HADAMARD AND LIOUVILLE TYPE RESULTS FOR FULLY NONLINEAR PARTIAL DIFFERENTIAL INEQUALITIES *

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Abstract

In this paper we prove some Hadamard and Liouville type properties for nonnegative viscosity supersolutions of fully non linear uniformly elliptic partial differential inequalities in the whole space.

1 Introduction

In this paper we consider the fully nonlinear partial differential inequality

$$F(x, u, Du, D^2u) \geq 0 \ \ , \ \ x \in \mathbb{R}^N \tag{1.1}$$

with the aim of identifying sufficient conditions on the uniformly elliptic function F which guarantee the validity of Liouville type results such as

(A) any nonnegative solution of (1.1) is a constant

or

(B) the only nonnegative solution of (1.1) is $u \equiv 0$.

This questions have been recently tackled in the framework of the theory of viscosity solutions (see [6] as a general reference on the subject), the natural one in view of the nonlinear dependence of F on second derivatives.
The first property has been established for the equation $F(D^2u) = 0$ as a consequence of the Krylov–Safonov–Harnack inequality for viscosity solutions in [3].

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In [7], the Liouville property for nonnegative viscosity solutions of the inequality $F(x, u, D^2u) \geq 0$ has been proved to hold true by a completely different method, relying in particular on a nonlinear extension of the classical Hadamard Three Spheres Theorem.

The main feature of the present paper, which borrows several ideas from [7], is that we allow F to depend on first order derivatives under the main condition that

$$F(x, 0, p, 0) \leq \sigma(|x|)|p|$$

where the radial function σ satisties some conditions which will be specified later.

Even under this restriction, the dependence of F on Du generates some new interesting phaenomena. Indeed, our version of the Hadamard Three Spheres Theorem (see Theorem 3.1) states that if u is a viscosity solution of (1.1), then the function

$$m(r) = \min_{r_1 \leq |x| \leq r} u(x)$$

is a concave function of

$$\psi(r) = \int_{r_1}^{r} s^{\frac{-\Lambda}{\lambda}(N-1)} \exp\left(-\frac{1}{\lambda}\int_{r_1}^{s} \sigma(\tau)d\tau\right) ds. \tag{1.2}$$

The function ψ in the above is a special non decreasing solution of the linear ordinary differential equation

$$\lambda\Phi'' + \left(\frac{\Lambda(N-1)}{r} + \sigma(r)\right)\Phi' = 0$$

which is connected with radial, convex, nondecreasing solutions of the Pucci maximal operator. In the case considered in [7], $\sigma \equiv 0$ and the function ψ has a finite limit as r tends to $+\infty$ as soon as $\lambda < \Lambda$ or $N > 2$. In the setting of the present paper, the possibility for ψ to diverge can also arise, due to the presence of the exponential term involving σ, namely when $\int_{r_1}^{+\infty} |\sigma(\tau)|d\tau = +\infty$.

Let us point out that our Theorem 3.1 applies in particular to non negative solutions of linear inequalities such as

$$-\operatorname{tr}(A(x)D^2u) + b(x) \cdot Du \geq 0$$

if $A(x)$ is positive definite, $|b(x)| \leq \sigma(|x|)$ (see [14] for classical results in this direction).

As an immediate consequence of the Hadamard theorem the function m satisfies

$$m(r) \geq m(r_1)\left(1 - \frac{\psi(r)}{\psi(r_2)}\right).$$

This implies that $m(r)$ or $\frac{m(r)}{L-\psi(r)}$ are nondecreasing functions respectively when ψ is divergent or ψ has a finite limit.

The two different possible behaviors at infinity of ψ have then some consequence from the point of view of the validity of the Liouville properties (A), (B) which amount to prove that m is a constant or $m \equiv 0$. Theorems 4.1 and 4.2 in Section 4 cover the two possible cases.

Since we are dealing here with fully nonlinear inequalities, both Theorem 4.1 and Theorem 4.2 can be regarded as a twofold extension of several classical and more recent results for linear and semilinear equations (see [14], [9], [10], [12] and [4]).

The proofs of Theorems 3.1, 4.1, 4.2 employ several basic facts as well as important results (namely, comparison and strong minimum principle) from the theory of viscosity solutions. For the convenience of the reader these are collected in Section 2.

2 Preliminaries

In this short section we recall the definition and some relevant properties of viscosity solutions of fully nonlinear uniformly elliptic inequalities as well as a few basic facts about the Pucci maximal operator. For further information we refer to [6], [3].

Let Ω be an open set in $\mathbb{R}^N$, $\mathcal{S}_N$ the set of $N \times N$ symmetric matrices and denote by $\mathrm{tr}\, Q$ the trace of a matrix Q. A continuous function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \to \mathbb{R}$ is uniformly elliptic if there exist constants $0 < \lambda \leq \Lambda$ such that

$$\lambda\, \mathrm{tr}(Q) \leq F(x,t,p,M) - F(x,t,p,M+Q) \leq \Lambda\, \mathrm{tr}(Q) \qquad (2.1)$$

for all $M, Q \in \mathcal{S}_N$ with Q nonnegative definite and for every fixed $t \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $x \in \Omega$.
A fundamental example of uniformly elliptic operator is the Pucci maximal operator

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} (-\mathrm{tr(AM)}) \quad \mathrm{M} \in \mathcal{S}_\mathrm{N} \ , \qquad (2.2)$$

where

$$\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}_N \ : \ \lambda|\xi|^2 \leq A\,\xi \cdot \xi \leq \Lambda|\xi|^2, \ \forall\, \xi \in \mathbb{R}^N\}$$

For $\lambda = \Lambda = 1$, the operator $\mathcal{M}^+_{\lambda,\Lambda}$ coincides with the Laplace operator $-\Delta$.
The following representation of $\mathcal{M}^+_{\lambda,\Lambda}(M)$ will be often used in the paper:

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = -\lambda \sum_{i \in I^+} e_i - \Lambda \sum_{i \in I^-} e_i \, , \qquad (2.3)$$

where e_i $(i = 1, \ldots, N)$ are the eigenvalues of M and I^+, I^- are, respectively, the sets of indexes corresponding to positive and negative eigenvalues of M.
From the above it follows that the inequality

$$F(x,t,p,B) \leq F(x,t,p,0) + \mathcal{M}^+_{\lambda,\Lambda}(B) \qquad (2.4)$$

holds for any F satisfying (2.1) and for all $x \in \Omega$, $t \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $B \in \mathcal{S}_N$.

A viscosity solution of the inequality

$$F(x,u,Du,D^2u) \geq 0 \ \ , \ \ x \in \Omega \qquad (2.5)$$

is a lower semicontinuous function $u : \Omega \to \mathbb{R}$ such that

$$F(x_0, u(x_0), D\zeta(x_0), D^2\zeta(x_0)) \geq 0$$

for all $\zeta \in C^2(\Omega)$ and all $x_0 \in \Omega$ such that $u - \zeta$ has a local minimum at x_0.

Viscosity solution of the inequality

$$F(x,u,Du,D^2u) \le 0 \ , \quad x \in \Omega \tag{2.6}$$

are similarly defined by replacing lower semicontinuity with upper semicontinuity and local minima with local maxima. Finally, u is a viscosity solution of the equation

$$F(x,u,Du,D^2u) = 0 \ , \quad x \in \Omega \tag{2.7}$$

if it is simultaneously a viscosity solution of (2.5) and (2.6).

We will consider in the paper functions F satisfying (2.1) for some fixed λ, Λ and

$$F(x,0,0,0) = 0 \tag{2.8}$$

$$F(x,t,p,0) \le \sigma(|x|)|p| + h(x)t^\alpha \qquad \forall (x,t,p) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \tag{2.9}$$

where $\alpha \ge 1$ and σ and h are continuous real valued functions such that

$$|x|\sigma(|x|) \ge -\Lambda(N-1) \tag{2.10}$$

$$h(x) \le 0 \tag{2.11}$$

for all $x \in \Omega$. Let us observe that if a function $u \ge 0$ is a viscosity solution of (2.5) with F as in (2.8), (2.9), then it is easy to deduce from (2.4) that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| + h(x)u^\alpha \ge 0 \ \ , x \in \Omega \tag{2.12}$$

in the viscosity sense. If moreover (2.11) holds, then u is also a viscosity solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| \ge 0 \ , \quad x \in \Omega. \tag{2.13}$$

A large class of fully nonlinear operators satisfying this set of assumptions is that of Bellman - Isaacs operators

$$\inf_{\gamma \in \mathcal{B}} \sup_{\beta \in \mathcal{A}} \{- \operatorname{tr}(A^{\beta,\gamma}(x)D^2u) + b^{\beta,\gamma}(x) \cdot Du + c^{\beta,\gamma}(x)u\} \tag{2.14}$$

arising in stochastic optimal control and differential games theory (see [8]). Indeed, it is easy to check this, provided that $A^{\beta,\gamma} \in \mathcal{A}_{\lambda,\Lambda}$ for some $0 < \lambda \le \Lambda$, the vectorfields $b^{\beta,\gamma}$ and functions $c^{\beta,\gamma}$ satisfy

$$|b^{\beta,\gamma}(x)| \le \sigma(|x|) \ , \ c^{\beta,\gamma}(x) \le h(x)$$

with h and σ as in (2.10), (2.11) for any (β, γ) in the give parameter sets $\mathcal{A}, \mathcal{B}$. The Pucci maximal operator can be represented in this form by taking $\mathcal{B}$ as a singleton, $\mathcal{A} = \mathcal{A}_{\lambda,\Lambda}$ and $b^{\beta,\gamma} = c^{\beta,\gamma} = 0$. Our assumptions (2.1), (2.8),(2.9), (2.10), (2.11) are satisfied, in particular, by linear operators in non divergence form

$$- \operatorname{tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u$$

if $A(x)$ is positive definite, $|b(x)| \le \sigma(|x|)$, $c(x) \le 0$.

The following two fundamental results from the theory of viscosity solutions hold for fully nonlinear operators satisfying our assumptions. Let us record them in the following versions suited for our later purposes the **Comparison Principle** : let u and Φ be, respectively, a viscosity and a classical solution of (2.5) in the annulus $D = \{x \in \mathbb{R}^N \; : r_1 < |x| < r_2\}$. If $u \geq \Phi$ on ∂D then $u \geq \Phi$ in $\overline{D}$, (see [11])

the **Strong Minimum Principle** : let $u \geq 0$ be a viscosity solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(x)|Du| \geq 0 \;\; , \;\; x \in \Omega \subset\subset \mathbb{R}^N \; .$$

If u attains its minimum on $\overline{\Omega}$ at some $x_0 \in \Omega$ then u is a constant, (see [1]).

3 A nonlinear Hadamard theorem

The object of this section is the following nonlinear version of the classical Hadamard Three Spheres Theorem.

Theorem 3.1 *Let u be a viscosity solution of*

$$u \geq 0\,, \qquad F(x,u,Du,D^2u) \geq 0 \;\; , \;\; x \in D \tag{3.1}$$

where D is the annulus $\{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$.
If F satisfies conditions (2.1), (2.8), (2.9), (2.10) and (2.11) for all $x \in D$, then the function

$$m(r) = \min_{r_1 \leq |x| \leq r} u(x) \;\; , \;\; r \in [r_1, r_2] \tag{3.2}$$

satisfies

$$m(r) \geq \frac{\psi(r)}{\psi(r_2)} m(r_2) + \left(1 - \frac{\psi(r)}{\psi(r_2)}\right) m(r_1) \;\; , \;\; \forall r \in [r_1, r_2] \tag{3.3}$$

where ψ is given by

$$\psi(r) = \int_{r_1}^{r} s^{\frac{-\Lambda}{\lambda}(N-1)} \exp\left(-\frac{1}{\lambda}\int_{r_1}^{s} \sigma(\tau)d\tau\right) ds\,. \tag{3.4}$$

Proof. The first step of the proof is to observe, see (2.13), that in our assumptions any viscosity solution u of (3.1) is also a viscosity solution of

$$u \geq 0\;,\; \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| \geq 0 \;\; , \;\; x \in D\;.$$

We look next for a smooth solution $\Phi = \Phi(|x|)$ of the Dirichlet problem

$$\begin{array}{l} \mathcal{M}^+_{\lambda,\Lambda}(D^2\Phi) + \sigma(|x|)|D\Phi| = 0 \;\; , \;\; x \in D \\ \Phi(x) = m(r_1) \text{ if } |x| = r_1, \quad \Phi(x) = m(r_2) \text{ if } |x| = r_2 \end{array} \tag{3.5}$$

where m is as in (3.2). At this purpose, note that the eigenvalues of the Hessian matrix

$$D^2\Phi(|x|) \equiv \frac{\Phi'(|x|)}{|x|} I_N + \left[\frac{\Phi''(|x|)}{|x|^2} - \frac{\Phi'(|x|)}{|x|^3}\right] x \otimes x$$

(here I_N denotes the $N \times N$ identity matrix) are $\Phi''(|x|)$ which is simple and $\frac{\Phi'(|x|)}{|x|}$ with multiplicity $N-1$ (see [7], Lemma 3.1). Hence, taking (2.3) into account, any radial, convex, non increasing solution of (3.5) must satisfy

$$\begin{array}{l} \lambda\Phi'' + \left(\frac{\Lambda(N-1)}{r} + \sigma(r)\right)\Phi' = 0, \;\; , \;\; r \in (r_1, r_2) \\ \Phi(r_1) = m(r_1), \quad \Phi(r_2) = m(r_2)\;. \end{array} \tag{3.6}$$

A simple computation shows that the solution of (3.6) is

$$\Phi(r) = \frac{\psi(r)}{\psi(r_2)} m(r_2) + \left(1 - \frac{\psi(r)}{\psi(r_2)}\right) m(r_1) \tag{3.7}$$

with $\psi(r)$ given by (3.4). Hence, $\Phi(x) = \Phi(|x|)$ is a smooth solution of (3.5). Since, by construction, $\Phi(x) \leq u(x)$ on ∂D, by the Comparison Principle we get

$$u(x) \geq \Phi(x) \qquad \text{in } \overline{D}. \tag{3.8}$$

Observe now that the claim (3.3) is trivial if u (and, consequently, m) is a constant. Assume then that u is not a constant. Therefore, by the Strong Minimum Principle, u must attain its minimum value on the boundary of the compact set $\{x \in \mathbb{R}^N : r_1 \leq |x| \leq r\}$, for each $r \in (r_1, r_2)$, that is

$$m(r) = \min\{\min_{|x|=r_1} u(x); \min_{|x|=r} u(x)\} \qquad \text{for } r \in (r_1, r_2).$$

If $m(r) = \min_{|x|=r_1} u(x)$, then the definition of m, the boundary condition on Φ and the fact that Φ is non increasing yield

$$m(r) = m(r_1) = \Phi(r_1) \geq \Phi(r)$$

On the other hand, if $m(r) = \min_{|x|=r} u(x) = u(x_r)$ for some $|x_r| = r$, then from inequality (3.8) we get $m(r) \geq \Phi(|x_r|) = \Phi(r)$ and the claim is proved in this case as well.
Therefore, $m(r) \geq \Phi(r)$ for all $r \in [r_1, r_2]$ and the proof is complete. □

An analogous result, which can be proved by the same technique as the previous one, holds for the reversed inequalities.

Theorem 3.2 *Let u be a viscosity solution of*

$$u \leq 0, \qquad F(x, u, Du, D^2u) \leq 0 \ , x \in D \tag{3.9}$$

where D is the annulus $\{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$.
If F satisfies conditions (2.1), (2.8) and

$$F(x, t, p, 0) \geq \sigma(|x|)|p| + h(x)t^\alpha \ , \ \alpha \geq 1$$

where σ and h are continuous, and satisfy, respectively,

$$|x|\sigma(|x|) \leq \lambda(N-1) \ , \ h(x) \geq 0 \ , \ x \in D.$$

Then $M(r) = \max_{r_1 \leq |x| \leq r} u(x)$, $r \in [r_1, r_2]$ satisfies

$$M(r) \leq \frac{\tilde{\psi}(r)}{\tilde{\psi}(r_2)} M(r_2) + \left(1 - \frac{\tilde{\psi}(r)}{\tilde{\psi}(r_2)}\right) M(r_1) \ , \ \forall r \in [r_1, r_2]$$

where

$$\tilde{\psi}(r) = \int_{r_1}^{r} s^{\frac{-\lambda}{\Lambda}(N-1)} \exp\left(\frac{1}{\Lambda}\int_{r_1}^{s} \sigma(\tau)d\tau\right) ds$$

4 The Liouville property

In this section we show how from Theorem 3.1 one can deduce some properties of Liouville type for non negative viscosity solutions of fully nonlinear inequalities in the whole space. The results depend on the behaviour as r tends to $+\infty$ of the function

$$\psi(r) = \int_{r_1}^{r} s^{\frac{-\Lambda}{\lambda}(N-1)} \exp\left(-\frac{1}{\lambda}\int_{r_1}^{s}\sigma(\tau)d\tau\right) ds$$

where $r_1 > 0$ is arbitrarily fixed. Observe that ψ is non decreasing; moreover,

$$\psi''(r) = -\exp\left(-\frac{1}{\lambda}\int_{r_1}^{s}\sigma(\tau)d\tau\right)\left(\frac{\Lambda}{\lambda}(N-1)r^{-\frac{\Lambda}{\lambda}(N-1)-1} + \frac{\sigma(r)}{\lambda}r^{-\frac{\Lambda}{\lambda}(N-1)}\right)$$

and therefore, by assumption (2.10), ψ is concave.

The first result is for the case $\psi(r) \to +\infty$ as $r \to +\infty$.

Theorem 4.1 *Let u be a viscosity solution of*

$$u \geq 0\,, \qquad F(x,u,Du,D^2u) \geq 0 \qquad in\ \mathbb{R}^N \tag{4.1}$$

with F satisfying (2.1),(2.8), (2.9), (2.10) and (2.11) for all $x \in \mathbb{R}^N$. If

$$\lim_{r\to+\infty} \psi(r) = +\infty\,, \tag{4.2}$$

then u is a constant. Moreover, if the strict inequality holds in (2.11) at some point $x_0 \in \mathbb{R}^N$, then $u \equiv 0$.

Proof. Let u be a viscosity solution of (4.1). A fortiori, u is a solution of the same inequality in any annulus D. So we can apply Theorem 3.1 to u in the annulus

$$r_1 \leq |x| \leq r_2$$

for arbitrary $0 < r_1 < r_2$. Since $m(r_2) \geq 0$, (3.3) reads

$$m(r) \geq m(r_1)\left(1 - \frac{\psi(r)}{\psi(r_2)}\right) \qquad \text{for every } r \in [r_1, r_2]\,. \tag{4.3}$$

Keeping r fixed and letting r_2 go to $+\infty$ in the above, using assumption (4.2) we obtain

$$m(r) \geq m(r_1) \qquad \text{for } r \geq r_1. \tag{4.4}$$

Since $m(r)$ is, by its very definition, a nonincreasing function, we conclude that $m(r) = m(r_1)$ for every $r \geq r_1 > 0$. Hence,

$$m(r) \equiv m(0) = u(0)$$

that is u attains its minimum on the closed ball $|x| \leq r$ at the interior point $x = 0$.
Since u is also a solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| \geq 0$$

in any annulus (see (2.13)), by the Strong Minimum Principle u is a constant and the first claim is proved.

Now, if C is a nonnegative constant solution of (4.1), then using (2.9),

$$0 \leq F(x_0, C, 0, 0) \leq h(x_0)\, C^\alpha$$

which implies $u \equiv 0$ if the strict inequality holds in (2.11) at some point x_0. This completes the proof. □

As a consequence of the Theorem, any solution of

$$u \geq 0, \qquad \mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + \sigma(|x|)|Du| \geq 0 \ \text{ in } \mathbb{R}^N \tag{4.5}$$

with σ as in (2.10) and such that (4.2) holds, is necessarily a constant. Observe that for inequality (4.5), condition (4.2) is also a necessary condition for the validity of the Liouville property. Actually, were

$$\lim_{r\to+\infty} \psi(r) = L < +\infty, \tag{4.6}$$

then the nonnegative function

$$u = \begin{cases} L - \psi(|x|) & \text{if } |x| \geq r_1 \\ L & \text{if } |x| < r_1 \end{cases} \tag{4.7}$$

with $r_1 > 0$ arbitrarily fixed, would be a strictly positive nonconstant viscosity solution of (4.5). Indeed, $L - \psi$ is a classical solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + \sigma(|x|)|Du| \equiv \lambda\psi''(|x|) + \psi'(|x|)[\frac{\Lambda(N-1)}{|x|} + \sigma(|x|)] = 0 \text{ in } \mathbb{R}^N \setminus \{0\}$$

as it is easy to check, whereas L is a solution of the same equation in the whole space. Therefore, well known stability properties of viscosity solutions imply (see [6]) that $u \equiv \min\{L; L - \psi(|x|)\}$ is a viscosity solution of (4.5).

Let us now discuss assumption (4.2) with respect to the behaviour of σ, the structural constants λ, Λ and the space dimension N.
A first remark in this direction is that if

$$\int_{r_1}^{+\infty} |\sigma(\tau)| d\tau = M \ < +\infty \tag{4.8}$$

then condition (4.2) holds in dimension $N = 1$ for any λ, Λ while for $N > 1$ it holds only if $N \leq 2$ and $\lambda = \Lambda = 1$. Indeed, if (4.8) holds then

$$\exp\left(-\frac{M}{\lambda}\right) \int_{r_1}^{r} s^{-\frac{\Lambda(N-1)}{\lambda}} ds \leq \psi(r) \leq \exp\left(\frac{M}{\lambda}\right) \int_{r_1}^{r} s^{-\frac{\Lambda(N-1)}{\lambda}} ds \tag{4.9}$$

For $N = 1$ we have then

$$\exp\left(-\frac{M}{\lambda}\right) r \leq \psi(r) \leq \exp\left(\frac{M}{\lambda}\right) r$$

and ψ is divergent for any $0 < \lambda \leq \Lambda$. For general N,

$$\psi(r) \simeq r^{1-\frac{\Lambda(N-1)}{\lambda}} \ \text{ or } \ \psi(r) \simeq \log r \ \text{ as } r \to +\infty$$

according to whether $\frac{\Lambda(N-1)}{\lambda} \neq 1$ or $\frac{\Lambda(N-1)}{\lambda} = 1$. Therefore, ψ is divergent if and only if $\frac{\Lambda(N-1)}{\lambda} \leq 1$, that is $N = 2$ and $\lambda = \Lambda$.

For example, using Theorem 4.1 we can conclude that if $\gamma < -1$ any viscosity solution of

$$u \geq 0 \,, \qquad -\Delta u + |x|^\gamma |Du| \geq 0 \qquad \text{in } \mathbb{R}^2$$

is a constant. Note that since $|x|^\gamma \geq 0$ this fact cannot be deduced as a trivial consequence of the classical Liouville property for superharmonic function in $\mathbb{R}^2$.
A further remark in the same direction is that condition (4.2) does not hold for any choice of λ, Λ, N when $\sigma(x) \equiv \sigma_0 > 0$ (observe that $\sigma_0 < 0$ does not satisfy (2.10)) since

$$\psi(r) = \int_{r_1}^{r} s^{-\frac{\Lambda}{\lambda}(N-1)} \exp\Big(-\frac{\sigma_0}{\lambda}(s-r_1)\Big) ds.$$

In view of the discussion preceding the proof of Theorem 4.1 one can conclude then that nonconstant supersolutions of

$$u \geq 0 \,, \qquad \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma_0 |Du| \geq 0 \qquad \text{in } \mathbb{R}^N,$$

do exist for every $N \,, \lambda \,, \Lambda$.

The range of applicability of Theorem 4.1 is wider in case

$$\int_{r_1}^{+\infty} |\sigma(\tau)| d\tau = +\infty \,.$$

A class of examples in this direction is provided by functions σ such that

$$\frac{-\Lambda(N-1)}{|x|} \leq \sigma(|x|) \leq \frac{\lambda - \Lambda(N-1)}{|x|} \qquad \text{for large } |x| \tag{4.10}$$

Actually, in this case ψ satisfies

$$\psi(r) \geq C \log r \,, \quad \text{for large } r,$$

for some constant $C > 0$, so that (2.10) and (4.2) hold.

In the last part of this section we present a Liouville type result for the other possible case of behavior of function ψ, namely when $\psi(r)$ has a finite limit as $r \to +\infty$.
This situation arise, for example, when the function σ satisfies

$$|x|\sigma(|x|) \geq \lambda - \Lambda(N-1) + \delta \qquad \text{for some } \delta > 0,$$

a more stringent condition than (2.10). Actually, if the above holds, then

$$\psi(r) \leq \int_{r_1}^{r} s^{-\frac{\Lambda(N-1)}{\lambda}} \exp\left(\frac{\Lambda(N-1) - \lambda - \delta}{\lambda} \int_{r_1}^{s} \tau^{-1} d\tau\right) ds \leq \int_{r_1}^{r} s^{-1-\frac{\delta}{\lambda}} ds \,,$$

yielding $\lim_{r\to+\infty} \psi(r) \leq \lim_{r\to+\infty} \int_{r_1}^{r} s^{-1-\frac{\delta}{\lambda}} ds < +\infty$.

We also need to impose, in addition to (2.10) and (2.11), a specific condition on the behaviour at infinity of σ and on the zero order term in the operator, precisely

$$\sup_{\mathbb{R}^N} |x|\sigma(|x|) = C < +\infty \,, \tag{4.11}$$

and

$$h(x) \leq -g(|x|)\,, \quad \text{for } r \text{ large, and} \quad \lim_{r\to+\infty} r^2 g(r)(L-\psi(r))^{(\alpha-1)} = +\infty \tag{4.12}$$

where $\alpha \geq 1$ is as in (2.9). Note that (4.12) implies $h < 0$ for large r, excluding therefore the case $h \equiv 0$.
Observe also that if $\int_{r_1}^{\infty} |\sigma(\tau)|d\tau < +\infty$, then $\lim_{r\to+\infty}\psi(r) = L < +\infty$ if and only if $\lambda < \Lambda(N-1)$ and in this case $\psi(r) \simeq -r^{1-\frac{\Lambda(N-1)}{\lambda}}$ (recall the estimate (4.9)).
Hence, the asymptotic condition in (4.12) becomes

$$\lim_{r\to+\infty} g(r) r^{2+(\alpha-1)\left(1-\frac{\Lambda(N-1)}{\lambda}\right)} = +\infty \tag{4.13}$$

Furthermore, let us remark that (4.11) yields the estimate

$$L-\psi(r) \geq r^{1-\frac{\Lambda(N-1)+C}{\lambda}}\,,$$

with C defined in (4.11). Thus (4.12) holds true, in particular, if g and $\alpha \geq 1$ satisfy

$$\lim_{r\to+\infty} g(r) r^{2+(\alpha-1)(1-\frac{\Lambda(N-1)+C}{\lambda})} = +\infty\,.$$

For example, if we take $g(|x|) = |x|^\gamma$, for $|x|$ large, with $\gamma > -2$ and

$$\sigma(|x|) = \begin{cases} C & \text{if } |x| < 1 \\ \dfrac{C}{|x|} & \text{if } |x| \geq 1\,, \end{cases}$$

with $C > \lambda - \Lambda(N-1)$ in such a way that $\psi(r) \simeq -r^{1-\frac{\Lambda(N-1)+C}{\lambda}} \to 0$, as $r \to +\infty$, then (4.12) holds true if $1 \leq \alpha < \frac{\beta+\gamma}{\beta-2}$ with $\beta = 1 + \frac{\Lambda(N-1)+C}{\lambda}$.

Theorem 4.2 *Let u be a viscosity solution of*

$$u \geq 0\,, \qquad F(x,u,Du,D^2u) \geq 0 \qquad \text{in } \mathbb{R}^N \tag{4.14}$$

with F satisfying (2.1), (2.8), (2.9), (2.10), (4.11), (2.11) and (4.12).
If $\lim_{r\to+\infty}\psi(r) = L < +\infty$, then $u \equiv 0$.

Proof. Let u be be a viscosity solution of (4.14). As in the proof of Theorem 4.1 it follows that

$$u \geq 0\,, \qquad \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| + h(x)u^\alpha \geq 0 \qquad \text{in } \mathbb{R}^N \tag{4.15}$$

and, a fortiori,

$$u \geq 0\,, \qquad \mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \sigma(|x|)|Du| \geq 0 \qquad \text{in } \mathbb{R}^N \tag{4.16}$$

in the viscosity sense. By the Strong Minimum Principle applied to (4.16) in a ball of arbitrary radius r, we deduce that either $u \equiv 0$ or $u > 0$ in $\mathbb{R}^N$.

Let us assume then by contradiction that $u > 0$ in $\mathbb{R}^N$. Applying Theorem 3.1 with arbitrarily fixed $r_2 > r_1 > 0$, we get

$$m(r) \geq m(r_1)\left(1 - \frac{\psi(r)}{\psi(r_2)}\right) \qquad \text{for every } r \in [r_1, r_2]\,. \tag{4.17}$$

Since $\psi(r) \to L < +\infty$, as $r \to +\infty$, we obtain, after letting r_2 go to infinity and keeping r fixed in (4.17)

$$m(r) \geq \left(1 - \frac{\psi(r)}{L}\right) m(r_1).$$

Since $\psi(r_1) = 0$, the above inequality implies that

$$r \mapsto \frac{m(r)}{L - \psi(r)} \qquad \text{is non decreasing on} \quad [r_1, +\infty)\,. \tag{4.18}$$

Since

$$\min_{|x| \leq r} u(x) = \min_{|x|=r} u(x) > 0, \qquad \text{for all } r > 0, \tag{4.19}$$

then

$$m(r) = \min_{|x| \leq r} u(x). \tag{4.20}$$

By assumption (4.12) there exists R_0 such that $h(x) < 0$ for $|x| \geq R_0$. Thus, there are no constant, positive solutions of (4.15) and the Strong Minimum Principle implies that $m(r)$ is strictly decreasing. Let us consider then the radial function

$$\zeta(|x|) = m(r) \left[1 - \frac{[(|x| - r)^+]^3}{(R - r)^3}\right],$$

where r, R are parameters such that $R \geq r \geq R_0$.
Observe that $\zeta(|x|) \leq 0 < u(x)$ for $|x| \geq R$ and that $\zeta(|x|) \equiv m(r) < u(x)$ for $|x| < r$, since m is strictly decreasing, and, moreover, $\zeta(|\hat{x}|) = u(\hat{x})$ for at some $\hat{x}$ with $|\hat{x}| = r$.
Therefore, $u - \zeta$ attains a nonpositive global minimum at some point x_R^r such that $r \leq |x_R^r| < R$. By definition of viscosity solution, from (4.15) it follows that

$$\mathcal{M}^+_{\lambda,\Lambda}\left(D^2\zeta(x_R^r)\right) + \sigma(|x_R^r|)|D\zeta(x_R^r)| \geq -h(x_R^r)u^\alpha(x_R^r). \tag{4.21}$$

Since ζ is radial, the same computation as performed in the proof of Theorem 3.1 based on the representation formula (2.3), shows that the left - hand side of the above is

$$\frac{3\Lambda m(r)}{(R-r)^3}\left[2 + \left(\frac{(N-1)}{|x_R^r|} + \frac{\sigma(|x_R^r|)}{\Lambda}\right)(|x_R^r| - r)^+\right](|x_R^r| - r)^+. \tag{4.22}$$

If $|x_R^r| = r \geq R_0$, using the fact that $h < 0$ for $|x| \geq R_0$ we immediately deduce from (4.21) and (4.22) that

$$h(x_R^r)u^\alpha(x_R^r) \geq 0\,,$$

contradicting the assumption $u > 0$ in $\mathbb{R}^N$.

Assume then $r < |x_R^r| < R$; from assumption (4.12) and inequalities (4.21) and (4.22) we get

$$g(|x_R^r|)u^\alpha(x_R^r) \leq 3m(r)\left[\frac{\Lambda(N+1)}{(R-r)^2} + \frac{\sigma(|x_R^r|)}{R-r}\right] \tag{4.23}$$

yielding

$$g(|x_R^r|)m^\alpha(R) \le 3m(r)\left[\frac{\Lambda(N+1)}{(R-r)^2}+\frac{\sigma(|x_R^r|)}{R-r}\right]. \tag{4.24}$$

Thanks to the monotonicity of $r \to \frac{m(r)}{L-\psi(r)}$ (recall (4.18)) we obtain

$$g(|x_R^r|)m^\alpha(R) \le 3m(R)\frac{L-\psi(r)}{L-\psi(R)}\left[\frac{\Lambda(N+1)+(R-r)\sigma(|x_R^r|)}{(R-r)^2}\right]. \tag{4.25}$$

Choose now $r=\frac{R}{2}$ in (4.25) and denote by x_R the corresponding minimum point $x_R^{\frac{R}{2}}$ of $u-\zeta$. Observe that (4.11) guarantees that $\int_{\frac{R}{2}}^{R}\sigma(\tau)d\tau \le C\log 2$; therefore,

$$\frac{\psi'(\frac{R}{2})}{\psi'(R)} = 2^{\frac{\Lambda}{\lambda}(N-1)}\exp\left(\frac{1}{\lambda}\int_{\frac{R}{2}}^{R}\sigma(\tau)d\tau\right) \le K, \text{ for some } K>0. \tag{4.26}$$

Then, by the De l'Hôpital rule,

$$\frac{L-\psi(\frac{R}{2})}{L-\psi(R)} \le \frac{K}{2}\,. \tag{4.27}$$

Consider now the case $\alpha=1$ in (4.25). Taking (4.27) into account we get the estimate

$$R^2 g(|x_R|) \le 3K[2\Lambda(N+1)+R\sigma(|x_R|)]. \tag{4.28}$$

for large enough R. Since $\frac{R}{2}\le|x_R|\le R$, by (4.11) we deduce that the right-hand side of the previous inequality is bounded. Hence, assumption (4.12) yields the contradiction

$$+\infty = \lim_{R\to+\infty} R^2 g(R) \le \limsup_{R\to+\infty} R^2 g(|x_R|) \le M \qquad \text{for some } M>0.$$

The statement is then proved in the case $\alpha=1$.

Assume now $\alpha>1$. From (4.25) with $r=\frac{R}{2}$ we get

$$R^2 g(|x_R|)m^{\alpha-1}(R) \le 3K[2\Lambda(N+1)+R\sigma(|x_R|)]$$

(x_R denotes as above the point $x_R^{\frac{R}{2}}$) and dividing by $L-\psi(R)$, we obtain

$$\frac{m(R)}{L-\psi(R)} \le \left(\frac{3K\,[2\Lambda(N+1)+R\sigma(|x_R|)]}{R^2 g(|x_R|)(L-\psi(R))^{\alpha-1}}\right)^{\frac{1}{\alpha-1}}. \tag{4.29}$$

Now, the left - hand side is, by (4.18), a positive non decreasing function. Thus, we get the contradiction if we show that the right - hand side of (4.29) tends to zero as $R\to\infty$.

At this purpose, observe that (4.11) yields as above the boundedness of $R\sigma(|x_R|)$; moreover, from (4.27),

$$L-\psi(R) \ge \frac{2}{K}\left(L-\psi\left(\frac{R}{2}\right)\right) \ge \frac{2}{K}\left(L-\psi(|x_R|)\right)\,,$$

as $L-\psi(r)$ is non increasing. Then applying (4.12), taking into account that $|x_R|\to+\infty$, as $R\to+\infty$, we get

$$R^2\, g(|x_R|)\,(L-\psi(R))^{\alpha-1} \ge \frac{1}{4}\,|x_R|^2\, g(|x_R|)\,(L-\psi(|x_R|))^{\alpha-1} \to +\infty$$

as $R \to +\infty$. Therefore, the right-hand side of (4.29) tends to zero as R diverges and we get the desired contradiction. □

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