Representations of Solutions of Hamilton - Jacobi Equations

Italo Capuzzo Dolcetta
Dipartimento di Matematica, Università di Roma - La Sapienza

1 Introduction

In this paper we report on some classical and more recent results about representation formulas for generalized solutions of the evolution partial differential equation

$$u_t + H(x, Du) = 0 \ , \ (x,t) \in \mathbb{R}^N \times (0, +\infty) \tag{1.1}$$

We consider here only the case where $H = H(x,p)$ is a convex function with respect to the p variable. In this setting, representation formulas can be obtained by exploiting the well - known connection existing via convex duality between the Hamilton - Jacobi equation (1.1) with Calculus of Variations or, more generally, Optimal Control problems.

In Section 1 we briefly review some known existence results and representation formulas for viscosity solutions of problem (1.1). In Section 2, the Hopf formula of Section 1.1 is revisited from quite a different point of view, pointing out some links with the classical vanishing viscosity method and with the closely related large deviation problem for the underlying stochastic processes. In particular, we sketch a non standard proof, based on Varadhan's Large Deviations Principle, of the fact that the inf-convolution of the initial datum g, namely

$$u(x,t) = \inf_{y \in \mathbb{R}^N} \left[g(y) + \frac{|x-y|^2}{2t} \right]$$

is the Hopf solution of

$$u_t + \frac{1}{2}|Du|^2 = 0 \ , \ (x,t) \in \mathbb{R}^N \times (0,+\infty) \ \ , \ \ u(x,0) = g(x) \ \ , \ x \in \mathbb{R}^N$$

Some indications to the connection with the Maslov's idempotent analysis approach to Hamilton - Jacobi equations are also given.

The final part of the paper comprises a description of a recent generalization, due to H. Ishii and the author, of the Hopf representation formula, covering some cases of state dependent equations, including possible degeneracies in the x dependence.

2 The Cauchy problem in the viscosity sense

We consider first order nonlinear evolution equations of the Hamilton - Jacobi type

$$u_t + H(x, Du) = 0 \ , \ (x,t) \in \mathbb{R}^N \times (0, +\infty) \tag{2.1}$$

equipped with the initial condition

$$u(x,0) = g(x) \ , \ x \in \mathbb{R}^N \tag{2.2}$$

Here, H is a given continuous scalar function of the variables $(x,p) \in \mathbb{R}^{2N}$ which we will always assume to be convex in the p variable and the initial datum g is given on $\mathbb{R}^N$. The notations u_t and Du stand, respectively, for the time derivative and the spatial gradient of the real - valued unknown function $u = u(x,t)$.

It is well - known that problem (2.1), (2.2) does not have, in general, global classical solutions, even for smooth data. The notion of viscosity solutions has proved to be appropriate for the analysis of the well - posedness of such nonlinear problems in a nondifferentiable framework, see [10], [14], [4], [3].
Let us recall for the convenience of the reader the Barron - Jensen [5] definition of lower semicontinuous viscosity solution (or bilateral supersolution in the terminology of [3]) of problem (2.1), (2.2). The definition below extends the classical Crandall - Lions definition to the lower semicontinuous case and coincides with it for continuous solutions, provided the Hamiltonian H is convex with respect to p. A lower semicontinuous function u is a viscosity solution of (2.1), (2.2) if

$$\lambda + H(x, \eta) = 0 \quad \forall (\eta, \lambda) \in D^- u(x,t) \ , \tag{2.3}$$

where $D^- u(x,t)$ is the subdifferential of u at (x,t), that is the closed, convex, possibly empty set whose elements are the vectors $(\eta, \lambda) \in \mathbb{R}^N \times \mathbb{R}$ such that

$$\liminf_{(y,s) \to (x,t)} \frac{u(y,s) - u(x,t) - \eta \cdot (y - x) - \lambda(s - t)}{|y - x| + |s - t|} \geq 0$$

We refer to [5], [4], [3] for a through discussion of this notion of solution and for existence, comparison and stability results.

We briefly report now on three different methods which produce existence results together with representation formulas for the viscosity solution of problem (2.1), (2.2).

2.1 The method of characteristics

The method of characteristics is the classical approach to construct local solutions to nonlinear first order partial differential equations, see [11] for a recent presentation. Consider for simplicity the Cauchy problem

$$u_t + H(Du) = 0 \ , \ (x,t) \in \mathbb{R}^N \times (0, +\infty) \ , \tag{2.4}$$

$$u(x,0) = g(x) \ , \ x \in \mathbb{R}^N \tag{2.5}$$

The associated characteristic system is

$$x'(t) = \frac{\partial H}{\partial p}(p(t)) \ \ , \ \ p'(t) = -\frac{\partial H}{\partial x}(p(t)) \equiv 0 \tag{2.6}$$

with the initial conditions

$$x(0) = x \ \ , \ \ p(0) = Dg(x) \tag{2.7}$$

The solution of (2.6), (2.7) is of course

$$x(t;x) = x + t\frac{\partial H}{\partial p}(Dg(x)) \ \ , \ \ p(t) \equiv Dg(x)$$

and so the candidate solution produced by the method of characteristics is

$$u(x,t) = g\left(x^{-1}(t;x)\right) + t\left(\frac{\partial H}{\partial p} \cdot Dg - H(Dg)\right)\left(x^{-1}(t;x)\right) \tag{2.8}$$

Here, x^{-1} is the inverse of the map $x \to x(t;x)$, which is defined, in general, only for small $t > 0$. As a consequence, function u is not globally defined by (2.8). However, under some restrictive assumptions on H and g, the map $x \to x(t;x)$ is globally invertible and the above formula defines u as a global solution of (2.4), (2.5).

A model global existence result taken from [14] is as follows:

Theorem 2.1 *Assume that H and g are in $C^2(\mathbb{R}^N)$ and convex. Then the function u given by* (2.8) *is a classical and, a fortiori, viscosity solution of* (2.4), (2.5).

Under the assumptions of the Theorem, we have indeed

$$det \left(I + t\frac{\partial^2 H}{\partial p^2}(Dg(x)) \right) D^2 g(x) \geq 1$$

for all $x \in \mathbb{R}^N$ and $t > 0$. So, u is globally defined and the fact that it is a viscosity solution is a simple, direct verification.

2.2 The optimal control method

We assume here that the convex function $p \to H(x, p)$ can be expressed as the envelope of a family of affine functions of p, namely

$$H(x, p) = \sup_{a \in A} [-F(x, a) \cdot p - L(x, a)] \tag{2.9}$$

where A is a closed subset of $\mathbb{R}^M$, $F : \mathbb{R}^N \times A \to \mathbb{R}^N$ and $L : \mathbb{R}^N \times A \to \mathbb{R}$ are Lipschitz continuous in the first variable, uniformly in a. This the typical situation in optimal control theory; note, however, that quite general functions H can be represented in this way, see [13].

Let us associate to H the autonomous control system

$$y'(t) = F(y(t), \alpha(t)) \ , \ y(0) = x \ , \tag{2.10}$$

where the control α is any measurable function of $t \in [0, +\infty)$ valued in A and the functional

$$J(x, t; \alpha) = \int_0^t L(y(s), \alpha(s))ds + g(y(t)) \tag{2.11}$$

where $y(s) = y(x, s; \alpha)$ is the solution trajectory of (2.10).
The minimization of J with respect to all possible controls defines the value function of the above Bolza optimal control problem as the function u given by

$$u(x, t) = \inf_{\alpha} J(x, t; \alpha) \tag{2.12}$$

Formula (2.12) provides a representation for the solution of (2.1), (2.2). Indeed, we have the following

Theorem 2.2 *Assume H as in* (2.9) *with F and L as above. Assume also that L is bounded and that g is bounded and uniformly continuous. Then, the value function (2.12) is a viscosity solution of* (2.1), (2.2). *Moreover, u is the unique bounded and continuous function on $\mathbb{R}^N \times [0, +\infty)$ solving* (2.1), (2.2) *in the viscosity sense.*

The proof of this existence and uniqueness result (and of some of its generalizations) can be found in [3]. Let us only mention here that the basic ingredient to prove that u is a viscosity solution of (2.1) is the Dynamic Programming Principle, a consequence of the nonlinear semigroup property of the control system (2.10):
the value function of the optimal control problem at hand satisfies the identity

$$u(x,t) = \inf_{\alpha} \int_0^{\tau} L(y(s), \alpha(s))ds + u(y(\tau), t - \tau)$$

for all $x \in R^N$ and all $0 < \tau \leq t$.
The value function solves equation (2.1) also according to other different weak notions of solution (e.g. Subbotin's or Dini's solutions); we refer again to [3] for more information on these topics.

2.3 The Hopf - Lax method

We assume again here that H does not depend on x and, moreover, that H is superlinear at infinity, namely,

$$\lim_{|p| \to +\infty} \frac{H(p)}{|p|} = +\infty$$

By a classical duality result in convex analysis, then

$$H(p) = \sup_{a \in \mathbb{R}^N} [a \cdot p - H^*(a)]$$

where $H^*(a) = \sup_{q \in \mathbb{R}^N} [q \cdot a - H(q)]$ is the Legendre - Fenchel transform of H. Therefore, the representation (2.9) holds in this case with

$$A = \mathbb{R}^N, \; F(x,a) = a, \; L(x,a) = H^*(a) \; .$$

The optimal control problem in the previous section becomes then the classical Bolza problem in the Calculus of Variations

$$\inf_{\alpha} J(x,t;\alpha) \equiv \inf_{\alpha} \int_0^t H^*(y(s))ds + g(y(t))$$

where α is any measurable functions taking values in $\mathbb{R}^N$ and

$$y(s) = x + \int_0^t \alpha(s)ds$$

is the solution of the control system (2.10) in the case under consideration.

The representation formula (2.12) has a simplified expression in the present setting, namely

$$u(x,t) = \inf_{y \in \mathbb{R}^N} \left[g(y) + tH^* \left(\frac{x-y}{t} \right) \right] \tag{2.13}$$

The right-hand side of the above is usually called the Hopf (or Hopf - Lax) function.

The proof of the equivalence between (2.12) and (2.13) relies on one side on the fact that the geodesics of the variational problem are just straight lines; this easily gives the inequality

$$\inf_{\alpha} \int_0^t H^*(y(s))ds + g(y(t)) \leq \inf_{y \in \mathbb{R}^N} \left[g(y) + tH^* \left(\frac{x-y}{t} \right) \right]$$

The reverse inequality follows by an application of the classical Jensen's convexity inequality, see [11].

As a particular case of Theorem 2.2, the Hopf function

$$u(x,t) = \inf_{y \in \mathbb{R}^N} \left[g(y) + tH^* \left(\frac{x-y}{t} \right) \right] , \tag{2.14}$$

is a viscosity solution of the Cauchy problem

$$u_t + H(Du) = 0 \ , \ (x,t) \in \mathbb{R}^N \times (0,+\infty) \tag{2.15}$$

$$u(x,0) = g(x) \ \ , \ x \in \mathbb{R}^N \ . \tag{2.16}$$

This result has been proved in [2], extending to the viscosity setting the original result of Hopf [12] and generalized later in several directions, see [1], [6].

The main issue to be pointed out here is that the Hopf solution given by formula (2.14) expresses the solution u of the state - independent Cauchy problem (2.15), (2.16) as the optimal value of the following family of unconstrained, static, finite dimensional optimization problems parametrized by (x, t)

$$\inf_{y \in \mathbb{R}^N} \left[g(y) + tH^* \left(\frac{x - y}{t} \right) \right] = \inf_{y \in \mathbb{R}^N} \sup_{q \in \mathbb{R}^N} \left[g(y) + q \cdot (x - y) - tH(q) \right] \; .$$

An alternative way, which makes no explicit reference to the associated variational problem, of deriving the Hopf function can be found in [12].
Since in Section 3 below we will exploit similar ideas in order to deal with the state - dependent case, let us describe briefly the Hopf's construction. This starts from the simple observations that, for affine initial datum $g(x) = q \cdot x + c$, the smooth solution of (2.15), (2.16) is

$$v(x, t) = g(x) - tH(Dg(x))$$

and that, for general g, the affine functions

$$v^{y,q}(x, t) = g(y) + q \cdot (x - y) - tH(q)$$

solve (2.15) for any choice of $(y, q) \in \mathbb{R}^N \times \mathbb{R}^N$ but do not satisfy (2.16).
The procedure proposed in [12] is then to build a significant solution of the Cauchy problem by means of the following envelope

$$\inf_{y \in \mathbb{R}^N} \sup_{q \in \mathbb{R}^N} v^{y,q}(x, t)$$

of the family $v^{y,q}$. It is easy to check that

$$\sup_{q \in \mathbb{R}^N} v^{y,q}(x, t) \; \equiv g(y) + tH^* \left(\frac{x - y}{t} \right)$$

so that the above defined function coincides indeed with u in (2.14).

The Hopf's original result illustrating the connection between function (2.14) and Hamilton - Jacobi equations is next:

Theorem 2.3 *Assume that H is convex and superlinear at infinity and that g is Lipschitz continuous. Then, the function*

$$u(x, t) = \inf_{y \in \mathbb{R}^N} \left[g(y) + tH^* \left(\frac{x - y}{t} \right) \right] \tag{2.17}$$

is Lipschitz continuous on $\mathbb{R}^N \times (0,+\infty)$*, satisfies equation* (2.15) *almost everywhere and*

$$\lim_{t\to 0^+} u(x,t) = g(x)$$

at any $x \in \mathbb{R}^N$.

3 Hopf's formula and convolutions

Let us recall that the inf - convolution g_t (sometimes also called Yosida - Moreau transform) of a function $g : \mathbb{R}^N \to \mathbb{R}$ is defined for $t > 0$ by

$$g_t(x) = \inf_{y\in\mathbb{R}^N} \left[g(y) + \frac{|x-y|^2}{2t} \right] \tag{3.1}$$

This is a well - known regularization procedure in convex and in nonsmooth analysis. Indeed, if g is continuous then g_t is Lipschitz continuous and also semiconcave, that is

$$g_t(x+h) - 2g_t(x) + g_t(x-h) \le \left(C + \frac{1}{t} \right) |h|^2 \tag{3.2}$$

holds for some constant $C = C(t) > 0$ and any $(x,h) \in \mathbb{R}^{2N}$ and $t > 0$.
Moreover, functions g_t converge to g locally uniformly as $t \to 0^+$. We refer to [3] for additional information on this topic.
In the special case when $H(p) = \frac{1}{2}|p|^2 = H^*(p)$, the Hopf's function (2.14) becomes

$$u(x,t) = \inf_{y\in\mathbb{R}^N} \left[g(y) + \frac{|x-y|^2}{2t} \right] \tag{3.3}$$

which is precisely the inf - convolution of the initial datum g.

An interesting but not evident relationship exists between the inf - convolution and another standard regularization method, namely the classical integral convolution procedure. Let us illustrate this with reference to the Cauchy problem

$$u_t + \frac{1}{2}|Du|^2 = 0\ ,\ (x,t) \in \mathbb{R}^N \times (0,+\infty) \tag{3.4}$$

$$u(x,0) = g(x)\ \ ,\ x \in \mathbb{R}^N \tag{3.5}$$

whose solution is given by (3.3) by the results reported in Section 2. Assume that g is continuous and bounded and consider the parabolic regularization of the Cauchy problem (3.4), (3.5), that is

$$u_t^\epsilon - \epsilon \Delta u^\epsilon + \frac{1}{2}|Du^\epsilon|^2 = 0\ , \quad u^\epsilon(x,0) = g(x) \tag{3.6}$$

where ϵ is a positive parameter. A direct computation shows that if u^ϵ is a smooth solution of the above, then its Hopf - Cole transform

$$w^\epsilon = e^{-\frac{u^\epsilon}{2\epsilon}}$$

satisfies the linear heat problem

$$w_t^\epsilon - \epsilon \Delta w^\epsilon = 0\ \ ,\ w^\epsilon(x,0) = g^\epsilon(x) = e^{-\frac{g(x)}{2\epsilon}} \tag{3.7}$$

By classical linear theory, see [11] for example, its solution w^ϵ can be expressed as the convolution $w^\epsilon = \Gamma \star g^\epsilon$ where Γ is the fundamental solution of the heat equation, that is

$$w^\epsilon(x,t) = (4\pi\epsilon t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4\epsilon t}} e^{-\frac{g(x)}{2\epsilon}} dy$$

Hence, by inverting the Hopf - Cole transform,

$$u^\epsilon(x,t) = -2\epsilon\ \log\left((4\pi\epsilon t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4\epsilon t}} e^{-\frac{g(x)}{2\epsilon}} dy \right) \tag{3.8}$$

turns out to be a solution of the quasilinear problem (3.6).

It is natural to expect that the solutions u^ϵ of (3.6) should converge, as $\epsilon \to 0^+$, to the solution of

$$u_t + \frac{1}{2}|Du|^2 = 0\ \ ,\ \ u(x,0) = g(x)$$

that is, to the Hopf's function (3.3).

We have indeed the following result which shows, in particular, how the inf - convolution can be regarded, roughly speaking, as a singular limit of integral convolutions:

Theorem 3.1 *Assume that g is bounded. Then,*

$$\lim_{\epsilon\to 0^+} -2\epsilon \, \log\left((4\pi\epsilon t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4\epsilon t}} e^{-\frac{g(x)}{2\epsilon}} dy\right) = \inf_{y\in\mathbb{R}^N}\left[g(y) + \frac{|x-y|^2}{2t}\right] \tag{3.9}$$

The proof can be obtained by a direct application of a general large deviations result by S.N. Varadhan. Consider at this purpose the family of probability measures $P^{\epsilon}_{x,t}$ defined on Borel subsets of $\mathbb{R}^N$ by

$$P^{\epsilon}_{x,t}(B) = (4\pi\epsilon t)^{-\frac{N}{2}} \int_B e^{-\frac{|x-y|^2}{4\epsilon t}} dy$$

and the function

$$I_{x,t}(y) = \frac{|x-y|^2}{4t} \; .$$

It is not hard to check that, for all fixed x and t, the family $P^{\epsilon}_{x,t}$ satisfies the large deviation principle, see Definition 2.1 in [19], with rate function $I_{x,t}$.
By Theorem 2.2 in [19], then

$$\lim_{\epsilon\to 0^+} \epsilon \, \log\left(\int_{\mathbb{R}^N} e^{\frac{F(y)}{\epsilon}} dP^{\epsilon}_{x,t}(y)\right) = \sup_{y\in\mathbb{R}^N}\left[F(y) - I(y)\right]$$

for any bounded continuous function F. The choice $F = -\frac{g}{2}$ in the above shows then the validity of the limit relation (3.9).

The same convergence result can be proved also by purely PDE methods. Uniform estimates for the solutions of (3.6) and compactness arguments show the existence of a limit function u solving (3.4), (3.5) in the viscosity sense. Uniqueness results for viscosity solutions allow then to identify the limit u as the Hopf's function, see [14], [3].

The way of deriving the Hopf function via the Hopf - Cole transform and the large deviations principle is closely related to the Maslov's approach [16], [17] to Hamilton - Jacobi equations based on idempotent analysis. In that approach, the base field $\mathbb{R}$ of ordinary calculus is replaced by the semiring $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$ with operations $a \oplus b = \min\{a, b\}$, $a \odot b = a + b$.
A more detailed description of this relationship is beyond the scope of this paper; let us only observe in this respect that the nonsmooth operation $a \oplus b$ has the smooth approximation

$$a \oplus b = \lim_{\epsilon\to 0^+} -\epsilon \log\left(e^{-\frac{a}{\epsilon}} + e^{-\frac{b}{\epsilon}}\right) .$$

A final remark is that the Hopf - Cole transform can be also used to deal with the parabolic regularization of more general Hamilton - Jacobi equations such as

$$u_t + \frac{1}{2}|\sigma(x)Du|^2 = 0$$

where σ is a given $M \times N$ matrix, provided the regularizing second order operator is chosen appropriately. Indeed, if one looks at the regularized problem

$$u_t^\epsilon - \epsilon \, \mathrm{div}\,(\sigma^*(x)\sigma(x)Du^\epsilon) + \frac{1}{2}|\sigma(x)Du^\epsilon|^2 = 0 \ ,$$

then the Hopf - Cole transform $w^\epsilon = e^{-\frac{u^\epsilon}{2\epsilon}}$ solves the linear equation

$$w_t^\epsilon - \epsilon \, \mathrm{div} div\,(\sigma^*(x)\sigma(x)Dw^\epsilon) = 0$$

This observation will be developed in the forthcoming work [8].

4 An Hopf formula for state dependent Hamiltonians

As described in Section 1.2 and 1.3, the value function representation and the Hopf function actually coincide when the Hamiltonian does not depend on the variable x. In this section we present a new Hopf type formula, obtained in collaboration with H. Ishii, see [9], for the viscosity solution of the state - dependent Cauchy problem

$$u_t + H(x, Du) = 0 \ , \ (x,t) \in \mathbb{R}^N \times (0, +\infty) \ , \tag{4.1}$$

$$u(x,0) = g(x) \ , \ x \in \mathbb{R}^N \tag{4.2}$$

It is not hard to realize that the Hopf envelope method of Section 2 does not work if H depends on x. Nonetheless, an Hopf type formula can be proved even in this more general case under the basic structural assumption that the Hamiltonian $H : \mathbb{R}^{2N} \mapsto \mathbb{R}$ is of the form

$$H(x,p) = \Phi\left(H_o(x,p)\right) \tag{4.3}$$

where H_o is a continuous function on $\mathbb{R}^{2N}$ satisfying the following conditions

$$p \mapsto H_o(x,p) \text{ is convex} \ , H_o(x, \lambda p) = \lambda H_o(x,p) \tag{4.4}$$

$$H_o(x,p) \geq 0, \quad |H_o(x,p) - H_o(y,p)| \leq \omega(|x-y|(1+|p|)) \tag{4.5}$$

for all x, y, p, for all $\lambda > 0$ and for some modulus ω such that $lim_{s\to 0^+}\omega(s) = 0$.

Concerning function Φ we assume

$$\Phi : [0,+\infty) \to [0,+\infty) \text{ , is convex, non decreasing , } \Phi(0) = 0 \text{ .} \tag{4.6}$$

The next result shows that the validity of an Hopf type formula for the solution of problem (4.1), (4.2) is guaranteed if the associated stationary eikonal problem

$$H_o(x, Dd) = 1 \text{ , } x \in \mathbb{R}^N \setminus \{y\} \text{ , } d(y) = 0 \tag{4.7}$$

has a solution $d(x) = d(x;y)$ for any value of the parameter $y \in \mathbb{R}^N$.
We shall describe below a setting in which this condition can be enforced.

Theorem 4.1 *Assume* (refA4), (4.4), (4.5), (4.6) *and*

$$g \text{ lower semicontinuous , } g(x) \geq -C(1+|x|) \text{ for some } C > 0 \text{ .} \tag{4.8}$$

Assume also that problem (4.7) *has a unique continuous viscosity solution* $d(x) = d(x;y)$ *for each* $y \in \mathbb{R}^N$ *. Then, the function*

$$u(x,t) = \inf_{y\in\mathbb{R}^N} \left[g(y) + t\Phi^* \left(\frac{d(x;y)}{t} \right) \right] \tag{4.9}$$

is the unique lower semicontinuous viscosity solution of (4.1) *which is bounded below by a function of linear growth and such that*

$$\liminf_{(y,t)\to(x,0^+)} u(y,t) = g(x)$$

In order to understand why the Hopf function (4.9) solves (4.1), let us proceed heuristically by assuming that (4.7) has a smooth solution $d(x)$ and look for special solutions of (4.1) of the form

$$v^y(x,t) = g(y) + t\Psi\left(\frac{d(x;y)}{t}\right)$$

where $y \in \mathbb{R}^N$ plays the role of a parameter and Ψ is a smooth function to be appropriately selected. Set now $\tau = \frac{d(x;y)}{t} > 0$ and compute

$$v^y_t = \Psi(\tau) + t\Psi'(\tau)\frac{-d^2}{t^2} = \Psi(\tau) - \tau\Psi'(\tau) \text{ ; } Dv^y = t\Psi'(\tau)\frac{D_x d}{t} = \Psi'(\tau)D_x d$$

Imposing that v^y solves (4.1) gives

$$\Psi(\tau) - \tau\Psi'(\tau) + \Phi(H_o(x, \Psi'(\tau)D_x d)) = 0$$

Therefore, if Ψ is strictly increasing, the positive homogeneity of H_o and the fact that d solves the eikonal equation yield

$$\Psi(\tau) - \tau\Psi'(\tau) + \Phi(\Psi'(\tau)) = 0\ .$$

Since the solution of this Clairaut's differential equation is $\Psi = \Phi^*$, the above heuristics leads then to formula (4.9). This formal arguments can be made rigorous by some duality arguments in convex analysis and by using the apparatus of comparison and stability methods of the theory of viscosity solutions. We refer to [9] for details.

The assumption that the eikonal equation has a unique continuous viscosity solution made in Theorem 4.1 is trivially satisfied with $d(x; y) = |x - y|$ for the simplest case $H_o(x, p) = |p|$ and, more generally, when the Hamiltonian H_o is of the form

$$H_o(x, p) = |A(x)p|$$

where $A(x)$ is a symmetric positive definite $N \times N$ matrix.
The associated eikonal equations are solved in this case by Riemannian metrics, see [14], [18] at this purpose.

In the examples above, the coercivity condition

$$\lim_{|p|\to+\infty} H_o(x, p) = +\infty \tag{4.10}$$

obviously holds true. Let us briefly discuss now the issue of finding sufficient conditions for the validity of the eikonal assumption in Theorem 4.1 even for degenerate situations when (4.10) may fail.
Consider for example the homogeneous Hamiltonian

$$H_o(x, p) = |\sigma(x)p|$$

where $\sigma(x)$ is an $M \times N$ (with $M \leq N$) matrix such that $x \to \sigma(x)$ is $C^\infty(R^N)$ and satisfies the Chow - Hormander rank condition of order k, see [3], [7]. Consider then the differential inclusion

$$\dot{X}(t) \in \partial H_o(X(t), 0) \tag{4.11}$$

and, for $x, y \in \mathbb{R}^N$, the set $F_{x,y}$ of all trajectories $X(\cdot)$ of (4.11) such that

$$X(0) = x,\ X(T) = y$$

for some $T = T(X(\cdot)) > 0$.
By the well - known Chow's Connectivity Theorem, see [7], the set $F_{x,y}$ is non empty and, consequently, the function

$$d(x; y) = \inf_{X(\cdot) \in F_{x,y}} T(X(\cdot)) \tag{4.12}$$

is finite for all x, y. Moreover, d is a sub - Riemannian metric of Carnot - Carathéodory type which compares locally with the euclidean distance $|x - y|$ on $\mathbb{R}^N$ as

$$C_1|x - y| \leq d(x; y) \leq C_2|x - y|^{\frac{1}{k}}$$

We refer to [9] for a detailed proof.

Let us observe that in the present non coercive setting the function d is not, in general, differentiable almost everywhere; the notion of viscosity solution seems therefore to be essential to interpret d as a solution of (4.7).
An interesting particular case (here $N = 3$ to simplify notations) is

$$H_o(x, p) = \sqrt{(p_1 - \frac{x_2}{2} p_3)^2 + (p_2 + \frac{x_1}{2} p_3)^2}$$

arising in connection with Carnot - Carathéodory on the Heisenberg group H^1. Our Hopf formula (4.9) coincides in this special case with the one recently found for this example by Manfredi - Stroffolini [15].

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Address: Italo Capuzzo Dolcetta
DIPARTIMENTO DI MATEMATICA, UNIVERSITÁ DI ROMA 1,
P.LE A. MORO 2, 00185, ROMA, ITALY
capuzzo@mat.uniroma1.it

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