Negative conclusion cases: further proposal for likelihood ratio evaluation

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Abstract

The use of Bayesian approach in forensic science requires the evaluation of likelihood ratio related to the crime scene evidence (denoted as event E) and the suspect characteristic (denoted as event C). This evaluation is trivially naught when the two events are disjoint, and it is a fraction with zero denominator when $E \subseteq C$.

In this paper we define an extended likelihood ratio, which is well-defined and different from zero for any pair E and C, using the theory of copulas. This theory allows us to extend our previous paper [1], that was restricted to Gaussian database, to a general database.

For different kinds of copulas (Fréchet, Cuadras-Augé and normal copulas), with a correlation coefficient r (with $0 < r < 1$), we show that the likelihood ratio has larger values when both the events are in the tail of the distribution, as expected. Moreover, it reduces to the standard one when r tends to 1, and its value is 1 in the case of independence ($r = 0$).

We propose three different approaches in choosing the parameter r in real cases. In the first, r is chosen as a fixed parameter (for instance $r = 0.95$); in the second, in order to over-estimate the extended likelihood ratio LR^r, the value of the parameter r should be that which corresponds to the supremum of LR^r. In the third approach, a choice of a maximal score K for the likelihood ratio should determine the value of the parameter r.

Application in the case of height is performed using Italian Carabinieri database.

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1 Introduction

The aim of forensic science is to support intelligence and the judge in making decision in Court by means of a scientific approach in evaluation of evidences. Accordingly, the forensic scientist has to present the results of his analysis in such appropriate way that no doubt of interpretation could arise.
The widespread accepted approach in the evaluation of evidences is the probabilistic one, based on the Bayesian model [1, 2, 3, 4, 5, 6]. For example, we can consider an event C related to a blood group of a biological stain, or to a refractive index (RI) of a glass fragment, or to a height, and so on, concerning the suspect of the crime; so the event C is usually related to the guilt of a person, or his presence on the crime scene, or his responsibility in breaking a glass window, and so on.
In a suitable probability space with a probability P, the *a priori* odds $O(C)$ in favor of the event C is defined by:

$$O(C) = \frac{P(C)}{P(\bar{C})} \tag{1}$$

where $\bar{C}$ is the complementary event of C, and so $P(\bar{C}) = 1 - P(C)$.
When information changes the scenario, new elements could be considered, and they could improve or not the evaluation of odds. If the event E represents the evidence, the judge must evaluate the *a posteriori* odds $O(C|E)$, which is defined by:

$$O(C|E) = \frac{P(C|E)}{P(\bar{C}|E)} \tag{2}$$

where the usual notation for the conditional probability of the event A with respect to the event B has been used, i.e.:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{3}$$

For example, the evidence E represents information that arises from the crime scene analysis, and concerns the same variable of C: for instance, they are both intervals of heights.

According to Bayes' theorem, the *a posteriori* odds $O(C|E)$ can be written with respect to the *a priori* odds $O(C)$ as:

$$O(C|E) = LR(C,E) \cdot O(C) \tag{4}$$

where the quantity:

$$LR(C,E) = \frac{P(E|C)}{P(E|\bar{C})} = \frac{P(E \cap C)}{P(E \cap \bar{C})} \cdot \frac{P(\bar{C})}{P(C)} \tag{5}$$

is called *likelihood ratio*.
Note that, if $E \subseteq C$, one has $P(E \cap \bar{C}) = 0$ and so the likelihood ratio is not defined.
If $LR(C,E) > 1$, the evidence E favors the event C, and the greater the value of the likelihood ratio is, the more determinant the evidence is. On the contrary, if $LR(C,E) < 1$, the evidence E favors the event $\bar{C}$. Finally, $LR(C,E) = 1$ implies the absolutely unconcern of the evidence about the two events C and $\bar{C}$.
Let us consider the following examples. Let the probability space be the set of the real numbers, $\Re$, and the events C are as in the following two cases:

> case (1): the height in centimeters of the suspect belongs to the interval $C = [170, 171]$;
>
> case (2): the height in centimeters of the suspect belongs to the interval $C = [160, 170]$.

Let us suppose now that the evidence E is:

> the height in centimeters of the crime perpetrator measured from the videotape recorded on the crime scene belongs to the interval $E = [164, 169]$.

In case (1), the trivial consideration about the possibility that C matches E is obviously nought ($C \cap E = \emptyset$), and so a negative conclusion follows about the fact that the measured height of the crime author from the recorded videotape can belong to the interval C.
The reasons for which this negative conclusion is unsatisfactory lie on the fact that an error of measurement could be done, or the evidence E could arise from non 100%-reliable information (see, for example, [4], where the information about the color of a taxi could be not 100%-reliable since it was detected in darkness time).
In case (2), the likelihood ratio can be given the infinite value since in the denominator $P(E \cap \bar{C}) = 0$.

We define the case $C = E$ as *perfect match*, while the case $E \subseteq C$ as *inclusion*. Attempts to overcome this match/non-match approach into a continuous one has been already done in case of interpretation of evidence for RI of glass fragments [7, 8]. Moreover, in a previous paper [1], we proposed a method to overcome the match/non-match approach in the case in which the measures were represented by intervals and the database was Gaussian.
In the present paper, we extend the proposed method for all kind of databases, and not only for Gaussian ones. The extension of the method is based on the copula theory, as explained below. We first recall in the next section the basic ideas of our definition of extended likelihood ratio, as in [1].

2 The extended likelihood ratio

Measurements in a database can be modeled as a random variable, which is characterized by a probability distribution from which it is straight forward to determine the probabilities of intervals of interest. For instance, the database of the heights of a population is described by a random variable, say X, with probability distribution P_X. The probability of events like $\{X \in A\}$, where A is for instance an interval of the real line $\Re$, is denoted as $P_X(A)$. Moreover, the distribution function $F_X(x) = P_X(X < x)$ can be also defined.
Our method is based on the idea that there is another random variable Y, with probability distribution P_Y equal to P_X, and that X is related to the characteristic, while Y to the evidence.
We consider a pair of variables (X, Y), with joint distribution P^r_{XY}, and joint distribution function $F^r_{XY}(x, y) = P^r_{XY}(X < x, Y < y)$, such that its two marginal distributions coincide with the distribution of X.
The strength of the dependence between the two variables can be quantified by their correlation index r, and it is well known that in general one has:

$$-1 < r < 1 \tag{6}$$

and that the case $r = 0$ corresponds to the case of uncorrelated X and Y. The major interest lies in the case r close to 1, i.e. when the bivariate density is concentrated along the straight line $\{x = y\}$, corresponding to a strong positive dependence between X and Y.
Our method is based on the following rule:

to replace in the likelihood ratio the univariate probability of the intersection of two events with the bivariate probability of the associated rectangle.

More precisely, we make the replacement:

$$P_X(A \cap B) \quad \rightarrow \quad P^r_{XY}(A \times B) \tag{7}$$

The proposed replacement is such that the bivariate representation computed on rectangles enjoys the properties of the univariate probability of intersection, but if $A \cap B = \emptyset$, while $P_X(A \cap B) = 0$ in the classical case, in our approach one has $P^r_{XY}(A \times B) > 0$. Also the case of non-empty intersection the calculation differs from the standard one, and one expects that $P^r_{XY}(A \times B)$ tends to $P_X(A \cap B)$ as r tends to 1, i.e. when the correlation takes its maximum value.
With these assumptions, we can provide a new version of the formula for computing the likelihood ratio. The extended likelihood ratio $LR^r(C, E)$ is:

$$LR^r(C, E) = \frac{P^r_{XY}(E \times C)}{P^r_{XY}(E \times \bar{C})} \cdot \frac{P_X(\bar{C})}{P_X(C)} \tag{8}$$

obtained by making the replacement (7) in the definition of the likelihood ratio in equation (5).
In order to construct the bivariate probability P^r_{XY}, from its known equal marginals P_X, we use the well-known theory of copulas, briefly described in the next section.

3 Copulas

We want to construct a pair of variables (X, Y), with joint distribution P_{XY}, such that its two marginal distributions coincide with the distribution of X, P_X.
According to theory of copulas [9, 10, 11], the joint distribution P_{XY} can be expressed as a function of its marginals; in particular, the joint distribution function $F_{XY}(x, y)$ can be written as:

$$F_{XY}(x, y) = \mathcal{C}(F_X(x), F_Y(y)) \tag{9}$$

where the function $\mathcal{C}$ (said *copula*) is a joint distribution of two uniform random variables U and V defined in the unit interval $\mathbf{I} = [0, 1]$.
So, to each pair of real numbers (x, y) we can associate three numbers: $F_X(x)$, $F_Y(y)$, and $F_{XY}(x, y)$. Since these three numbers lie in the unit interval $\mathbf{I}$, we can say that to each pair of real numbers (x, y) we can associate the pair $(F_X(x), F_Y(y))$ in the unit square $\mathbf{I}^2$, and this ordered pair in turn corresponds to a number $F_{XY}(x, y)$ in $\mathbf{I}$. The correspondence, which assigns the value of

the joint distribution function to each ordered pair of values of the individual distribution functions, is indeed a function, denoted as copula.
In other words, copulas allow to express and generate the joint distribution function F_{XY}, given their marginals F_X and F_Y.
Note that for each copula $\mathcal{C}$, $\forall(u,v) \in \mathbf{I}^2$:

$$\mathcal{C}(u,0) = 0 \ ; \ \mathcal{C}(0,v) = 0 \ ; \ \mathcal{C}(u,1) = u \ ; \ \mathcal{C}(1,v) = v \ ; \ \mathcal{C}(1,1) = 1 \qquad (10)$$

In this scenario, if $C = [x_C, y_C]$ and $E = [x_E, y_E]$ are two intervals, the joint probability is computed as:

$$\begin{aligned} P^r_{XY}(E \times C) &= F_{XY}(y_E, y_C) - F_{XY}(x_E, y_C) - F_{XY}(y_E, x_C) + F_{XY}(x_E, x_C) \\ &= \mathcal{C}(F_X(y_E), F_X(y_C)) - \mathcal{C}(F_X(x_E), F_X(y_C)) \\ &\quad -\mathcal{C}(F_X(y_E), F_X(x_C)) + \mathcal{C}(F_X(x_E), F_X(x_C)) \end{aligned} \qquad (11)$$

and the other quantities in likelihood ratio, defined in equation (8), can be easily computed in a similar way.
Since different choices of copulas can be made, in this paper we focus our attention on the three different kind of them: Fréchet, Cuadras-Augé and normal copulas. A brief panorama on these main families of copulas is reported in Appendix.
In applications, the value of r has to be chosen in the interval $(0, 1)$ depending on the specific case-work under consideration. The question of an optimal choice of r is an important problem which is discussed in section 5.
Note that, for all choices of the copula, for $r = 0$ the two variables are independent, so the likelihood ratio reduces to 1 for all pair of events; otherwise, in the limit $r \to 1$, the likelihood ratio reduces to the standard one, defined in equation (5).

4 A case-work: evaluation of extended likelihood ratio for the height

We apply the proposed approach to a case-work in which the measurements of the height of the suspect is the interval $C = [x_C, y_C]$ and the evidence is the interval $E = [x_E, y_E]$. We use the database of about 200,000 people filed in the police archive of Italian Carabinieri. This distribution is very close to a Gaussian one with mean $\mu = 170.73$ cm and standard deviation $\sigma = 8.86$ cm, and its histogram is reported in figure 1. However, note that the database is not symmetric and in particular that the left tail of the distribution is longer than the right one.

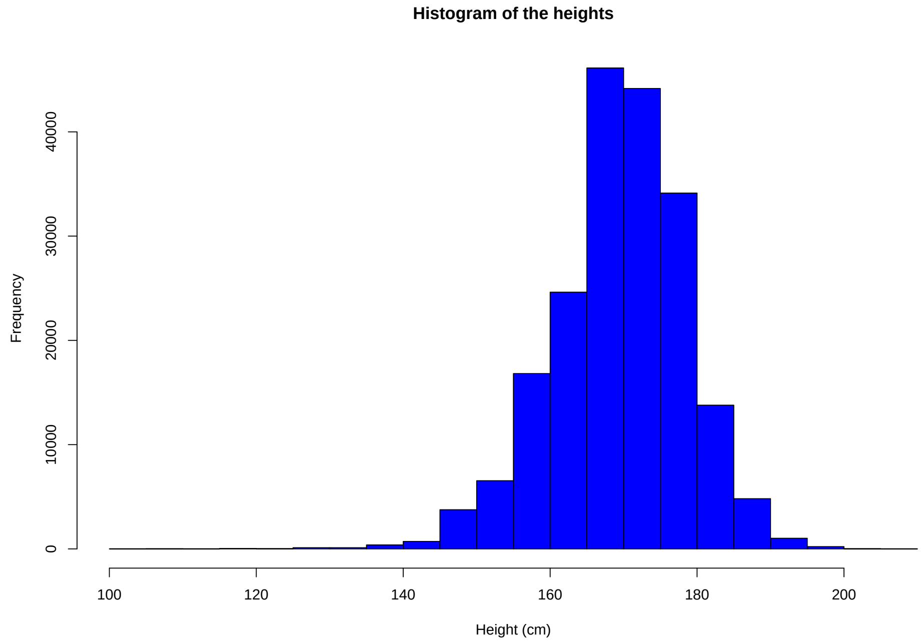

Figure 1: Histogram of the heights (in centimeters) of about 200,000 people filed in the police archive of Italian Carabinieri.

Using the free statistical software R [12], with the appropriate packages, the likelihood ratio in the three case of Fréchet (LR_F^r), Cuadras-Augé (LR_C^r) and normal (LR_N^r) copulas can be easily calculated, by means of equations (8) and (11).
We have investigate the behavior of the likelihood ratio as a function of its arguments, which are the two intervals C and E and the correlation parameter r. We have also investigated the dependence on the copula.
As to the choice of copula, we have found that the qualitative behavior is equal in the case of the three kinds of copulas, and so we show the results in the case of normal copula.
The dependence on C and E has been performed in the following.
For example, choosing four pairs of intervals C and E (all the combination between $C = [x-1, x+1]$, $C = [x-5, x+5]$, and $E = [y-1, y+1]$, $E = [y-5, y+5]$), we have plotted the extended likelihood ratio, as function of x and y, in the case of normal copula, with $r = 0.95$ (see figure 2).
We have chosen these intervals in order to consider small (2 cm) and large (10 cm) intervals.
The value of the extended likelihood ratio is very large when the amplitude of the two intervals C and E is 2 cm, and they are close, i.e. the point (x, y)

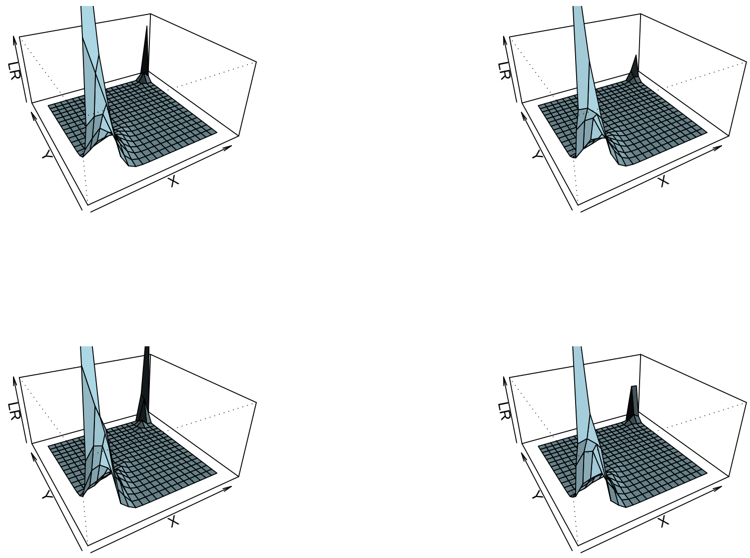

Figure 2: Plots of the extended likelihood ratio $LR_N^r(C,E)$, calculated in the case of normal copula with $r = 0.95$, as a function of x and y, for all combination of pairs for the intervals $C = [x-1, x+1]$ and $C = [x-5, x+5]$ (rows), and $E = [y-1, y+1]$ and $E = [y-5, y+5]$ (columns). For example, the graphic in the first row and in the second column refers to $C = [x-1, x+1]$ and $E = [y-5, y+5]$. The x and y variables range from 100 to 210; the z variable ranges from 0 to 2000 in all plots.

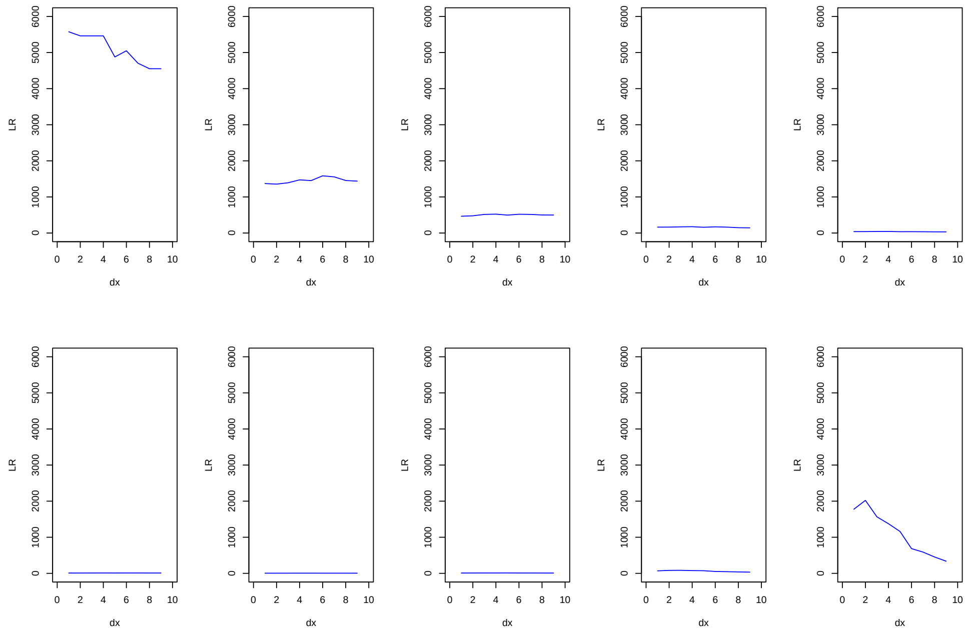

Figure 3: Plots of the extended likelihood ratio $LR_N^r(C,E)$, for $r = 0.95$, in case of perfect match, for the intervals $C = E = [x - dx, x + dx]$, as a function of dx for 10 particular values of x (from 110 to 200, step 10), and for dx (from 1 to 9, step 1).

nearby the diagonal line $x = y$, and in the tails of the height distribution, as expected. More precisely, this value is larger in the left tail case rather than in the right tail one. Of course, this last effect can not be detected using a Gaussian approximation of the database.

In order to further analyze the behavior of the likelihood ratio, we have plotted the case of perfect match $C = E$, where the center of the intervals x and the semi-amplitude dx vary. In figure 3, the first panel in the first row shows that the likelihood ratio assume its maximum value when x is in the left tail of the distribution and dx is equal to 1 cm, i.e. $C = E = [109, 111]$. Note that the other plots range on a smaller scale.

We have also investigated the case of inclusion $E \subseteq C$, for $C = [x - 5, x + 5]$ with x ranging from 100 to 200, step 10, and $E = [x - dx, x + dx]$, with dx ranging from 1 to 5, step 1. In all these cases the standard approach gives a likelihood ratio with null denominator. The plots are very similar to those of figure 3; in particular the extended likelihood ratio assumes its maximum value when $x = 110$ and $dx = 1$, i.e. for $C = [105, 115]$ and $E = [109, 111]$; this value is very close to the previous case $C = E = [109, 111]$.

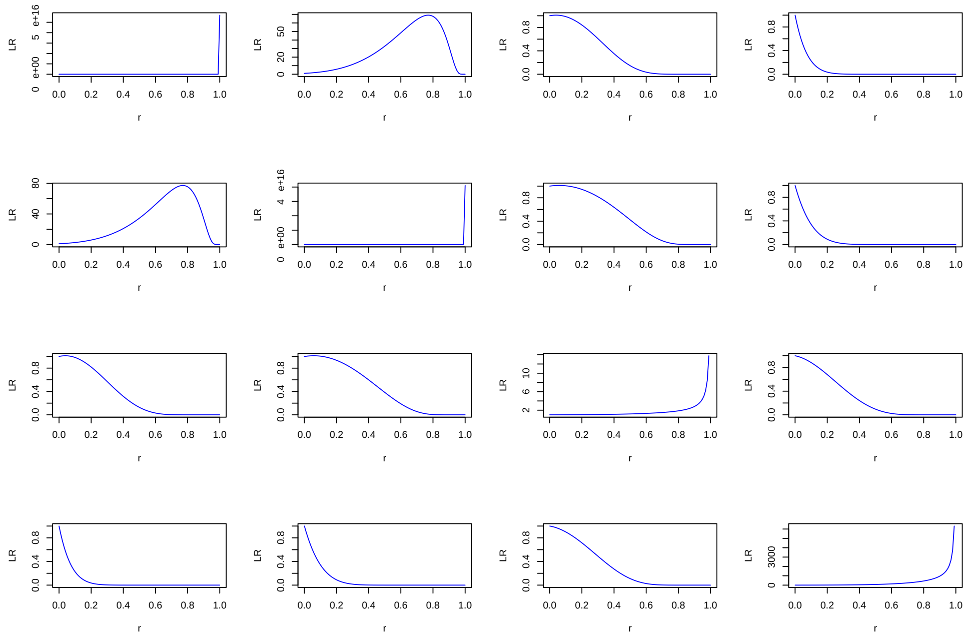

Figure 4: Plots of the extended likelihood ratio $LR_N^r(C, E)$ as a function of r, for particular values of $C = [x-1, x+1]$ (columns) and $E = [y-1, y+1]$ (rows) for all the combination of x and y in the values 110, 140, 170 and 200. For example, the graphic in the third column and in the second row refers to $C = [169, 171]$ and $E = [139, 141]$. Note the different scales in ordinate axis.

Finally, we have investigated the dependence on r. We have fixed the two intervals C and E, and plotted the value of the extended likelihood ratio $LR^r(C, E)$ as function of the parameter r, for $r \in (0, 1)$.
We have represented the cases of small (2 cm) intervals, in the tails and nearby the average of the height distribution.
The plots are reported, in the case of normal copula, in figure 4.
In the case of perfect match $C = E$ (referring to the figure, the diagonal from the top left to bottom right), the plots clearly show divergence when $r \to 1$. This is consistent with the formula of standard evaluation of likelihood ratio, in which the denominator is zero.
Our finding are consistent with the expected relationships:

$$\lim_{r \to 0} LR^r(C, E) = 1 \tag{12}$$

for all pair of events C and E; moreover, if $E \subseteq C$ (inclusion):

$$\lim_{r \to 1} LR^r(C, E) = +\infty \tag{13}$$

5 Choice of the parameter r

The choice of the parameter r is a focal point in this theory, and we propose three different approaches.

Fixed r

Firstly, we propose to fix the value of r, for instance $r = 0.95$, and to compute the extended likelihood ratio using this value. This is the simplest approach and the theory can be used in forensic evaluation straightforwardly.

Supremum

Secondly, we propose to over-estimate the likelihood ratio, and for this reason we consider, for all pairs of intervals C and E, the value:

$$LR^*(C,E) = \sup_{r\in(0,1)} LR^r(C,E) \tag{14}$$

This choice finds its reason in the fact that this theory arises from the intention to overcome the null value of the likelihood ratio; so to the objection that the standard value (naught) is increased, $LR^*(C,E)$ represents the maximum value that the extended likelihood ratio reaches. In other words, it corresponds to the maximum over-estimation of LR^r.
Assuming regularity of the function LR^r, its maximum value coincides either with the limit in one of the two extremes $r = 0$ or $r = 1$, or in the values calculated in those points in which its derivative is zero:

$$\frac{d}{dr}LR^r(C,E) = 0 \tag{15}$$

and, since:

$$\frac{d}{dr}LR^r(C,E) = \frac{1}{[P^r_{XY}(E\times\bar{C})]^2}\cdot\frac{P_X(\bar{C})}{P_X(C)}\cdot\frac{d}{dr}P^r_{XY}(E\times C) \tag{16}$$

equation (15) reduces to:

$$\frac{d}{dr}P^r_{XY}(E\times C) = 0 \tag{17}$$

This approach has a limitation, as the standard one, in the case of inclusion $E \subseteq C$ in which $LR^*(C,E) = +\infty$. However, we notice that in the standard case the likelihood ratio was not defined at all, while in our approach the supremum is infinite, which corresponds to the intuitive point of view.

Maximal score

The third approach consists in choosing a maximal score K related to the weight of the evidence. Different kinds of evidence can have different maximal scores: for example, the height evidence should have a smaller maximal score rather than other kinds of evidence (such as DNA, fingerprint).
Our investigation suggests that, for any value of r, the maximum value for the extended likelihood ratio is attained when the following three conditions hold:

perfect match;

intervals in the left tail of the distribution;

minimum amplitude of the intervals (2 cm).

Denoting with $\hat{C}$ and $\hat{E}$ the intervals which satisfy the above conditions (note obviously that $\hat{C} = \hat{E}$), our proposal consists in determining the value $\hat{r}$ such that $LR^{\hat{r}}(\hat{C}, \hat{C}) = K$. For all other intervals, the extended likelihood ratio can be calculated with the value $\hat{r}$, and obviously one has $LR^{\hat{r}}(C, E) \leq K$.
With the assumptions that $LR^{r}(\hat{C}, \hat{C})$ is a function of r monotonic increasing and continuous in a left neighborhood of 1, since:

$$\lim_{r \to 1} LR^{r}(\hat{C}, \hat{C}) = +\infty \qquad (18)$$

$\hat{r}$ exists and it is unique.
Practically one proceeds as follows: in figure 3, the first plot shows the pair of intervals such that $LR^{r}(C, E)$ is the largest one, for all r, i.e. $C = E = [109, 111]$. Hence the value $\hat{r}$ is computed intersecting the profile relative to this pair in the figure 4 (which is the first one) with the value K.
The main advantage of this approach is that $LR^{\hat{r}}$ is finite for all pairs of events.

6 Examples

In order to make some examples in the choice of r, we take into consideration the cases as in the previous paragraphes.

Standard case

First of all, we have calculated the likelihood ratio in the standard case, i.e. for $r \to 1$. The values are reported in scientific notation in the following table. Note that the value "Inf" is to be considered in the calculation limit of the statistical software R.

	$[109,111]$	$[139,141]$	$[169,171]$	$[199,201]$
$[109,111]$	5.7 E+16	0	0	0
$[139,141]$	0	4.1 E+16	0	0
$[169,171]$	0	0	Inf	0
$[199,201]$	0	0	0	Inf

Table 1. Values of $LR(C,E)$, i.e. of $LR^r(C,E)$ for $r \to 1$, for particular values of C (columns) and E (rows).

Fixed r

We can compare the values of the standard case with those of LR^r calculated for the fixed value of $r = 0.95$, reported in the following table.

	$[109,111]$	$[139,141]$	$[169,171]$	$[199,201]$
$[109,111]$	5.6 E+03	3.9	0	0
$[139,141]$	3.9	1.6 E+02	1.1 E-12	0
$[169,171]$	0	1.3 E-12	4.4	0
$[199,201]$	0	0	0	1.8 E+03

Table 2. Values of $LR^r(C,E)$ for fixed value of $r = 0.95$, for particular values of C (columns) and E (rows).

Supremum

In the present case, we have calculated the values of the parameter r which correspond to the maximum of $LR^r(C,E)$.

	$[109,111]$	$[139,141]$	$[169,171]$	$[199,201]$
$[109,111]$	1	7.7 E-01	5.0 E-02	9.9 E-03
$[139,141]$	7.7 E-01	1	1.1 E-12	9.9 E-03
$[169,171]$	5.0 E-02	1.2 E-12	1	9.9 E-03
$[199,201]$	9.9 E-03	9.9 E-03	9.9 E-03	1

Table 3a. Values of r for which $LR^r(C,E)$ has a maximum, for particular values of C (columns) and E (rows).

Then, for each case, we have calculated the value of $LR^*(C,E)$, reported in the following table.

	$[109, 111]$	$[139, 141]$	$[169, 171]$	$[199, 201]$
$[109, 111]$	5.6 E+16	7.7 E+01	1.0	1
$[139, 141]$	6.9 E+01	4.1 E+16	1.0	1
$[169, 171]$	1.0	1.0	Inf	1
$[199, 201]$	1	1	1	Inf

Table 3b. Values of $LR^r(C, E)$ for the value of r reported in table 3a, for particular values of C (columns) and E (rows).

Maximal score

For the present case, we have first settled a maximal score $K = 1,000$, and we have reported in the following table the values of r for which $LR^r(C, E) = K$, in case of perfect match $C = E$.

	$[109, 111]$	$[139, 141]$	$[169, 171]$	$[199, 201]$
$[109, 111]$	7.7 E-01	-	-	-
$[139, 141]$	-	1	-	-
$[169, 171]$	-	-	1	-
$[199, 201]$	-	-	-	9.1 E-01

Table 4a. Values of r for which $LR^r(C, E) = K$, with $K = 1,000$, for particular values of C (columns) and E (rows).

Then, from the previous table we have choose the value of $\hat{r} = 0.77$, which correspond to the case $C = E = [109, 111]$. Hence we have calculated $LR^{\hat{r}}(C, E)$, reporting the values in the following table.

	$[109, 111]$	$[139, 141]$	$[169, 171]$	$[199, 201]$
$[109, 111]$	1.0 E+03	7.7 E+01	1.1 E-04	0
$[139, 141]$	6.9 E+01	4.7 E+01	1.2 E-02	0
$[169, 171]$	1.3 E-04	1.4 E-02	1.7	1.1 E-04
$[199, 201]$	0	0	9.4 E-05	3.6 E+02

Table 4b. Values of $LR^{\hat{r}}(C, E)$ for the value of $1hatr = 0.77$, deduced by table 4a, for particular values of C (columns) and E (rows).

7 Concluding remarks

The forensic problem of negative conclusion cases of the likelihood ratio evaluation has been analyzed by steps.
The first step concerns with the definition of the extended likelihood ratio which overcomes the macht/non-match approaches. The theory of copulas

allows to define the likelihood ratio for all kinds of distributions. We have used three families of copulas, depending on a parameter r, and we have obtained similar results. We have shown an application in the particular case of height, basing on Italian Carabinieri database, that differs from the Gaussian one for asymmetric tails.
The second step concerns with the choice of the correlation parameter r. We have made three proposals: fixed r, supremum and maximal score K. The first one is the simplest approach and the theory can be straightforwardly applied. The second one represents an over-estimation, independent on r, but however in inclusion cases it gives an infinite value. In the third one it is possible to choose the value of the maximal score K weighted on the type of the evidence (height, DNA, fingerprint, etc.).
The theory can be further investigated in several directions, among which we mention the following two: the case of non-unimodal univariate database and that of multivariate database (this last case is important, for instance, in analysis of voice frequencies).

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Appendix

In this appendix, we briefly introduce the main families of copulas [9, 10, 11], studying their properties.
In particular, we have calculated and reported also a widely used index of dependence in the theory of copulas: the so-called Kendall correlation τ, defined by:

$$\tau = 4 \cdot E[\mathcal{C}(U,V)] - 1 \tag{19}$$

where E denotes the expectation. Note that for independent variables, for which $\mathcal{C}(u,v) = uv$, one has $\tau = 0$; otherwise, for perfectly correlated variables $U = V$, one has $\tau = 1$.

Upper Fréchet copula $\mathcal{C}^+$

The *upper Fréchet copula* $\mathcal{C}^+(u,v)$ is defined by:

$$\mathcal{C}^+(u,v) = \min(u,v) \tag{20}$$

and its graphic is reported figure 5 (left side); its Kendall correlation index is $\tau^+ = 1$. In this case, the two variables U and V are completely correlated, and for intervals it reduces to the standard intersection.

Lower Fréchet copula $\mathcal{C}^-$

The *lower Fréchet copula* $\mathcal{C}^-(u,v)$ is defined by:

$$\mathcal{C}^-(u,v) = \max(u+v-1, 0) \tag{21}$$

and its graphic is reported figure 5 (right side), its Kendall correlation index is $\tau^- = -1$. In this case, the two variables U and V are completely negative correlated.

It can be demonstrated that, for all copulas $\mathcal{C}$ and for all $(u,v) \in \mathbf{I}^2$:

$$\mathcal{C}^-(u,v) \leq \mathcal{C}(u,v) \leq \mathcal{C}^+(u,v) \tag{22}$$

In this sense, $\mathcal{C}^-$ and $\mathcal{C}^+$ are the lower and the upper bounds for all copulas.

Product copula $\mathcal{C}^\perp$

The *product copula* $\mathcal{C}^\perp(u,v)$ is defined by:

$$\mathcal{C}^\perp(u,v) = uv \tag{23}$$

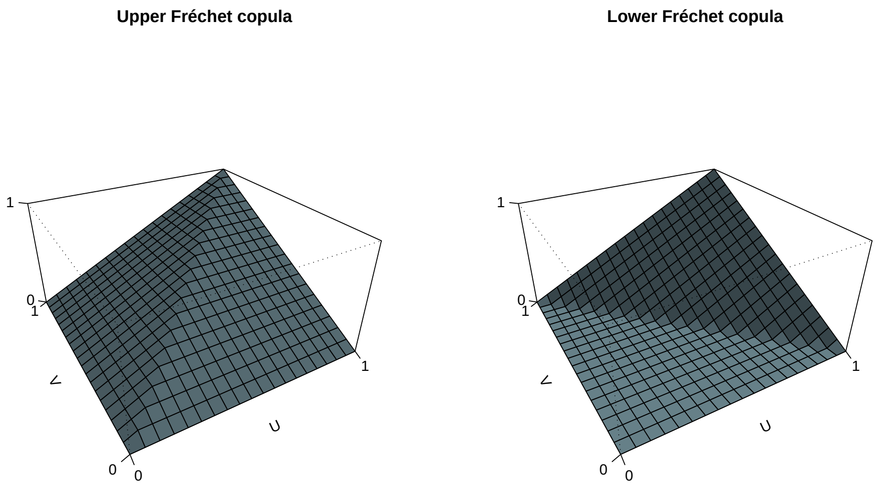

Figure 5: Upper Fréchet copula $\mathcal{C}^+$ (left), and Lower Fréchet copula $\mathcal{C}^-$ (right).

and the graphics of it and its density are reported figure 6; its Kendall correlation index is $\tau^\perp = 0$. In this case, the two variables U and V are independent.

Fréchet family of copulas $\mathcal{C}_F^r$

The *Fréchet family of copulas* $\mathcal{C}_F^r(u, v)$, with $0 \le r \le 1$, is defined by:

$$\mathcal{C}_F^r(u, v) = r \cdot \mathcal{C}^+(u, v) + (1 - r) \cdot \mathcal{C}^\perp(u, v) \tag{24}$$

and its graphic (for $r = 0.7$) is reported figure 7 (left side).
Its Kendall correlation index is:

$$\tau_F(r) = \frac{1}{3}r^2 + \frac{2}{3}r \tag{25}$$

its inversion, for $r \in (0, 1)$, gives $r = \sqrt{1 + 3\tau_F} - 1$.
Note that for $r = 0$ the two variables are independent, and for $r = 1$ we obtain the upper Fréchet copula $\mathcal{C}^+$.

Cuadras-Augé family of copulas $\mathcal{C}_C^r$

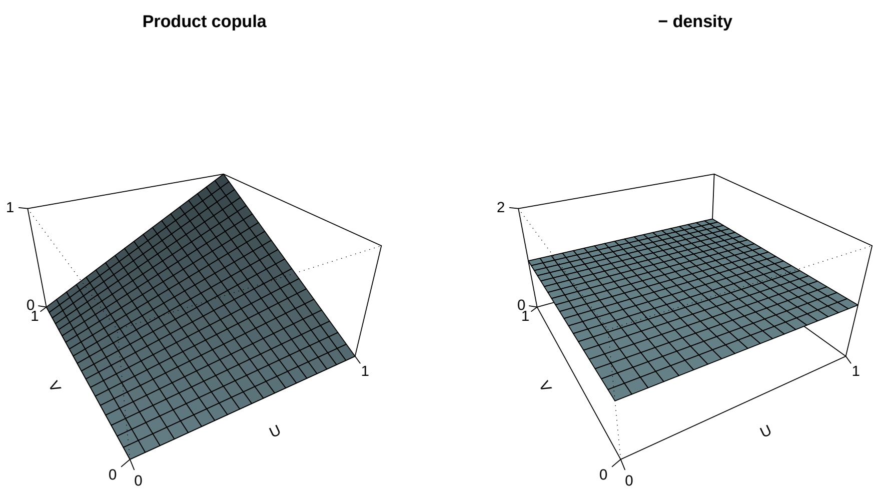

Figure 6: Product copula $\mathcal{C}^{\perp}$ (left), and its density (right).

The *Cuadras-Augé family of copulas* $\mathcal{C}_C^r(u, v)$, with $0 \leq r \leq 1$, is defined by a weighted geometric mean of $\mathcal{C}^+$ and $\mathcal{C}^{\perp}$:

$$\mathcal{C}_C^r(u, v) = [\mathcal{C}^+(u, v)]^r \cdot [\mathcal{C}^{\perp}(u, v)]^{1-r} \tag{26}$$

and its graphic (for $r = 0.7$) is reported figure 7 (right side).
Its Kendall correlation index is:

$$\tau_C(r) = \frac{r}{2 - r} \tag{27}$$

its inversion, for $r \in (0, 1)$, gives $r = \frac{2\tau}{1+\tau_C}$.
Note that, as in the previous case, for $r = 0$ the two variables are independent, and for $r = 1$ we obtain the upper Fréchet copula $\mathcal{C}^+$.

Normal copula $\mathcal{C}_N^r$

Denoting by $B(x, y; r)$ the bivariate normal distribution function with correlation index r (with $-1 < r < 1$), and by $\Phi(u)$ the percentile function for the standard normal distribution, then the *normal copula* $\mathcal{C}_N^r$ is defined by:

$$\mathcal{C}_N^r(u, v) = B(\Phi(u), \Phi(v); r) \tag{28}$$

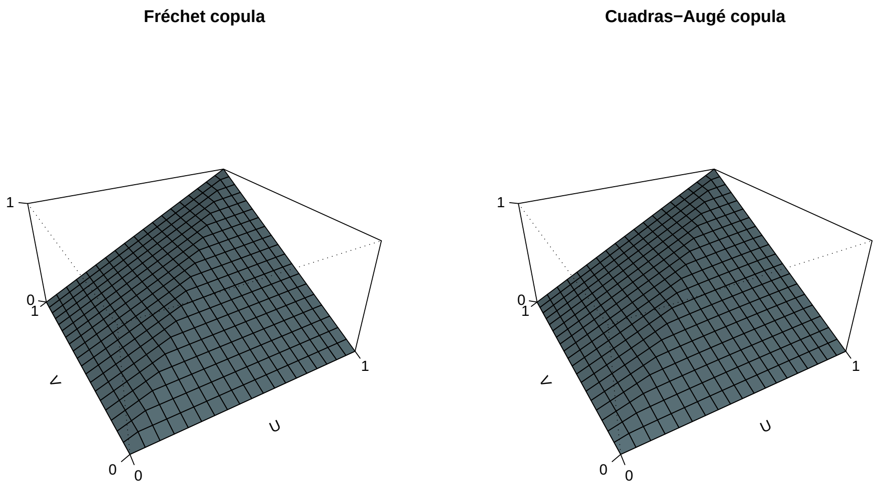

Figure 7: Fréchet copula $\mathcal{C}_F^r(u,v)$ with $r = 0.7$ (left), and Cuadras-Augé copula $\mathcal{C}_C^r(u,v)$ with $r = 0.7$ (right).

The graphics of the normal copula (for $r = 0.95$) and its density are reported in figure 8.
Its Kendall correlation index is:

$$\tau_N(r) = \frac{2}{\pi} \arcsin(r) \tag{29}$$

its inversion $\tau_N(r)$, for $r \in (0,1)$, gives $r = \sin \frac{\pi}{2} \tau_N$.

It can be demonstrated that, for all normal copula $\mathcal{C}_N^r$ and for all $(u,v) \in \mathbf{I}^2$:

$$\begin{aligned} \mathcal{C}^-(u,v) = \mathcal{C}_N^{-1}(u,v) \le \mathcal{C}_{N(-1<r<0)}^r(u,v) \le \\ \le \mathcal{C}_N^0(u,v) = \mathcal{C}^\perp(u,v) \le \\ \le \mathcal{C}_{N(0<r<1)}^r(u,v) \le \mathcal{C}_N^1(u,v) = \mathcal{C}^+(u,v) \end{aligned} \tag{30}$$

In this sense, in the family of normal copulas independence ($r = 0$) and complete correlation ($r \to 1$) are two specific cases.

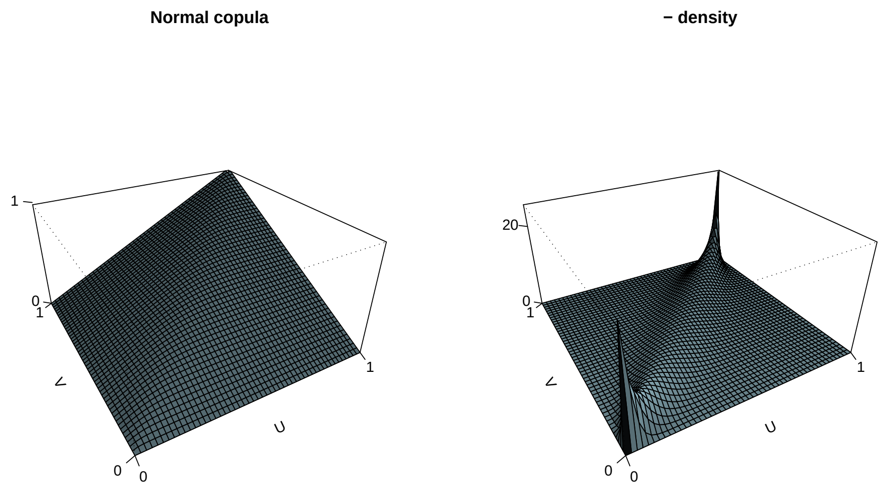

Figure 8: Normal copula $\mathcal{C}_N^r$ with $r = 0.95$ (left), and its density (right).