Approximation of solutions of Hamilton-Jacobi equations on the Heisenberg group

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Abstract

We propose and analyze numerical schemes for viscosity solutions of time-dependent Hamilton-Jacobi equations on the Heisenberg group. The main idea is to construct a grid compatible with the noncommutative group geometry. Under suitable assumptions on the data, the Hamiltonian and the parameters for the discrete first order scheme, we prove that the error between the viscosity solution computed at the grid nodes and the solution of the discrete problem behaves like $\sqrt{h}$ where is h is the mesh step. Such an estimate is similar to those available in the Euclidean geometrical setting. The theoretical results are tested numerically on some examples for which semi-analytical formulas for the computation of geodesics are known. Other simulations are presented, for both steady and unsteady problems.

1 Introduction

This paper is concerned with the approximation of solutions of Cauchy problems for some first order degenerate Hamilton-Jacobi partial differential equation, of the form

$$\begin{array}{rcll} \frac{\partial u}{\partial t} + \Phi(|D_H u|) & = & 0, & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) & = & u_0(x), & \text{in } \mathbb{R}^3, \end{array} \tag{1}$$

where Φ is a positive, continuous and convex function on $\mathbb{R}_+$ (we shall make further assumption on Φ later), and D_H is defined as follows:

$$D_H u = \begin{pmatrix} \partial_{x_1} u + 2x_2 \partial_{x_3} u \\ \partial_{x_2} u - 2x_1 \partial_{x_3} u \end{pmatrix}, \tag{2}$$

and $|D_H u|$ stands for the Euclidean norm of the vector $D_H u$, namely

$$|D_H u| = \sqrt{(\partial_{x_1} u + 2x_2 \partial_{x_3} u)^2 + (\partial_{x_2} u - 2x_1 \partial_{x_3} u)^2}\,.$$

If D is the standard gradient operator in $\mathbb{R}^3$, we have

$$D_H = \sigma(x) D \quad , \quad \sigma(x) = \begin{pmatrix} 1 & 0 & 2x_2 \\ 0 & 1 & -2x_1 \end{pmatrix}. \tag{3}$$

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The degeneracy of equation (1) comes from the fact that the matrix σ has rank two at any point $x \in \mathbb{R}^3$. Problem (1) is strongly associated to the asymptotic behavior of the heat kernel on the Heisenberg group $(\mathbb{R}^3, \oplus)$, where

$$y \oplus x = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_1 y_2 - x_2 y_1)), \tag{4}$$

see [7]. Problem (1) is also related to the dynamic programming approach to optimal control problems for the Brockett system, see [10, 3], and to the level set approach to front propagation [27].

Under suitable assumptions on Φ and u_0, a Hopf-Lax type representation formula (see (15) below) for the viscosity solution of (1) has been established by Manfredi and Stroffolini [26], see also [13].

For nondegenerate Hamilton-Jacobi equations, Crandall and Lions [16] studied finite difference schemes for the approximation of viscosity solutions: for monotone and consistent schemes on a uniform grid, they proved convergence and optimal error estimates. In this direction, further developments were proposed by Osher and Sethian [27, 29] inspired by the Engquist-Osher scheme for conservation laws. Osher and Sethian also proposed fast marching methods for the eikonal equation, see [29]. In the same context, Lagrangian methods were proposed and analyzed in [11, 15, 19, 20]. Higher order schemes have been proposed in e.g. [28, 20]. For finite element methods, see for example [22, 30]. Finally, finite difference schemes for degenerate Hamilton-Jacobi-Bellman equations have been studied in for e.g. [6, 5, 4, 24, 25].

The purpose of the present paper is to propose and analyze finite difference schemes for the approximation of viscosity solutions of (1). The main idea is to construct a grid compatible with the translations (4) in such a way that it inherits the geometrical properties of the Heisenberg group. More precisely, the grid nodes are chosen to be $\xi_{i,j,k} = (ih, jh, (4k + 2ij)h^2)$, where h is the grid step, and i, j, k are integers. Such a grid has been introduced in [1] for designing a finite difference scheme for the Dirichlet problem with the Kohn Laplacian on the Heisenberg group. Once the grid is constructed, it is natural to implement the above mentioned schemes. Inspired by [16], we show that, under suitable assumptions on Φ, the initial data u_0 and the parameters for the discrete scheme, the error between the viscosity solution computed at the grid nodes and the solution of the discrete problem is bounded by $C\sqrt{h}$, which is precisely the estimate obtained by Crandall and Lions in the nondegenerate case.

In order to test the theoretical results, we use the scheme for both (1) and the associated static eikonal equation. In particular, we test the numerical method against the semi-analytical formulas provided by Beals et al [7] for the Carnot-Carathéodory distance.

The results in the present paper certainly hold for the higher dimensional version of equation (1) in $\mathbb{R}^{2n+1}$. In this setting, D_H is replaced by $D_{H^n} = \sigma(x)D$ with

$$\sigma(x) = \begin{pmatrix} I & 0 & 2x'' \\ 0 & I & -2x' \end{pmatrix},$$

where $x', x'' \in \mathbb{R}^n$ and $x = (x', x'', x_{2n+1})$.

Similar methods, with appropriate changes in the choice of the grid, may work for more general problems like

$$\begin{aligned} \frac{\partial u}{\partial t} + \Phi(|\sigma(x)Du|) &= 0 && \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) &= u_0(x) && \text{in } \mathbb{R}^n, \end{aligned} \tag{5}$$

under the assumption that the columns of the $m \times n$ matrix $\sigma(x)$ satisfy the Hörmander-Chow rank condition at some order k at all points $x \in \mathbb{R}^n$, see [8].

2 Basic facts on the Heisenberg group and viscosity solutions

Let us start by recalling relevant properties of the Heisenberg group $H = (\mathbb{R}^3, \oplus)$, where

$$y \oplus x = \big(x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_1 y_2 - x_2 y_1)\big) .$$

It is obvious that, in general, $x \oplus y \neq y \oplus x$. Note that $x \oplus y = y \oplus x$ if and only if $x_1 y_2 - x_2 y_1 = 0$.
The operator D_H commutes with left translations, i.e. for all $y \in \mathbb{R}^3$, calling $\tau_y^L u$ the function $x \mapsto u(y \oplus x)$,

$$D_H(\tau_y^L u) = \tau_y^L (D_H u). \tag{6}$$

On the contrary, calling $\tau_y^R u$ the function $x \mapsto u(x \oplus y)$,

$$\left(D_H(\tau_y^R u)\right)(x) = \left(\tau_y^R (D_H u)\right)(x) + 4((\partial_{x_3} u)(x \oplus y)) \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix} .$$

Let α be a nonnegative parameter, the dilation of x by α is defined by

$$\alpha \cdot x = (\alpha x_1, \alpha x_2, \alpha^2 x_3). \tag{7}$$

One can verify that $\alpha \cdot (x \oplus y) = \alpha \cdot x \oplus \alpha \cdot y$.
The operator D_H has the following behavior with respect to dilatations: calling $u \circ \alpha$ the function $x \mapsto u(\alpha \cdot x)$, we have

$$D_H(u \circ \alpha) = \alpha \ (D_H u) \circ \alpha. \tag{8}$$

Observe that for all $x \in \mathbb{R}^3$ and $y = (y_1, y_2, 0)$, one has

$$x \oplus ty = x(t), \tag{9}$$

where $x(t)$ is the solution of the ordinary differential equation

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x_2(t) & -2x_1(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

with the initial value $x(0) = x$.
For any fixed $y \in \mathbb{R}^3$, the stationnary eikonal type problem

$$|D_H w_y(x)| = 1 \quad \text{ in } \mathbb{R}^3 \backslash \{y\}, \qquad w_y(y) = 0 \tag{10}$$

has a unique viscosity solution satisfying

$$\begin{aligned} w_y(x) \geq 0, \quad \forall x, y \in \mathbb{R}^3, \\ \lim_{|x-y| \to \infty} w_y(x) = +\infty, \\ w_y(x) + w_z(y) \geq w_z(x), \quad \forall x, y, z \in \mathbb{R}^3, \end{aligned}$$

see [2, 3], where $| \cdot |$ is the standard Euclidean norm in $\mathbb{R}^2$.
We use the notation $d(x; y) = w_y(x)$ the so-called Carnot-Carathéodory distance. It follows easily from the left invariance and homogeneity of D_H, see (6) and (8), that

$$d(z \oplus x; z \oplus y) = d(x; y), \quad \text{and} \quad d(\alpha \cdot x; \alpha \cdot y) = \alpha d(x; y). \tag{11}$$

It is also well-known, see [8], that for any $R > 0$ there exists a constant $K(R) > 0$ such that

$$d(x;y) \leq K(R)\,|x-y|^{\frac{1}{2}} \quad \text{for all} \quad x,y \in \mathbb{R}^3\,,\ |x-y| \leq R\,. \tag{12}$$

We denote by $|\cdot|_K$ the Koràny homogeneous norm in $\mathbb{R}^3$, which is naturally associated with the Heisenberg group:

$$|x|_K = \big((x_1^2+x_2^2)^2 + x_3^2\big)^{\frac{1}{4}}\,. \tag{13}$$

It is clear that

$$|x|_K = \sqrt{x_1^2+x_2^2} = |x|$$

for any horizontal vector $x = (x_1, x_2, 0)$. Note also that for each $\alpha \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^3$, $|\alpha \cdot x|_K = \alpha |x|_K$ and $|-y \oplus x|_K = |-x \oplus y|_K$. It is proved in [23] that $(x,y) \mapsto |-y \oplus x|_K$ defines a metric in $\mathbb{R}^3$. It can be seen that $x \mapsto |-y \oplus x|_K$ is a viscosity subsolution of (10). We also recall that there exist two positive constants $c_1 < c_2$ such that

$$c_1|-x \oplus y|_K \leq d(x;y) \leq c_2|-x \oplus y|_K, \tag{14}$$

see [8]. For what follows, we define the Carnot-Carathéodory balls

$$B_C(r) = \{x \in \mathbb{R}^3,\ d(x;0) \leq r\},$$

and the Koràny balls

$$B_K(r) = \{x \in \mathbb{R}^3,\ |x|_K \leq r\}.$$

We shall say that u is Lipschitz continuous with respect to the left translations with a constant L if, for all $y \in \mathbb{R}^3$,

$$\sup_{z \in \mathbb{R}^3} |u(y \oplus z) - u(z)| \leq L|y|_K.$$

Similarly, u is Lipschitz continuous with respect to the right translations with a constant L if, for all $y \in \mathbb{R}^3$,

$$\sup_{z \in \mathbb{R}^3} |u(z \oplus y) - u(z)| \leq L|y|_K.$$

For example, for any real valued Lipschitz continuous function χ on $\mathbb{R}_+$, $x \mapsto \chi(|x|_K)$ is Lipschitz continuous w.r.t. right translations. Also, any bounded subsolution of $|D_H w| \leq 1$ in $\mathbb{R}^3$ is Lipschitz continuous with respect to right translations, see [9].
If the initial datum u_0 is bounded and continuous and $\Phi : [0, +\infty) \to \mathbb{R}$ is convex nondecreasing with $\Phi(0) = 0$, then, introducing the conjugate function

$$\Phi^*(q) = \sup_{p \geq 0} \big(p\,q - \Phi(p)\big),$$

the Hopf-Lax formula

$$u(x,t) = \inf_{y \in \mathbb{R}^3} \left(u_0(y) + t\Phi^*\left(\frac{d(x;y)}{t}\right)\right), \tag{15}$$

see [26, 12, 13, 14], provides the unique continuous viscosity solution of problem (1), see [17].
It is simple to verify that Φ^* is convex and nondecreasing with $\Phi(0) = 0$.
In what follows, we make the following assumption on Φ:

Assumption 1 *The function Φ is convex and nondecreasing, and $\Phi(0) = 0$ and the conjugate function Φ^* is such that*

$$\lim_{q \to +\infty} \frac{\Phi^*(q)}{q} = +\infty. \tag{16}$$

The assumption is fulfilled for example by $\Phi(p) = \frac{1}{\alpha}p^\alpha$ with $\alpha \geq 1$. If Assumption 1 holds, then

$$u(x,t) = \min_{y\in\mathbb{R}^3}\left(u_0(y) + t\Phi^*\left(\frac{d(x;y)}{t}\right)\right). \tag{17}$$

For each $t \geq 0$, let $S(t)$ be the time t map associated with (1), i.e. $S(t)u_0(x) = u(x,t)$ where u is the viscosity solution of (1). In the following proposition we summarize several useful properties of $S(t)$.

Proposition 1 *Let Φ satisfy Assumption 1. Then, for u_0 and v_0 continuous in $\mathbb{R}^3$,*

1. $\|(S(t)u_0 - S(t)v_0)^+\|_\infty \leq \|(u_0 - v_0)^+\|_\infty$.
2. $\|S(t)u_0 - S(t)v_0\|_\infty \leq \|u_0 - v_0\|_\infty$.
3. $\inf_{\mathbb{R}^3} u_0 \leq S(t)u_0 \leq \sup_{\mathbb{R}^3} u_0$.
4. $\|\tau_y^L(S(t)u_0) - S(t)u_0\|_\infty \leq \|\tau_y^L(u_0) - u_0\|_\infty$.
5. *If u_0 is Lipschitz continuous with respect to left translations with a constant L_1, then so is $S(t)u_0$.*
6. $S(t+\tau)u_0 \leq S(t)u_0\,, \forall \tau > 0$.
7. *If u_0 is Lipschitz continuous with respect to right translations with a constant L, then for $K = \Phi\left(\frac{L}{c_1}\right)$, where c_1 appears in (14),*

$$\|S(t)u_0 - S(t')u_0\| \leq K|t-t'|, \quad \forall t,t' \geq 0. \tag{18}$$

8. *If* supp $(u_0) \subset B_C(R_0)$, *then $S(t)u_0$ is compactly supported and there exists a function $R : \mathbb{R}_+ \to \mathbb{R}_+$, nondecreasing, which only depends on Φ^* and on $\|u_0^-\|_\infty$, such that*

$$\text{supp }(S(t)u_0) \subset B_C(R_0 + R(t)) \subset B_K(\frac{1}{c_1}(R_0 + R(t))). \tag{19}$$

9. *If*
   - *u_0 is supported in the Carnot-Carathéodory ball $B_C(R_0)$,*
   - *u_0 is Lipschitz continuous with respect to left translations with a constant L_1,*
   - *there exists L_2 such that $\|u_0(\cdot \oplus \delta e_3) - u_0(\cdot)\| \leq L_2|\delta|$, for all $\delta > 0$,*

   then $S(t)u_0$ is Lipschitz continuous with respect to right translations with a constant

$$L(t) = L_1 + \frac{4L_2(R_0 + R(t))}{c_1}. \tag{20}$$

Proof. To prove points 1 and 2, set $v(x,t) = S(t)v_0, u(x,t) = S(t)u_0$ and from (17), let $\bar{y}$ be such that $v(x,t) = v_0(\bar{y}) + t\Phi^*\left(\frac{d(x;\bar{y})}{t}\right)$.
Then,

$$u(x,t) - v(x,t) \leq u_0(\bar{y}) + t\Phi^*\left(\frac{d(x;\bar{y})}{t}\right) - v_0(\bar{y}) - t\Phi^*\left(\frac{d(x;\bar{y})}{t}\right).$$

The above gives

$$u(x,t) - v(x,t) \le (u_0 - v_0)(\bar{y}) \le \|u_0 - v_0\|_\infty,$$

and also

$$u(x,t) - v(x,t) \le (u_0 - v_0)^+(\bar{y}).$$

From the above, it follows that

$$(u-v)^+(x,t) \le (u_0 - v_0)^+(\bar{y}) \le \|(u_0 - v_0)^+\|_\infty .$$

The proof of points 1 and 2 is now completed by exchanging the roles of u and v.
To verify the right-hand side inequality at point 3 it is enough to take $y = x$ in the representation formula (17); on the other hand, since $\Phi^* \ge 0$ we have that

$$u_0(y) + t\Phi^*\left(\frac{d(x;y)}{t}\right) \ge \inf_{y\in\mathbb{R}^3} u_0,$$

which implies the left-hand side inequality at point 3.
Point 4 stems from point 2 and from the fact that $\tau_y^L(S(t)u_0) = S(t)\left(\tau_y^L u_0\right)$. The last identity comes from (17) and (11), because

$$\begin{aligned}
\left(\tau_y^L(S(t)u_0)\right)(x) = u(y\oplus x, t) &= \min_{z\in\mathbb{R}^3}\left(u_0(z) + t\Phi^*\left(\frac{d(y\oplus x; z)}{t}\right)\right) \\
&= \min_{z'\in\mathbb{R}^3}\left(u_0(y\oplus z') + t\Phi^*\left(\frac{d(y\oplus x; y\oplus z')}{t}\right)\right) \\
&= \min_{z'\in\mathbb{R}^3}\left(u_0(y\oplus z') + t\Phi^*\left(\frac{d(x; z')}{t}\right)\right) \\
&= \min_{z'\in\mathbb{R}^3}\left(\tau_y^L u_0(z') + t\Phi^*\left(\frac{d(x; z')}{t}\right)\right) \\
&= \left(S(t)\left(\tau_y^L u_0\right)\right)(x).
\end{aligned}$$

Point 5 is an immediate consequence of point 4.
For proving point 6, observe that since $\Phi \ge 0$,

$$(t+\tau)\Phi^*\left(\frac{d(x;y)}{t+\tau}\right) = \sup_{p\ge 0}\left(p\,d(x;y) - (t+\tau)\Phi(p)\right) \le \sup_{p\ge 0}\left(p\,d(x;y) - t\Phi(p)\right) = t\Phi^*\left(\frac{d(x;y)}{t}\right),$$

and the claim follows from the Hopf-Lax formula.
Let us prove point 7: let $\bar{y} = \bar{y}(x,t)$ be such that $u(x,t) = u_0(\bar{y}) + t\Phi^*\left(\frac{d(x;\bar{y})}{t}\right)$.
Then, by (17),

$$u(x,t') - u(x,t) \le u_0(y) + t'\Phi^*\left(\frac{d(x;y)}{t'}\right) - u_0(\bar{y}) - t\Phi^*\left(\frac{d(x;\bar{y})}{t}\right), \quad \forall y \in \mathbb{R}^3. \tag{21}$$

Using now the Lipschitz continuity w.r.t. right translations and (14), we obtain from (21)

$$u(x,t') - u(x,t) \le \frac{L}{c_1} d(y;\bar{y}) + (t'-t)\Phi^*\left(\frac{d(x;\bar{y})}{t}\right). \tag{22}$$

With no restriction, we can assume that $t' < t$. Consider the geodesic connecting x to $\bar{y}$, and choose y on the geodesic such that

$$\frac{d(x;y)}{t'} = \frac{d(x;\bar{y})}{t}. \tag{23}$$

It is clear that $d(x;\bar{y}) = d(x;y) + d(y;\bar{y})$. Thus, from (22),

$$
\begin{aligned}
u(x,t') - u(x,t) &\leq \frac{L}{c_1}(d(x;\bar{y}) - d(x,y)) + (t'-t)\Phi^*\left(\frac{d(x;\bar{y})}{t}\right) \\
&= (t-t')\left(\frac{L}{c_1}\frac{d(x;\bar{y})}{t} - \Phi^*\left(\frac{d(x;\bar{y})}{t}\right)\right) \\
&\leq (t-t')\Phi(\frac{L}{c_1}),
\end{aligned}
$$

where the second line comes from (23), and the third line comes from Fenchel's inequality.
For the proof of point 8, assume that u_0 is supported in the Carnot-Carathéodory ball $B_C(R_0)$. We are going to use the representation formula (15) to prove that for each $t > 0$, $x \mapsto u(x,t)$ has a bounded support. We first observe that Φ^* is a nonnegative function. We proceed in two steps:

first step: from (16), we see that there exists a positive number ξ such that

$$
\Phi^*(q) \geq q\|u_0^-\|_\infty, \quad \forall q \geq \xi. \tag{24}
$$

If $d(x;0) > R_0 + \max(1,\xi t)$, then for all $y \in$ supp (u_0), we have $d(x;y) \geq d(x;0) - d(y;0) > \max(1,\xi t)$. Thus, from (24), $u_0(y) + t\Phi^*(\frac{d(x;y)}{t}) \geq u_0(y) + \|u_0^-\|_\infty \geq 0$. Note also that for $y \notin$ supp (u_0), $u_0(y) + t\Phi^*(\frac{d(x;y)}{t}) \geq 0$ because Φ^* is nonnegative. This and the representation formula (17) imply that if $d(x;0) > R_0 + \max(1,\xi t)$, then $u(x,t) \geq 0$.

second step: if $d(x;0) > R_0 + \max(1,\xi t)$, take a sequence $y_n \notin$ supp (u_0) such that $\lim_{n\to\infty} y_n = x$. We have

$$
0 \leq u(x,t) \leq u_0(y_n) + t\Phi^*(\frac{d(x;y_n)}{t}) = t\Phi^*(\frac{d(x;y_n)}{t}).
$$

We have $\lim_{n\to\infty} d(x;y_n) = 0$ thanks to (12), which yields $u(x,t) = 0$ since $\Phi^*(0) = 0$.
Point 8 is proved with $R(t) = \max(1,\xi t)$.
To prove point 9, we see that

$$
u(x \oplus y,t) - u(x,t) = u(y \oplus x,t) - u(x,t) + u(x \oplus y,t) - u(y \oplus x,t).
$$

But $x \oplus y = 4(x_2y_1 - x_1y_2)e_3 \oplus y \oplus x$. Therefore,

$$
|u(x \oplus y,t) - u(x,t)| \leq |u(y \oplus x,t) - u(x,t)| + |u(4(x_2y_1 - x_1y_2)e_3 \oplus y \oplus x,t) - u(y \oplus x)|,
$$

We make out two cases:

1. if $\sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} > \frac{1}{c_1}(R_0 + R(t))$, then $u(x \oplus y,t) = u(y \oplus x,t) = 0$ and we have

$$
|u(x \oplus y,t) - u(x,t)| = |u(y \oplus x,t) - u(x,t)| \leq L_1|y|_K,
$$

   from point 5.

2. If $\sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} \leq \frac{1}{c_1}(R_0 + R(t))$, then

$$
|x_2y_1 - x_1y_2| = |(x_2+y_2)y_1 - (x_1+y_1)y_2| \leq \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2}|y|_K \leq \frac{1}{c_1}(R_0+R(t))|y|_K
$$

and we have that

$$\begin{aligned}|u(x \oplus y,t)-u(x,t)| &\le |u(y\oplus x,t)-u(x,t)| + |u(4(x_2y_1-x_1y_2)e_3\oplus y\oplus x,t)-u(y\oplus x)|\\ &\le \|\tau^L_y(u_0)-u_0\|_\infty + \|\tau^L_{4(x_2y_1-x_1y_2)e_3}(u_0)-u_0\|_\infty\\ &\le L_1|y|_K + 4L_2|x_2y_1-x_1y_2|\\ &\le \left(L_1+\frac{4L_2(R_0+R(t))}{c_1}\right)|y|_K,\end{aligned}$$

where we have used successively point 4, the definitions of L_1 and L_2 and the estimate on $|x_2y_1-x_1y_2|$ obtained just above.

Therefore $S(t)u_0$ is Lipschitz continuous with respect to right translations with the constant $L(t)$ defined in (20). ■

Remark 1 *In Theorem 1, the assumptions of point 9 imply the fact that u_0 is Lipschitz continuous with respect to the right translations with a constant $L = L_1 + 4\frac{L_2R}{c_1}$. Therefore the assumptions of point 9 imply point 7 with*

$$K = \Phi\left(L_1 + 4\frac{L_2R}{c_1}\right). \tag{25}$$

Remark 2 *We do not know if the asssumptions of point 9 are optimal. For example, one may wonder if the asssumption that u_0 is Lipschitz continuous w.r.t. right translations would be enough to reach the same conclusions.*

3 Finite difference schemes

Let T be a positive time. We are interested in approximating u for times $t \le T$. Let P be a positive integer and $\Delta t = \frac{T}{P}$. Let h be a positive real number. Hereafter, we assume that there exists a constant C such that

$$\Delta t \le Ch. \tag{26}$$

For three integers i, j, k we define the nodes $\xi_{i,j,k} = (ih, jh, (4k+2ij)h^2)$, and for a nonnegative integer n, we define $t_n = n\Delta t$. This lattice was first introduced in [1], as the key ingredient for a second order finite difference scheme for the Kohn Laplacian on the Heisenberg group. Calling (e_1, e_2, e_3) the canonical basis of $\mathbb{R}^3$, we have

$$\begin{aligned}\xi_{i,j,k}\oplus \pm he_1 &= \xi_{i\pm1,j,k},\\ \xi_{i,j,k}\oplus \pm he_2 &= \xi_{i,j\pm1,k\mp i}.\end{aligned} \tag{27}$$

More generally,

$$\xi_{\ell,m,n}\oplus\xi_{i,j,k} = \xi_{\ell+i,m+j,k+n-j\ell}. \tag{28}$$

Formulas (27) and (28) clearly show between the grid and the group operations $\oplus$ and $\cdot$. Since $\xi_{i,j,k}\oplus\xi_{\ell,m,n} = \xi_{\ell+i,m+j,k+n-im}$, we see that $\xi_{i,j,k}\oplus\xi_{\ell,m,n}$ and $\xi_{\ell,m,n}\oplus\xi_{i,j,k}$ coincide if and only if $im = j\ell$.

Capital letters U, V,... will stand for discrete functions defined on the lattice $\{\xi_{i,j,k}, i,j,k\in\mathbb{Z}\}$ and their values at $\xi_{i,j,k}$ will be written $U_{i,j,k}$, $V_{i,j,k}$,... The notations Δ^1_+U and Δ^2_+U will be used for the discrete functions:

$$(\Delta^1_+U)_{i,j,k} = U_{i+1,j,k} - U_{i,j,k}, \qquad (\Delta^2_+U)_{i,j,k} = U_{i,j+1,k-i} - U_{i,j,k}.$$

The value of the numerical approximation of $u(\xi_{i,j,k}, t_n)$ will be written $U^n_{i,j,k}$.
We shall consider numerical schemes

$$U^{n+1}_{i,j,k} = G(U^n_{i,j,k}, U^n_{i+1,j,k}, U^n_{i-1,j,k}, U^n_{i,j+1,k-i}, U^n_{i,j-1,k+i}), \tag{29}$$

such that there exists a continuous function $g : \mathbb{R}^4 \to \mathbb{R}$, called the *numerical Hamiltonian*, with

$$\begin{aligned} &G(U_{i,j,k}, U_{i+1,j,k}, U_{i-1,j,k}, U_{i,j+1,k-i}, U_{i,j-1,k+i}) \\ =&U_{i,j,k} - \Delta t g\left(\frac{1}{h}(\Delta^1_+ U)_{i,j,k}, \frac{1}{h}(\Delta^1_+ U)_{i-1,j,k}, \frac{1}{h}(\Delta^2_+ U)_{i,j,k}, \frac{1}{h}(\Delta^2_+ U)_{i,j-1,k+i}\right). \end{aligned} \tag{30}$$

For the scheme (29) to be consistent with the Hamilton-Jacobi equation, we must have

$$g(a, a, b, b) = \Phi\left(\left|\left(\begin{array}{c} a \\ b \end{array}\right)\right|\right). \tag{31}$$

We will say that (29) is monotone if G is a nondecreasing function of each of its five arguments.
We will say that (29) is monotone on $[-R, R]$ if, for all $i, j, k \in \mathbb{Z}$,
$G(U_{i,j,k}, U_{i+1,j,k}, U_{i-1,j,k}, U_{i,j+1,k-i}, U_{i,j-1,k+i})$ is a nondecreasing function of each of its five arguments as long as $(\Delta^1_+ U)_{\ell,m,n}$ $(\Delta^2_+ U)_{\ell,m,n}$ are contained in $[-R, R]$ for all $\ell, m, n \in \mathbb{Z}$.
For brevity, we will use the notation $\vec{G}(U) = (G(U)_{i,j,k})_{i,j,k\in\mathbb{Z}}$. We will also use the notation $\|U\|_\infty = \sup_{i,j,k\in\mathbb{Z}} |U_{i,j,k}|$. We will say that $U \in \ell^\infty(\mathbb{Z}^3)$ if $\|U\|_\infty < +\infty$.
For $\Lambda > 0$, we call $\mathcal{C}_\Lambda$ the set

$$\mathcal{C}_\Lambda = \{U \in \ell^\infty(\mathbb{Z}^3),\ |(\Delta^1_+ U)_{i,j,k}| < \Lambda h,\ |(\Delta^2_+ U)_{i,j,k}| < \Lambda h,\ \forall i, j, k \in \mathbb{Z}\}. \tag{32}$$

Finally, for $(\ell, m, n) \in \mathbb{Z}^3$, we note $\tau^L_{(\ell,m,n)} U$ the discrete function defined by

$$(\tau^L_{(\ell,m,n)} U)_{i,j,k} = U_{\ell+i, m+j, k+n-4j\ell}.$$

Proposition 2 *Assume that the scheme (29) is consistent and monotone on $[-\Lambda, \Lambda]$. Then*

1. *Identifying $\lambda \in \mathbb{R}$ with the constant function λ on $\mathbb{Z}^3$, we have $\vec{G}(U + \lambda) = \vec{G}(U) + \lambda$, for all discrete function U.*

2. *For U and V in $\mathcal{C}_\Lambda$,*
$$\|(\vec{G}(U) - \vec{G}(V))^+\|_\infty \le \|(U - V)^+\|_\infty. \tag{33}$$

3. *For U and V in $\mathcal{C}_\Lambda$ such that $U \le V$, $\vec{G}(U) \le \vec{G}(V)$.*

4. *For U and V in $\mathcal{C}_\Lambda$,*
$$\|\vec{G}(U) - \vec{G}(V)\|_\infty \le \|U - V\|_\infty. \tag{34}$$

5. *The operator $\vec{G}$ commutes with the left lattice translations: for $(\ell, m, n) \in \mathbb{Z}^3$,*
$$\vec{G}(\tau^L_{(\ell,m,n)} U) = \tau^L_{(\ell,m,n)} \vec{G}(U). \tag{35}$$

6. *If $U^0 \in \mathcal{C}_\Lambda$ and if there exists a positive number L_1 such that for all $(\ell, m, n) \in \mathbb{Z}^3$, $\|\tau^L_{(\ell,m,n)} U^0 - U^0\|_\infty \le L_1 |\xi_{\ell,m,n}|_K$, then for all $p \ge 0$, $U^p = \vec{G}^p(U^0)$ has the same property.*

7. *If the discrete function U^0 satisfies: there exist two positive integers I_0 and J_0 and two positive real numbers L_1 and L_2 such that*

- $U^0_{i,j,k} = 0$ *if* $|i| > I_0$ *and* $|j| > J_0$,
- *for all* $(\ell, m, n) \in \mathbb{Z}^3$, $\|\tau^L_{(\ell,m,n)} U^0 - U^0\|_\infty \le L_1 |\xi_{\ell,m,n}|_K$,
- *for all* $k \in \mathbb{Z}$, $\|\tau^L_{(0,0,k)} U^0 - U^0\|_\infty \le 4L_2 |k| h^2$,
- $L_1 + 4L_2(P + \max(I_0, J_0))h < \Lambda$,

then for all $p \ge 0$, $U^p = \vec{G}^p(U^0)$ *is such that*

$$\begin{array}{rcl} \|\Delta^1_+ U^p\|_\infty & \le & (L_1 + 4L_2(p + J_0)h)h, \\ \|\Delta^2_+ U^p\|_\infty & \le & (L_1 + 4L_2(p + I_0)h)h. \end{array} \tag{36}$$

8. *Under the assumptions of point 7 on* U^0, *there exists a constant* K' *depending on* L_1, L_2, Ph, $(I_0 + J_0)h$ *such that, for all* $p < P$,

$$\|U^{p+1} - U^p\|_\infty \le K' \Delta t. \tag{37}$$

Proof. Point 1 is a direct consequence of (30).
If $V \in \mathcal{C}_\Lambda$, then for all constant α, $V + \alpha \in \mathcal{C}_\Lambda$. Thus, if the two lattice functions U and V belong to $\mathcal{C}_\Lambda$, then U and $V + \|(U - V)^+\|_\infty$ belong to $\mathcal{C}_\Lambda$. From this, the mononicity of G, and the inequality $U_{i,j,k} \le V_{i,j,k} + \|(U - V)^+\|_\infty$, for all $(i, j, k) \in \mathbb{Z}^3$, we deduce that $\vec{G}(U) \le \vec{G}(V) + \|(U - V)^+\|_\infty$. This implies (33).
Point 3 is straightforward consequence of (33). Also from (33), we see that

$$\begin{aligned} \|\vec{G}(U) - \vec{G}(V)\|_\infty =& \|(\vec{G}(U) - \vec{G}(V))^+ - (\vec{G}(U) - \vec{G}(V))^-\|_\infty \\ =& \max(\|(\vec{G}(U) - \vec{G}(V))^+\|_\infty, \|(\vec{G}(V) - \vec{G}(U))^+\|_\infty) \\ \le& \max(\|(U - V)^+\|_\infty, \|(V - U)^+\|_\infty) \\ =& \|U - V\|_\infty, \end{aligned}$$

and we have proved (34).
Identity (35) comes from straightforward calculus.
We have $\|\tau^L_{(\ell,m,n)} \vec{G}(U^0) - \vec{G}(U^0)\|_\infty = \|\vec{G}(\tau^L_{(\ell,m,n)} U^0) - \vec{G}(U^0)\|_\infty$ from (35). It is simple to verify that if U^0 belongs to $\mathcal{C}_\Lambda$, then, for all $(\ell, m, n) \in \mathbb{Z}^3$, $\tau^L_{(\ell,m,n)} U^0 \in \mathcal{C}_\Lambda$. Thus, we can use (34), and we obtain that

$$\|\tau^L_{(\ell,m,n)} \vec{G}(U^0) - \vec{G}(U^0)\|_\infty \le \|\tau^L_{(\ell,m,n)} U^0 - U^0\|_\infty \le L_1 |\xi_{\ell,m,n}|_K.$$

This proves point 6 for $p = 1$. For $p > 1$, we proceed by induction.
To prove point 7, we first observe that U^0 belongs to $\mathcal{C}_\Lambda$, because

$$\begin{aligned} (\Delta^1_+ U^0)_{i,j,k} &= U^0_{i+1,j,k} - U^0_{i,j,k} = U^0_{i+1,j,k} - U^0_{i+1,j,k-j} + U^0_{i+1,j,k-j} - U^0_{i,j,k}, \\ (\Delta^2_+ U^0)_{i,j,k} &= U^0_{i,j+1,k-i} - U^0_{i,j,k} = U^0_{i,j+1,k-i} - U^0_{i,j+1,k} + U^0_{i,j+1,k} - U^0_{i,j,k}. \end{aligned}$$

Moreover, if $|i| > I_0$, then $U^0_{i,j+1,k-i} - U^0_{i,j+1,k} = 0$, and if $|j| > J_0$, $U^0_{i+1,j,k} - U^0_{i+1,j,k-j} = 0$. This, together with the other assumptions on U^0 imply that $|U^0_{i,j+1,k-i} - U^0_{i,j+1,k}| \le 4L_2 |i| h^2 \le 4L_2 I_0 h^2$. Similarly, $|U^0_{i+1,j,k} - U^0_{i+1,j,k-j}| \le 4L_2 J_0 h^2$. Therefore

$$\begin{aligned} |(\Delta^1_+ U^0)_{i,j,k}| &\le |U^0_{i+1,j,k} - U^0_{i+1,j,k-j}| + |U^0_{i+1,j,k-j} - U^0_{i,j,k}| \le (L_1 + 4L_2 J_0 h)h < \Lambda h, \\ |(\Delta^2_+ U^0)_{i,j,k}| &\le |U^0_{i,j+1,k-i} - U^0_{i,j+1,k}| + |U^0_{i,j+1,k} - U^0_{i,j,k}| \le (L_1 + 4L_2 I_0 h)h < \Lambda h. \end{aligned}$$

Assume now that for some q, $0 \le q < P$, (36) is true for all p, $0 \le p \le q$. Then, for $p \le q$, $U^p \in \mathcal{C}_\Lambda$ because $L_1 + 4L_2(P + \max(I_0, J_0))h < \Lambda$. We also verify that $U^p_{i,j,k} = 0$ if $|i| > I_0 + p$ or if $|j| > J_0 + p$. Moreover, we know from points 5 and 4 that

$$\begin{aligned} \|\tau^L_{(\ell,m,n)} U^p - U^p\|_\infty &\le L_1 |\xi_{\ell,m,n}|, \quad \forall (\ell,m,n) \in \mathbb{Z}^3, \\ \|\tau^L_{(0,0,n)} U^p - U^p\|_\infty &\le 4L_2 |n| h^2, \quad \forall n \in \mathbb{Z}. \end{aligned} \tag{38}$$

We wish to study the properties of $U^{q+1} = \vec{G}(U^q)$: we see immediately that $U^{q+1}_{i,j,k} = 0$ if $|i| > I_0 + q + 1$ or if $|j| > J_0 + q + 1$ and we deduce from (35),(34) and from $U^q \in \mathcal{C}_\Lambda$ that

$$\begin{aligned} \|\tau^L_{(\ell,m,n)} U^{q+1} - U^{q+1}\|_\infty &\le L_1 |\xi_{\ell,m,n}|, \quad \forall (\ell,m,n) \in \mathbb{Z}^3, \\ \|\tau^L_{(0,0,n)} U^{q+1} - U^{q+1}\|_\infty &\le 4L_2 |n| h^2, \quad \forall n \in \mathbb{Z}. \end{aligned} \tag{39}$$

Now, we use exactly the same arguments as those we just used for U^0 and prove that

$$\begin{aligned} |(\Delta^1_+ U^{q+1})_{i,j,k}| &\le |U^{q+1}_{i+1,j,k} - U^{q+1}_{i+1,j,k-j}| + |U^{q+1}_{i+1,j,k-j} - U^{q+1}_{i,j,k}| \le (L_1 + 4L_2(J_0 + q + 1)h)h < \Lambda h, \\ |(\Delta^2_+ U^{q+1})_{i,j,k}| &\le |U^{q+1}_{i,j+1,k-i} - U^{q+1}_{i,j+1,k}| + |U^{q+1}_{i,j+1,k} - U^{q+1}_{i,j,k}| \le (L_1 + 4L_2(I_0 + q + 1)h)h < \Lambda h. \end{aligned}$$

We have proved (36) by induction.
For proving the last point, we call $\tilde{\Lambda} = (L_1 + 4L_2(\max(I_0, J_0) + P))$: we have $\tilde{\Lambda} < \Lambda$; from (36) and from the monotonicity of the scheme, we see that

$$\begin{aligned} G(U^p_{i,j,k}, U^p_{i,j,k} - \tilde{\Lambda}h, U^p_{i,j,k} - \tilde{\Lambda}h, U^p_{i,j,k} - \tilde{\Lambda}h, U^p_{i,j,k} - \tilde{\Lambda}h) &\le U^{p+1}_{i,j,k}, \\ U^{p+1}_{i,j,k} \le G(U^p_{i,j,k}, U^p_{i,j,k} + \tilde{\Lambda}h, U^p_{i,j,k} + \tilde{\Lambda}h, U^p_{i,j,k} + \tilde{\Lambda}h, U^p_{i,j,k} + \tilde{\Lambda}h). \end{aligned}$$

From (30), we see that

$$-\Delta t g(-\tilde{\Lambda}, \tilde{\Lambda}, -\tilde{\Lambda}, \tilde{\Lambda}) \le U^{p+1}_{i,j,k} - U^p_{i,j,k} \le -\Delta t g(\tilde{\Lambda}, -\tilde{\Lambda}, \tilde{\Lambda}, -\tilde{\Lambda})$$

This yields (37) with $K' = \max(|g(-\tilde{\Lambda}, \tilde{\Lambda}, -\tilde{\Lambda}, \tilde{\Lambda})|, |g(\tilde{\Lambda}, -\tilde{\Lambda}, \tilde{\Lambda}, -\tilde{\Lambda})|)$. ■

4 Examples

4.1 An upwind scheme

The equation $\frac{\partial u}{\partial t} + |D_H u| = 0$ We first consider the simpler case when Φ is the identity. We choose the level set scheme proposed by Osher and Sethian in [27], see also [29]. This scheme is connected with the Engquist-Osher scheme for conservation laws, see [18]. The scheme is given by (29), with (30) and

$$g(u_1, u_2, v_1, v_2) = \left(\min(u_1, 0)^2 + \max(u_2, 0)^2 + \min(v_1, 0)^2 + \max(v_2, 0)^2\right)^{\frac{1}{2}}. \tag{40}$$

From the inequality: for any $x \in \mathbb{R}^4$, $\sum_{i=1}^4 |x_i| \le 2 \left(\sum_{i=1}^4 |x_i|\right)^{\frac{1}{2}}$, and after some algebra, we see that the scheme is monotone if $2\Delta t \le h$.

The general case of equation (1) Take equation (1) with Φ satisfying Assumptions 1. The upwind scheme proposed by Osher and Sethian reads (29), with (30) and

$$g(u_1, u_2, v_1, v_2) = \Phi\left(\left(\min(u_1, 0)^2 + \max(u_2, 0)^2 + \min(v_1, 0)^2 + \max(v_2, 0)^2\right)^{\frac{1}{2}}\right). \tag{41}$$

From the hypothesis on Φ, we see that the scheme is monotone on $[-\Lambda, \Lambda]$ if $1 - \frac{2\Delta t}{h}\Phi'(2\Lambda) \ge 0$.

4.2 The Lax-Friedrichs scheme

The analogue of the Lax-Friedrichs scheme for equation (1) is (29), with (30) and

$$g(u_1, u_2, v_1, v_2) = \Phi\left(\left(\left(\frac{u_1+u_2}{2}\right)^2 + \left(\frac{v_1+v_2}{2}\right)^2\right)^{\frac{1}{2}}\right) - \theta\frac{h}{\Delta t}\left(u_1 - u_2 + v_1 - v_2\right), \quad (42)$$

where θ is a positive constant. It can be verified that the scheme is monotone on $[-\Lambda, \Lambda]$ provided $0 < \theta < \frac{1}{4}$ and $\theta - \frac{\Delta t}{2h}\Phi'(\sqrt{2}\Lambda) \geq 0$.

5 Error estimate

We now give the main theorem:

Theorem 1 *Under the following assumptions:*

1. *Φ satisfies Assumption 1,*
2. *the difference scheme (29) is in the form (30), monotone on $[-\Lambda, \Lambda]$ and consistent with (1),*
3. *the function u_0 satisfies the assumptions in point 9 of Proposition 1, and the interpolation U^0 of u_0 on the lattice $(ih, jh, (4k+2ij)h^2)$, $i, j, k \in \mathbb{Z}$, satisfies the assumptions in point 7 of Proposition 2,*
4. *$L(T)$ defined by (20) satisfies $L(T) < \Lambda$,*
5. *the numerical Hamiltonian g is locally Lipschitz continuous,*
6. *for a positive constant C, $\Delta t \leq Ch$,*

there exist two positive constants H and c (independent of h) such that for $h < H$,

$$|U^p_{i,j,k} - u(\xi_{i,j,k}, t_p)| \leq ch^{\frac{1}{2}}, \quad (43)$$

for all $0 \leq p \leq P$ and $i, j, k \in \mathbb{Z}$.

6 Proof of Theorem 1

6.1 General strategy and preliminary lemmas

The strategy for proving Theorem 1 is similar to that of [16]. We seek to estimate

$$\sup_{\substack{i,j,k\in\mathbb{Z}\\ 0\leq p\leq P}} |U^p_{i,j,k} - u(\xi_{i,j,k}, p\Delta t)|.$$

For that purpose, we will assume

$$\sup_{\substack{i,j,k\in\mathbb{Z}\\ 0\leq p\leq P}} \left(u(\xi_{i,j,k}, p\Delta t) - U^p_{i,j,k}\right) = \sigma > 0, \quad (44)$$

and look for an upper bound on σ. Were $\inf_{\substack{i,j,k \in \mathbb{Z} \\ 0 \leq p \leq P}} \left(u(\xi_{i,j,k}, p\Delta t) - U^p_{i,j,k} \right) = -\sigma < 0$, we could estimate σ exactly in the same way, so we have bounds from below and from above. For that, we define

$$M = \|u_0\|_{L^\infty(\mathbb{R}^3)} + 1. \tag{45}$$

Note that Propositions 1 and 2 above imply that

$$|u| \leq M \quad \text{on } Q, \qquad \text{and} \qquad \|U^p\|_\infty \leq M \quad 0 \leq p \leq P. \tag{46}$$

For simplifying the notations, we call $Q = \mathbb{R}^3 \times [0, T]$ and $Q^d = \{(\xi_{i,j,k}, p\Delta t), i, j, k \in \mathbb{Z}, 0 \leq p \leq P\}$. The main ingredient for obtaining the desired estimate will be a function $\Psi : Q \times Q^d \to \mathbb{R}$,

$$\Psi(\eta, t, \xi, s) = u(\eta, t) - U^p_{i,j,k} + (5M + \frac{\sigma}{2})\beta_\epsilon(-\xi \oplus \eta, t - s) - \frac{\sigma(t - s)}{4T} \tag{47}$$

where $\xi = \xi_{i,j,k}$, $s = p\Delta t$ and $\beta_\epsilon(x, t) = \beta(|\frac{1}{\epsilon} \cdot x|_K, \frac{t}{\epsilon})$, with ϵ is a positive real number and β a smooth function on $\mathbb{R} \times \mathbb{R}$, satisfying

$$\beta(0, 0) = 1, \qquad 0 \leq \beta \leq 1, \qquad \beta(r, t) = 0 \quad \text{if } r^4 + t^4 > 1. \tag{48}$$

Lemma 1 (Crandall-Lions) *Under the assumptions of Theorem 1, there is a point $(\eta_0, t_0, \xi_0, s_0) \in Q \times Q^d$ such that*

1. $\Psi(\eta_0, t_0, \xi_0, s_0) \geq \Psi(\eta, t, \xi, s)$, $\forall (\eta, t, \xi, s) \in Q \times Q^d$,
2. $\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) \geq 3/5$.

Proof. The proof is exactly the same as for Lemma 4.1 in [16]. ■

Lemma 2 *Let $(\eta_0, t_0, \xi_0, s_0)$ be the same as that in Lemma 1, and $L(T)$ be given by (20). We have*

$$(5M + \frac{\sigma}{2})\, |(D_H\beta_\epsilon)(-\xi_0 \oplus \eta_0, t_0 - s_0)| \leq L(T), \tag{49}$$

and

$$(5M + \frac{\sigma}{2})\, |\partial_3\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)| \leq L_2. \tag{50}$$

If $t_0 > 0$, then

$$-(5M + \frac{\sigma}{2})D_t\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) \leq K - \frac{\sigma}{4T}, \tag{51}$$

with K given by (25).
If $0 < t_0 < T$, then

$$(5M + \frac{\sigma}{2})|D_t\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)| \leq K + \frac{\sigma}{4T}. \tag{52}$$

Proof. The mapping

$$\eta \mapsto u(\eta, t_0) + (5M + \frac{\sigma}{2})\beta_\epsilon(-\xi_0 \oplus \eta, t_0 - s_0)$$

is maximized at η_0, so

$$(5M + \frac{\sigma}{2})\, (\beta_\epsilon(-\xi_0 \oplus \eta, t_0 - s_0) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)) \leq u(\eta_0, t_0) - u(\eta, t_0) \leq L(T)|-\eta_0 \oplus \eta|_K. \tag{53}$$

On the other hand, choosing $\eta = \eta_0 \oplus r_1 e_1 \oplus r_2 e_2$, we have that

$$\beta_\epsilon(-\xi_0 \oplus \eta, t_0 - s_0) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) = (D_H \beta_\epsilon)(-\xi_0 \oplus \eta_0, t_0 - s_0) \cdot (-\eta_0 \oplus \eta) + o(|-\eta_0 \oplus \eta|_K).$$

The last observation and (53) yield (49).
We also know that

$$(5M + \frac{\sigma}{2}) \left(\beta_\epsilon(-\xi_0 \oplus \eta_0 + re_3, t_0 - s_0) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)\right) \le u(\eta_0, t_0) - u(\eta_0 + re_3, t_0) \le L_2 r. \tag{54}$$

On the other hand

$$\beta_\epsilon(-\xi_0 \oplus \eta_0 + re_3, t_0 - s_0) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) = r\partial_3 \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) + o(r).$$

The last observation and (54) yield (50).
Similarly, the function

$$t \mapsto u(\eta, t) - \frac{\sigma t}{4T} + (5M + \frac{\sigma}{2}) \beta_\epsilon(-\xi_0 \oplus \eta, t - s_0)$$

is maximized at t_0. If $t_0 > 0$, then for a small r,

$$(5M + \frac{\sigma}{2}) \left(\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0 - r) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)\right) \le u(\eta_0, t_0) - u(\eta_0, t_0 - r) - \frac{\sigma r}{4T} \le Kr - \frac{\sigma r}{4T},$$

where the last inequality comes from (18). Then (51) follows immediately.
If $T > t_0 > 0$, then one obtains (52) in a similar way. ■
In what follows, we shall choose

$$\epsilon = h^{\frac{3}{8}}, \tag{55}$$

and the function β such that there exists a smooth function $b : \mathbb{R}_+ \to [0, 1]$, with

$$\begin{array}{rcll} \beta(x,t) & = & b(|x|_K^4 + t^4), & \\ b(z) & = & 1 - z, & \text{if } z \le \frac{1}{2}, \\ b(z) & = & 0, & \text{if } z \ge 1, \\ b(z) & \le & \frac{1}{2}, & \text{if } z \ge \frac{1}{2}. \end{array} \tag{56}$$

The following formulas can be obtained by standard calculus: we take (x, t) such $|x|_K^4 + t^4 < \frac{1}{2}$:

$$D_H \beta(x,t) = -4 \begin{pmatrix} (x_1^2 + x_2^2) x_1 + x_2 x_3 \\ (x_1^2 + x_2^2) x_2 - x_1 x_3 \end{pmatrix}, \qquad \text{and} \qquad |D_H \beta(x,t)| = 4|x|_H^2 \sqrt{x_1^2 + x_2^2}, \tag{57}$$

and

$$\partial_3 \beta(x,t) = -2x_3, \qquad \text{and} \qquad \partial_3^2 \beta(x,t) = -2. \tag{58}$$

Lemma 3 *There exist two positive constants $\bar{h}$ and C such that, for $h \le \bar{h}$, calling i_0, j_0, k_0 the integers such that $\xi_0 = (i_0 h, j_0 h, (4k_0 + 2i_0 j_0) h^2)$,*

$$\begin{array}{l} |\frac{1}{h}(\Delta_+^1(\beta_\epsilon(-\cdot \oplus \eta_0, t_0 - s_0 + \Delta t)))_{i_0,j_0,k_0} + (D_H \beta_\epsilon)_1(-\xi_0 \oplus \eta_0, t_0 - s_0)| \le Ch^{\frac{1}{2}}, \\ |\frac{1}{h}(\Delta_+^1(\beta_\epsilon(-\cdot \oplus \eta_0, t_0 - s_0 + \Delta t)))_{i_0-1,j_0,k_0} + (D_H \beta_\epsilon)_1(-\xi_0 \oplus \eta_0, t_0 - s_0)| \le Ch^{\frac{1}{2}}, \\ |\frac{1}{h}(\Delta_+^2(\beta_\epsilon(-\cdot \oplus \eta_0, t_0 - s_0 + \Delta t)))_{i_0,j_0,k_0} + (D_H \beta_\epsilon)_2(-\xi_0 \oplus \eta_0, t_0 - s_0)| \le Ch^{\frac{1}{2}}, \\ |\frac{1}{h}(\Delta_+^2(\beta_\epsilon(-\cdot \oplus \eta_0, t_0 - s_0 + \Delta t)))_{i_0,j_0-1,k_0+i_0} + (D_H \beta_\epsilon)_2(-\xi_0 \oplus \eta_0, t_0 - s_0)| \le Ch^{\frac{1}{2}}. \end{array} \tag{59}$$

Also, if $t_0 > 0$, then

$$\left| \frac{1}{\Delta t} \left(\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0 + \Delta t)\right) + D_t \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) \right| \le Ch^{\frac{1}{2}}. \tag{60}$$

Proof. The second point of Lemma 1 and the choice of β yield that

$$|-\xi_0 \oplus \eta_0|_K^4 + |t_0 - s_0|^4 \le \frac{2\epsilon^4}{5}. \tag{61}$$

Therefore, in a neighborhood of $(\eta_0, t_0, \xi_0, s_0)$, the function $(\eta, t, \xi, s) \mapsto \beta_\epsilon(-\xi\oplus\eta, t-s)$ coincides with $1 - \frac{1}{\epsilon^4}(|-\xi \oplus \eta|^4 + |t-s|^4)$. From this, we can compute $(D_H\beta_\epsilon)(-\xi_0 \oplus \eta_0, t_0 - s_0)$ from (57), and see that

$$\begin{aligned} |(D_H\beta_\epsilon)(-\xi_0 \oplus \eta_0, t_0 - s_0)|^2 &= \frac{16}{\epsilon^8}\left((\xi_{0,1} - \eta_{0,1})^2 + (\xi_{0,2} - \eta_{0,2})^2\right)|\xi_0 - \eta_0|_K^2, \\ |\partial_3\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)|^2 &= \frac{4}{\epsilon^8}|\xi_{0,3} - \eta_{0,3} + 2(\eta_{0,2}\xi_{0,1} - \eta_{0,1}\xi_{0,2})|^2. \end{aligned} \tag{62}$$

Then, (49) and (50) yield that

$$\begin{array}{rcl} (\xi_{0,1} - \eta_{0,1})^2 + (\xi_{0,2} - \eta_{0,2})^2 & \le & \left(\frac{L(T)\epsilon^4}{20M+2\sigma}\right)^{\frac{2}{3}}, \\ |\xi_{0,3} - \eta_{0,3} + 2(\eta_{0,2}\xi_{0,1} - \eta_{0,1}\xi_{0,2})| & \le & \frac{L_2}{10M+\sigma}\epsilon^4. \end{array} \tag{63}$$

To summarize, as $\epsilon \to 0$, we have

$$\begin{aligned} (\xi_{0,1} - \eta_{0,1})^2 + (\xi_{0,2} - \eta_{0,2})^2 &\lesssim \epsilon^{\frac{8}{3}}, \\ |\xi_{0,3} - \eta_{0,3} + 2(\eta_{0,2}\xi_{0,1} - \eta_{0,1}\xi_{0,2})| &\lesssim \epsilon^4. \end{aligned} \tag{64}$$

Let us focus on the first inegality in (59), because the other three are obtained in the same manner. The first thing is to notice that

$$-\xi_{i_0+1,j_0,k_0} \oplus \eta_0 = -\xi_{i_0,j_0,k_0} \oplus \eta_0 \oplus (-he_1) + 4h(\eta_{0,2} - \xi_{0,2})e_3,$$

and that $\eta_{0,2} - \xi_{0,2} = O(\epsilon^{\frac{4}{3}})$, because of (63). Thus

$$-\xi_{i_0+1,j_0,k_0} \oplus \eta_0 = -\xi_{i_0,j_0,k_0} \oplus \eta_0 \oplus (-he_1) + \lambda e_3, \tag{65}$$

where $\lambda = O(\epsilon^{\frac{4}{3}})h$. Thus,

$$\begin{aligned} &\frac{1}{h}(\Delta_+^1(\beta_\epsilon(-\cdot\oplus\eta_0, t_0 - s_0 + \Delta t)))_{i_0,j_0,k_0} + (D_H\beta_\epsilon)_1(-\xi_0 \oplus \eta_0, t_0 - s_0) = I + II, \\ &I = \frac{1}{h}\left(\beta_\epsilon(-\xi_{i_0+1,j_0,k_0} \oplus \eta_0, t_0 - s_0 + \Delta t) - \beta_\epsilon(-\xi_{i_0,j_0,k_0} \oplus \eta_0 \oplus (-he_1), t_0 - s_0 + \Delta t)\right), \\ &II = \begin{pmatrix} \frac{1}{h}\left(\beta_\epsilon(-\xi_0 \oplus \eta_0 \oplus (-he_1), t_0 - s_0 + \Delta t) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0 + \Delta t)\right) \\ +(D_H\beta_\epsilon)_1(-\xi_0 \oplus \eta_0, t_0 - s_0). \end{pmatrix} \end{aligned}$$

In order to estimate I, we first observe that if h and Δt are small enough, then the straight line segment $[(-\xi_{i_0+1,j_0,k_0} \oplus \eta_0, t_0 - s_0 + \Delta t), (-\xi_{i_0,j_0,k_0} \oplus \eta_0 \oplus (-he_1), t_0 - s_0 + \Delta t)]$ is contained in the region $|x|_K^4 + t^4 \le \frac{1}{2}\epsilon^4$. A first order Taylor expansion yields that $I = \frac{\lambda}{h}\partial_3\beta_\epsilon(\theta, t_0 - s_0 + \Delta t)$ where θ lies in the above mentioned line segment. Therefore $|I| = \frac{2|\lambda||\theta_3|}{h}$, which yields that $|I| \lesssim \frac{(\epsilon^4 + h\epsilon^{\frac{4}{3}})}{\epsilon^4}\epsilon^{\frac{4}{3}} \lesssim \epsilon^{\frac{4}{3}} = h^{\frac{1}{2}}$.

In order to estimate II, we first observe that if h and Δt are small enough, then $\{(-\xi_0 \oplus \eta_0 \oplus \lambda he_1, t_0 - s_0 + \Delta t), \lambda \in [0,1]\}$ is contained in the region $|x|_K^4 + t^4 \le \frac{1}{2}$. Therefore $II = \frac{1}{h}\left(\beta_\epsilon(-\xi_0 \oplus \eta_0 \oplus (-he_1), t_0 - s_0) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0)\right) + (D_H\beta_\epsilon)_1(-\xi_0 \oplus \eta_0, t_0 - s_0)$. A second

order Taylor expansion yields that $|II|$ is less that $c\frac{h(\epsilon^{\frac{8}{3}}+h\epsilon^{\frac{4}{3}}+h^2)}{\epsilon^4} \lesssim h^{\frac{1}{2}}$.
At this point, we have proved (59). In order to prove (60), we distinguish the case when $0 < t_0 < T$ and the case when $t_0 = T$.
If $0 < t_0 < T$, we obtain from (52), (61) and the definition of b, that

$$|t_0 - s_0| \lesssim \epsilon^{\frac{4}{3}}. \tag{66}$$

This implies that for h small enough, the line segment $[-\xi_0 \oplus \eta_0, t_0 - s_0, -\xi_0 \oplus \eta_0, t_0 - s_0 + \Delta t]$ is contained in the region $|x|_K^4 + t^4 \leq \frac{1}{2}\epsilon^4$. In that region $D_t^2 \beta_\epsilon(x,t) = -\frac{12t^2}{\epsilon^4}$. A second order Taylor expansion in t yields that

$$\begin{aligned} &\left| \frac{1}{\Delta t} \left(\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0 + \Delta t) \right) + D_t \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) \right| \\ =& \frac{12(t_0 - s_0 + \tau)^2 \Delta t}{\epsilon^4}, \end{aligned} \tag{67}$$

with $0 < \tau < \Delta t$. Thus

$$\begin{aligned} &\left| \frac{1}{\Delta t} \left(\beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) - \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0 + \Delta t) \right) + D_t \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) \right| \\ \lesssim& (\epsilon^{\frac{8}{3}} + h\epsilon^{\frac{4}{3}} + h^2) \frac{h}{\epsilon^4} \lesssim h^{\frac{1}{2}}, \end{aligned} \tag{68}$$

and (60) is proved.
In the case $t_0 = T$, we use (51), which can be written

$$(5M + \frac{\sigma}{2}) \frac{(T - s_0)^3}{\epsilon^4} \leq K - \frac{\sigma}{4T}, \tag{69}$$

which shows that $\frac{\sigma}{4T} \leq K$ and that $0 \leq T - s_0 \lesssim \epsilon^{\frac{4}{3}}$. Then (60) is proved exactly as above. ■
There are now several cases to be considered, namely

- $t_0, s_0 > 0$;
- $t_0 \geq 0$, $s_0 = 0$,
- $t_0 = 0$, $s_0 > 0$.

6.2 The case when $t_0 > 0$, $s_0 > 0$

The point (η_0, t_0) is a maximum of the function

$$(\eta, t) \mapsto u(\eta, t) + (5M + \frac{\sigma}{2}) \beta_\epsilon(-\xi_0 \oplus \eta, t - s_0) - \frac{\sigma(t - s_0)}{4T}.$$

By the definition of the viscosity solution of (1), we have

$$\frac{\sigma}{4T} - (5M + \frac{\sigma}{2}) D_t \beta_\epsilon(-\xi_0 \oplus \eta_0, t_0 - s_0) + \Phi\left((5M + \frac{\sigma}{2}) |(D_H \beta_\epsilon)(-\xi_0 \oplus \eta_0, t_0 - s_0)| \right) \leq 0. \tag{70}$$

The analogous estimate on the discrete side is obtained as follows: (ξ_0, s_0) ($\xi_0 = \xi_{i_0,j_0,k_0}$, $s_0 = p_0 \Delta t$) minimizes

$$(i,j,k,p) \mapsto U^p_{i,j,k} + (5M + \frac{\sigma}{2}) \beta_\epsilon(-\xi_{i,j,k} \oplus \eta_0, t_0 - p\Delta t) + \frac{\sigma(t_0 - p\Delta t)}{4T}.$$

Thus

$$U^p_{i,j,k} \geq U^{p_0}_{i_0,j_0,k_0} + \frac{\sigma(p_0-p)\Delta t}{4T} - (5M+\frac{\sigma}{2})\left(\beta_\epsilon(-\xi_0\oplus\eta_0,t_0-s_0) - \beta_\epsilon(-\xi_{i,j,k}\oplus\eta_0,t_0-p\Delta t)\right).$$

Let us consider the lattice function $(i,j,k)\mapsto B_{i,j,k}$, where

$$B_{i,j,k} = U^{p_0}_{i_0,j_0,k_0} + \frac{\sigma\Delta t}{4T} - (5M+\frac{\sigma}{2})\left(\beta_\epsilon(-\xi_0\oplus\eta_0,t_0-s_0) - \beta_\epsilon(-\xi_{i,j,k}\oplus\eta_0,t_0-s_0+\Delta t)\right).$$

Assuming $h\leq\bar{h}$, from (59), (49) and the fact that $L(T)<\Lambda$, $M>1$, we obtain that the lattice function B belongs to $\mathcal{C}_\Lambda$ for h small enough, say $h\leq H_1$. From this, the monotonicity of G, and the fact that $U^{p_0-1}_{i,j,k}\geq B_{i,j,k}$ for all $i,j,k\in\mathbb{Z}$, we deduce that

$$U^{p_0}_{i_0,j_0,k_0} \geq (\vec{G}(B))_{i_0,j_0,k_0},$$

which is equivalent to

$$\begin{aligned} U^{p_0}_{i_0,j_0,k_0} \geq\ & U^{p_0}_{i_0,j_0,k_0} + \tfrac{\sigma\Delta t}{4T} - (5M+\tfrac{\sigma}{2})\left(\beta_\epsilon(-\xi_0\oplus\eta_0,t_0-s_0) - \beta_\epsilon(-\xi_0\oplus\eta_0,t_0-s_0+\Delta t)\right) \\ & -\Delta t g\begin{pmatrix} \frac{5M+\frac{\sigma}{2}}{h}(\Delta^1_+(\beta_\epsilon(-\cdot\oplus\eta_0,t_0-s_0+\Delta t)))_{i_0,j_0,k_0}, \\ \frac{5M+\frac{\sigma}{2}}{h}(\Delta^1_+(\beta_\epsilon(-\cdot\oplus\eta_0,t_0-s_0+\Delta t)))_{i_0-1,j_0,k_0}, \\ \frac{5M+\frac{\sigma}{2}}{h}(\Delta^2_+(\beta_\epsilon(-\cdot\oplus\eta_0,t_0-s_0+\Delta t)))_{i_0,j_0,k_0}, \\ \frac{5M+\frac{\sigma}{2}}{h}(\Delta^2_+(\beta_\epsilon(-\cdot\oplus\eta_0,t_0-s_0+\Delta t)))_{i_0,j_0-1,k_0+i_0} \end{pmatrix}. \end{aligned} \tag{71}$$

Going back to (71), we replace each finite difference in the arguments of g by the corresponding coordinate of $-(5M+\frac{\sigma}{2})(D_H\beta_\epsilon)(-\xi_0\oplus\eta_0,t_0-s_0)$, thereby creating errors that can be estimated in terms of $h^{\frac{1}{2}}$, thanks to (59) and the locally Lipschitz character of g. We obtain

$$\begin{aligned} \frac{\sigma}{4} \leq & (5M+\frac{\sigma}{2})\frac{\beta_\epsilon(-\xi_0\oplus\eta_0,t_0-s_0) - \beta_\epsilon(-\xi_0\oplus\eta_0,t_0-s_0+\Delta t)}{\Delta t} \\ & +\Phi\left(\left|(5M+\frac{\sigma}{2})(D_H\beta_\epsilon)(-\xi_0\oplus\eta_0,t_0-s_0)\right|\right) + Ch^{\frac{1}{2}}. \end{aligned} \tag{72}$$

Making similar arguments on the t-difference above, we further deduce from (60) that

$$\frac{\sigma}{4} \leq (5M+\frac{\sigma}{2})D_t\beta_\epsilon(-\xi_0\oplus\eta_0,t_0-s_0) + \Phi\left(\left|(5M+\frac{\sigma}{2})(D_H\beta_\epsilon)(-\xi_0\oplus\eta_0,t_0-s_0)\right|\right) + Ch^{\frac{1}{2}}. \tag{73}$$

with a new constant C. Taken together, (73) and (70) yield

$$\sigma \lesssim h^{\frac{1}{2}}. \tag{74}$$

6.3 The case when $t_0\geq 0$ and $s_0=0$

In this case, (71) cannot be used. Yet, the proof of (74) is simpler. The estimates (64) and (66) are true, because in the proof of Lemma 3, we did not use the fact that $s_0>0$. Note that (66) becomes $t_0\lesssim\epsilon^{\frac{4}{3}}$.

We have that

$$\sup_{Q\times Q^d}\psi \geq \sup_{i,j,k\in\mathbb{Z},n\geq 0}\left(u(\xi_{i,j,k},t_n) - U^n_{i,j,k}\right) + 5M = \sigma + 5M.$$

From this, (20), (18) and the choice of β, we see that

$$\begin{aligned} 5M+\sigma \leq \Psi(\eta_0,t_0,\xi_0,0) &\leq |u(\eta_0,t_0)-u(\xi_0,t_0)| + |u(\xi_0,t_0)-u(\xi_0,0)| + (5M+\frac{\sigma}{2})\beta_\epsilon(-\xi_0\oplus\eta_0,t_0) \\ &\leq L(T)|-\xi_0\oplus\eta_0|_K + Kt_0 + 5M + \frac{\sigma}{2}. \end{aligned} \tag{75}$$

This yields immediately (74).

6.4 The case when $t_0 = 0$ and $s_0 > 0$

In this case, we can neither use (70), nor (51) and (52). As for (75), we obtain that

$$5M + \sigma \le \Psi(\eta_0, 0, \xi_0, s_0) \le L(T)|-\xi_0 \oplus \eta_0|_K + Ks_0 + 5M + \frac{\sigma}{2},$$

which implies that

$$\sigma \le 2L(T)|-\xi_0 \oplus \eta_0|_K + 2Ks_0. \tag{76}$$

To estimate s_0, we use the fact that $\Psi(\eta_0, 0, \xi_0, s_0) \ge \Psi(\eta_0, 0, \xi_0, s_0 - \Delta t)$, so

$$\begin{aligned} -U^{p_0}_{i_0,j_0,k_0} - \frac{\sigma}{4T}s_0 + (5M + \frac{\sigma}{2})\beta_\epsilon(-\xi_0 \oplus \eta_0, -s_0) \\ \ge -U^{p_0-1}_{i_0,j_0,k_0} - \frac{\sigma}{4T}(s_0 - \Delta t) + (5M + \frac{\sigma}{2})\beta_\epsilon(-\xi_0 \oplus \eta_0, -s_0 + \Delta t). \end{aligned}$$

From (61), we can see that for h small enough, we can replace $\beta_\epsilon(x,t)$ by $1 - |\frac{1}{\epsilon}x|^4_K - \frac{t^4}{\epsilon^4}$ in the identity above, which becomes

$$\frac{5M + \frac{\sigma}{2}}{\epsilon^4}\left(s_0^4 - (s_0 - \Delta t)^4\right) \le U^{p_0-1}_{i_0,j_0,k_0} - U^{p_0}_{i_0,j_0,k_0} - \frac{\sigma}{4T}\Delta t \le K\Delta t.$$

This yields that

$$\frac{5M + \frac{\sigma}{2}}{\epsilon^4}s_0^2\,(4s_0 - 6\theta\Delta t) \le K, \quad \text{for some } 0 < \theta < 1,$$

and finally

$$s_0 \lesssim \epsilon^{\frac{4}{3}} = h^{\frac{1}{2}}.$$

From this, (76) and (63), we deduce the desired result.

7 Numerical implementation

7.1 The initial value problem

In what follow, we assume that the function Φ is a one to one increasing function from $\mathbb{R}_+$ onto $\mathbb{R}_+$, and we present the two schemes that we have tested for approximating the solution to (1). The first tested scheme is the first order one proposed in (29), (30) and (41). We have seen above that under a stability condition, this scheme is convergent and that it produces an error of $O(h^{\frac{1}{2}})$.

Alternatively, we shall test the second order scheme proposed in [27], see also [29]: the basic trick is to build a switch that turns itself off if a singularity is detected; otherwise, it will use a higher order approximation to the neighboring values on the grid by means of a higher order polynomial using an ENO construction (see [21], [27]). The scheme is as follows:

$$U^{n+1}_{i,j,k} = U^{n+1}_{i,j,k} - \Delta t\Phi\left(\left(\max(A,0)^2 + \min(B,0)^2 + \max(C,0)^2 + \min(D,0)^2\right)^{\frac{1}{2}}\right), \tag{77}$$

with

$$\begin{aligned} A &= \tfrac{1}{h}\left((\Delta^1_- U)_{i,j,k} + \tfrac{1}{2}m((\Delta^1_{-,-}U)_{i,j,k}, (\Delta^1_{+,-}U)_{i,j,k})\right), \\ B &= \tfrac{1}{h}\left((\Delta^1_+ U)_{i,j,k} + \tfrac{1}{2}m((\Delta^1_{-,+}U)_{i,j,k}, (\Delta^1_{+,+}U)_{i,j,k})\right), \\ C &= \tfrac{1}{h}\left((\Delta^2_- U)_{i,j,k} + \tfrac{1}{2}m((\Delta^2_{-,-}U)_{i,j,k}, (\Delta^2_{+,-}U)_{i,j,k})\right), \\ D &= \tfrac{1}{h}\left((\Delta^2_+ U)_{i,j,k} + \tfrac{1}{2}m((\Delta^2_{-,+}U)_{i,j,k}, (\Delta^2_{+,+}U)_{i,j,k})\right), \end{aligned} \tag{78}$$

where the second order finite differences are given by

$$\begin{array}{rcl}(\Delta^1_{++}U)_{i,j,k} &=& U_{i+2,j,k} - 2U_{i+1,j,k} + U_{i,j,k},\\ (\Delta^1_{--}U)_{i,j,k} &=& U_{i-2,j,k} - 2U_{i-1,j,k} + U_{i,j,k},\\ (\Delta^2_{++}U)_{i,j,k} &=& U_{i,j+2,k-2i} - 2U_{i,j+1,k-i} + U_{i,j,k},\\ (\Delta^2_{--}U)_{i,j,k} &=& U_{i,j-2,k+2i} - 2U_{i,j-1,k+i} + U_{i,j,k},\end{array} \quad (79)$$

$$\begin{array}{rcl}(\Delta^1_{+-}U)_{i,j,k} = (\Delta^1_{-+}U)_{i,j,k} &=& U_{i+1,j,k} - 2U_{i,j,k} + U_{i-1,j,k},\\ (\Delta^2_{+-}U)_{i,j,k} = (\Delta^1_{-+}U)_{i,j,k} &=& U_{i,j+1,k-i} - 2U_{i,j,k} + U_{i,j-1,k+i},\end{array} \quad (80)$$

and where the switch function m is

$$m(a,b) = \begin{cases} \begin{cases} a & \text{if} \quad |a| \le |b|, \\ b & \text{if} \quad |a| > |b|, \end{cases}, & \text{if} \quad ab \ge 0, \\ \qquad 0, & \text{if} \quad ab < 0. \end{cases} \quad (81)$$

7.2 The static Hamilton-Jacobi equation

Here, we discuss the numerical methods for solving the static Hamilton-Jacobi equation

$$\begin{array}{rcll}\Phi(|D_H u|) &=& f, & \text{in } \mathbb{R}^3\backslash\overline{\omega},\\ u(x) &=& u_0(x), & \text{in } \overline{\omega},\end{array} \quad (82)$$

where ω is a given subset of $\mathbb{R}^3$. For solving (82), the analogue of the scheme proposed in § 4.1, (due to Osher and Sethian, [27]) is

$$\begin{array}{rcll}\Phi\left(\left(\begin{array}{l}\min(\frac{1}{h}(\Delta^1_+U)_{i,j,k},0)^2 + \max(\frac{1}{h}(\Delta^1_+U)_{i-1,j,k},0)^2\\ +\min(\frac{1}{h}(\Delta^2_+U)_{i,j,k},0)^2 + \max(\frac{1}{h}(\Delta^2_+U)_{i,j-1,k+i},0)^2\end{array}\right)^{\frac{1}{2}}\right) &=& f_{i,j,k}, & \xi_{i,j,k} \notin \overline{\omega},\\ U_{i,j,k} &=& 0 & \xi_{i,j,k} \in \overline{\omega}.\end{array} \quad (83)$$

As explained in [29], a slightly different upwind scheme will turn out to be more convenient:

$$\begin{array}{rcll}\Phi\left(\left(\begin{array}{l}\max(-\frac{1}{h}(\Delta^1_+U)_{i,j,k}, \frac{1}{h}(\Delta^1_+U)_{i-1,j,k},0)^2\\ +\max(-\frac{1}{h}(\Delta^2_+U)_{i,j,k}, \frac{1}{h}(\Delta^2_+U)_{i,j-1,k+i},0)^2\end{array}\right)^{\frac{1}{2}}\right) &=& f_{i,j,k}, & \xi_{i,j,k} \notin \overline{\omega},\\ U_{i,j,k} &=& 0 & \xi_{i,j,k} \in \overline{\omega}.\end{array} \quad (84)$$

Assuming Φ is a one to one mapping from $\mathbb{R}_+$ onto $\mathbb{R}_+$, $\Phi^{-1}(f_{i,j,k})$ can be computed by a Newton method and the equation in (84) is equivalent to the quadratic equation

$$\max(-(\Delta^1_+U)_{i,j,k}, (\Delta^1_+U)_{i-1,j,k},0)^2 + \max(-(\Delta^2_+U)_{i,j,k}, (\Delta^2_+U)_{i,j-1,k+i},0)^2 = \left(h\Phi^{-1}\left(f_{i,j,k}\right)\right)^2. \quad (85)$$

For solving (84), we use the fast marching method advocated by Sethian [29]. The central idea behind it is to systematically construct U using upwind values only. Indeed, the upwind difference structure of (84) allows us to propagate information one way, i.e. from the smaller values of U to larger values. Therefore, the fast marching method consists of building the solution to (84) always stepping downwind: there are two zones, the zone where the solution is already computed or known and the zone where the solution remains to be computed. After the initialization step, the first zone only contains the boundary nodes where the solution is known, whereas the values of U in the zone where the solution is not known are set to some large and positive real number. Following Sethian, we consider a thin zone of trial nodes around the existing front between the two previously mentioned zones: by and large, the fast marching method consists of the loop:

- while loop: as long as the set of trial nodes is not empty,
  1. let $\xi_{\ell,m,n}$ be the trial node corresponding to the smallest value of U : $U_{\ell,m,n} = \min\left(U_{i,j,k} :\ \xi_{i,j,k} \text{ is a trial node}\right)$.
  2. Add $\xi_{\ell,m,n}$ to the set of nodes for which the corresponding value of U is known. Remove $\xi_{\ell,m,n}$ from the set of trial nodes.
  3. All the neighbors of $\xi_{\ell,m,n}$, (i.e. the nodes $\xi_{\ell\pm1,m,n}$, $\xi_{\ell,m\pm1,n\mp\ell}$), for which the corresponding value of U is not known yet, become trial nodes.
  4. Recompute the values of U at the trial nodes $\xi_{i,j,k}$ by solving the quadratic equation (85). It is important to realize that these new values $U_{i,j,k}$ only depend on the known values of U.

The details of the implementation are well explained in [29], in particular the initialization of U and of the trial zone, as well as the use the min-heap data structure with backpointers to store the values of U.
It is possible to obtain a more accurate fast marching method by using a higher order scheme where it is possible to use already computed values: The idea is to define the boolean variables $\mathrm{switch}^{\pm,\ell}_{\mathrm{i,j,k}}$, $\ell = 1, 2$, by

$$\begin{aligned}
\mathrm{switch}^{\pm,1}_{\mathrm{i,j,k}} &= \begin{cases} 1 & \text{if } U_{i\pm2,j,k} \text{ and } U_{i\pm1,j,k} \text{ are known and } U_{i\pm2,j,k} \le U_{i\pm1,j,k}, \\ 0 & \text{otherwise,} \end{cases} \\
\mathrm{switch}^{\pm,2}_{\mathrm{i,j,k}} &= \begin{cases} 1 & \text{if } U_{i,j\pm2,k\mp2i} \text{ and } U_{i,j\pm1,k\mp i} \text{ are known and } U_{i,j\pm2,k\mp2i} \le U_{i,j\pm1,k\mp i}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{86}$$

With $(\Delta^1_{++}U)_{i,j,k}$, $(\Delta^1_{--}U)_{i,j,k}$, $(\Delta^2_{++}U)_{i,j,k}$, $(\Delta^2_{--}U)_{i,j,k}$ the second order finite differences in (79), and $I^1_{i,j,k}$, $I^2_{i,j,k}$ the two numbers

$$\begin{aligned}
I^1_{i,j,k} &= \max\left(-\left((\Delta^1_+U)_{i,j,k} - \tfrac{1}{2}\mathrm{switch}^{+,1}_{\mathrm{i,j,k}}(\Delta^1_{++}\mathrm{U})_{\mathrm{i,j,k}}\right), (\Delta^1_+U)_{i-1,j,k} + \tfrac{1}{2}\mathrm{switch}^{-,1}_{\mathrm{i,j,k}}(\Delta^1_{--}\mathrm{U})_{\mathrm{i,j,k}}, 0\right)^2, \\
I^2_{i,j,k} &= \max\left(-\left((\Delta^2_+U)_{i,j,k} - \tfrac{1}{2}\mathrm{switch}^{+,2}_{\mathrm{i,j,k}}(\Delta^2_{++}\mathrm{U})_{\mathrm{i,j,k}}\right), (\Delta^2_+U)_{i,j-1,k+i} + \tfrac{1}{2}\mathrm{switch}^{-,2}_{\mathrm{i,j,k}}(\Delta^2_{--}\mathrm{U})_{\mathrm{i,j,k}}, 0\right)^2,
\end{aligned} \tag{87}$$

the new scheme is

$$\Phi\left(\frac{\sqrt{I^1_{i,j,k} + I^2_{i,j,k}}}{h}\right) = f_{i,j,k}. \tag{88}$$

This scheme attempts to use a second order stencil when the nodes are available and reverts to a first order one in the other cases. It is compatible with a fast marching method.

8 Numerical results

8.1 The eikonal equation

To test the methods against semi-analytical results, we first consider the eikonal equation (10) for which a complete theory is available. We first aim at computing numerically the Carnot-Caratheodory distance to the origin, that is the solution u of problem (82) with $\Phi(s) = s$, $f = 1$, $\overline{\omega} = \{(0,0,0)\}$ and $u_0 = 0$. As shown in Beals, Gaveau and Greiner [7], the geodesics or Hamiltonian paths relative to the origin and a point $x = (x_1, x_2, x_3)$ such that $x_1^2 + x_2^2 > 0$, (which satisfy $x(0) = 0$, $x(t) = x$, for some $t > 0$), are given by

$$\begin{aligned} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} &= \frac{\sin(2s\theta)}{\sin(2t\theta)} e^{(s-t)\theta\Xi} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \text{with} \quad \Xi = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \\ x_3 - x_3(s) &= \frac{4(t-s)\theta - \sin(2t\theta) + \sin(2s\theta)}{2\sin^2(2t\theta)} (x_1^2 + x_2^2), \end{aligned} \tag{89}$$

where θ is a solution to

$$\mu(2t\theta) = \frac{x_3}{x_1^2 + x_2^2}, \tag{90}$$

and where we have set

$$\mu(\phi) = \frac{\phi}{\sin^2 \phi} - \cot \phi. \tag{91}$$

It is proved that (90) has a unique solution $2t\theta$ in the interval $[0, \pi)$, and that the square of the Carnot-Caratheodory distance $d^2(x; 0)$ is the action integral corresponding to the Hamiltonian curve:

$$\begin{aligned} d^2(x;0) &= \frac{4t^2\theta^2}{2t\theta + \sin^2(2t\theta) - \sin(2t\theta)\cos(2t\theta)} \left(|x_3| + x_1^2 + x_2^2\right) && \text{if} \quad \theta \neq 0, \\ d^2(x;0) &= \left(x_1^2 + x_2^2\right) && \text{if} \quad \theta = 0. \end{aligned} \tag{92}$$

Thus if $x_1^2 + x_2^2 > 0$, computing $d(x; 0)$ requires solving the one dimensional nonlinear equation (90) in $[0, \pi)$, which can be done numerically with Newton's method for example. If, on the contrary $x_1^2 + x_2^2 = 0$, the Carnot-Carathéodory distance is given by

$$d(x; 0) = \sqrt{\pi |x_3|}. \tag{93}$$

Let u be the solution to the eikonal equation $|D_H u(x)| = 1$ for $x \neq 0$ and $u(0) = 0$, then the geodesic curve joining x to the origin is computed as follows:

- set $t = u(x)$.
- Compute $x(s)$, $s \in [0, t]$, by solving the Cauchy problem:

$$\begin{aligned} \frac{dx}{dt}(s) &= -\frac{1}{|D_H u(x(s))|^2} (\sigma(x(s)))^T D_H u(x(s)) \quad 0 < s < t, \\ x(0) &= x. \end{aligned} \tag{94}$$

We have tested the fast marching method with the schemes (84) and (88). Table 1 contains the error $\max_{\xi_{i,j,k} \in [-\frac{1}{2}, \frac{1}{2}]^3} |U_{i,j,k} - d_(\xi_{i,j,k}; 0)|$ where U has been computed with the fast marching method and either the first order scheme (84) or the first/second order scheme (88). The first line of the table contains the number of unknowns, i.e. $\frac{1}{4h^4}$. On Figure 1, we have plotted the error versus h in logarithmic scale. We see that the error produced by scheme (84) behaves like $O(\sqrt{h})$, in agreement with the theory above. The error produced by scheme (88) is smaller, and the slope (in logarithmic scale) of the curve lies between $\frac{1}{2}$ and 1.
On Figure 2, we have plotted some Carnot-Carathéodory spheres centered at 0, intersected with the planar region $\{0\} \times [-0.5, 0.5]^2$: these spheres are obtained as the level sets of U computed by scheme (88) with $h = 1/100$. We very well see that the spheres have a conical singularity near the axis $x_1 = x_2 = 0$, with an angle that gets sharper as $|x_3|$ grows. Note that, for obvious

reasons, the grid used for representing the Carnot Carathéodory spheres is coarser than the one used for computation, and corresponds to $h = 1/60$.
On Figure 3, we have plotted the Carnot-Carathéodory geodesic curve between the point $(0.15, 0.15, 0.3)$ and the origin, computed by the semi-analytic formula (89) or by a discrete solution to (94):

- the parameter h is 1/120.
- in (94) $D_H u$ is first approximated at the grid nodes by a second order difference formula applied to U, where U has been computed with scheme (88).
- for a point x not on the grid, $D_H u(x)$ is computed by a bilinear interpolation of the values previously computed at the grid nodes.
- A second order midpoint scheme is used for integrating (94).

On Figure 3, we see that the geodesic curve is well approximated by the discrete method.
On Figure 4, we have computed the Carnot-Carathéodory distance to some compact sets $\overline{\omega}$, by solving the boundary value problem (82) with scheme (88) and $h = 1/120$. On the left of figure, we choose $\bar{\omega}$ as the convex set $\{x;\ |x_1| + |x_2| + |x_3| \le 0.2\}$. On the right of the figure, $\bar{\omega}$ is nonconvex, and has the shape of a three-dimensional cross.

$1/h$	20	40	60	80	100	120
size	4.10^4	$6.4\ 10^5$	$3.24\ 10^6$	$1.024\ 10^7$	$2.5\ 10^7$	$5.184\ 10^7$
scheme (84)	0.121287	0.0769367	0.060584	0.0497205	0.0446911	0.0405027
scheme (88)	0.0996706	0.0499173	0.0361482	0.0286559	0.0244234	0.0218842

Table 1: L^∞ Error between the theoretical and computed values of $d(x; 0)$ for $x \in [-\frac{1}{2}, \frac{1}{2}]^3$ vs. h

8.2 A case with nonuniform speed

We still solve (82) with $\Phi(d) = d$, but we choose

$$\begin{aligned} f(x) &= \frac{1}{\min(d(x;0), d(x;A) + 0.001)}, \\ u_0(x) &= d(x;0), \end{aligned} \tag{95}$$

with ω is the Koràny ball centered at the origin with radius $r = 0.05$, and $A = (0, 0, 1/4)$. The contours of the solution in the plane $x_1 = 0$ is plotted on Figure 5.

8.3 The initial value problem

We consider the following boundary value problem (1), with $\Phi(d) = d$ and

$$\begin{aligned} u_0(x) &= -0.5 \quad \text{if } |x|_K \le 0.1, \\ u_0(x) &= 0.5 - \exp(-10^3 |x|_K^4 + 0.1) \quad \text{if } |x|_K > 0.1. \end{aligned} \tag{96}$$

We have discretized this equation for $x \in (-1, 1)^2 \times (-\frac{1}{2}, \frac{1}{2})$, $t \in (0, 1)$, with the scheme (29) (30), (40). We have taken $h = 1/100$; the lattice in the x variable has $200^2 \times 100^2/4 = 10^8$

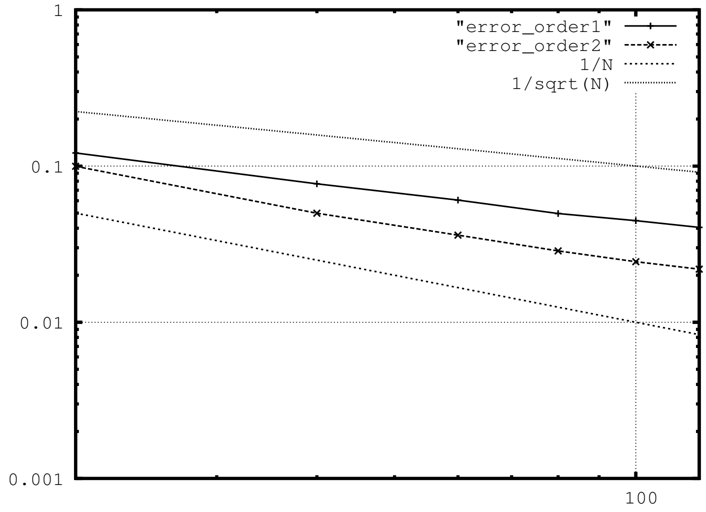

Figure 1: L^∞ Error between the theoretical and computed values of $d(x;0)$ for $x \in [-\frac{1}{2}, \frac{1}{2}]^3$ vs. $N = \frac{1}{h}$

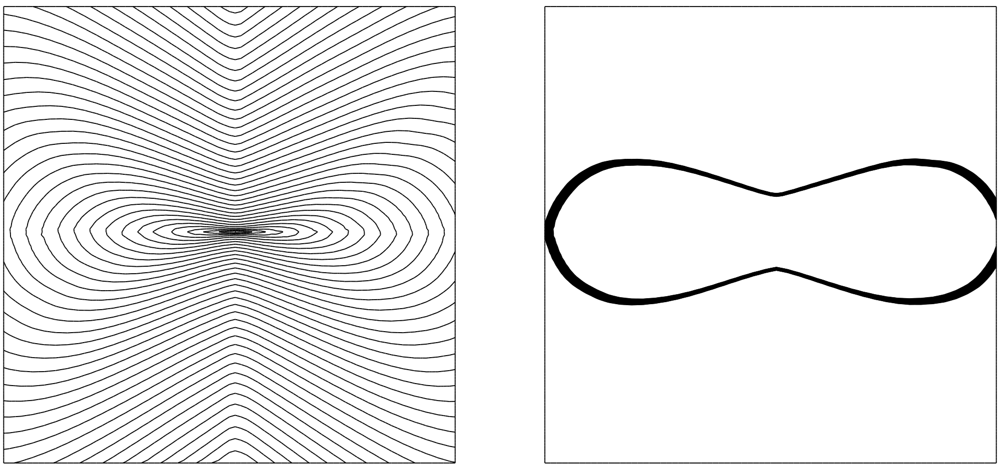

Figure 2: Left: Carnot-Carathéodory spheres ∂B_C intersected with the plane $x_1 = 0$, found as the level sets of U computed with (88) and $h = 1/100$. Right: some Carnot-Carathéodory spheres with radius close to 0.5.

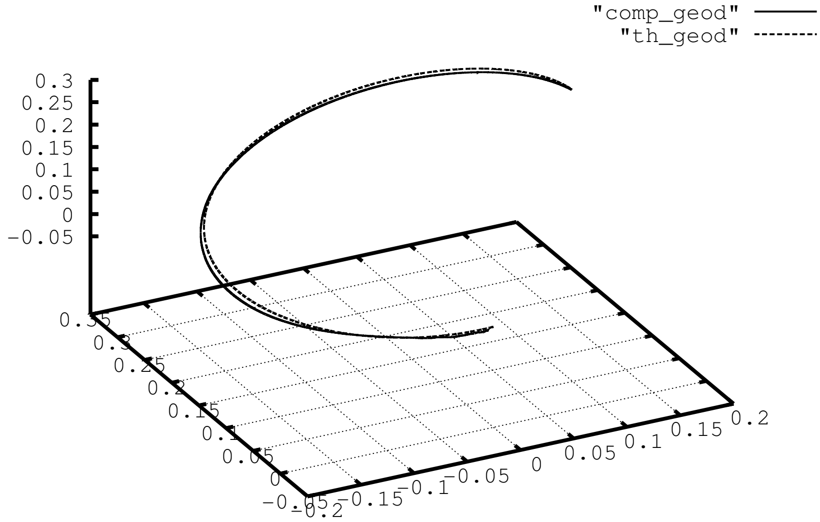

Figure 3: Comparison between the Carnot-Carathéodory geodesic joining $(0.15, 0.15, 0.3)$ and the origin, computed either by (89) (90) or by (94), with u computed by scheme (88) on a grid with $120 \times 120 \times \frac{(120^2)}{4}$ nodes.

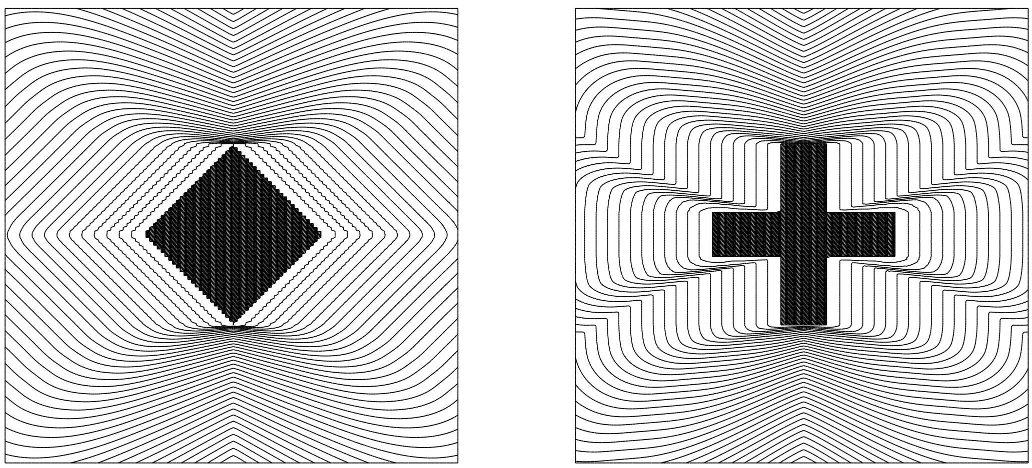

Figure 4: Level sets (intersected with the plane $x_1 = 0$) of the Carnot-Caratheodory distance to a convex set (the set $|x_1| + |x_2| + |x_3| \leq 0.2$) and to a nonconvex set

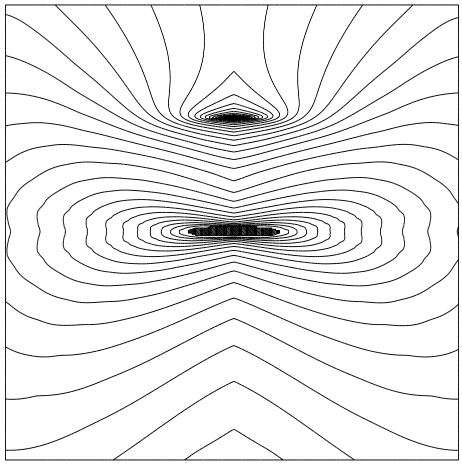

Figure 5: Contours of the solution to the static Hamilton-Jacobi equation (82) with (95), in the plane $x_1 = 0$

nodes. We have chosen $\Delta t = 1/200$, so the first order scheme in (29), (30), (40) is monotone. Alternatively, we have used the second-order scheme (77) (78) in the space variable. On Figure 6, we have plotted five level sets of u at time $t = 0.25$, around the front $u = 0$, corresponding to $u = -0.2, -0.1, 0, 0.1, 0.2$, computed by using the second order scheme described above. We see clearly the singular behavior of u around the axis $x_1 = x_2 = 0$.

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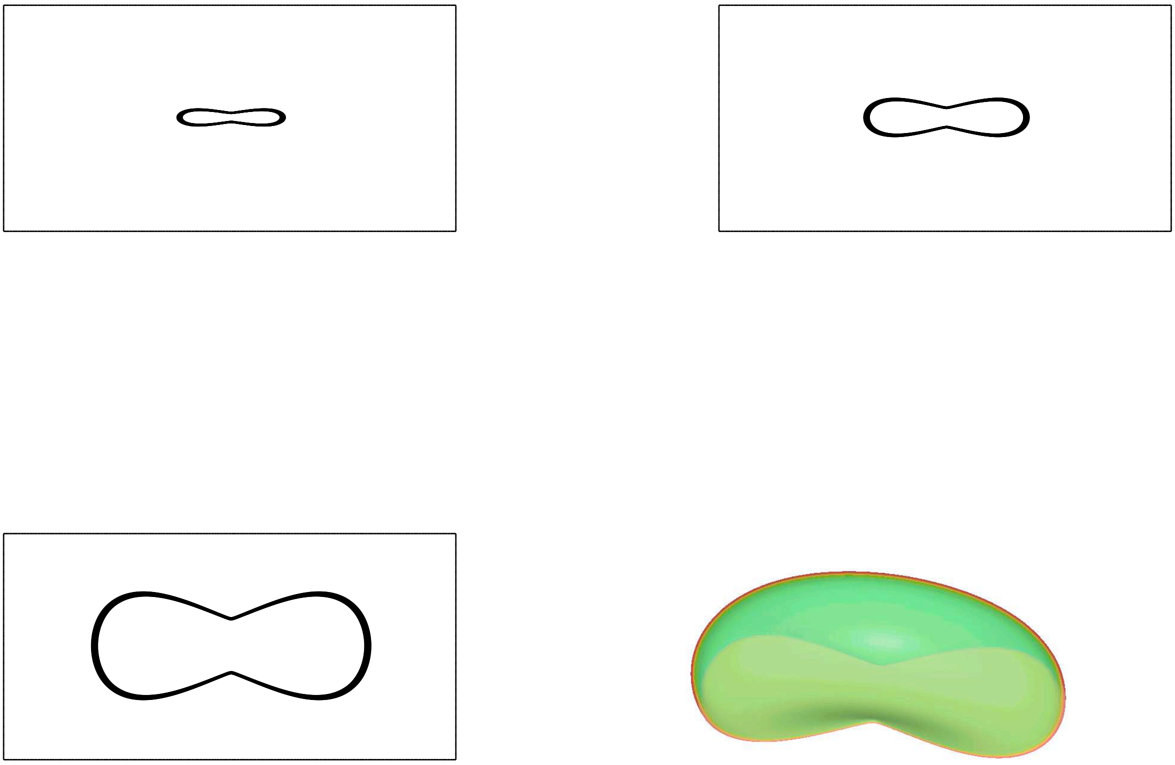

Figure 6: The front of the solution to (96) intersected with the plane $x_1 = 0$ at times $t = 0.125$, $t = 0.25$ and $t = 0.5$. The second order scheme has been used. At the bottom right, 3D view of the front at $t = 0.5$, in the half space $x_1 < 0$.

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