

# A Qualitative Phragmén-Lindelöf Theorem for Fully Nonlinear Elliptic Equations

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**Abstract** We establish qualitative results of Phragmén-Lindelöf type for upper semicontinuous viscosity solutions of fully nonlinear partial differential inequalities of the second order in general unbounded domains of  $\mathbb{R}^n$ .

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## 1 Introduction and Results

One form of the classical Phragmén-Lindelöf theorem for subharmonic functions  $w$  in an unbounded angular sector  $\Omega \subset \mathbb{R}^2$  of opening  $\frac{\pi}{\alpha}$  states that if  $w \leq 0$  on  $\partial\Omega$  and  $w(x) = O(|x|^\alpha)$  as  $|x| \rightarrow +\infty$ , then  $w \leq 0$  on  $\Omega$ . See [2] for extensions of this result to higher dimensions. Several variants and extensions of this result to smooth solutions of linear and nonlinear elliptic inequalities in more general unbounded domains of  $\mathbb{R}^n$  can be found in the literature, see for example [16, 9, 14, 15, 12, 13, 1, 3].

In the present paper we establish a qualitative result of Phragmén-Lindelöf type for upper semicontinuous viscosity solutions, see [8], of the fully nonlinear partial

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differential inequality

$$F(x, Dw, D^2w) \geq 0 \text{ in } \Omega \quad (1.1)$$

in a general unbounded domain  $\Omega$  of  $\mathbb{R}^n$ .

Here  $Dw$  and  $D^2w$  are, respectively, the gradient and the hessian of the unknown function  $w$  and  $F = F(x, p, M)$  is a given continuous real valued mapping defined on  $\Omega \times \mathbb{R}^n \times \mathcal{S}^n \rightarrow \mathbb{R}$ ,  $\mathcal{S}^n$  denoting the set of  $n \times n$  real symmetric matrices equipped with the partial order  $\geq$  induced by the cone of nonnegative definite matrices.

Our basic assumptions on  $F$  are (degenerate) ellipticity, that is

$$F(x, p, M) \geq F(x, p, N) \quad (1.2)$$

for all  $x \in \Omega$ ,  $t \in \mathbb{R}$ ,  $p \in \mathbb{R}^n$  and  $M, N \in \mathcal{S}^n$  with  $M \geq N$ , and the structure condition

$$F(x, p, M) \leq \mathcal{P}_{\lambda, \Lambda}^+(M) + b(x)|p| \quad (1.3)$$

where  $b > 0$  is a continuous function and

$$\mathcal{P}_{\lambda, \Lambda}^+(M) = \Lambda \operatorname{Tr} M^+ - \lambda \operatorname{Tr} M^-$$

is the Pucci maximal operator.

Let us point out that conditions (1.2) and (1.3) do not imply, in general, the uniform ellipticity of  $M \rightarrow F(x, p, M)$ , see Section 2 for further comments about this point. As for the domain, we will assume that  $\Omega$  satisfies the geometric condition  $\mathbf{G}^*$  which will be stated precisely in Section 3. Let us note for now that the notion of  $\mathbf{G}^*$  domain is a weak version of the one of  $\mathbf{G}$  domain introduced in [4] and is satisfied by a wider class of domains, comprising for example  $n$ -dimensional cones, including the cut plane in  $\mathbb{R}^2$ , and sets which are the complements of graphs of sublinear functions defined on  $(n - 1)$ -dimensional cones.

Two Phragmén-Lindelöf type results for conical and cylindrical domains can be derived from our main result Theorem 3. We refer to the monograph [10] for analogous results for strong solutions of linear inequalities in domains of conical or cylindrical type.

**Theorem A** *Assume that  $\Omega$  is a  $\mathbf{G}^*$  domain of  $\mathbb{R}^n$  of conical type and that  $F$  satisfies (1.2) and (1.3) with*

$$|b(x)| \leq \frac{b_0}{(1 + |x|^2)^{\frac{1}{2}}}.$$

Under these conditions, there exists  $\alpha > 0$ , depending on  $F$  and  $\Omega$ , such that if  $w \in USC(\Omega)$  is a viscosity solution of

$$F(x, Dw, D^2w) \geq 0 \text{ in } \Omega$$

satisfying  $w \leq 0$  on  $\partial\Omega$  and  $w(x) = O(|x|^\alpha)$  as  $|x| \rightarrow +\infty$ , then  $w \leq 0$  in  $\Omega$ . The same conclusion holds for solutions of

$$F(x, Dw, D^2w) + c(x)w \geq 0 \text{ in } \Omega$$

if  $c^+(x) \leq \frac{c_0}{1+|x|^2}$  for small enough  $c_0 > 0$ .

**Theorem B** Assume that  $\Omega$  is a  $\mathbf{G}^*$  domain of  $\mathbb{R}^n$  of cylindrical type and that  $F$  satisfies (1.2) and (1.3) with

$$|b(x)| \leq b_0$$

Under these conditions, there exists  $\alpha > 0$ , depending on  $F$  and  $\Omega$ , such that if  $w \in USC(\Omega)$  is a viscosity solution of

$$F(x, Dw, D^2w) \geq 0 \text{ in } \Omega$$

satisfying  $w \leq 0$  on  $\partial\Omega$  and  $w(x) = O(e^{\alpha|x|})$  as  $|x| \rightarrow +\infty$ , then  $w \leq 0$  in  $\Omega$ .

Theorems A and B extend the classical results of [10] Section 1.5 in the direction of more general unbounded domains as well as of viscosity solutions of non necessarily uniformly elliptic fully nonlinear differential inequalities containing lower order terms.

The proof of these theorems makes use of a reduction of the partial differential inequality to a standard form via the change of unknown  $w = u\xi$  for a suitably chosen positive  $C^2$  function  $\xi$  (see Lemma 1) and of an appropriate version of the Alexandrov-Bakelman-Pucci estimate for bounded viscosity solutions of

$$F(x, Dw, D^2w) \geq f(x)$$

in a  $\mathbf{G}^*$  domain (see Theorem 1). The techniques employed to establish the result, which rely in an essential way on the Caffarelli-Cabré [5] boundary weak Harnack inequality and the local maximum principle for viscosity solutions, are partially mutated from the papers [17, 7]. In both papers the focus is on general unbounded domains, in [17] the target being the Phragmén-Lindelöf principle for

linear elliptic equations while [7] deals with the ABP Maximum Principle for fully nonlinear equations satisfying conditions (1.2) and (1.3).

The general geometric condition  $\mathbf{G}^*$  is precisely defined in Section 3, with examples and counterexamples, and generalized by an iteration process, starting from the globalization of a local geometric condition.

The Phragmén-Lindelöf theorem for the considered domains, established in Section 4, turns out to be a consequence of some auxiliary results of autonomous interest which are extensions of classical estimates to the viscosity solution framework: the Krylov-Safonov Growth Lemma via the boundary weak Harnack inequality (see Lemma 2, Lemma 3 and Remark 3), the ABP estimate (see Theorem 1) and the stability of the Maximum Principle under a small zero-order perturbation of a fully nonlinear operator (see Lemma 4).

## 2 Some preliminary facts

In this section we recall some basic facts about viscosity solutions of fully nonlinear elliptic equations and prove a calculus lemma, which will be used later for reducing the partial differential inequality to a standard form, as well as a version of the boundary weak Harnack inequality.

For given numbers  $0 < \lambda \leq \Lambda$ , the Pucci maximal operator  $\mathcal{P}_{\lambda,\Lambda}^+$  is defined as

$$\mathcal{P}_{\lambda,\Lambda}^+(M) = \Lambda \operatorname{Tr} M^+ - \lambda \operatorname{Tr} M^-$$

where we denoted by  $\operatorname{Tr}$  the trace of a matrix and the matrices  $M^+$  and  $M^-$  are such that

$$M = M^+ - M^- \quad M^+ \geq O \quad M^- \geq O \quad M^+ M^- = O$$

The operator  $\mathcal{P}_{\lambda,\Lambda}^+$  is uniformly elliptic, that is

$$\lambda \operatorname{Tr} Q \leq \mathcal{P}_{\lambda,\Lambda}^+(N + Q) - \mathcal{P}_{\lambda,\Lambda}^+(N) \leq \Lambda \operatorname{Tr} Q \quad \text{for all } N, Q \in \mathcal{S}^n \text{ with } Q \geq 0,$$

positively homogeneous and subadditive, i.e.

$$\mathcal{P}_{\lambda,\Lambda}^+(\alpha M) = \alpha \mathcal{P}_{\lambda,\Lambda}^+(M) \quad , \quad \mathcal{P}_{\lambda,\Lambda}^+(M + N) \leq \mathcal{P}_{\lambda,\Lambda}^+(M) + \mathcal{P}_{\lambda,\Lambda}^+(N)$$

for all  $\alpha > 0$  and  $M, N \in \mathcal{S}^n$ .

The Pucci minimal operator  $\mathcal{P}_{\lambda,\Lambda}^-$  is defined in a symmetric way as

$$\mathcal{P}_{\lambda,\Lambda}^-(M) = \lambda \operatorname{Tr} M^+ - \Lambda \operatorname{Tr} M^- ,$$

see [5] for these properties and further informations on the Pucci operators.

We will always assume that the function  $F$  involved in the partial differential inequality (1.1) satisfies conditions (1.2) and (1.3) in the Introduction.

Let us briefly comment on this point. Our assumptions (1.2), (1.3) are satisfied by any uniformly elliptic  $F$  growing at most linearly with respect to the  $p$  variable, that is if  $F$  satisfies the following conditions:

$$\lambda \operatorname{Tr} Q \leq F(x, p, N + Q) - F(x, p, N) \leq \Lambda \operatorname{Tr} Q \quad (2.1)$$

$$F(x, p, O) \leq b(x) |p| \quad (2.2)$$

for some  $0 < \lambda \leq \Lambda$ , for all  $(x, p, N) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n$  and for all  $Q \in \mathcal{S}^n, Q \geq O$ . Indeed, condition (2.1) obviously implies (1.2). To check (1.3), observe that (2.1) yields

$$F(x, p, M^+ - M^-) - F(x, p, -M^-) \leq \Lambda \operatorname{Tr} M^+$$

$$F(x, p, O) - F(x, p, -M^-) \geq \lambda \operatorname{Tr} M^- .$$

Hence,

$$F(x, p, M) - F(x, p, O) \leq \Lambda \operatorname{Tr} M^+ - \lambda \operatorname{Tr} M^-$$

and (1.3) follows taking (2.2) into account.

Let us observe explicitly that assumptions (1.2), (1.3) allow nonlinear, possibly degenerate, elliptic operators of the form

$$F(M) = \Lambda \left( \sum_{i=1}^n \varphi(\mu_i^+) \right) - \lambda \left( \sum_{i=1}^n \psi(\mu_i^-) \right)$$

where  $\mu_i, i = 1, \dots, n$ , are the eigenvalues of the matrix  $M \in \mathcal{S}^n$  and  $\varphi, \psi : [0, +\infty) \rightarrow [0, +\infty)$  are continuous and nondecreasing functions such that  $\varphi(s) \leq s \leq \psi(s)$ .

Observe also that if the principal part  $x \rightarrow F(x, 0, M)$  of  $F$  is linear and satisfies (1.2) and (1.3), then  $F(x, 0, M)$  is uniformly elliptic. Indeed, using (1.3) with  $M = \pm Q$  with  $Q \geq O$  yields

$$F(x, 0, Q) \leq \mathcal{P}_{\lambda, \Lambda}^+(Q) = \Lambda \operatorname{Tr} Q, \quad F(x, 0, -Q) \leq \mathcal{P}_{\lambda, \Lambda}^+(-Q) = -\lambda \operatorname{Tr} Q$$

so that, by linearity,

$$\lambda \operatorname{Tr}(Q) \leq F(x, 0, Q) \leq \Lambda \operatorname{Tr} Q \quad \forall Q \geq O$$

and (2.1) holds. In particular, for viscous Hamilton-Jacobi operators of the form  $\Delta w + H(x, Dw)$ , conditions (1.2) and (1.3) are satisfied with  $\lambda = \Lambda = 1$  if

$H(x, p) \leq b(x)|p|$  where  $b > 0$  is a continuous function. The same is true if  $\Delta w$  is replaced by a general uniformly elliptic operator in non divergence form with continuous coefficients .

We denote by  $USC(\Omega)$  the set of upper semicontinuous functions defined on  $\Omega$ . Let us recall for convenience that a function  $w \in USC(\Omega)$  is a viscosity solution of the partial differential inequality (1.1) provided that

$$F(x_0, D\varphi(x_0), D^2\varphi(x_0)) \geq f(x_0)$$

at any point  $x_0 \in \Omega$  and for all  $\varphi \in C^2(\Omega)$  such that  $(w - \varphi)(x_0) = 0$  and  $(w - \varphi)(x) \leq 0$  in a neighborhood of  $x_0$ , see [8, 5].

Note that for  $F(x, p, M) = \text{Tr}M$ , the notion of viscosity solution of (1.1) coincides with that of subharmonic function, see for example [6], and also that any  $C^2$  function satisfying (1.1) in the viscosity sense is a classical solution of (1.1).

**Lemma 1** (reduction to standard form) *Let  $w \in USC(\Omega)$  be a viscosity solution of*

$$F(x, Dw, D^2w) \geq f(x) \text{ in } \Omega \quad (2.3)$$

*Assume that  $f \in C(\Omega)$  and that conditions (1.2), (1.3) hold. If  $\xi \in C^2(\Omega)$  is such that*

$$\xi > 0 \quad , \quad |D\xi| \leq k_1(x)\xi \quad , \quad |D^2\xi| \leq k_2(x)\xi \text{ in } \Omega$$

*for some continuous positive functions  $k_1, k_2$ , then  $u = \frac{w}{\xi}$  is a viscosity solution of*

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2u) + \gamma_1(x)|Du| + \gamma_2(x)u^+ \geq \frac{f(x)}{\xi(x)} \text{ in } \Omega \quad (2.4)$$

*where  $\gamma_1(x) = 2h_1bn\Lambda k_1(x) + b(x)$ ,  $\gamma_2(x) = h_2n\Lambda k_2(x) + k_1(x)b(x)$ .*

**Proof.** Let  $\varphi \in C^2(\Omega)$  and  $x_0 \in \Omega$  be such that  $0 = (u - \varphi)(x_0) \geq (u - \varphi)(x)$  in a neighborhood of  $x_0$ . Since  $u = \frac{w}{\xi}$  with  $\xi > 0$ , it follows that

$$w(x) - \xi(x)\varphi(x) \leq w(x_0) - \xi(x_0)\varphi(x_0) = 0.$$

Since  $\xi \in C^2(\Omega)$  and  $w$  is a viscosity solution of (2.3), then

$$F(x_0, D(\xi\varphi)(x_0), D^2(\xi\varphi)(x_0)) \geq f(x_0).$$

A direct, elementary computation shows then that at point  $x_0$  the following inequality holds

$$F(x_0, \xi D\varphi + \varphi D\xi, \xi D^2\varphi + 2D\xi \times D\varphi + \varphi D^2\xi) \geq f$$

where we denoted by  $D\xi \times D\varphi$  the symmetrized product  $\frac{1}{2}(D\xi \otimes D\varphi + D\varphi \otimes D\xi)$ .

Using the structure condition (1.3) we obtain,

$$\mathcal{P}_{\lambda,\Lambda}^+ (\xi D^2\varphi + 2D\xi \times D\varphi + \varphi D^2\xi) + \xi b |D\varphi| + |D\xi| b \varphi^+ \geq f$$

at  $x_0$ . Hence, by positive homogeneity,

$$\mathcal{P}_{\lambda,\Lambda}^+ \left( D^2\varphi + 2\frac{D\xi}{\xi} \times D\varphi + \varphi \frac{D^2\xi}{\xi} \right) + b |D\varphi| + \frac{|D\xi|}{\xi} b \varphi^+ \geq \frac{f}{\xi}.$$

We use now the assumptions made on function  $\xi$  to obtain the matrix inequality

$$2\frac{D\xi}{\xi} \times D\varphi + \frac{D^2\xi}{\xi} \varphi^+ \leq (2h_1k_1|D\varphi| + h_2k_2\varphi^+) I$$

for some positive constants  $h_1, h_2$ . From ellipticity and subadditivity of  $\mathcal{P}_{\lambda,\Lambda}^+$  we deduce

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2\varphi) + \mathcal{P}_{\lambda,\Lambda}^+((2h_1k_1|D\varphi| + h_2k_2\varphi^+)I) + b|D\varphi| + k_1b\varphi^+ \geq \frac{f}{\xi}$$

Using the positive homogeneity of  $\mathcal{P}_{\lambda,\Lambda}^+$  once more we obtain

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2\varphi) + (2h_1n\Lambda k_1 + b)|D\varphi| + (h_2n\Lambda k_2 + k_1b)\varphi^+ \geq \frac{f}{\xi}$$

at  $x_0$ , which proves the validity of inequality (2.4) in the viscosity sense.  $\square$

As mentioned in the Introduction, one fundamental tool which will be employed is a boundary weak Harnack inequality for viscosity solutions in annular domains, see [4], Remark 3.2, which is stated below in a convenient form for our purposes.

Let  $T$  and  $T'$  be domains of  $\mathbb{R}^n$  such that, for some positive integer  $N$  and positive real numbers  $\mu$  and  $\rho$ ,

$$T \subset \bigcup_{i=1}^N B_\rho^i \subset \bigcup_{i=1}^N B_{2\rho}^i \subset T'$$

where  $B_\rho^i, B_{2\rho}^i$  are balls of radius  $\rho$  and  $2\rho$ , respectively, satisfying

$$|T| \geq \mu\rho^n, \quad |B_\rho^i \cap B_\rho^{i+1}| \geq \mu\rho^n.$$

Consider next a domain  $A$  of  $\mathbb{R}^n$  such that both  $T \cap A$  and  $T' \setminus A$  are nonempty. For  $v \in LSC(\bar{A})$ ,  $v \geq 0$ , the lower semicontinuous extension  $v_m^-$  of  $v$  is defined by

$$v_m^-(x) = \begin{cases} \min\{v(x), m\} & \text{if } x \in A \\ m & \text{if } x \notin A \end{cases}$$

where

$$m = \inf_{x \in T' \cap \partial A} v(x).$$

In this setting we have:

**Lemma 2** (a boundary weak Harnack inequality)

Let  $T$ ,  $T'$  and  $A$  as above. Assume that  $N \leq N_0$ ,  $\mu \geq \mu_0$ ,  $\rho \leq \rho_0$  for positive constants  $N_0, \mu_0$  and  $\rho_0$ . If  $v \in LSC(\bar{A})$ ,  $v \geq 0$ , is a viscosity solution of

$$P^-(D^2v(x)) - \gamma(x)|Dv(x)| \leq g(x) \text{ in } A,$$

with  $g \in C(A) \cap L^\infty(A)$  and  $\gamma \in C(A)$  satisfies

$$\|\gamma\|_{L^\infty(T' \cap A)} \leq \gamma_0,$$

then

$$\left( \frac{1}{|T|} \int_T (v_m^-)^p \right)^{1/p} \leq C \left( \inf_{T \cap A} v + \rho \|g\|_{L^\infty(T' \cap A)} \right) \quad (2.5)$$

where  $p$  and  $C$  are positive constants depending on  $n, \lambda, \Lambda, N_0, \mu_0$  and  $\rho_0 \gamma_0$ .

**Remark 1** The Lemma above is the analogue for viscosity supersolutions of fully nonlinear operators of Theorem 3.1 of [4] for strong supersolutions of linear operators. It can be deduced from the fully nonlinear version of the boundary weak Harnack inequality in balls established in [7] using a covering argument, along the same lines of the proof of the above mentioned result of [4].

**Remark 2** We will use the boundary weak Harnack inequality on the annular domains  $T = B_R(y) \setminus \bar{B}_{2\varepsilon R}(0)$  and  $T' = B_{R/\tau}(y) \setminus \bar{B}_{\varepsilon R}(0)$  with positive constants  $\tau < 1$  and  $\varepsilon < 1/2$ . In this case we take  $N_0 = N_0(n, \tau, \varepsilon)$ ,  $\mu_0 = \mu_0(n, \tau, \varepsilon)$ ,  $\rho_0 = O_{n, \tau, \varepsilon}(R)$ .



### 3 Geometric conditions on $\Omega$

We will establish in the next section some local and global Alexandrov-Bakelman-Pucci (ABP, in short) type estimates for bounded above solutions  $w$  of the partial differential inequality

$$F(x, Dw, D^2w) \geq f(x) \text{ in } \Omega. \quad (3.1)$$

The local estimates, see Lemma 3 below, will be obtained at those points of the domain  $\Omega$  which satisfy some specified local geometric condition.

**Definition 1** (local geometric conditions) Let  $\sigma, \tau \in (0, 1)$ .

(i) a point  $y \in \mathbb{R}^n$  satisfies condition  $\mathbf{G}_{\sigma, \tau}$  in  $\Omega$  if there exists a ball  $B$  of radius  $R = R(y)$  such that

$$y \in B, \quad |B \setminus \Omega_{y, B, \tau}| \geq \sigma |B| \quad (3.2)$$

where  $\Omega_{y, B, \tau}$  is the connected component of  $B_{R/\tau} \cap \Omega$  containing  $y$ .

(ii) a point  $y \in \mathbb{R}^n$  satisfies condition  $\mathbf{G}_{\sigma, \tau}^{R_0, \eta}$  in  $\Omega$  if  $y$  satisfies  $\mathbf{G}_{\sigma, \tau}$  in  $\Omega$  with  $R(y) \leq R_0 + \eta|y|$  for some positive constants  $R_0, \eta$ .

**Example 1** To illustrate the above local geometric condition (i), let  $\Omega$  be the "cut plane" in  $\mathbb{R}^2$ , that is  $\Omega = \mathbb{R}^2 \setminus \{x = (x_1, x_2) : x_1 \geq 0, x_2 = 0\}$ . A point  $y = (y_1, y_2) \in \Omega$  with  $|y_2| < \frac{\eta_1}{2}$  satisfies condition  $\mathbf{G}_{\frac{1}{2}, \frac{1}{2}}$ . Conversely, if  $|y_2| > \frac{\eta_1}{2}$ , then  $y$  cannot satisfy  $\mathbf{G}_{\frac{1}{2}, \tau}$  for  $\tau < \frac{1}{5}$ . Note also that a point on the negative  $x_1$ -axis cannot satisfy condition  $\mathbf{G}_{\sigma, \tau}$  no matter how the parameters  $\sigma$  and  $\tau$  are chosen. Indeed, for any circle  $B$  containing such a point, the set  $B \cap \Omega$  turns out to be connected.

To get the uniform ABP estimate in Theorem 1 we need to globalize the local geometric condition. Let us observe that condition  $\mathbf{G}_{\sigma, \tau}^{R_0, \eta}$  is stronger than condition  $\mathbf{wG}$  introduced in [CV] which requires  $\mathbf{G}_{\sigma, \tau}$  to be satisfied at all points of  $\Omega$ .

**Definition 2** (global geometric conditions)

(j) a domain  $\Omega$  satisfies condition  $\mathbf{G}^*$  if  $\mathbf{G}_{\sigma, \tau}^{R_0, \eta}$  holds at point of  $\Omega$

(jj) a domain  $\Omega$  is piecewise  $\mathbf{G}^*$  if there exists  $H \subset \Omega$  such that all connected components of  $\Omega \setminus H$  satisfy  $\mathbf{G}^*$  with the same parameters  $\sigma, \tau, R_0, \eta$  and, moreover, any  $y \in H$  satisfy condition  $\mathbf{G}_{\sigma, \tau}^{R_0, \eta}$  in  $\Omega$ . (jjj) a domain  $\Omega$  is piecewise  $\mathbf{G}^*$

reducible if there exists  $H \subset \Omega$  such that all connected components of  $\Omega \setminus H$  are piecewise  $\mathbf{G}^*$  with the same parameters  $\sigma, \tau, R_0, \eta$  and, moreover, any  $y \in H$  satisfy condition  $\mathbf{G}_{\sigma, \tau}^{R_0, \eta}$  in  $\Omega$ .

It is worth to notice, for computations, that condition  $\mathbf{G}_{\sigma, \tau}^{R_0, \eta}$  imply condition  $\mathbf{G}_{\sigma', \tau'}^{R'_0, \eta'}$  if  $\sigma' \leq \sigma, \tau' \geq \tau, R'_0 \leq R_0$  and  $\eta' \geq \eta$ . For completeness, we also remark that a subdomain of a (piecewise, iteratively)  $\mathbf{G}^*$  domain is (piecewise, iteratively)  $\mathbf{G}^*$ .

**Example 2** Condition (j) in definition 2 with  $\eta = 0$  is condition  $\mathbf{G}$  of [4] which is satisfied for example by domains with finite Lebesgue measure, cylinders and also by the whole space with periodic holes with periodic holes having non-empty interior (see Fig.1).

Proper open cones in  $\mathbb{R}^2$  and complements of logarithmic spirals, for instance  $r = e^\theta$  in polar coordinates, are  $\mathbf{G}^*$  but not  $\mathbf{G}$ , while the cut plane of Example 1 is a piecewise  $\mathbf{G}^*$  domain but not  $\mathbf{G}^*$ . More generally, open cones in  $\mathbb{R}^n$  whose closure is different from  $\mathbb{R}^n$  are  $\mathbf{G}^*$ ; the complements of hypersurfaces which are graphs of continuous functions with at most linear growth on  $(n-1)$ -dimensional cones, are piecewise  $\mathbf{G}^*$ .

Finally, considering the  $2^{n-1}$ -hyperplane  $Q = \{x = (x', 0) \in \mathbb{R}^n / x'_j > 0, j = 1, \dots, n-1\}$ , let  $\Omega$  be a domain which is obtained from  $\mathbb{R}^n$  removing all balls, having the some fixed radius, centered at the points of  $Q$  with integer coordinates. Using  $H = \bar{Q} \cap \Omega$ , one recognizes that  $\Omega$  is piecewise  $\mathbf{G}^*$  reducible (see Fig.1).

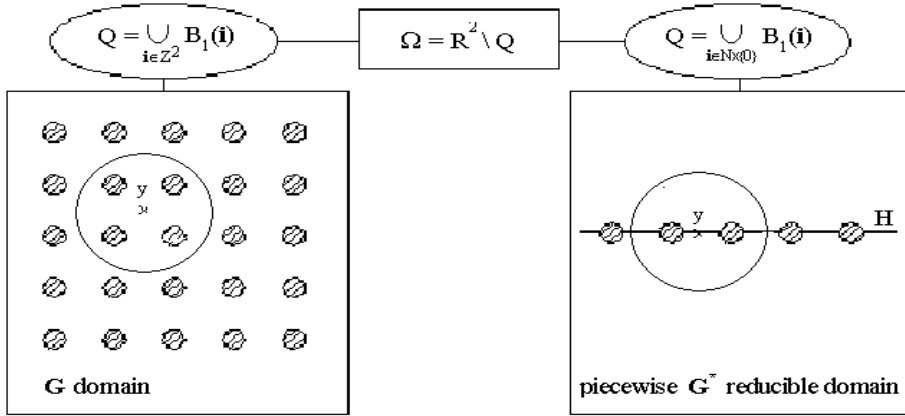


Fig.1 – Schemes for global geometric conditions

Fig.1

## 4 The ABP estimate and the Phragmén-Lindelöf Theorem

The next Lemma provides a pointwise estimate for viscosity solutions of

$$F(x, Dw, D^2w) \geq f(x) \quad \text{in } \Omega \quad (4.1)$$

at those points  $y \in \Omega$  where the geometric condition  $\mathbf{G}_{\sigma, \tau}$  in  $\Omega$  holds. As it will be seen in the subsequent remark, this estimate yields in particular a viscosity solutions version of the well-known Krylov-Safonov Growth Lemma.

As for notations, with reference to Definition 1 we will denote by  $B_R$  a ball of radius  $R = R(y)$  containing  $y$ , a concentric ball of radius  $R/\tau$  will be denoted by  $B_{R/\tau}$ , while  $A_{\varepsilon R}^{R/\tau}$  will denote the annular set  $B_{R/\tau} \setminus \overline{B_{\varepsilon R}}(0)$ . Also,  $\chi_c^+$  will be the characteristic function of the set  $]c, +\infty[$ , i.e.  $\chi_c^+ = 1$  in  $]c, +\infty[$  and  $\chi_c^+ = 0$  outside, and  $\chi_c^- = 1 - \chi_c^+$ .

**Lemma 3** *Let  $w \in USC(\Omega)$  be a viscosity solution of (4.1) with  $f \in C(\Omega)$ . Assume that the structure condition (1.3) holds with  $b \in C(\Omega)$  such that*

$$0 < b(x) \leq b_0$$

for some  $b_0 > 0$ . Then, at any  $y \in \Omega$  satisfying condition  $\mathbf{G}_{\sigma, \tau}^{R_0, \eta}$  the following inequality holds

$$w(y) \leq \kappa \sup_{B_{R/\tau} \cap \Omega} w^+ + (1 - \kappa) \limsup_{x \rightarrow B_{R/\tau} \cap \partial \Omega} w^+ + \chi_{R_0}^-(|y|) R_0 \hat{f} + \chi_{R_0}^+(|y|) R \tilde{f} \quad (4.2)$$

where  $\hat{f} = \|f\|_{L^n(B_{R/\tau} \cap \Omega)}$ , with  $R \leq (1 + \eta)R_0$ , and  $\tilde{f} = \|f\|_{L^n(A_{\varepsilon R}^{R/\tau} \cap \Omega)}$ , for positive constants  $\varepsilon = \varepsilon(\sigma, \eta)$  and  $\kappa = \kappa(n, \lambda, \Lambda, b_0, \sigma, \tau, R_0, \eta, R \|b\|_{L^\infty(A_{\varepsilon R}^{R/\tau} \cap \Omega)}) < 1$ .

**Proof.** Thanks to (1.3),  $w$  satisfies

$$\mathcal{P}_{\lambda, \Lambda}^+(D^2w(x)) + b(x)|Dw(x)| \geq f(x) \quad x \in \Omega.$$

It is easy to check that  $v(x) = \sup_{B_{R/\tau} \cap \Omega} w^+ - w(x)$  is a viscosity solution of

$$\mathcal{P}_{\lambda, \Lambda}^-(D^2v(x)) - b(x)|Dv(x)| \leq f^-(x) \quad x \in \Omega.$$

Take now a ball  $B = B_R$  as in Definition 2. If  $|y| > R_0$ , we set  $T = A_{2\varepsilon R}^R$  and  $T' = A_{\varepsilon R}^{R/\tau}$ , with  $0 < \varepsilon < \min(\frac{1}{2(1+\eta)}, \frac{\sigma}{4})$ , and consider the component  $A$  of  $T' \cap \Omega$

containing  $y$ . In fact, in this case  $R \leq (1 + \eta)|y|$  and, with the above choice of  $\varepsilon$ ,  $y \in A_{2\varepsilon R}^R$ . Also,

$$\begin{aligned} |T \setminus A| &\geq |T \setminus \Omega_{y,\tau}| \geq |B_R \setminus \Omega_{y,\tau}| - |B_{2\varepsilon R}| \geq \\ &\geq \sigma |B_R| - (2\varepsilon)^n |B_R| \geq \frac{\sigma}{2} |B_R| \geq \frac{\sigma}{2} |T|. \end{aligned}$$

If, on the contrary,  $|y| \leq R_0$ , we have  $R \leq (1 + \eta)R_0$ . In this case we set  $T = B_R$ ,  $T' = B_{R/\tau}$  and  $A = \Omega_{y,B,\tau}$ .

Suppose temporarily that  $w_0 \equiv \limsup_{x \rightarrow B_{R/\tau} \cap \partial\Omega} w^+ \leq 0$ . Since

$$T' \cap \partial A \subset T' \cap \partial(T' \cap \Omega) \subset T' \cap (\partial T' \cup \partial\Omega) \subset T' \cap \partial\Omega$$

then

$$\liminf_{x \rightarrow T' \cap \partial A} v(x) = \bar{w} - \limsup_{x \rightarrow T' \cap \partial A} w(x) \geq \bar{w} - \limsup_{x \rightarrow T' \cap \partial\Omega} w^+(x) \geq \bar{w} \quad (4.3)$$

where

$$\bar{w} = \sup_{B_{R/\tau} \cap \Omega} w^+.$$

Since  $y \in T \cap A$ , then

$$\inf_{T \cap A} v \leq v(y) = \bar{w} - w(y) \quad (4.4)$$

Set  $m = \liminf_{x \rightarrow T' \cap \partial A} v(x)$  and use (4.3) and (4.4) together with Lemma 2 to obtain

$$\begin{aligned} \left(\frac{\sigma}{2}\right)^{1/p} \bar{w} &\leq \left(\frac{|T \setminus A|}{|T|}\right)^{1/p} \bar{w} \leq \left(\frac{1}{|T|} \int_{T \setminus A} m^p\right)^{1/p} \leq \left(\frac{1}{|T|} \int_T (v_m^-)^p\right)^{1/p} \leq \\ &\leq C \left( \inf_{T \cap A} v + R \|f^-\|_{L^n(T' \cap A)} \right) \leq C(\bar{w} - w(y) + R \|f^-\|_{L^n(A_{\varepsilon R}^{R/\tau} \cap \Omega)}). \end{aligned} \quad (4.5)$$

Recalling the dependence of the constants  $C, p$  (see Lemma 2) and  $\varepsilon$  (see Remark 2), we use inequality (4.2) with  $\kappa = 1 - \frac{(\sigma/2)^{1/p}}{\max(C,1)}$  is established in the case  $w_0 \leq 0$ . To obtain (4.2) in its generality, it suffices to consider the function  $w(x) - w_0$ .  $\square$

**Remark 3** Let  $w \in C(\bar{\Omega})$ ,  $w \leq 0$  on  $\partial\Omega$  be a viscosity solution of

$$\mathcal{P}_{\lambda,\Lambda}^+(D^2w(x)) + b(x)|Dw(x)| \geq 0 \quad x \in \Omega$$

Consider a ball  $B_R$  of radius  $R \leq R_0$  such that  $|B_R \cap \Omega| \leq t|B_R|$  with  $0 < t < 1$ . Then, each point  $y \in B_R \cap \Omega$  satisfies condition  $\mathbf{G}_{\sigma,\tau}^{R_0,0}$  in  $\Omega$  for any positive  $\sigma < 1 - t$  and  $\tau \rightarrow 1^-$ . Hence, as a consequence of Lemma 3, we get

$$\sup_{B_R \cap \Omega} w \leq \kappa \sup_{B_{R/\tau} \cap \Omega} w^+ \quad (4.6)$$

for a positive constant  $\kappa = \kappa(n, \lambda, \Lambda, b_0, t, \tau, R_0) < 1$ .

This local estimate is well-known for strong subsolutions of uniformly elliptic linear equations as the Krylov-Safonov Growth Lemma, see [11],[13],[10].

If the domain satisfies the global geometric (iteratively piecewise)  $\mathbf{G}^*$  condition, see Definition 2, Lemma 3 above can be used to obtain an ABP estimate for bounded above solutions of 4.1 in such domains .

**Theorem 1** *Let  $F$  and  $f$  be as in Lemma 3. If  $w \in USC(\Omega)$  is a bounded above viscosity solution of*

$$F(x, Dw, D^2w) \geq f(x) \text{ in } \Omega$$

where  $\Omega$  is an (iteratively piecewise)  $\mathbf{G}^*$  domain. Then,

$$\sup_{\Omega} w \leq \limsup_{x \rightarrow \partial\Omega} w^+ + C(R_0 \sup_{y \in \Omega} \hat{f} + \chi_0^+(\eta) \sup_{y \in \Omega, |y| > R_0} R\tilde{f}), \quad (4.7)$$

where  $\hat{f} = \|f\|_{L^n(B_{R/\tau} \cap \Omega)}$ , with  $R \leq (1 + \eta)R_0$ , and  $\tilde{f} = \|f\|_{L^n(A_{\varepsilon R}^{R/\tau} \cap \Omega)}$ , for some  $\varepsilon = \varepsilon(\sigma, \eta)$  and  $C = C\left(n, \lambda, \Lambda, b_0, \sigma, \tau, R_0, \eta, R\|b\|_{L^\infty(A_{\varepsilon R}^{R/\tau} \cap \Omega)}\right)$ .

**Proof.** Consider first the case where  $w_0 = \limsup_{x \rightarrow \partial\Omega} w^+ \leq 0$ . If condition  $\mathbf{G}^*$  is satisfied, then, using the pointwise estimate (4.2) and taking the supremum over  $y \in \Omega$ , we get at once

$$\sup_{\Omega} w \leq C(R_0 \sup_{y \in \Omega} \hat{f} + \chi_0^+(\eta) \sup_{y \in \Omega, |y| > R_0} R\tilde{f}). \quad (4.8)$$

Next, consider a piecewise  $\mathbf{G}^*$  domain. The above argument, when applied to  $w - \sup_H w^+$  in each connected component of  $\Omega \setminus H$ , yields, by j) of Definition 2,

$$\sup_{\Omega} w \leq \sup_H w^+ + C(R_0 \sup_{y \in \Omega} \hat{f} + \chi_0^+(\eta) \sup_{y \in \Omega, |y| > R_0} R\tilde{f}). \quad (4.9)$$

For  $x \in H$ , using part jj) of Definition 2, the pointwise estimate (4.2) implies

$$w(x) \leq \kappa \sup_{\Omega} w^+ + R_0 \sup_{y \in H} \hat{f} + \chi_0^+(\eta) \sup_{y \in H, |y| > R_0} R\tilde{f}$$

with  $\kappa < 1$ .

When inserted in (4.9), the above inequality extends (4.8) to piecewise  $\mathbf{G}^*$  domains. Finally, for a piecewise  $\mathbf{G}^*$  reducible domain, a similar reduction to components, by virtue of the result just obtained for piecewise  $\mathbf{G}^*$  domains, provides

(4.7) when  $w_0 \leq 0$ . The case of a general upper bound on the boundary easily follows by considering the function  $w(x) - w_0$ .  $\square$

In the case of slabs  $\Omega = S \times \mathbb{R}^k \subset \mathbb{R}^n$ , where  $S$  is a bounded open set in  $\mathbb{R}^h$  and  $h + k = n$ , we can get  $\mathbf{G}^*$  with  $\sigma = 1/2, \tau = 1/2, R_0 = \text{diam}S \equiv \delta$  and  $\eta = 0$ . Therefore the ABP estimate (4.7) implies

$$\sup_{\Omega} w \leq \limsup_{x \rightarrow \partial\Omega} w^+ + CR_0 \sup_{y \in \Omega} \|f^-\|_{L^n(B_{2\delta}(y) \cap \Omega)} \quad (4.10)$$

with  $C$  depending on  $n, \lambda, \Lambda, b_0, \delta$ .

On the other side, when  $\eta > 0$  and  $\sup_{y \in \Omega} R(y) = +\infty$ , we need a suitable decay of the first-order coefficient at infinity, to keep the cross term  $Rb$  bounded, and thus  $C$  finite.

For proper circular cones  $\Omega \subset \mathbb{R}^n$  of opening  $\phi$  with vertex in the origin, we can obtain  $\mathbf{G}^*$  with  $\sigma = \sigma(\phi), \tau = 1/2, R_0 = 0$  and  $\eta = 2$ . In this case, supposing  $|b(x)| \leq b_0/(1 + |x|^2)^{1/2}$ , the above ABP estimate (4.7) yields

$$\sup_{\Omega} w \leq \limsup_{x \rightarrow \partial\Omega} w^+ + C \sup_{R>0} R \|f^-\|_{L^n((B_{2R}(y) \setminus B_{\varepsilon R}(0)) \cap \Omega)}, \quad (4.11)$$

with  $\varepsilon = \varepsilon(\phi)$  and  $C = C(n, \lambda, \Lambda, b_0, \phi)$ .

As an easy consequence of Theorem 1, for our general domains we have

**Corollary 2** *Suppose, in addition to the assumptions of Theorem 1, that*

$$b(x) \leq \frac{b_0}{(1 + \chi_0^+(\eta)|x|^2)^{1/2}}, \quad (4.12)$$

*then (4.7) holds with  $\varepsilon = \varepsilon(\sigma, \eta)$  and  $C = C(n, \lambda, \Lambda, b_0, \sigma, \tau, R_0, \eta)$ .*

**Remark 4** In the case  $\eta = 0$ , the above result is a viscosity version of the "improved" ABP estimate of [4] for cylinders and in general for  $\mathbf{G}$  domains (see Example 2). Analogously, for  $\eta > 0$  the above extends the "variant" of ABP estimate of [17] for cones and for the much more general class of  $\mathbf{G}^*$  domains (see again Example 2).

Inequality (4.7) for  $f \equiv 0$  implies the validity of the weak Maximum Principle for bounded above viscosity solutions of

$$F(x, Dw, D^2w) \geq 0$$

in a piecewise  $\mathbf{G}^*$  reducible domain  $\Omega$ . The next result shows that the validity of the weak Maximum Principle is preserved under an additive perturbation with a sufficiently small positive zero-order term. This fact will be used next to derive our qualitative Phragmèn-Lindelöf principles.

**Lemma 4** *Let  $\Omega$  be a piecewise  $\mathbf{G}^*$  reducible domain. Assume that  $F$  satisfies condition (1.3) with  $b \in C(\Omega)$  such that*

$$0 < b(x) \leq \frac{1}{(1 + \chi_0^+(\eta)|x|^2)^{1/2}}.$$

*Let  $c \in C(\Omega)$  and  $w \in USC(\Omega)$  be a bounded above viscosity solution of*

$$F(D^2w(x), Dw(x), x) + c(x)w^+(x) \geq 0 \quad \text{in } \Omega \quad (4.13)$$

*such that*

$$\limsup_{x \rightarrow \partial\Omega} w(x) \leq 0.$$

*If*

$$c^+(x) \leq \frac{c_0}{1 + \chi_0^+(\eta)|x|^2} \quad (4.14)$$

*for some sufficiently small positive constant  $c_0$ , depending on the structure data and the geometric parameters, then  $w \leq 0$  in  $\Omega$ .*

**Proof.** Using the structure assumptions (1.3) it is easy to check that  $w$  is a viscosity solution of

$$P_{\lambda, \Lambda}^+(D^2w(x)) + b(x)|Dw(x)| \geq -c^+(x)w^+(x)$$

We apply now Theorem 1 with  $f = c^+w^+$ . At this purpose we estimate the right-hand side of inequality (4.7) using condition (4.14). This yields

$$R_0 \|c^+w^+\|_{L^n(B_{R/\tau} \cap \Omega)} \leq \tau^{-1} \omega_n^{1/n} (1 + \eta) R_0^2 c_0 \sup_{\Omega} w^+$$

$$R \|c^+w^+\|_{L^n(A_{\varepsilon R}^{R/\tau} \cap \Omega)} \leq \tau^{-1} \omega_n^{1/n} c_0 \varepsilon^{-2} \sup_{\Omega} w^+$$

where  $\omega_n$  is volume of the unit ball in  $\mathbb{R}^n$ . From (4.7) it follows then that

$$\sup_{\Omega} w \leq K c_0 \sup_{\Omega} w^+$$

for some constant  $K > 0$  independent of  $w$  and the statement follows.  $\square$  We are now in position to prove our main result:

**Theorem 3** Assume that  $\Omega$  is a piecewise  $\mathbf{G}^*$  reducible domain with parameters  $\sigma, \tau, R_0, \eta$ . Let  $w \in USC(\Omega)$  be a viscosity solution of

$$F(x, Dw(x), D^2w(x)) \geq 0 \text{ in } \Omega$$

such that

$$\limsup_{x \rightarrow \partial\Omega} w(x) \leq 0$$

where  $F$  satisfies the structure condition (1.3) with  $b \in C(\Omega)$  such that

$$0 < b(x) \leq \frac{1}{(1 + \chi_0^+(\eta)|x|^2)^{1/2}}.$$

Then there exists a positive  $\alpha$ , depending on the structure data and the geometric parameters, such that if

$$w^+(x) = O(e^{\alpha(\log|x| + \chi_0^-(\eta)|x|)}) \quad (4.15)$$

as  $|x| \rightarrow \infty$ , then  $w \leq 0$  in  $\Omega$ .

**Proof.** Consider, for  $\alpha > 0$  to be chosen later, the positive smooth function

$$\xi(x) = \chi_0^+(\eta)(1 + |x|^2)^{\alpha/2} + \chi_0^-(\eta) e^{\alpha(1+|x|^2)^{1/2}}$$

If  $w$  grows at infinity as prescribed by condition (4.15), the function  $u(x) = \frac{w(x)}{\xi(x)}$  is bounded above and obviously  $\limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$ .

A straightforward calculation shows now that

$$\frac{|D\xi|}{\xi} \leq \frac{\alpha}{2(1 + \chi_0^+(\eta)|x|^2)^{1/2}}, \quad \frac{|D^2\xi|}{\xi} \leq \frac{2n\alpha}{1 + \chi_0^+(\eta)|x|^2}$$

for a sufficiently small  $\alpha$ . Thus, from Lemma 1, using (4.12), we deduce that

$$P_{\lambda, \Lambda}^+(D^2u(x)) + \gamma_1(x)|Du(x)| + \gamma_2(x)u^+(x) \geq 0$$

with

$$\gamma_1(x) = \frac{2h_1n\Lambda\alpha + b_0}{2(1 + \chi_0^+(\eta)|x|^2)^{1/2}}, \quad \gamma_2(x) = \frac{\alpha(2h_2n^2\Lambda + b_0)}{1 + \chi_0^+(\eta)|x|^2}$$

For sufficiently small  $\alpha > 0$  the coefficient  $\gamma_2$  satisfies condition (4.14). Hence, by Lemma 4,  $u \leq 0$ . This concludes the proof of the theorem.  $\square$



**Remark 5** To obtain the assert of Theorem 3 we do not need the admissible growth (4.15) on all spherical sections of  $\Omega$ , but only on an increasing sequence of spherical sections  $|x| = R_k$  such that  $R_k \rightarrow \infty$  as  $k \rightarrow \infty$ . Indeed, using a typical Phragmén-Lindelöf argument (see [16]), we can assume, instead of (4.15), that

$$\liminf_{k \rightarrow \infty} \frac{M_k}{e^{\alpha(\log R_k + \chi_0^-(\eta) R_k)}} < +\infty,$$

where  $M_k = \sup_{\Omega \cap \partial B_{R_k}(0)} w^+$ . This is a refinement of Theorem 3 along the lines of classical results, which turn out to be extended to viscosity subsolutions with exponential ( $\eta = 0$ ) and polynomial growth ( $\eta > 0$ ) in the above piecewise  $\mathbf{G}^*$  reducible domains of cylindrical type ( $\eta = 0$ ) and conical type ( $\eta > 0$ ), respectively.

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