Hamiltonian is

$$H(s) = - \sum_{\{i,j\} \in B} J_{ij} s_i s_j$$

where the interaction J_{ij} is assumed to be ferromagnetic, i.e.

$$J_{ij} \geq 0$$

The Gibbs measure μ at inverse temperature β with open boundary conditions is defined as

$$(1) \qquad \mu(s) = \frac{e^{-\beta H(s)}}{\sum_{s \in S_V} e^{-\beta H(s)}}$$

We denote by C the set of unit lines (or surfaces in $d > 2$ dimensions) which separate the first neighbors pairs; if $c \in C$, we denote the neighboring pair that is separated by c by $\{c_1, c_2\}$. Given the spins configuration s, the associated contours configuration γ_s is the subset of C defined as

$$\gamma_s = \{c \in C | s_{c_1} \neq s_{c_2}\}$$

The contours configurations γ_s are elements of $\Omega_C = \{0, 1\}^C$; the set of contours configurations is

$$\Gamma = \{\gamma \in \Omega_C | \gamma = \gamma_s \ \text{ for some } \ s \in S_V\}$$

In order to simplify the notations we consider an interaction, denoted by J, which is not dependent on the pair, but our results do not depend on this assumption. On the contours configurations one can define the probability measure

$$(2) \qquad \lambda_\Gamma(\gamma) = \frac{e^{-2\beta J|\gamma|}}{\sum_{\gamma \in \Gamma} e^{-2\beta J|\gamma|}}$$

where $|\gamma|$ denotes the length of the contours configuration γ. The Gibbs measure can be expressed in terms of the contours by means of the obvious equation

$$(3) \qquad \mu(s) = \lambda_\Gamma(\gamma_s)$$

If C is any finite set, Ω_C has a natural group structure: the product of two sets is their symmetric difference and the identity is the empty set. The contours configurations set Γ is in turn a subgroup of Ω_C: the product of two contours configurations γ_1 and γ_2, denoted by $\gamma_1 \cdot \gamma_2$, is a contours configuration and the identity is the empty contours configuration. For other properties see sec. 3.

The group structure of the classical lattice systems has been widely investigated in a general context (see for instance [7]). This structure has been recognized to be useful in connection with the correlation inequalities in [8] and used in [1]. In [3] the Gibbs measure was investigated as a Bernoulli one conditioned to a group and some correlation inequalities and monotonicity properties were proved; applications to the Ising model were also provided. In particular there was considered the extension of the measure in eq. (2), considering any subgroup G of Ω_C, where C is any finite set.

The conditional probability with respect to G is defined putting for any $E \subset G$

$$\lambda_G(E) = \sum_{\gamma \in E} \lambda_G(\gamma)$$

In [3] the following correlation inequalities were proved.

Proposition 1.1. *Let G be a subgroup of Ω_C and let E, F be subgroups of G and T a coset of E; then*

$$(4) \qquad \lambda_G(E) \ \geq \ \lambda_G(T)$$

$$(5) \qquad \lambda_G(E \cap F) \ \geq \ \lambda_G(E) \, \lambda_G(F)$$

The measure λ_G can be considered as a Bernoulli measure on Ω_C conditioned to the set G. If G is a subgroup, the inequality (5) states a positive dependence among pairs of events that are subgroups of G. It is reminiscent of the well known FKG inequality [4] for monotone events, but the kind of events here considered is very different. The aim of this paper is to discuss some applications of these inequalities using the contours representation of the Ising model. In particular we shall give an improved version of the Griffiths' inequalities [6]. These essentially state that the spins variables are positively correlated. We shall prove that this is also true for the Gibbs measure conditioned to a subgroup. We remark that our results depend on symmetry arguments (the external field is zero) and on the assumption of first neighbors interaction between two valued variables. We use them to analyze a new type of problems. In particular we give examples of positive and negative correlations among the spins for the measure conditioned to a contour or to the separation line between the phases. The next section is devoted to these applications; the other one contain a self consistent proof of the inequalities. The proof, which tries to simplify the one given in [3], is based on several ingredients. The first one is an expansion of the Gibbs weight based on the ferromagnetic assumption; the second one exploits the group properties of the contours configurations; the third one makes use of the FKG theorem. This theorem has been used as a powerful tool for proving both known and new inequalities. We refer for instance to [9] and [1].

2. Applications

We discuss the two dimensional Ising model, but our arguments are independent on the dimensionality. We first consider open boundary conditions. The unit lines in C are incident in vertexes of semiinteger coordinates. Let us denote by I the set of vertexes which are surrounded by four sites of V. More precisely, if (i_1, i_2) is a vertex of semiinteger coordinates, it belongs to I if the four sites $(i_1 \pm 1/2, i_2 \pm 1/2)$ belong to V. The group Γ is characterized by the following parity condition: a contours configuration γ belongs to Γ if the number of lines of γ that are incident

in any vertex of I is even (zero, two or four). This group can be considered as the intersection of the local groups K_i just defined:

$$\Gamma = \cap_{i \in I} K_i$$

Let us denote by ∂I the set of vertexes which are surrounded by a number of sites of V greater than zero and smaller than four (boundary vertexes). In ∂I the contours configurations do not satisfy the parity condition (the incident lines are zero or one). Hence the contours can be 'open' in ∂I. The spins configurations set S_V has also a natural group structure: the product of two spins configurations s and t is given by the ordinary pointwise product, that we denote by $s \times t$, with the obvious identity. We notice that

$$\gamma_s \cdot \gamma_t = \gamma_{s \times t}$$

This implies that the map $s \rightarrow \gamma_s$ maps subgroups into subgroups. This map is not injective since $\gamma_s = \gamma_{-s}$. The cylindrical event $\{s | s_i = 1\}$ is a subgroup but the event $\{s | s_i = -1\}$ is not. We are interested in the conditional measure with respect to a given subset G of S_V, defined as

$$(6) \qquad\qquad \mu_G(s) = \frac{e^{-\beta H(s)}}{\sum_{s \in G} e^{-\beta H(s)}}$$

if $s \in G$ and zero otherwise. We consider only events which are closed under the spin flip operation. If E is such an event, we denote by $\tilde{E}$ its image in the contours configurations. Hence the Gibbs measure of E conditioned to G is represented in terms of a conditioned measure on the contours by

$$(7) \qquad\qquad \mu_G(E) = \lambda_{\tilde{G}}(\tilde{E})$$

Comparison with the Griffiths' inequalities

We now consider the relationship between inequality (5) and the Griffiths' correlation inequalities [6]

$$(8) \qquad\qquad \langle s_A \rangle \geq 0$$

$$(9) \qquad\qquad \langle s_A s_B \rangle \geq \langle s_A \rangle \langle s_B \rangle$$

where $\langle \ \rangle$ denotes the expectation with respect to the Gibbs measure and $s_A = \prod_{i \in A} s_i$. We also denote by $\langle s_A \rangle_G$ the conditional expectation of s_A with respect to G and if G is any subgroup we state an improved version of the inequalities.

Proposition 2.1. *Let G be any subgroup closed under the spin flip of the spins configurations space and A , B disjoint lattice sets of even cardinality; then*

$$(10) \qquad\qquad \langle s_A \rangle_G \geq \langle s_A \rangle$$

and so

$$(11) \qquad\qquad \langle s_A \rangle_G \geq 0$$

furthermore

$$(12) \qquad\qquad \langle s_A s_B \rangle_G \geq \langle s_A \rangle_G \langle s_B \rangle_G$$

Proof. Let us denote by χ_A the indicator function of the event $E_A = \{s | s_A = 1\}$. This event is a group and is closed under the spin flip. One has

$$s_A = 2\chi_A(s) - 1$$

and

$$\langle s_A \rangle_G = 2\mu_G(E_A) - 1 = 2\lambda_{\tilde{G}}(\tilde{E}_A) - 1$$

where $\tilde{G}$ is a subgroup of Γ. Using the inequality (5) one has

$$\lambda_{\tilde{G}}(\tilde{E}_A) = \frac{\lambda_\Gamma(\tilde{E}_A \cap \tilde{G})}{\lambda_\Gamma(\tilde{G})} \geq \lambda_\Gamma(\tilde{E}_A)$$

and so

$$\langle s_A \rangle_G \geq \langle s_A \rangle \geq 0$$

We also have

$$\langle s_A s_B \rangle_G = \langle (2\chi_A - 1)(2\chi_B - 1) \rangle_G = 4\lambda_{\tilde{G}}(\tilde{E}_A \cap \tilde{E}_B) - 2\lambda_{\tilde{G}}(\tilde{E}_A) - 2\lambda_{\tilde{G}}(\tilde{E}_B) + 1$$

Since the events $\tilde{E}_A$ and $\tilde{E}_B$ are subgroups, the inequality (5) gives

$$\lambda_{\tilde{G}}(\tilde{E}_A \cap \tilde{E}_B) \geq \lambda_{\tilde{G}}(\tilde{E}_A)\lambda_{\tilde{G}}(\tilde{E}_B)$$

and then the inequality (12) follows. $\qquad\square$

Remark The inequalities (11) and (12) can be obtained also using the standard method of proof in the natural spins representation (see [5]). According to this method the main point is now to prove that for any subgroup G and any lattice subset A one has

$$(13) \qquad\qquad \sum_{s \in G} s_A \geq 0$$

If s' is a duplicated spins configuration, one obviously has

$$\sum_{s \in G} s_A \sum_{s' \in G} s'_A \geq 0$$

Since s and s' are both elements of the group G, there is a unique $t \in G$ such that $s' = ts$. Hence the left hand side is

$$\sum_{(s,s') \in G \times G} s_A s'_A = \sum_{(s,t) \in G \times G} s_A t_A s_A = |G| \sum_{t \in G} t_A$$

and this proves the inequality (13). If L is a coset of G it is natural to ask for the sign of

(14)
$$\sum_{s \in L} s_A$$

Since $L = l \times G$ for some $l \in L$, one has

$$\sum_{s \in L} s_A = l_A \sum_{s \in G} s_A$$

and in general there is no simple inequality for this sum. Hence we are led to consider particular cosets and this can be naturally done in the contours description.

Conditioning to a contour or to a separation line

We are interested in conditioning to an event which is a cylinder in the contours. Given $\alpha \subset C$ the cylinder

$$L(\alpha) = \{\gamma \in \Gamma | \gamma \cap \alpha = \alpha\}$$

is the set of contours configurations that occupy the lines in α; given $\eta \subset C$ the cylinder

$$G(\eta) = \{\gamma \in \Gamma | \gamma \cap \eta = \emptyset\}$$

is the set of configurations that occupy only lines outside η. In the 0, 1 language, $L(\alpha)$ is a cylinder of 1's and $G(\eta)$ is a cylinder of 0's. It is easy to check that $G(\eta)$ is a subgroup of Γ and $L(\alpha)$ is a coset of $G(\alpha)$. If $\alpha \cap \eta = \emptyset$ we also consider the cylinder $G(\eta) \cap L(\alpha)$, which is a coset of $G(\alpha \cup \eta)$. In order to simplify the notations we denote the corresponding sets in the spins by the same notations.

We want to study the Gibbs measure conditioned to $G(\eta) \cap L(\alpha)$. This can be done using the local character of these events and the additivity of the Hamiltonian in the contours configurations. Hence in order to have a 0 conditioning on η and a 1 conditioning on α it is sufficient to restrict the space of contours to those which lie outside $\alpha \cup \eta$. One has just to guarantee that these contours are compatible with α, in the sense that they satisfy the parity conditions. What follows is just a formal exposition of the above ideas.

We consider the subset of $\Omega_{C\setminus(\alpha\cup\eta)}$ defined as

$$\Gamma(\alpha \cup \eta) = \{\rho \in \Omega_{C\setminus(\alpha\cup\eta)} \mid \rho \cup \alpha \in \Gamma\}$$

Obviously there is a one to one map between $\Gamma(\alpha \cup \eta)$ and $L(\alpha) \cap G(\eta)$ given by $\gamma = \rho \cup \alpha$ and one has

$$|\gamma| = |\rho| + |\alpha|$$

If one chooses $\alpha \in \Gamma$, then $\Gamma(\alpha \cup \eta)$ is a subgroup of $\Omega_{C\setminus(\alpha\cup\eta)}$ with the usual product. Actually, the empty subset of $C \setminus (\alpha \cup \eta)$ is the identity and considering the elements $\rho_1, \rho_2 \in \Gamma(\alpha \cup \eta)$ one has $\rho_1 \cdot \rho_2 \in \Gamma(\alpha \cup \eta)$ since $(\rho_1 \cdot \rho_2) \cup \alpha \in \Gamma$. The group $\Gamma(\alpha \cup \eta)$ can also be considered as the intersection of local groups, according to the equation

$$\Gamma(\alpha \cup \eta) = \cap_{i\in I} K_i(\alpha \cup \eta)$$

where the parity condition now depends on the fact that both the unit lines of α and η are not to be occupied. We remark that this new space of contours configurations is corresponding to spins ones only if α is a contours configuration. The Gibbs measure conditioned to $L(\alpha) \cap G(\eta)$ is simply given by the measure on $\Gamma(\alpha \cup \eta)$ defined as

$$(15) \qquad \lambda_{\Gamma(\alpha\cup\eta)}(\rho) = \frac{e^{-2\beta J|\rho|}}{\sum_{\rho\in\Gamma(\alpha\cup\eta)} e^{-2\beta J|\rho|}}$$

Now the correspondence between the spins and the contours which takes the place of (3) is

$$(16) \qquad \mu_{L(\alpha)\cap G(\eta)}(s) = \lambda_{\Gamma(\alpha\cup\eta)}(\rho_s)$$

where ρ_s is such that $\gamma_s = \rho_s \cup \alpha$.

Let us now consider the particular case of conditioning to $L(\alpha)$. Just to fix the ideas choose α to be the a simple loop, say the boundary of a square. In this case we say that i and j are separated by α if they can be joined by a path which intersects α only one time, and they are not separated if there is a path which does not intersect α. For any contours configuration α we say that the sites are separated by α if the number of intersections is odd and they are not separated if this number is even. We can prove the following.

Proposition 2.2. *Given two sites i and j and a contours configuration α, one has*

(17) $$\langle s_i s_j \rangle_{L(\alpha)} \leq 0$$

if the two sites are separated by α and

(18) $$\langle s_i s_j \rangle_{L(\alpha)} \geq 0$$

if they are not separated.

Proof. Conditioning to $L(\alpha)$, the set $s_i = s_j$ corresponds to 'i and j are separated by an odd number of contours', since one separating contour is given by α. Denoting by $F_{ij} = $ 'i and j are separated by an even number of contours', with obvious notations one has

$$\langle s_i s_j \rangle_{L(\alpha)} = \lambda_{\Gamma(\alpha)}(F_{ij}^c) - \lambda_{\Gamma(\alpha)}(F_{ij})$$

The event F_{ij}^c is a coset of the group F_{ij}. Using eq. (4) in the configurations space

$\Gamma(\alpha)$ one has

$$\lambda_{\Gamma(\alpha)}(F_{ij}) \geq \lambda_{\Gamma(\alpha)}(F_{ij}^c)$$

and the inequality

$$\langle s_i s_j \rangle_{L(\alpha)} \leq 0$$

follows. The other stated inequality can be proved using a similar argument.   □

This provides an example of negative dependence; we refer to [10] where related problems and recent results are discussed.

A simple consequence is the following one. Given two lattice sites i and j we denote by $R_0(i,j)$ the event 'i and j belong to the same cluster', where, as usual, a cluster is a maximal connected set of sites having the same spin. We also denote by $R_1(i,j)$ the complement, i.e. 'the sites belong to different clusters'. We also denote by $\langle s_i s_j \rangle_0$ and $\langle s_i s_j \rangle_1$ the conditional expectations with respect to $R_0(i,j)$ and $R_1(i,j)$. One obviously has $\langle s_i s_j \rangle_0 = 1$ and it is natural to ask for the sign of $\langle s_i s_j \rangle_1$. We can prove the following

Proposition 2.3. *The expectation of $s_i s_j$ conditioned to 'i and j belong to different clusters' is non positive, i.e.*

(19) $$\langle s_i s_j \rangle_1 \leq 0$$

Proof. Fix a site i and a spin configuration s. It is so defined the set η of the lines which separate the points of the cluster to which i belongs in s and the set α of the lines which separate the points of the cluster from the outside. If $s \in R_1(i,j)$ there exist α and η such that $s \in L(\alpha) \cap G(\eta)$ and one can so define a partition of $R_1(i,j)$. We now have

(20) $$\langle s_i s_j \rangle_1 = \sum_{\alpha,\eta} \langle s_i s_j \rangle_{L(\alpha) \cap G(\eta)} \, \mu(L(\alpha) \cap G(\eta))$$

where the sum is over all the elements of the partition. We also have

$$\langle s_i s_j \rangle_{L(\alpha) \cap G(\eta)} = \lambda_{\Gamma(\alpha \cup \eta)}(F_{ij}^c) - \lambda_{\Gamma(\alpha \cup \eta)}(F_{ij})$$

and using the argument used in the above proposition we get

$$\langle s_i s_j \rangle_{L(\alpha) \cap G(\eta)} \le 0$$

and the inequality (19) follows. □

We now consider the separation line between the phases in the two dimensional Ising model and, as usual, we put + and - boundary conditions respectively on the upper half and on the lower half of a square box. Let us denote by a and b the unit lines that separate the + spins from the - ones on the boundary. We define the separation line ξ as the maximal connected component of the contours configuration that joins a and b. Given two lattice sites i and j and a spin configuration s we say that they are separated by ξ if i can be connected, say, to the upper half boundary without crossing ξ and j can be connected to the lower half boundary without crossing ξ. The event 'the points are (not) separated' is the set of spin configurations such that there is a line which (not) separates the points and it is denoted by L_1 (L_0).

Proposition 2.4. *In the model with separation line, given two lattice points i and j, one has*

$$(21) \qquad\qquad \langle s_i s_j \rangle_{L_1} \le 0$$

and

$$(22) \qquad\qquad \langle s_i s_j \rangle_{L_o} \ge 0$$

Proof. Let us denote by $L(\xi)$ the event in the spins configurations that the line is ξ. Conditioning to this event is equivalent to consider the reduced contours configurations $\Gamma(\xi)$. If i and j are separated by ξ, the event $s_i = s_j$ corresponds to 'i and j are separated by an odd number of contours' in $\Gamma(\xi)$. Actually any path that joins i and j crosses ξ an odd number of times. With the same notations used before one has

$$\langle s_i s_j \rangle_{L(\xi)} = \lambda_{\Gamma(\xi)}(F_{ij}^c) - \lambda_{\Gamma(\xi)}(F_{ij})$$

and so for any ξ this gives

$$\langle s_i s_j \rangle_{L(\xi)} \le 0$$

Similarly one has

$$\langle s_i s_j \rangle_{L(\xi)} \ge 0$$

if the sites are not separated by ξ. Using the same arguments of the above proposition the result follows. □

3. Proof of the inequalities

An expansion for the ferromagnetic Ising measure

We provide an expansion of the measure λ_Γ based on the factorization of the Gibbs weight of the contours configurations and on the ferromagnetic character of the interaction. We define $x = e^{-2\beta J}/(1 + e^{-2\beta J})$ and write the measure λ_Γ as

$$\text{(23)} \qquad \lambda_\Gamma(\gamma) = \frac{x^{|\gamma|}(1-x)^{|\gamma^c|}}{\sum_{\gamma \in \Gamma} x^{|\gamma|}(1-x)^{|\gamma^c|}}$$

where γ^c denotes the complement of γ in C.

This equation states that the Gibbs measure can be considered as a product one on the space Ω_C conditioned to the subset Γ of contours configurations. We now define the probability measure on Ω_C

$$\text{(24)} \qquad \nu_\Gamma(\omega) = \frac{(1-2x)^{|\omega|} x^{|C \setminus \omega|} \, |\Gamma_0^\omega|}{\sum_{\omega \subset C}(1-2x)^{|\omega|} x^{|C \setminus \omega|} \, |\Gamma_0^\omega|}$$

were we have used the notation

$$E_0^\omega = \{\gamma | \gamma \in E, \gamma \cap \omega = \emptyset\}$$

Using this measure we can state the following representation.

Proposition 3.1. *The ferromagnetic Ising measure λ_Γ can be represented in terms of the measure ν_Γ on Ω_C as*

$$\text{(25)} \qquad \lambda_\Gamma(E) = \sum_{\omega \subset C} \nu_\Gamma(\omega) \frac{|E_0^\omega|}{|\Gamma_0^\omega|}$$

Proof. From the ferromagnetic hypothesis, $J \geq 0$, one has $x \leq 1/2$ and so the following expansion

$$(1-x)^{|\gamma^c|} = \sum_{\omega \subset \gamma^c} (1-2x)^{|\omega|} x^{|\gamma^c \setminus \omega|}$$

has only non negative summands. We also get

$$x^{|\gamma|}(1-x)^{|\gamma^c|} = \sum_{\omega \subset \gamma^c}(1-2x)^{|\omega|}x^{|C\setminus\omega|}$$

Let us compute the λ_Γ probability of an event E:

$$\lambda_\Gamma(E) = \frac{\sum_{\gamma \in E} x^{|\gamma|}(1-x)^{|\gamma^c|}}{\sum_{\gamma \in \Gamma} x^{|\gamma|}(1-x)^{|\gamma^c|}}$$

Using the above expansion the numerator is given by

(26)
$$\sum_{\gamma \in E}\sum_{\omega \subset \gamma^c}(1-2x)^{|\omega|}x^{|C\setminus\omega|} = \sum_{\omega \subset C}(1-2x)^{|\omega|}x^{|C\setminus\omega|}\,|E_0^\omega|$$

where we have used

$$\sum_{\gamma \in E,\,\gamma \subset \omega^c} 1 \;=\; |E_0^\omega|$$

and the denominator by

(27)
$$\sum_{\gamma \in \Gamma}\sum_{\omega \subset \gamma^c}(1-2x)^{|\omega|}x^{|C\setminus\omega|} = \sum_{\omega \subset C}(1-2x)^{|\omega|}x^{|C\setminus\omega|}\,|\Gamma_0^\omega|$$

Using the definition of ν_Γ, this completes the proof. $\qquad\square$

We also write
$$\lambda_\Gamma(E) = \nu_\Gamma(f_E)$$
where the right hand side denotes the average with respect to ν_Γ of the function f_E on Ω_C defined by

$$f_E(\omega) = \frac{|E_0^\omega|}{|\Gamma_0^\omega|}$$

We notice that while the contours configurations measure λ_Γ is a measure conditioned to the subset Γ, the measure ν_Γ is defined in all the space Ω_C.

The group structure of the contours

We now recall some of the properties of the group Ω_C (for a general reference see for instance [2]). We shall use that the group is commutative and that $\omega^{-1} = \omega$. If G is a subgroup the binary relation $\sim$ defined by $\omega_1 \sim \omega_2$ if and only if $\omega_1 \cdot \omega_2 \in G$ is an equivalence relation. The elements of the partition of Ω_C so defined are the cosets of G. The subgroup itself is an element of the partition. Any coset L different from G is so disjoint from G ad is given by

$$L = \sigma \cdot G = \{\alpha \in \Omega_C | \alpha = \sigma \cdot \omega, \, \omega \in G\}$$

for any $\sigma \in L$. We shall use that G and L have the same cardinality: $|G| = |L|$. Given two cosets H, L the set

$$H \cdot L = \{\alpha \in \Omega_C | \, \alpha = \sigma \cdot \omega, \, \sigma \in H, \, \omega \in L\}$$

is a coset. The set of the cosets of a group G is itself a group with respect to the product above defined, the identity being G. If F is a subgroup of G the quotient G/F is a group whose elements are the cosets of F. Hence its cardinality is given by the equation

$$(28) \qquad\qquad\qquad |G/F| = \frac{|G|}{|F|}$$

We shall use the following result

Lemma 3.2. *For any two subgroups E, F of the group G one has*

$$(29) \qquad\qquad |E \cdot F||E \cap F| = |E||F|$$

Proof. We notice that both $E \cdot F$ and $E \cap F$ are subgroups of G. We first consider the case $E \cap F = \{\emptyset\}$, i.e. the two groups have in common only the identity. In this case from the definitions it follows easily that

$$|E \cdot F| = |E||F|$$

In the general case we consider the quotient with respect to $E \cap F$ of the groups $E, F, E \cdot F$. These quotients that we denote by $E/(E \cap F), F/(E \cap F), (E \cdot F)/(E \cap F)$, are groups, the identity being $E \cap F$. Since the two first have in common only the identity, the above equation gives

$$|(E \cdot F)/(E \cap F)| = |E/(E \cap F)| \, |F/(E \cap F)|$$

From the eq. (28) it follows that

$$\frac{|E \cdot F|}{|E \cap F|} = \frac{|E|}{|E \cap F|} \frac{|F|}{|E \cap F|}$$

and this proves the lemma.

If $\sigma \subset C$ we denote

$$(30) \qquad E_\alpha^\sigma = \{\omega \in \Omega_C | \, \omega \in E, \, \omega \cap \sigma = \alpha\}$$

In the sequel we shall use the following property, whose proof is a direct consequence of the definitions: G_0^σ is a group and G_α^σ are its cosets. As a consequence one has

$$(31) \qquad |G_\alpha^\sigma| = |G_0^\sigma|$$

Furthermore for any α and β

$$(32) \qquad G_\alpha^\sigma \cdot G_\beta^\sigma = G_{\alpha \cdot \beta}^\sigma$$

Proof of the inequality (4)

We use the representation (25) for $\lambda_G(E)$ and $\lambda_G(T)$ and the obvious fact that if T_0^ω is non empty it is a coset of E_0^ω. Hence $|E_0^\omega| \geq |T_0^\omega|$ and one easily gets the result. $\square$

The FKG structure

The set Ω_C has a natural order structure based on the partial order

$$\omega_1 \leq \omega_2 \quad \text{if} \quad \omega_1 \subset \omega_2$$

A function f on Ω_C is called 'increasing' if

$$\omega_1 \leq \omega_2 \quad \Rightarrow \quad f(\omega_1) \leq f(\omega_2)$$

A similar definition is given for decreasing functions. A probability measure μ on Ω_C is said to be 'positively associated' or to have the FKG property if for any two increasing (or decreasing) functions f, g the following inequality holds for the expectations with respect to μ

$$(33) \qquad \mu(fg) \geq \mu(f) \, \mu(g)$$

An event is called 'increasing' if its indicator function is so. Hence if two events A, B are both increasing, it follows

$$(34) \qquad \mu(A \cap B) \geq \mu(A) \, \mu(B)$$

A sufficient condition for positive association is [4]

$$(35) \qquad \mu(\omega_1 \cup \omega_2) \, \mu(\omega_1 \cap \omega_2) \geq \mu(\omega_1) \, \mu(\omega_2)$$

Using the spins language version of this condition one can get, as it is well known, that the Ising ferromagnetic measure is associated. We are looking for a similar property in the contours language. We will first show that the probability measure ν_Γ has the FKG property. We then show that if E is a group then the function f_E is monotonic and finally we use the representation (25). This will be sufficient to deduce the correlation inequality.

Proposition 3.3. *The probability measure ν_Γ is FKG.*

Proof. We shall prove the more general statement for the measure ν_G where G is any subgroup of Ω_C. We shall check the sufficient condition

$$\nu_G(\omega_1 \cup \omega_2)\nu_G(\omega_1 \cap \omega_2) \geq \nu_G(\omega_1)\nu_G(\omega_2)$$

which is equivalent to

$$(36) \qquad\qquad |G_0^{\omega_1\cup\omega_2}||G_0^{\omega_1\cap\omega_2}| \geq |G_0^{\omega_1}||G_0^{\omega_2}|$$

We first consider the case $\omega_1 \cap \omega_2 = \emptyset$ in which $G_0^{\omega_1\cap\omega_2} = G$. Hence we have to prove

$$(37) \qquad\qquad |G_0^{\omega_1\cup\omega_2}||G| \geq |G_0^{\omega_1}||G_0^{\omega_2}|$$

We have

$$|G_0^{\omega_1}| = \sum_{\alpha_2\subset\omega_2} |G_{\alpha_2}^{\omega_1\cup\omega_2}|; \quad |G_0^{\omega_2}| = \sum_{\alpha_1\subset\omega_1} |G_{\alpha_1}^{\omega_1\cup\omega_2}|; \quad G = \sum_{\alpha_1\subset\omega_1}\sum_{\alpha_2\subset\omega_2} |G_{\alpha_1\cup\alpha_2}^{\omega_1\cup\omega_2}|$$

The sets which appear in the sums, if non empty, are cosets of the group $G_0^{\omega_1\cup\omega_2}$. From the eq. (32) if $\alpha_1 \subset \omega_1, \alpha_2 \subset \omega_2$, it easily follows that

$$G_{\alpha_1}^{\omega_1\cup\omega_2} \neq \emptyset, \quad G_{\alpha_2}^{\omega_1\cup\omega_2} \neq \emptyset \quad \Rightarrow \quad G_{\alpha_1\cdot\alpha_2}^{\omega_1\cup\omega_2} = G_{\alpha_1}^{\omega_1\cup\omega_2} \cdot G_{\alpha_2}^{\omega_1\cup\omega_2} \neq \emptyset$$

and obviously one has $\alpha_1 \cdot \alpha_2 = \alpha_1 \cup \alpha_2$. Since all the sets that appear in the sums have the same cardinality (if non empty) one gets

$$|G_{\alpha_1}^{\omega_1\cup\omega_2}| \, |G_{\alpha_2}^{\omega_1\cup\omega_2}| \leq |G_0^{\omega_1\cup\omega_2}| \, |G_{\alpha_1\cup\alpha_2}^{\omega_1\cup\omega_2}|$$

Using this inequality we easily get eq. (37).

We now consider the case $\omega_1 \cap \omega_2 = \tau \neq \emptyset$. We put $\tau_1 = \omega_1 \setminus \tau, \tau_2 = \omega_2 \setminus \tau$ and since $\tau_1 \cap \tau_2 = \emptyset$ we apply the above argument to the group G_0^τ in place of G, and this completes the proof. $\qquad\square$

Proposition 3.4. *If E is a subgroup of G, the function defined on Ω_C by*

$$f_E(\omega) = \frac{|E_0^\omega|}{|G_0^\omega|}$$

is increasing.

Proof. We have to prove that for each ω and $i \in C \setminus \omega$ one has

$$(38) \qquad\qquad f_E(\omega) \leq f_E(\omega \cup \{i\})$$

We use the notation

$$E_{01}^{\omega i} = \{\rho \in E | \rho \cap \omega = \emptyset, \rho \cap \{i\} = \{i\}\}$$

and the similar one for $E_{00}^{\omega i}$ and for G. Hence

$$f_E(\omega \cup \{i\}) = \frac{|E_{00}^{\omega i}|}{|G_{00}^{\omega i}|}$$

and the inequality (38) is equivalent to

(39) $$|G_0^{\omega}|\,|E_{00}^{\omega i}| \geq |E_0^{\omega}|\,|G_{00}^{\omega i}|$$

We have

$$|E_0^{\omega}| = |E_{00}^{\omega i}| + |E_{01}^{\omega i}|; \quad |G_0^{\omega}| = |G_{00}^{\omega i}| + |G_{01}^{\omega i}|$$

so the above inequality is equivalent to

(40) $$|G_{01}^{\omega i}||E_{00}^{\omega i}| \geq |E_{01}^{\omega i}||G_{00}^{\omega i}|$$

If $E_{01}^{\omega i} = \emptyset$ this inequality is trivially true. Suppose that this set is non empty. It is a coset of the group $E_{00}^{\omega i}$ and so it has the same cardinality; in addition it follows that also the coset $G_{01}^{\omega i}$ is non empty and it has the same cardinality of $G_{00}^{\omega i}$. In this case the eq. (40) holds as an equality. We notice that if $G_{01}^{\omega i}$ were empty, also $E_{01}^{\omega i}$ would be so, since by hypothesis $E \subset G$. □

Proof of the correlation inequality (5)

We prove the more general statement for any group G. From the representation (25) one gets

$$\lambda_G(E \cap F) = \sum_{\omega \subset C} \nu_G(\omega) \frac{|(E \cap F)_0^{\omega}|}{|G_0^{\omega}|}$$

We shall use for each $\omega \subset C$ the following inequality

(41) $$\frac{|(E \cap F)_0^{\omega}|}{|G_0^{\omega}|} \geq \frac{|E_0^{\omega}|}{|G_0^{\omega}|}\,\frac{|F_0^{\omega}|}{|G_0^{\omega}|}$$

and since the two functions at the right hand side are both increasing, the FKG theorem proves the statement. From the eq. (29), since $E \cdot F \subset G$, one has

(42) $$|G||E \cap F| \geq |E||F|$$

We now use that $(E \cap F)_0^{\omega} = E_0^{\omega} \cap F_0^{\omega}$ and the fact that $E_0^{\omega}, F_0^{\omega}$ are subgroups of G_0^{ω}. The above inequality then gives ineq. (41). □

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