ERROR ESTIMATES FOR THE APPROXIMATION OF THE EFFECTIVE HAMILTONIAN

FABIO CAMILLI, ITALO CAPUZZO DOLCETTA AND DIOGO A.GOMES

Abstract. We study approximation schemes for the cell problem arising in homogenization of Hamilton-Jacobi equations. We prove several error estimates concerning the rate of convergence of the approximation scheme to the effective Hamiltonian, both in the optimal control setting and as well as in the calculus of variations setting.

1. Introduction

Starting from [23], where the basic approach to periodic homogenization via viscosity solutions was outlined, the homogenization theory for Hamilton-Jacobi equations has received a considerable interest both from theoretical, as well as from an applied viewpoint (see, for example, [3], [7], [9], [15], [24], [25]).

An essential step in the homogenization procedure is the identification of the structure of the Hamiltonian of the limit problem, the so-called effective Hamiltonian $\overline{H}(P)$. The function $\overline{H}(P)$ is the unique value of the parameter λ for which the cell problem

$$H(x, Du + P) = \lambda \qquad x \in \mathbb{T}^N$$

($\mathbb{T}^N$ is the N-dimensional torus) admits a (periodic) viscosity solution u, called the corrector.

Since this equation, in general, cannot be explicitly solved except in special cases, see [8], it is important to design numerical schemes to approximate the solution. From the numerical point of view, this is a very difficult task since it requires the approximation of a first order Hamilton-Jacobi equation in which the unknowns are both the viscosity solution u and the constant $\overline{H}(P)$. Moreover, while $\overline{H}(P)$ is uniquely identified by the cell problem, the corresponding solution u is, in general, not unique.

A numerical scheme for the computation of the effective Hamiltonian $\overline{H}(P)$ was proposed in [20], where a convergence theorem and some error estimates were proved. This scheme relies on an inf-sup representation formula for $\overline{H}(P)$ and therefore it does not require the solution of the cell problem. On the other hand, since the computation of the approximate value of $\overline{H}(P)$, for any fixed P, requires the solution of a minimax optimization problem, this scheme is computationally very expensive. Furthermore, the scheme in [20] does not approximate the solution.

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A different approach, called small-δ method, has been proposed in [26]: it consists in discretizing by a finite-difference method a regularized version of cell problem, namely

$$\delta u_\delta + H(x, Du_\delta) = 0, \qquad x \in \mathbb{T}^N. \tag{1.1}$$

for $\delta > 0$. It is well known in fact that $\|\delta u_\delta + \overline{H}(p)\|_\infty$ converges to 0 for δ going to 0 (see [23]). There is a considerable work related to the approximation of (1.1) and the solution of the discrete equation can be computed by one of the several numerical solvers available in the literature, see [11].

To prove the convergence of the scheme, both δ and the discretization step h have to be sent to 0. Since the limit problem does not have a unique viscosity solution, it is not possible to apply the Barles-Souganidis theorem [1] and, to our knowledge, there is no theoretical result providing a rigorous justification for the convergence of this class of schemes.

The first objective of this paper is to prove the convergence of a scheme based on a semi-lagrangian discretization of (1.1) (but the idea is general and applies to other class of schemes), in particular, to provide an estimate for the rate of convergence to $\overline{H}(P)$. We address both the classical control setting and as well the calculus of variations case.

The idea to study convergence steams from a remark in [4] and consists in putting together two estimates available in viscosity solution theory, the one for $\|\delta u_\delta + \overline{H}(P)\|_\infty$ in [7] and the one for the semi-discrete approximation of (1.1) in [6]. The key point is that, thanks to the coercivity assumption on the Hamiltonian needed to prove the existence of a solution to the cell problem, the constant C_δ in the estimate $C_\delta h^{1/2}$ provided in [6], which in general blows up for δ going to 0 with an arbitrary rate, in this case is of the type C/δ. Taking into account that the quantity which converges to $\overline{H}(P)$ is δu_δ, this fact is sufficient to get an error estimate of order $h^{1/2}$ which is optimal in viscosity solution theory.

We also provide a similar error estimate for the other method described in [26], the so called large -T method, based on an evolutionary approximation of the cell problem.

We also present an interpretation of the semi-lagrangian discretization in terms of generalized Aubry-Mather measures. The use of Mather measures enabled us to compare, not only the value of approximate effective Hamiltonian, but also the derivatives of the viscosity solutions. These estimates build upon the ideas in [10] and use the recent construction of generalized Mather measures that is presented in [17].

Finally, we describe a fully discrete scheme derived from the semi-discrete one which is feasible to perform numerical computations. Also for this approximation step we provide an error estimate.

2. The Cell problem

We consider the following *cell problem*: for each $P \in \mathbb{R}^N$, find $\lambda \in \mathbb{R}$ such that there exists a viscosity solution to

$$H(x, Du + P) = \lambda, \qquad x \in \mathbb{T}^N. \tag{2.1}$$

We assume that the Hamiltonian H is of the form

$$H(x,p) = \sup_{a\in A}\{-f(x,a)\cdot p - L(x,a)\}, \tag{2.2}$$

where A is a compact subset of $\mathbb{R}^M$, $f : \mathbb{T}^N \times A \to \mathbb{R}^N$, $L : \mathbb{T}^N \times A \to \mathbb{R}$ are continuous functions such that

$$|f(x,a) - f(y,a)| \le C|x-y| \tag{2.3}$$

$$|L(x,a) - L(y,a)| \le C|x-y|, \qquad |L(x,a)| \le C \tag{2.4}$$

$$B(0,\nu) \subset \overline{\text{co}} f(x,A) \tag{2.5}$$

for some positive constant C, ν and for x, $y \in \mathbb{T}^N$, $a \in A$ ($\overline{\text{co}}$ is the closure of the convex hull). Note that the previous assumptions imply that the Hamiltonian is convex and coercive in p. In this case, it is well known that the cell problem admits a solution (see [23], [15]). Hamiltonians of this form are the typical ones arising in the Dynamic Programming approach to deterministic optimal control problems, see [2].
An important different setting is that of the calculus of variations, where $A = \mathbb{R}^N$ and $f(x,a) = a$. To compensate the lack of compactness of A we require in this case $L(x,a)$ to be strictly convex, $D^2_{aa}L \ge \gamma > 0$, coercive, that is,

$$\lim_{|a|\to\infty} \frac{L(x,a)}{|a|} = +\infty,$$

and

$$L(x,a) \le C + C|a|^2. \tag{2.6}$$

We may assume further, by adding to L a suitable constant, that $L \ge 0$. Let us recall the following fundamental result due to Lions, Papanicolaou and Varadhan, see [23] :

Theorem 2.1. *Under the hypothesis (2.3)-(2.5) or in the calculus of variations setting, for any $P \in \mathbb{R}^N$, there exists a unique number $\overline{H}(P)$ and a function u which satisfy* (2.1) *in viscosity sense. Furthermore, the following characterization holds*

$$\overline{H}(P) = \inf\{\lambda \in \mathbb{R} : \ (2.1) \ \textit{admits a viscosity subsolution}\} \tag{2.7}$$

The identity (2.7) is sometimes called the minimax formula, see [22], as it is equivalent to the following identity:

$$\overline{H}(P) = \inf_{\varphi\in C^1(\mathbb{T}^N)} \sup_{x\in\mathbb{T}^N} H(x, D\varphi + P). \tag{2.8}$$

The function $\overline{H}$ is called the *effective Hamiltonian*, while a solution u of (2.1) with $\lambda = \overline{H}(P)$ is called a corrector. Note the corrector is, in general, not unique.

We consider two quite natural approximations of (2.1), namely the ergodic approximation

$$\delta u + H(x, Du + P) = 0 \qquad x \in \mathbb{T}^N \tag{2.9}$$

for a given $\delta > 0$, and the evolutionary approximation

$$u_t + H(x, Du + P) = 0 \qquad x \in \mathbb{T}^N \times (0, +\infty). \tag{2.10}$$

These approximations are motivated also by optimal control theory where they are known, respectively, as the small discount and the long-run average approximations, see for example [2]. Theoretical convergence results are known for both approximation procedures. For the convenience of the reader we collect them in the next statement.

Theorem 2.2.
i) *Let u_δ be a solution of* (2.9). *Then $-\delta u_\delta$ converges to $\overline{H}(P)$ as δ goes to* 0. *Moreover, for a fixed $x_0 \in \mathbb{T}^N$, $u_\delta - u_\delta(x_0)$ converges, up to a subsequence, to a solution of* (2.1).
ii) *Let $u(x,t)$ be a solution of* (2.10). *Then $-\frac{1}{T}u(\cdot, T)$ converges to $\overline{H}(P)$ as T goes to $+\infty$. Moreover, for a fixed $x_0 \in \mathbb{T}^N$, $u(\cdot, T) - u(x_0, T)$ converges, up to a subsequence, to a solution of* (2.1).

The result in i) is due to Lions, Papanicolaou and Varadhan [23], whereas the one in ii) is due to Fathi [14].

We should also mention that other interesting approximations to (2.1) can be considered and ideas similar to those in the present paper could also be developed in these different settings. Examples of such approximations are the implicit discretization of (2.10),

$$\frac{u_{n+1} - u_n}{\sigma} + H(x, P + Du_{n+1}) = 0,$$

and the vanishing viscosity approximation [18]

$$-\epsilon \Delta u + H(x, P + Du) = \overline{H}_\epsilon.$$

3. The approximation scheme

In this section we study an approximation scheme for the cell problem (2.1) obtained via discretization of the ergodic approximation. We discretize (2.9) by means of a (first order) semi-Lagrangian scheme as described, for example, in [6], [11]. The approximating equation is

$$\delta U(x) + \sup_{a \in A} \left\{ -(1 - \delta h) \frac{U(x + hf(x,a)) - U(x)}{h} - L(x,a) - P \cdot f(x,a) \right\} = 0, \tag{3.1}$$

for $x \in \mathbb{T}^N$. The previous scheme is, in the terminology of [26], a small-δ approximation of (2.1).

Equation (3.1) is the dynamic programming equation of a discrete-time infinite horizon discounted control problem with dynamics

$$\begin{cases} y_{n+1} = y_n + hf(y_n, a_n) \\ y_0 = x \end{cases} \tag{3.2}$$

for some control sequence $\{a_n\} \subset A$, and cost functional

$$J_{h\delta}(x, \{a_n\}) = \sum_{n=0}^{\infty} h(1 - h\delta)^n (L(y_n, a_n) + f(y_n, a_n) \cdot P).$$

It is well known, see for instance [6], that the value function of the control problem

$$U_{h\delta}(x) = \inf_{\{a_n\}} J_{h\delta}(x, \{a_n\}), \qquad x \in \mathbb{T}^N,$$

is the unique solution to (3.1).

Equation (3.1) can be interpreted also as an ergodic regularization of the discrete cell problem

$$\sup_{a \in A} \left\{ -\frac{U(x + hf(x, a)) - U(x)}{h} - L(x, a) - P \cdot f(x, a) \right\} = \lambda, \qquad x \in \mathbb{T}^N, \tag{3.3}$$

which is a semi-lagrangian approximation of the cell problem (2.1). An equation similar to (3.3) has been introduced in [19] in the study of the analogue of Aubry-Mather theory for discrete multi-dimensional maps.

3.1. **Existence of stationary solutions.** As a first step, we prove the existence of a solution to the discrete cell problem (3.3) and a representation formula for the discrete effective Hamiltonian similar to (2.7).

Proposition 3.1. *For any $P \in \mathbb{R}^N$ there exists a unique number $\overline{H}_h(P)$ and a function U_h, in general not unique, which satisfy equation* (3.3)*. The following characterization holds*

$$\overline{H}_h(P) = \inf\{\lambda \in \mathbb{R} : \text{ (3.3) } \textit{admits a subsolution}\}. \tag{3.4}$$

Note that (3.4) can be rewritten as

$$\overline{H}_h = \inf_{\varphi} \sup_{x,a} \left\{ -\frac{\varphi(x + hf(x, a)) - \varphi(x)}{h} - L(x, a) - P \cdot f(x, a) \right\},$$

which is the discrete version of (2.8).

We would like to point out that in this discrete setting we also have:

$$\overline{H}_h = \sup_{\varphi} \inf_{x} \sup_{a} \left\{ -\frac{\varphi(x + hf(x, a)) - \varphi(x)}{h} - L(x, a) - P \cdot f(x, a) \right\},$$

which is an useful formula to prove lower bounds on the effective Hamiltonian.

For the proof of Prop.3.1, we need two preliminary lemmas.

Lemma 3.2. *Suppose there exists a subsolution U and a supersolution V of* (3.3) *with* $\lambda = \lambda_1$, *and* $\lambda = \lambda_2$, *respectively, i.e.,*

$$\sup_{a\in A}\left\{-\frac{U(x+hf(x,a))-U(x)}{h} - L(x,a) - P\cdot f(x,a)\right\} \le \lambda_1, \qquad x\in\mathbb{T}^N,$$

and

$$\sup_{a\in A}\left\{-\frac{V(x+hf(x,a))-V(x)}{h} - L(x,a) - P\cdot f(x,a)\right\} \ge \lambda_2, \qquad x\in\mathbb{T}^N.$$

then

$$\lambda_2 \le \lambda_1. \tag{3.5}$$

Proof. Assume by contradiction that $\lambda_2 - \lambda_1 = 2\varepsilon$, for some positive ε. Set

$$M = \sup_{\mathbb{T}^N}\{U - V\},$$

and define $W = V + M$. Let x_ε be such that $(U-V)(x_\varepsilon) \ge M - h\varepsilon$. Then

$$W(x_\varepsilon) = V(x_\varepsilon) + M \le V(x_\varepsilon) + U(x_\varepsilon) - V(x_\varepsilon) + h\varepsilon \le U(x_\varepsilon) + h\varepsilon, \tag{3.6}$$

and

$$W(x) \ge V(x) + U(x) - V(x) = U(x). \tag{3.7}$$

Moreover observe that W is still a supersolution of (3.3) with $\lambda = \lambda_2$. This, together with (3.6) and (3.7), implies that

$$\begin{aligned} U(x_\varepsilon) + h\varepsilon &\ge W(x_\varepsilon) \ge \inf_{a\in A}\{W(x_\varepsilon + hf(x_\varepsilon,a)) + hL(x_\varepsilon,a) + hf(x,a)\cdot P\} + h\lambda_2 \\ &\ge \inf_{a\in A}\{U(x_\varepsilon + hf(x_\varepsilon,a)) + hL(x_\varepsilon,a) + hf(x,a)\cdot P\} + h\lambda_1 + 2h\varepsilon \\ &\ge U(x_\varepsilon) + 2h\varepsilon. \end{aligned}$$

The contradiction yields claim (3.5). □

Lemma 3.3. *For any* $\delta > 0$,

$$\|\delta U_{h\delta}\|_\infty \le C_1 \tag{3.8}$$

$$|U_{h\delta}(x) - U_{h\delta}(y)| \le C_2|x-y| \qquad x,y\in\mathbb{T}^N, \tag{3.9}$$

for suitable constants C_1, C_2 *independent of* δ.

Proof. Estimate (3.8) immediately follows by

$$|J_{h\delta}(x,\{a_n\})| \le \frac{\|L\|_\infty + C|P|}{\delta},$$

where the constant C is such that $|f(x,a)| \le C$ for any $a \in A$.

Given $x,\ y \in \mathbb{T}^N$, we set $N = [|x-y|/\,h\nu]$ (here $[\cdot]$ stands for the integer part). By (2.5), for any $v \in \mathbb{R}^N$ and any $z \in \mathbb{T}^N$, we can find a control $a \in A$ such that $f(z,a)$ is parallel to v. We take a_1, a_2,..., a_{N-1} in such a way that, if we define y_n by (3.2), we have $y_0 = x$, $y_{i+1} - y_i$ parallel to $y - x$, and $|y_{i+1} - y_i| = h\nu$ for $i = 0, 1, ...N-2$. Finally, we take $\overline{a} \in A$ such that $y_{N-1} + hf(y_{N-1}, \overline{a}) = y$.

Let $\{a_n^y\}$ be an $\varepsilon-$optimal control for $U_{h\delta}(y)$, i.e. $U_{h\delta}(y) \ge J_{h\delta}(y, \{a_n^y\}) - \varepsilon$ and define a control by

$$\alpha_n = \begin{cases} a_n & n \le N-2 \\ \overline{a} & n = N-1 \\ a_{n-N}^y & n \ge N. \end{cases}$$

If y_n is the trajectory corresponding to the control $\{\alpha_n\}$, observing that $y_N = y$, we have

$$\begin{aligned} U_{h\delta}(x) &\le J_{h\delta}(x,\{\alpha_n\}) = \sum_{n=0}^{N-1} h(1-h\delta)^n (L(y_n,\alpha_n) + f(y_n,\alpha_n)\cdot P) + \\ &+ \sum_{n=N}^{+\infty} h(1-h\delta)^n (L(y_n,\alpha_n) + f(y_n,\alpha_n)\cdot P) \le \\ &\le (\|L\|_\infty + C|P|)hN + (1-h\delta)^N U_{h\delta}(y) + \varepsilon \le \frac{C}{\nu}|x-y| + U_{h\delta}(y) + \varepsilon, \end{aligned}$$

using in the last inequality the fact that $U_{h\delta} \ge 0$ since $L \ge 0$. Since ε is arbitrary, we get $U_{h\delta}(x) - U_{h\delta}(y) \le C|x-y|$. Exchanging the role of x and y, we get estimate (3.9). □

Proof of Prop. 3.1. Fix a reference point x_0. Estimates (3.8) and (3.9) imply that there exists $\lambda \in \mathbb{R}$ such that, for some subsequence δ_n converging to 0, $-\delta_n U_{h\delta_n}(x_0)$ converges to λ. Moreover, there exists a Lipschitz-continuous function U such that the sequence $U_n(x) = U_{h\delta_n}(x) - U_{h\delta_n}(x_0)$ converges to U. The function U_n satisfies

$$\begin{aligned} \delta_n U_n(x) \quad &+ \sup_{a\in A}\left\{-(1-\delta_n h)\frac{U_n(x+hf(x,a)) - U_n(x)}{h} - L(x,a) - P\cdot f(x,a)\right\} \\ &= -\delta_n U_{h\delta_n}(x_0). \end{aligned}$$

Letting $n \to \infty$, we see that the couple (U,λ) satisfies the equation (2.1). Because of Lemma 3.2, there is only one value of λ for which (3.3) has a solution.

To show formula (3.4), by Lemma 3.2 it cannot exist a subsolution to (3.3) for $\lambda < \overline{H}_h(P)$. On the other side, a solution of (3.3) at the critical level $\lambda = \overline{H}_h(P)$ provides a subsolution to (3.3) for any $\lambda \ge \overline{H}_h(P)$. □

Proposition 3.4. *For any $h > 0$, the function $p \mapsto \overline{H}_h(P)$ is convex and satisfies*

$$\nu|P| - C \leq \overline{H}_h(P) \leq \nu|P| + C, \tag{3.10}$$

hence it is coercive.

Proof. Take P_1, P_2 and $\theta \in (0,1)$. Let U_i, $i = 1, 2$, be a solution of

$$\sup_{a\in A}\left\{-\frac{U(x+hf(x,a)) - U(x)}{h} - L(x,a) - P_i \cdot f(x,a)\right\} = \overline{H}_h(P_i).$$

Then $\theta U_1 + (1-\theta)U_2$ is a subsolution of

$$\sup_{a\in A}\{-\frac{U(x+hf(x,a)) - U(x)}{h} - L(x,a) - (\theta P_1 + (1-\theta)P_2)\cdot f(x,a)\} =$$
$$\theta\overline{H}_h(P_1) + (1-\theta)\overline{H}_h(P_2)$$

which shows, by (3.4), that $\overline{H}_h$ is convex.

To prove the estimate

$$\overline{H}_h(P) \leq \nu|P| + C,$$

observe that

$$\overline{H}_h(P) \leq \max_{\mathbb{T}^N}[\sup_{a\in A}\{-f(x,a)\cdot P - L(x,a)\}]$$

otherwise the null function is a strict subsolution of (3.3) with $\lambda = \overline{H}_h(P)$, which is impossible by (3.4). By (2.4) and (2.5), we then get the estimate.

To prove the other estimate in (3.10), let U be a solution of (3.3) with $\lambda = \overline{H}_h(P)$ and x_0 be a maximum point of U in $\mathbb{T}^N$. Then we get

$$\overline{H}_h(P) \geq \sup_{a\in A}\{-f(x_0,a)\cdot P - L(x_0,a)\}$$

and, by (2.4) and (2.5), the estimate.

□

3.2. **Lipschitz estimate in the Calculus of Variations.** Most of what was discussed in the previous section carries through in the calculus of variations setting. However, the Lipschitz estimate does not follow from Lemma 3.3 and requires a different proof, which we will discuss next.

Proposition 3.5. *Suppose U is continuous, periodic and satisfies*

$$\sup_a\left[\frac{U(x) - U(x+ha)}{h} - L(x,a)\right] \leq C.$$

Then U is Lipschitz.

Note that, since we may assume $L \geq 0$, we have $\delta U_{h\delta} \geq 0$, and so the previous proposition implies that for $\delta \geq 0$ the corresponding discrete solutions to the cell problem are Lipschitz.

Proof. Let y be an arbitrary point. Then either

$$U(x) - U(y) - M|x-y| \leq 0,$$

in which case $U(x) - U(y) \leq M|x-y|$, or there exists a point of maximum $x \neq y$. In this last case,

$$U(x) - U(y) - M|x-y| \geq U(x+ha) - U(y) - M|x+ha-y|,$$

which implies

$$C \geq \sup_a \left[\frac{U(x) - U(x+ha)}{h} - L(x,a)\right] \geq \sup_a \left[\frac{M|x-y| - M|x+ha-y|}{h} - L(x,a)\right].$$

Since $x \neq y$, we have, by convexity, $|x-y| \geq |x-y+ha| - h\frac{x-y}{|x-y|}a$. Thus, by (2.6),

$$C \geq \sup_a \left[-M\frac{x-y}{|x-y|}a - L(x,a)\right] \geq -C + \sup_a \left[-M\frac{x-y}{|x-y|}a - C|a|^2\right] \geq -C + C|M|^2,$$

which is a contradiction is M is large enough. □

3.3. **Convergence as $\delta \to 0$.** We prove a rate of convergence of the ergodic approximation to (3.3), which is analogous to the one proved in [7] for the continuous problem (see Prop. 4.1 in the next section).

Theorem 3.6. *Let $h > 0$ be fixed. Then, for any δ,*

$$\|\delta U_{h\delta} + \overline{H}_h(P)\|_\infty \leq C(1+|P|)\delta. \tag{3.11}$$

Proof. For simplicity, we give the proof in the case $P = 0$ and we set $\overline{H}_h = \overline{H}_h(0)$. We claim that if V (resp. W) is a bounded subsolution (resp., supersolution) of (3.1), i.e.

$$\delta V(x) + \sup_{a\in A}\left\{-(1-\delta h)\frac{V(x+hf(x,a)) - V(x)}{h} - L(x,a)\right\} \leq 0$$

$$\left(\text{resp.} \qquad \delta W(x) + \sup_{a\in A}\left\{-(1-\delta h)\frac{W(x+hf(x,a)) - W(x)}{h} - L(x,a)\right\} \geq 0\right)$$

then

$$\begin{gathered} V \leq U_{h\delta} \\ (\textit{resp.} \qquad W \geq U_{h\delta}) \end{gathered} \tag{3.12}$$

We prove the claim in the subsolution case. We rewrite the subsolution condition as

$$V(x) \quad \leq \quad \inf_{a\in A}\{(1-h\delta)V(x+hf(x,a)) + hL(x,a)\} \qquad x \in \mathbb{T}^N$$

For any sequence $\{a_n\}$ and the corresponding trajectory $\{y_n\}$, we have

$$\begin{aligned} V(x) &\leq (1-h\delta)V(y_1)+hL(y_0,a_0) \leq (1-h\delta)[(1-h\delta)V(y_2)+hL(y_1,a_1)]+hL(y_0,a_0) = \\ &\leq \dots \leq \sum_{i=0}^{n} h(1-h\delta)^i L(y_i,a_i)+(1-h\delta)^{n+1}V(y_{n+1}). \end{aligned}$$

Sending $n \to \infty$, we get $V(x) \leq \sum_{n=0}^{\infty} h(1-h\delta)^n L(y_n,a_n)$ and therefore, recalling the definition of $U_{h\delta}$, claim (3.12) follows.

Let U be a solution of (3.3), see Prop. 3.1, and define $W_\delta = U - \overline{H}_h/\delta$. Substituting in (3.1), we get

$$\begin{aligned} \delta W_\delta(x) &+ \sup_{a\in A}\left\{-(1-\delta h)\frac{W_\delta(x+hf(x,a))-W_\delta(x)}{h} - L(x,a) - p\cdot f(x,a)\right\} \\ &= \delta \sup_{a\in A}\{U(x+hf(x,a))\} \end{aligned}$$

Hence, if $M = \|U\|_\infty$, $W^\delta - M$ and $W^\delta + M$ are respectively a subsolution and a supersolution of (3.1), therefore by (3.12)

$$W^\delta - M \leq U_{h\delta} \leq W^\delta + M.$$

Thus by the definition of W^δ, we get the estimate (3.11). □

4. Error estimates for the effective Hamiltonian

The error estimates for the effective Hamiltonian we prove in this section are based on two ingredients: an error estimate for the approximation of the effective Hamiltonian via the ergodic regularization (2.9), and an error estimate for the discretization of (2.9). By the juxtaposition of these two estimates we will obtain an error estimate for the approximation of $\overline{H}$.

The first result that we need was proved in [7, Theorem 1.2]:

Proposition 4.1. *Let u_δ be the unique viscosity solution of* (2.9). *Then*

$$\|\delta u_\delta + \overline{H}(P)\|_\infty \leq C(1+|P|)\delta \tag{4.1}$$

for some positive constant C independent of δ.

The second estimate we need is an error estimate for the approximation of (2.9), which we are going to discuss in the next subsection.

4.1. **Discretization error.**

Proposition 4.2. *Let u_δ be the viscosity solution of* (2.9) *and $U_{h\delta}$ be the solution of* (3.1). *Then*

$$\|\delta U_{h\delta} - \delta u_\delta\|_\infty \leq C(1+|P|)h^{1/2} \tag{4.2}$$

for some positive constant C independent of h, δ.

Proof. The proof is essentially the same of the corresponding result in [6] (see also [2, Ch.VI]). The key point is that in the estimate proved in [6], i.e.,

$$\|U_{h\delta} - u_\delta\|_\infty \le C_\delta h^{1/2}$$

the dependence of the constant C_δ on δ is of the type C/δ. This follows by the well-known fact that, because of the coercivity assumption (2.5), the functions u_δ are Lipschitz continuous with a constant independent of δ. For reader's convenience, we give a sketch of the proof of (4.2) in the case $P = 0$.

Define a function $\Psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by

$$\Psi(x, y) = U_{h\delta}(x) - u_\delta(y) - \frac{|x - y|^2}{2\varepsilon}$$

where ε is a positive constant. Let (x_0, y_0) be a maximum point for Ψ (which exists since both $U_{h\delta}(x)$ and $u_\delta(y)$ are periodic). Since y_0 is a minimum point for the function $-\Psi(x_0, y)$, by the definition of viscosity supersolution we get

$$\delta u_\delta(y_0) \ge \sup_a \left\{ f(y_0, a)\frac{x_0 - y_0}{\varepsilon} + L(y_0, a) \right\}. \tag{4.3}$$

Because $\Psi(x_0 + hf(x_0, a), y_0) \le \Psi(x_0, y_0)$, we get

$$\delta U_{h\delta}(x_0) \le \sup_a \{(1 - h\delta)f(x_0, a)\frac{x_0 - y_0}{\varepsilon} + L(x_0, a) + (1 - h\delta)\frac{h}{\varepsilon}|f(x_0, a)|^2\}. \tag{4.4}$$

Let a be a control which realizes the maximum in (4.4). By (4.3) and (4.4) we get

$$\delta U_{h\delta}(x_0) - \delta u_\delta(y_0) \le (f(x_0, a) - f(y_0, a))\frac{x_0 - y_0}{\varepsilon} + C|x_0 - y_0| + C\frac{h}{\varepsilon}. \tag{4.5}$$

Since $\Psi(x_0, y_0) \ge \Psi(y_0, y_0)$, we have

$$|x_0 - y_0| \le C_0\varepsilon, \tag{4.6}$$

where the constant C_0 depends only on the Lipschitz constants of u_δ. Taking $\varepsilon = h^{1/2}$ and, using (4.6) in (4.5), it follows that

$$\delta U_{h\delta}(x) - \delta u_\delta(x) \le \delta U_{h\delta}(x_0) - \delta u_\delta(y_0) \le Ch^{\frac{1}{2}}. \tag{4.7}$$

Exchanging the role of $U_{h\delta}$ and u_δ, we get the estimate (4.2). □

Under additional hypothesis, the previous estimate can be improved. For instance, we quote the following result from [6]):

Proposition 4.3. *Assume* (2.3)–(2.5) *and that, uniformly in* a, *for any* x, $y \in \mathbb{T}^N$, $a \in A$, f *satisfies*

$$|f(x + y, a) + f(x - y, a) - 2f(x, a)| \le C|y|^2,$$

and L *satisfies the semiconcavity estimate in* x,

$$L(x + y, a) + L(x - y, a) - 2L(x, a) \le C|y|^2.$$

Then

$$\|\delta U_{h\delta} + \overline{H}(P)\|_\infty \leq C(1+|P|)(\delta + h).$$

Proof. It is sufficient to observe that, because of the semiconcavity of f and L, the estimate (4.1) can be improved to (see [6])

$$\|\delta U_{h\delta} - \delta u_\delta\|_\infty \leq C(1+|P|)h.$$

□

Another way to improve the error estimate (4.1) is to use high-order approximation schemes (see [12]), which give better estimates for $\|U_{h\delta} - u_\delta\|_\infty$. But for these estimates, some additional regularity properties for u_δ, which are not true in general under assumptions (2.3)-(2.5), are required.

4.2. **Discretization error in the calculus of variations setting.** In the calculus of variations setting it is possible to give an elementary proof of Proposition 4.3, assuming that

$$|D_x L| \leq CL + C. \tag{4.8}$$

This last condition is a standard one in order to prove the existence of C^2 solutions to the corresponding Euler-Lagrange equations which are minimizing (see [16], for instance).

Theorem 4.4. *In the calculus of variations setting we have:*

$$\|U_{h\delta} - u_\delta\|_\infty \leq C\frac{h}{\delta}(1+|P|).$$

Proof. For simplicity, we consider in the proof in the case $P = 0$. Observe that there exists a point x_0 such that

$$|U_{h\delta}(x_0) - u_\delta(x_0)| = \|U_{h\delta} - u_\delta\|_\infty.$$

Firstly, consider the case in which

$$U_{h\delta}(x_0) - u_\delta(x_0) = \|U_{h\delta} - u_\delta\|_\infty,$$

and let $x(\cdot)$ be a C^2 optimal trajectory for u_δ. Define a sequence

$$a_n = \frac{x((n+1)h) - x(nh)}{h},$$

and let y_n be the corresponding trajectory given by (3.2). Observe that $|x(t) - y_n| \leq Ch$ and $|\dot{x}(t) - a_n| \leq Ch$, for $nh \leq t \leq (n+1)h$ and all n.

Set $T = Nh$. Then

$$U_{h\delta}(x_0) - u_\delta(x_0) \leq \sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \left|(1-h\delta)^n L(y_n, a_n) - e^{-\delta t} L(x, \dot{x})\right| dt + $$
$$+ e^{-\delta T}\|U_{h\delta} - u_\delta\|_\infty.$$

From the estimate

$$\sum_{n=0}^{N-1} \int_{nh}^{(n+1)h} \left|(1-h\delta)^n L(y_n, a_n) - e^{-\delta t} L(x, \dot{x})\right| dt \le Ch \int_0^T e^{-\delta t} dt \le \frac{Ch(1-e^{-\delta T})}{\delta},$$

we have

$$\|U_{h\delta} - u_\delta\|_\infty \le C\frac{h}{\delta}.$$

In alternative,

$$u_\delta(x_0) - U_{h\delta}(x_0) = \|U_{h\delta} - u_\delta\|_\infty.$$

In this case, consider an optimal trajectory (y_n, a_n) for $U_{h\delta}$ and construct a piecewise linear trajectory $x(\cdot)$ interpolating linearly between y_n at $t = nh$ and y_{n+1} at $t = (n+1)h$. Observe that for $nh \le t \le (n+1)h$ we have $|x(t) - y_n| \le Ch$ and $\dot{x}(t) = a_n$. Then

$$\begin{aligned} u_\delta(x_0) - U_{h\delta}(x_0) \le & \sum_{n=0}^{N} \int_{nh}^{(n+1)h} \left|(1-h\delta)^n L(y_n, a_n) - e^{-\delta t} L(x, \dot{x})\right| dt + \\ & + e^{-\delta T} \|U_{h\delta} - u_\delta\|_\infty, \end{aligned}$$

and thus a similar estimate follows. □

4.3. **Approximation error.** Now, by combining the previous estimates, we prove the error estimate for the approximation scheme.

Theorem 4.5. *For any h, $\delta > 0$*

$$\|\delta U_{h\delta} + \overline{H}(P)\|_\infty \le C(1+|P|)(h^{1/2} + \delta). \tag{4.9}$$

Proof. By estimates (4.1) and (4.2),

$$\|\delta U_{h\delta} + \overline{H}(P)\| \le \|\delta U_{h\delta} - \delta u_\delta\|_\infty + \|\delta u_\delta + \overline{H}(P)\|_\infty \le C(1+|P|)h^{1/2} + C(1+|P|)\delta.$$

□

Remark 4.6. If f, L are only continuous, but (2.5) still holds, we have that in any case the functions u_δ are Lipschitz continuous, uniformly in δ. It is well known (see [23], [24]) that this is sufficient to guarantee that $\|\delta u_\delta + \overline{H}(p)\|_\infty \to 0$ for δ going to 0^+. The argument used in Prop. 4.2 gives $\|\delta U_{h\delta} - \delta u_\delta\|_\infty \le \omega(h^{\frac{1}{2}})$ where ω is the maximum between the modulus of continuity of f and of L. Hence, assuming only the continuity of the coefficients, we get that, for any P, $\|\delta U_{h\delta} + \overline{H}(P)\|_\infty$ converges to 0 for δ, h going to 0.

By estimates (3.11), (4.1) and (4.2), taking $\delta = h^{\frac{1}{2}}$, it also follows that:

Proposition 4.7. *For any h,*

$$\|\overline{H}_h(P) - \overline{H}(P)\|_\infty \le C(1+|P|)h^{1/2}.$$

This estimate, even if theoretically interesting, is less relevant from a numerical point of view. The numerical solution of equation (3.3), in which the unknowns are $\overline{H}_h(P)$ and U, is very difficult and requires in any case some sort of approximation by a more regular equation. The problem (3.1), for fixed δ, can be easily solved by rewriting the equation in the equivalent form

$$U(x) = \inf_{a \in A} \{(1 - h\delta)U(x + hf(x, a)) + hL(x, a) + P \cdot f(x, a)\} \tag{4.10}$$

and observing that the right hand side of previous equation generates a contraction map in the space of continuous functions (see Section 6 for more details).

We have presented the error estimate for the semi-lagrangian scheme (3.1). But error estimates such as (4.2) can be obtained for other finite-difference schemes, see [27], and therefore applied to approximation schemes for the effective Hamiltonian such as the one described in [26]. The approach described in this section is in some sense a general procedure to obtain an error estimate for the effective Hamiltonian by a "good" estimate for the numerical approximation of equation (2.9).

Remark 4.8. In [7] an error estimate analogous to (4.1), but with a convergence rate $\delta^{1/3}$, is given for the ergodic approximation of the cell problem

$$H(x, \xi, Du(x) + P) = \overline{H}(\xi, P) \qquad x \in \mathbb{T}^N. \tag{4.11}$$

for $(\xi, p) \in \mathbb{R}^N \times \mathbb{R}^N$ fixed, where $H(x, \xi, v) = \sup_{a \in A}\{-f(x, \xi, a) \cdot v - L(x, \xi, a)\}$. In this case f and L are assumed to be Lipschitz-continuous in ξ and x. The corresponding approximation scheme is

$$\delta U(x) + \sup_{a \in A} \left\{ -(1 - \delta h) \frac{U(x + hf(x, \xi, a)) - U(x)}{h} - L(x, \xi, a) - P \cdot f(x, \xi, a) \right\} = 0,$$

for $x \in \mathbb{T}^N$, $(\xi, p) \in \mathbb{R}^N \times \mathbb{R}^N$ fixed. An error estimate analogous to (4.9), with a rate $\delta^{1/3} + h^{1/2}$, can be proved also in this case.

4.4. **The large-T approximation.** Another approximation scheme for the cell problem can be obtained by discretization of equation (2.10). This corresponds, in the terminology of [26], to a large-T approximation of the cell problem. Coupling a forward Euler scheme for u_t with a semi-lagrangian one for the Hamiltonian, we end up with the explicit scheme

$$\frac{U^{n+1}(x) - U^n(x)}{h} + \sup_{a \in A} \left\{ -\frac{U^n(x + hf(x, a)) - U^n(x)}{h} - L(x, a) - P \cdot f(x, a) \right\} = 0,$$

for $x \in \mathbb{T}^N$, where $U^0(x) = g(x)$ for a given $g \in C^0(\mathbb{T}^N)$.

Theorem 4.9. *For any $N \in \mathbb{N}$*

$$\left\| \frac{1}{Nh} U^N(\cdot) + \overline{H}(P) \right\|_\infty \leq C(1 + |P|)(h^{1/2} + \frac{1}{Nh}) \tag{4.12}$$

Proof. An argument similar to the one used for the estimate (4.1) gives

$$\left\| \frac{1}{T} u(\cdot, T) + \overline{H}(P) \right\|_\infty \leq C(1+|P|)\frac{1}{T} \tag{4.13}$$

where u is the solution of (2.10). In the calculus of variations case this estimate can be improved to

$$\left\| \frac{1}{T} u(\cdot, T) + \overline{H}(P) \right\|_\infty \leq \frac{C}{T},$$

where C does not depend on P.

Moreover, as in the proof of Prop. 4.2 (see [13] for more details), it is possible to estimate for $T = Nh$

$$\left\| u(\cdot, T) - U^N \right\|_\infty \leq CT(1+|P|)h^{1/2} \tag{4.14}$$

with a constant C independent of h and T. □

5. Discounted Mather measures and L^2 estimates

In this section we consider the calculus of variations setting and, for $P = 0$, we describe a variational interpretation of the corresponding ergodic problem in terms of generalized Mather measures and give some applications.

Note that in this case, equation (2.1) reads as

$$\delta u + \sup_{a \in \mathbb{R}^N} \{a\, Du - L(x,a)\} = 0, \qquad x \in \mathbb{T}^N. \tag{5.1}$$

In order to simplify the calculations, we consider a slight modification of the approximating equation (3.1), i.e.

$$\delta U(x) + \sup_{a \in A} \left\{ -\frac{U(x+ha) - U(x)}{h} - L(x,a) \right\} = 0. \tag{5.2}$$

but the estimates of Section 4 still hold for this scheme. Moreover we denote by $\overline{H}$ and $\overline{H}_h$ the value of the continuous and the discrete effective Hamiltonians for $P = 0$.

For a given probability measure ν, consider the problem of minimizing the action

$$\int_{\mathbb{T}^N \times A} L(x,a) d\mu, \tag{5.3}$$

over all probability measures μ in $\mathbb{T}^N \times A$, that satisfy, in the continuous case, the constraint

$$\int_{\mathbb{T}^N \times A} [aD\varphi - \delta\varphi]\, d\mu = -\delta \int_{\mathbb{T}^N} \varphi d\nu,$$

for all $\varphi \in C^1(\mathbb{T}^n)$, while, in the discrete one,

$$\int_{\mathbb{T}^N \times A} \left[\frac{\varphi(x+ha) - \varphi(x)}{h} - \delta\varphi \right] d\mu = -\delta \int_{\mathbb{T}^N} \varphi d\nu,$$

for all $\varphi \in C(\mathbb{T}^n)$. The measure ν is called the trace of μ.

As described in [17], this problem admits a dual problem which, in the continuous case is

$$\inf_{\varphi} \sup_{x} \left[\delta \left(\varphi - \int \varphi d\nu \right) + H(x, D\varphi) \right],$$

whereas the discrete case it is

$$\inf_{\varphi} \sup_{x,a} \left[\delta \left(\varphi - \int \varphi d\nu \right) - \frac{\varphi(x+ha) - \varphi(x)}{h} - L(x,a) \right].$$

Furthermore, the value of this infimum, $-\overline{H}^{\delta}$ in the continuous case, and $-\overline{H}_h^{\delta}$ in the discrete one, is given by

$$-\delta \int w d\nu,$$

with w equal to the unique viscosity solution of (5.1) in the continuous case, and, in the discrete setting, to the unique solution to (5.2). By the estimates of the previous section it is clear that, as $\delta \to 0$, $\overline{H}^{\delta} \to \overline{H}$ and $\overline{H}_h^{\delta} \to \overline{H}_h$.

5.1. **Minimizing measures for calculus of variations problems.** Using the same approach as in [10], for the continuous case, or [19], for the discrete problem, we have:

Theorem 5.1. *Both in the continuous and discrete settings, for every probability measure ν there exist corresponding discounted Mather measures μ_δ and $\mu_{h\delta}$.*

If a_0 and a_h are controls which realizes the maximum in (5.1) *and, respectively, in* (5.2)*, then*

$$a_0 = -D_p H(x, Du_\delta), \qquad \mu_\delta \text{ a.e.}, \tag{5.4}$$

and, respectively,

$$D_x U_{h\delta}(x) = -h D_x L(x, a_h) - D_a L(x, a_h), \qquad \mu_{h\delta} \text{ a.e.}. \tag{5.5}$$

Furthermore, for h sufficiently small,

$$a_h = -D_p H(x, DU_{h\delta}) + O(h), \qquad \mu_{h\delta} \text{ a.e.}. \tag{5.6}$$

The equations (5.4) and (5.6) assert, not only that the optimal controls are characterized by the derivative of the corresponding viscosity solution but also that μ_δ and $\mu_{h\delta}$ are supported in the graphs (x, a_0) or (x, a_h), respectively.

Corollary 5.2. *For all $\varphi \in C^1(\mathbb{T}^n)$, we have, in the continuous case:*

$$\int \delta\varphi + D_p H(x, Du_\delta) D\varphi d\mu_\delta = \int \delta\varphi d\nu,$$

and, in the discrete case:

$$\int \delta\varphi - \frac{\varphi(x+ha) - \varphi(x)}{h} d\mu_{h\delta} = \int \delta\varphi d\nu,$$

where $a = -D_pH(x, DU_{h\delta}) + O(h)$.

5.2. **Invariant measures.** A discounted Mather measure $\mu_{h\delta}$ is called invariant provided that its trace agrees with the projection in the x variable of $\mu_{h\delta}$, that is, in the discrete case,

$$\int \frac{\varphi(x+ha_h)-\varphi(x)}{h} d\mu_{h\delta} = 0, \tag{5.7}$$

for all $\varphi \in C(\mathbb{T}^N)$, whereas in the continuous case invariance reads

$$\int aD\varphi(x)d\mu_\delta = 0. \tag{5.8}$$

for all $\varphi \in C^1(\mathbb{T}^N)$.

Proposition 5.3. *Both in the continuous and discrete cases, there exists a (possibly non unique) invariant discounted Mather measure.*

Proof. To prove that a measure μ that satisfies the conditions (5.7) or (5.8) is a discounted Mather measure it suffices to check that its action has is minimal, that is

$$\int L d\mu = -\delta \int w d\mu,$$

with $w = u_\delta$, $\mu = \mu_\delta$ in the continuous case or $w = U_{h\delta}$, $\mu = \mu_{h\delta}$ in the discrete case.

First we deal with the discrete case. To construct such measures, consider a trajectory x_n such that

$$x_{n+1} = x_n + ha_n,$$

where a_n is such that

$$\delta U_{h\delta}(x_n) + \frac{U_{h\delta}(x_n) - U_{h\delta}(x_n + ha_n)}{h} + L(x_n, a_n) = 0.$$

Construct a measure μ by taking a weak limit through some subsequence:

$$\int \phi(x,a)d\mu = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi(x_n, a_n).$$

Then, immediately, one has (5.7), and also

$$\int L d\mu = -\delta \int U_{h\delta} d\mu.$$

To handle the continuous case we use a similar procedure, that is, we consider solutions to the equation

$$\dot{x} = -D_pH(x, Du_\delta(x)),$$

and define the measures by taking a limit of an average

$$\int \phi(x,a)d\mu = \lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(x(t), \dot{x}(t))dt.$$

□

Furthermore we have the following invariance property:

Proposition 5.4. *In the continuous case any invariant discounted Mather measure is invariant under the dynamics*

$$\dot{x} = -D_pH(x,p), \qquad \dot{p} = \delta p + D_xH(x,p),$$

with $p = Du_\delta(x)$.

In the discrete case, any invariant discounted Mather measure is invariant under the dynamics

$$\frac{x_{n+1} - x_n}{h} = -D_pH(x_n, p_{n+1}), \qquad \frac{p_{n+1} - p_n}{h} = \delta p_n + D_xH(x_n, p_{n+1}),$$

with $p_n = DU_{h\delta}(x_n)$.

5.3. **Semiconcavity and (local) semiconvexity.** In this section we prove that the solution to the discrete cell problem is uniformly semiconcave and locally semiconvex on the support of the discounted Mather measure. We make the following semiconcavity assumption: for all compact set K, there exist positive constants C_1 and C_2, such that for all $x, y, z \in \mathbb{R}^n$, and $v \in K$ we have

$$L(x+y, a+z) - 2L(x,a) + L(x-y, a-z) \leq C_1|y|^2 + C_2|z|^2. \tag{5.9}$$

Note that since L is $\mathbb{Z}^n$ periodic in x, it suffices to check the previous inequality for y bounded.

Proposition 5.5. *Let* $U_{h\delta}$ *be a solution to*

$$\delta U + \sup_a \left[-\frac{U(x+ha) - U(x)}{h} - L(x,a) \right] = \lambda_\delta, \tag{5.10}$$

where $\lambda_\delta = 0$ *if* $\delta > 0$ *and* $\lambda_\delta = \overline{H}_h$ *if* $\delta = 0$*. Then* $U_{h\delta}$ *is semiconcave, that is*

$$U_{h\delta}(x+y) - 2U_{h\delta}(x) + U_{h\delta}(x-y) \leq C|y|^2,$$

in which the constant, for small h *and* δ*, is independent of* h *and* δ*.*

Proof. Consider the case $\delta = 0$, as the other is similar. To simplify notation we will denote the solution by U. First observe that by Proposition 3.5, U is Lipschitz, and therefore the optimal control a given by (5.5) is uniformly bounded.

Let $0 \leq \theta \leq 1$. We have

$$\begin{aligned} U(x+y) - 2U(x) + U(x-y) \leq & h \left[L(x+y, a - \theta\frac{y}{h}) - 2L(x,a) + L(x-y, a+\theta\frac{y}{h}) \right] \\ & + U(x+a+(1-\theta)y) - 2U(x+a) + U(x+a-(1-\theta y). \end{aligned}$$

Suppose an estimate Λ_n for the semiconcavity modulus of U is known, that is

$$U(x+y) - 2U(x) + U(x-y) \leq \Lambda_n |y|^2$$

Then, using (5.9), we have

$$U(x+y) - 2U(x) + U(x-y) \leq hC_1|y|^2 + C_2\theta^2 \frac{|y|^2}{h} + \Lambda_n(1-\theta)^2|y|^2.$$

Therefore, by optimizing over θ we obtain:

$$\Lambda_{n+1} \leq hC_1 + \frac{C_2\Lambda_n}{C_2 + \Lambda_n h}.$$

In the limit, this yields a universal bound

$$\Lambda = \frac{C_1 h + \sqrt{C_1^2 h^2 + 4C_1C_2}}{2}.$$

□

Proposition 5.6. *Let $U_{h\delta}$ be a solution to* (5.10). *Suppose x is such that there exists an optimal trajectory x_n for $n \leq 0$ ending at x, that is $x_0 = x$. Then $U_{h\delta}$ is locally semiconvex at x, that is*

$$U_{h\delta}(x+y) - 2U_{h\delta}(x) + U_{h\delta}(x-y) \geq -C|y|^2,$$

in which the constant, for small h and δ, is independent of h and δ.

Remark 5.7. For any point x one can construct forward optimal trajectories. But only special points are endpoints of optimal trajectories. Another way to rephrase the assumption of this theorem is to assume that for any N there is a point x_0 and an optimal trajectory for the discrete problem such that $x_N = x$. However, it is easier to allow for negative indices, with the obvious meaning.

Proof. The proof uses a similar iterative argument using the following estimate: let $0 \leq \theta \leq 1$.

$$\begin{aligned} U(x_{n-1} &+ (1-\theta)y) - 2U(x_{n-1}) + U(x_{n-1} - (1-\theta)y) \\ &\leq h\left[L(x_{n-1} + (1-\theta)y, a + \theta\frac{y}{h}) - 2L(x_{n-1}, a) + L(x_{n-1} - (1-\theta)y, a - \theta\frac{y}{h})\right] \\ &\quad + U(x_n + y) - 2U(x_n) + U(x_n - y). \end{aligned}$$

This inequality yields an estimate for Λ_n, the (local) semiconvexity modulus at x_n in terms of Λ_{n-1} the local semiconvexity modulus at x_{n-1}. □

Finally, we quote a result from [10]:

Proposition 5.8. *Suppose U is a semiconcave function, and let x be a point at which U locally semiconvex. Then there exists a constant C such that*

$$|u(x) - u(y) - Du(x)(y-x)| \leq C|x-y|^2,$$

furthermore

$$|Du(x) - Du(y)| \leq C|x-y|,$$

for almost every y.

5.4. L^2 **estimates.**

Proposition 5.9. *Suppose u_δ is a solution of (5.1) and u is a solution of*

$$H(x, Du) = \overline{H}. \tag{5.11}$$

Let μ_δ be a discounted Mather measure, η_ϵ a standard mollifier, and $u^\epsilon = u * \eta_\epsilon$. Then

$$\int_{\mathbb{T}^N} |Du_\delta - Du^\epsilon|^2 d\mu_\delta \leq C\delta\|u\|_\infty + C\|\overline{H} + \delta u_\delta\|_\infty + O(\epsilon).$$

Proof. We have, μ_δ almost everywhere,

$$H(x, Du^\epsilon) - H(x, Du_\delta) \leq \overline{H} + \delta u_\delta + O(\epsilon).$$

Using convexity,

$$H(x, Du^\epsilon) - H(x, Du_\delta) \geq D_pH(x, Du_\delta)D(u^\epsilon - u_\delta) + C|Du_\delta - Du^\epsilon|^2.$$

Then, since

$$\int \delta(u^\epsilon - u_\delta) + D_pH(x, Du_\delta)D(u^\epsilon - u_\delta)d\mu_\delta = \delta \int (u^\epsilon - u_\delta)d\nu,$$

we have

$$\int |Du_\delta - Du^\epsilon|^2 \leq \delta \int (u^\epsilon - u_\delta)d(\mu_d - \nu) + \int \overline{H} + \delta u_\delta d\mu_\delta,$$

which implies

$$\int_{\mathbb{T}^N} |Du_\delta - Du^\epsilon|^2 d\mu_d \leq C\delta\|u\|_\infty + C\|\overline{H} + \delta u_\delta\|_\infty + O(\epsilon).$$

□

With a similar proof one would obtain that

$$\int_{\mathbb{T}^N} |Du_\delta^\epsilon - Du|^2 d\mu \leq C\delta\|u\|_\infty + C\|\overline{H} + \delta u_\delta\|_\infty + O(\epsilon),$$

where μ is the Mather measure corresponding to $\delta = 0$, and $u_\delta^\epsilon = u_\delta * \eta_\epsilon$.

Proposition 5.10. *Suppose u_δ is a solution of* (5.1) *and μ_δ a corresponding discounted Mather measure. Suppose $U_{h\delta}$ is a solution of* (5.2)*. Let $DU_{h\delta}(x)$ denote a measurable selection of the superdifferential of $U_{h\delta}$ in the support of μ_δ. Then*

$$\int_{\mathbb{T}^N} |Du_\delta - DU_{h\delta}|^2 d\mu_\delta \leq Ch.$$

Proof. By semiconcavity we have

$$U_{h\delta}(x+ha) \leq U_{h\delta}(x) + hDU_{h\delta}a + O(h^2).$$

Therefore

$$\sup_a \left\{ -\frac{U_{h\delta}(x+ha) - U_{h\delta}(x)}{h} - L(x,a) \right\} \geq H(x, DU_{h\delta}) + O(h).$$

Thus we have

$$\begin{aligned}
&\delta(U_{h\delta} - u_\delta) + \sup_a \left\{ -\frac{U_{h\delta}(x+ha) - U_{h\delta}(x)}{h} - L(x,a) \right\} - H(x, Du_\delta) \\
&\geq \delta(U_{h\delta} - u_\delta) + H(x, DU_{h\delta}) - H(x, Du_\delta) + O(h) \\
&\geq \delta(U_{h\delta} - u_\delta) + D_pH(x, Du_\delta)(DU_{h\delta} - Du_\delta) + C|DU_{h\delta} - Du_\delta|^2 + O(h)
\end{aligned}$$

Therefore, taking into account that $\delta(U_{h\delta} - u_\delta) = O(h)$, by theorem 4.4, we have

$$\int |DU_{h\delta} - Du_\delta|^2 d\mu_\delta \leq Ch.$$

□

Therefore, to summarize these estimates, in what concerns the approximation of the graph of the derivative of u by the discrete discounted approximation we have:

Corollary 5.11. *We have*

$$\int |Du^\epsilon - DU_{h\delta}|^2 d\mu_\delta \leq O(\delta) + O(h) + O(\epsilon).$$

Finally, we prove an estimate with respect to the discrete discounted Mather measure:

Proposition 5.12. *Suppose u_δ is a solution of* (5.1)*, $U_{h\delta}$ is a solution of* (5.2) *and $\mu_{h\delta}$ is the corresponding discounted Mather measure. Then*

$$\int_{\mathbb{T}^N} |Du_\delta^\epsilon - DU_{h\delta}|^2 d\mu_{h\delta} \leq Ch + O(\epsilon).$$

Proof. In this proof we will omit the mollification step and set $\epsilon = 0$, as it is similar as before. By Proposition 5.8 we have $\mu_{h\delta}$ almost everywhere

$$\sup_a \left\{ -\frac{U(x+ha) - U(x)}{h} - L(x,a) \right\} = H(x, DU_{h\delta}(x+ha^*)) + O(h),$$

where

$$D_aL(x,a^*) = DU_{h\delta}(x+ha^*).$$

Then

$$\begin{aligned} 0 =& \delta(u_\delta - U_{h\delta}) + H(x,Du_\delta) - \sup_a\left\{-\frac{U(x+ha)-U(x)}{h} - L(x,a)\right\} \\ =& \delta(u_\delta - U_{h\delta}) + H(x,Du_\delta) - H(x, DU_{h\delta}(x+ha^*)) + O(h). \end{aligned}$$

Now, note that

$$\begin{aligned} &H(x,Du_\delta) - H(x,DU_{h\delta}(x+ha^*)) \\ &\qquad \geq D_pH(x,DU_{h\delta})D(u_{h\delta}-U_{h\delta}) + C|D(U_{h\delta}-u_\delta)|^2. \end{aligned}$$

Thus, μ_δ almost everywhere,

$$C|D(U_{h\delta}-u_\delta)|^2 \leq -D_pH(x,DU_{h\delta})D(u_\delta - U_{h\delta}) - \delta(u_\delta - U_{h\delta}) + O(h).$$

Recall that u is semiconcave, and so

$$u(x+w) \leq u(x) + wDu(x) + C|w|^2.$$

Thus

$$-D_pH(x,DU_{h\delta})Du_\delta \leq \frac{u(x) - u(x+hD_pH(x,DU_{h\delta}))}{h} + O(h).$$

$\mu_{h\delta}$ almost everywhere, by Proposition 5.8 we have

$$D_pH(x,DU_{h\delta})DU_{h\delta} \leq \frac{U_{h\delta}(x) - U_{h\delta}(x-hD_pH(x,DU_{h\delta}))}{h} + O(h).$$

By Theorem 4.4, we have

$$\delta(u_\delta - U_{h\delta}) = O(h).$$

Therefore

$$\begin{aligned} C\int |D(U_{h\delta}-u_\delta)|^2 d\mu_\delta \leq& \int \frac{u(x)-u(x+hD_pH(x,DU_{h\delta}))}{h} d\mu_\delta \\ &+ \int \frac{U_{h\delta}(x) - U_{h\delta}(x-hD_pH(x,DU_{h\delta}))}{h} d\mu_\delta + O(h). \end{aligned}$$

Note that

$$\int \frac{U_{h\delta}(x) - U_{h\delta}(x-hD_pH(x,DU_{h\delta}))}{h} = 0,$$

and, using the invariance property,

$$\begin{aligned} \int \frac{u(x)-u(x+hD_pH(x,DU_{h\delta}))}{h} d\mu_\delta =& \int \frac{u(x-hD_pH(x,DU_{h\delta})) - u(x+O(h^2))}{h} d\mu_\delta \\ = \int \frac{u(x-hD_pH(x,DU_{h\delta})) - u(x)}{h} d\mu_\delta + O(h) = O(h). \end{aligned}$$

□

6. A fully discrete scheme

Note that the approximating equation (3.1) is still continuous in the space variable x. In this section we give a outline of the construction of a fully discrete scheme, feasible for numerical computations, from the semi-discrete one.

We introduce a space discretization which transforms (3.1) into a finite dimensional problem. For this purpose we choose a grid Γ covering $\mathbb{T}^N$ consisting of simplexes S_j, $j = 1, \ldots, M$ with nodes x_i, $i = 1, \ldots, N$. The discretization parameter k is the maximal diameter of the splices S_j. We look for an approximate solution of (3.1) in the space

$$\mathcal{P}_1 := \{w \in C(\mathbb{T}^N) \,|\, \nabla w \equiv \text{const on } S_j\}$$

of continuous linear finite elements on Γ. Rewriting (3.1) in the equivalent form (4.10), we end up with the fully discrete scheme

$$U(x_i) = \inf_{a \in A} \{(1 - h\delta)U(x_i + hf(x_i, a)) + h(L(x_i, a) + P \cdot f(x_i, a))\}, \tag{6.1}$$

for all nodes $x_i \in \Gamma$ and linear interpolation in the simples of the triangulation. Existence and uniqueness of a solution $U_{k\delta} \in \mathcal{P}_1$ to (6.1) for any δ is proved in [11]. A solution can be computed using the iterative methods proposed in [11], [21].

Proposition 6.1. *For any h, k, $\delta > 0$*

$$\|\overline{H}(P) + \delta U_{k\delta}\|_\infty \le C(1 + |P|)(\delta + h^{1/2} + \frac{k}{h}), \tag{6.2}$$

for some positive constant C independent of δ, h, k.

Proof. Recalling (4.9), it is sufficient to prove that

$$\|\delta U_{h\delta} - \delta U_{k\delta}\|_\infty \le C(1 + |P|)\frac{k}{h}. \tag{6.3}$$

The estimate (6.3) can be proved as in [11], observing that $U_{h\delta}$ is Lipschitz continuous uniformly in δ, see Prop 3.3, and the constant C_δ in the estimate

$$\|U_{h\delta} - U_k\|_\infty \le C_\delta \frac{k}{h}$$

is of the type C/δ. □

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Dip. di Matematica Pura e Applicata, Univ. dell'Aquila, loc. Monteluco di Roio, 67040 l'Aquila, Italy
E-mail address: camilli@ing.univaq.it

Dip. di Matematica, Univ. di Roma "La Sapienza", P.le Aldo Moro 2, 00185 Roma, Italy
E-mail address: capuzzo@mat.uniroma1.it

Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa.
E-mail address: dgomes@math.ist.utl.pt