On the rate of convergence in homogenization of Hamilton-Jacobi equations

I. Capuzzo Dolcetta*

Dipartimento di Matematica - Università di Roma "La Sapienza"

H. Ishii†

Department of Mathematics - Tokyo Metropolitan University

Abstract

We consider the homogenization problem for fully nonlinear first order scalar partial differential equations of Hamilton-Jacobi type such as

$$u^\epsilon(x) + H\left(x, \frac{x}{\epsilon}, Du^\epsilon(x)\right) = 0, \quad x \in \mathbf{R}^N ,$$

where ϵ is a small positive parameter and H is a periodic function of the second variable. Our main results (Theorems 1.1 and 1.2 below) give estimates on the rate of convergence of u^ϵ to the solution u of the homogenized problem

$$u(x) + \bar{H}(x, Du(x)) = 0, \quad x \in \mathbf{R}^N .$$

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1 Introduction

The homogenization of partial differential equations can be regarded as an asymptotic process in which one looks at the limiting behaviour of solutions of a PDE with a rapidly oscillating structure as the frequency of oscillations tends to infinity (see [6] as a general reference on the subject).

We shall consider here fully nonlinear first order scalar partial differential equations of Hamilton-Jacobi type such as

$$(\mathrm{HJ})^\epsilon \qquad u^\epsilon(x) + H(x, \frac{x}{\epsilon}, Du^\epsilon(x)) = 0, \quad x \in \mathbf{R}^N,$$

where ϵ is a small positive parameter, under the basic structural assumption that the Hamiltonian $H : \mathbf{R}^{3N} \to \mathbf{R}$ is periodic in the second variable: i.e.,

$$(\mathrm{H1}) \qquad \xi \mapsto H(x, \xi, p) \text{ is } \mathbf{Z}^N\text{-periodic for each } (x, p) \in \mathbf{R}^{2N}.$$

Partial differential equations of type $(\mathrm{HJ})^\epsilon$ arise, for example, in the dynamic programming approach to optimization and differential games problems for non linear control systems with rapidly oscillating dynamics. Let us describe here an example taken from optimal control, namely the discounted infinite horizon problem

$$(1.1) \qquad u^\epsilon(x) := \inf_{\alpha \in \mathcal{A}} \int_0^{+\infty} L(y^\epsilon_{(x,\alpha)}(s), \alpha(s)) e^{-s} ds\,,$$

where $\mathcal{A} = \{\alpha : (0, +\infty) \to A,\ \alpha \text{ measurable}\}$, with A being a subset of $\mathbf{R}^M$, and $y^\epsilon_{(x,\alpha)}(\cdot)$ is the trajectory of the nonlinear control system

$$y'(s) = f\left(\frac{y(s)}{\epsilon}, \alpha(s)\right), \qquad s > 0,$$

corresponding to the initial condition $y(0) = x \in \mathbf{R}^N$ and the measurable control $\alpha \in \mathcal{A}$. Assume that f and L satisfy the conditions:

$$\xi \mapsto f(\xi, a) \text{ is } \mathbf{Z}^N\text{-periodic},$$

$$|f\xi, a) - f(\eta, a)| \leq C|\xi - \eta|,$$

$$|L(x, a) - L(y, a)| \leq C|x - y|\,; \quad |L(x, a)| \leq C,$$

$$B(0,\nu) \subseteq \overline{\mathrm{co}}\, f(\xi, A),$$

for some positive constants C, ν and all $x, y, \xi, \eta \in \mathbf{R}^N$, $a \in A$. Here and henceforth $\overline{\mathrm{co}}\, C$ denotes closed convex hull of the set $C \subset \mathbf{R}^N$ and $B(z,r)$ denotes the closed ball of $\mathbf{R}^N$ with radius $r > 0$ and center at z.

It is well-known (see [4]) that under these assumptions the value function u^ϵ defined in (1.1) is the unique continuous $\mathbf{Z}^N$ -periodic viscosity solution of equation $(\mathrm{HJ})^\epsilon$ with

$$H(x, \xi, p) = \sup_{a \in A} \{-f(\xi, a) \cdot p - L(x, a)\}.$$

It is not hard to check that the above conditions on f, L imply that H fulfills (H1) as well as all the other assumptions of our Theorem 1.1 below. Note, in particular, that the controllability condition $B(0,\nu) \subseteq \overline{\mathrm{co}}\, f(\xi, A)$ and the boundedness of L yield

$$H(x, \xi, p) \geq \sup_{q \in \overline{\mathrm{co}}\, f(\xi, A)} \{-q \cdot p\} - C \geq \sup_{q \in B(0,\nu)} \{-q \cdot p\} - C = \nu |p| - C,$$

which implies the coercivity condition (H4).

At our knowledge, the first general results on the homogenization of Hamilton-Jacobi equations are due to P.-L. Lions, G. Papanicolau and S.R.S. Varadhan [12] who established, under quite general assumptions on H, that the limit problem of equation $(\mathrm{HJ})^\epsilon$ is given by

$$(\overline{\mathrm{HJ}}) \qquad u(x) + \bar{H}(x, Du(x)) = 0, \quad x \in \mathbf{R}^N,$$

where the effective Hamiltonian $\bar{H}$ is obtained by solving for $(\lambda, v) \in \mathbf{R} \times \mathrm{BUC}(\mathbf{R}^N)$ the cell problem:

$$(\mathrm{CP}) \qquad H(x, \xi, p + Dv(\xi)) = \lambda, \quad x \in \mathbf{R}^N,$$

where (x,p) is fixed in $\mathbf{R}^{2N}$. Here and henceforth $\mathrm{BUC}(\mathbf{R}^N)$ denotes the space of all bounded, uniformly continuous functions on $\mathbf{R}^N$. Indeed (see [12] and [9, 10]), there is a unique value $\lambda = \lambda(x,p)$ of the real constant λ for which (CP) has a bounded solution $v = v(\xi; x, p)$. The effective Hamiltonian $\bar{H} : \mathbf{R}^{2N} \to \mathbf{R}$ is then defined by setting

$$\bar{H}(x,p) = \lambda.$$

The next major contributions to the subject are due to L. C. Evans [9, 10] who developed the perturbed test functions methods for studying the homogenization problem in the framework of the theory of viscosity solutions. More recent research in this direction is reported in [5, 7, 8, 11, 1, 2].
Let us point out here that the methodology of viscosity solutions of Hamilton-Jacobi equations (see [13, 3, 4]) seems to be a convenient one in the investigation of homogenization problems for such equations since the comparison results and the weak limit technique allow to establish in a simple way uniform estimates in the sup norm for the solutions of $(\mathrm{HJ})^\epsilon$ (and their gradients) and to interpret their limit as the solution of the homogenized equation $(\overline{\mathrm{HJ}})$.

The question of estimating the rate of the uniform convergence of solutions of equations $(\mathrm{HJ})^\epsilon$ to the solution of equation $(\overline{\mathrm{HJ}})$ in terms of ϵ has not been tackled up to now as far as we know. The purpose of this paper is to present the following quite general result in this direction :

Theorem 1.1 *Assume that H satisfies the conditions:*

(H1) $$\xi \mapsto H(x,\xi,p) \ \ \textit{is} \ \ \mathbf{Z}^N\textit{-periodic for each} \ \ (x,p) \in \mathbf{R}^{2N};$$

for each $R > 0$ there is a constant $C(R) > 0$ such that for all $x, y, \xi, \eta \in \mathbf{R}^N$, $p, q \in B(0,R)$,

(H2) $$|H(x,\xi,p) - H(y,\eta,q)| \leq C(R)\,(|x-y| + |\xi-\eta| + |p-q|)\,;$$

(H3) $$\exists\, C_1 > 0 : \ |H(x,\xi,0)| \leq C_1, \quad (x,\xi) \in \mathbf{R}^{2N};$$

(H4) $$\lim_{R\to+\infty} \inf \Big\{ H(x,\xi,p) \mid x,\xi,p \in \mathbf{R}^N, \ |p| \geq R \Big\} = +\infty.$$

Then there is a constant $C > 0$, independent of $\epsilon \in (0,1)$, such that

$$\sup_{x\in\mathbf{R}^N} |u^\epsilon(x) - u(x)| \leq C\epsilon^{1/3}\,,$$

where u^ϵ, $u \in \mathrm{BUC}(\mathbf{R}^N)$ are, respectively, the viscosity solutions of equations $(\mathrm{HJ})^\epsilon$ *and* $(\overline{\mathrm{HJ}})$.

The rate of convergence of u^ϵ to u can be improved in some special case. For example, as the following theorem shows, the rate is of order 1 if the Hamiltonian $H(x, \xi, p)$ does not depend on the first variable.

Theorem 1.2 *Assume that $H(x, \xi, p)$ is independent of x and satisfies* (H1), (H2), (H4). *Let u^ϵ, $u \in \mathrm{BUC}(\mathbf{R}^N)$ be, respectively, the viscosity solutions of* $(\mathrm{HJ})^\epsilon$ *and* $(\overline{\mathrm{HJ}})$. *Then there is a constant $C > 0$, independent of $\epsilon \in (0, 1)$, such that*

$$\sup_{x \in \mathbf{R}^N} |u^\epsilon(x) - u(x)| \leq C\epsilon .$$

We will see in Section 3 that the solution u of $(\overline{\mathrm{HJ}})$ in the above theorem is indeed a constant, which allows us to obtain the better rate of convergence.

In Section 2 we collect some preliminary estimates on equations $(\mathrm{HJ})^\epsilon$ and on a useful approximate version of (CP). Section 3 contains the proof of Theorems 1.1 and 1.2. In the whole paper we will assume, sometimes without explicit mention, that H satisfies conditions (H1), (H2), (H3), (H4). In the proofs we will frequently make use of some basic results about viscosity solutions for which we refer the unexperienced reader to [3, 4].

2 Preliminary facts and estimates

It is well-known (see, for example, [13, 3, 4]) that, under the assumptions made, for each $\epsilon \in (0, 1)$ the equation

$$(\mathrm{HJ})^\epsilon \qquad u^\epsilon(x) + H\left(x, \tfrac{x}{\epsilon}, Du^\epsilon(x)\right) = 0, \quad x \in \mathbf{R}^N,$$

has a unique $\mathbf{Z}^N$-periodic viscosity solution $u^\epsilon \in \mathrm{BUC}(\mathbf{R}^N)$. Moreover, uniform Lipschitz estimates for u^ϵ hold, as shown in the next lemma.

Lemma 2.1 *There is a constant $C_2 > 0$, independent of $\epsilon \in (0, 1)$, such that*

$$(2.1) \qquad \max\{\|u^\epsilon\|_\infty, \|Du^\epsilon\|_\infty\} \leq C_2.$$

Proof. We first note that from assumption (H3) it follows easily that C_1 and $-C_1$ are, respectively, a supersolution and a subsolution of equation $(\mathrm{HJ})^\epsilon$. Hence, by a standard comparison result, $|u^\epsilon(x)| \le C_1$ for all $x \in \mathbf{R}^N$. Thus, in particular, u^ϵ satisfies

$$-C_1 + H\left(x, \frac{x}{\epsilon}, Du^\epsilon(x)\right) \le 0, \qquad x \in \mathbf{R}^N, \tag{2.2}$$

in the viscosity sense. By assumption (H4), there is a constant $C_2 > 0$ such that

$$-C_1 + H(x, \xi, p) > 0$$

for all $x, \xi, p \in \mathbf{R}^N$ with $|p| > C_2$. Hence, in view of (2.2), u^ϵ satisfies

$$|Du^\epsilon(x)| \le C_2, \qquad x \in \mathbf{R}^N,$$

in the viscosity sense. This proves that u^ϵ is Lipschitz continuous with Lipschitz constant $\le C_2$. □

As a consequence of the uniform estimates in Lemma 2.1 we have

Lemma 2.2 *There exists $\widetilde{H} : \mathbf{R}^{3N} \to \mathbf{R}$ such that, for some constants $\nu > 0$ and $C_3 > 0$,*

$$\nu|p| - C_3 \le \widetilde{H}(x, \xi, p) \le \nu|p| + C_3, \quad \forall\, x, \xi, p \in \mathbf{R}^N, \tag{2.3}$$

$$\|D\widetilde{H}\|_\infty \le C_3, \tag{2.4}$$

where D denotes the gradient with respect to all variables, and

$$u^\epsilon(x) + \widetilde{H}\left(x, \frac{x}{\epsilon}, Du^\epsilon(x)\right) = 0, \qquad x \in \mathbf{R}^N, \tag{2.5}$$

where u^ϵ is the solution of equation $(\mathrm{HJ})^\epsilon$.

Proof. Let $C_2 > 0$ be the constant from Lemma 2.1. By (H2) and (H3) we have

$$|H(x, \xi, p)| \le C_1 + C_2 C(C_2),$$

for all $x, \xi, p \in \mathbf{R}^N$ with $|p| \leq C_2$. In view of (H4) we can therefore select a constant $L \geq C_2 + 1$ so that

$$H(x, \xi, p) \geq 1 + C_1 + C_2 C(C_2)$$

for all $x, \xi, p \in \mathbf{R}^N$ with $|p| \geq L$. Consider now the function $\widetilde{H}$ defined on $\mathbf{R}^{3N}$ by

$$\widetilde{H}(x, \xi, p) = \begin{cases} H(x, \xi, p) & \text{if} \quad |p| \leq C_2, \\ H(x, \xi, p) \wedge \left(\dfrac{|p| - C_2}{L - C_2} + M \right) & \text{if} \quad C_2 \leq |p| \leq L, \\ \dfrac{|p| - C_2}{L - C_2} + M & \text{if} \quad |p| \geq L. \end{cases}$$

with $M = C_1 + C_2 C(C_2)$. It is easy to check that $\widetilde{H}$ satisfies (2.3), (2.4). Now it is immediate to conclude from the definition of $\widetilde{H}$ that u^ϵ is a solution of (2.5) since we know by Lemma 2.1 that $|Du^\epsilon(x)| \leq C_2$ for all $x \in \mathbf{R}^N$. □

Conditions (2.3), (2.4) imply of course that (H2), (H3), (H4) hold true. Thus, as far as the solutions u^ϵ are concerned, we may assume without loss of generality, and we shall do so from now on, that H satisfies (2.3), (2.4).

As mentioned in the Introduction, the limiting behaviour, as ϵ tends to 0, of the solutions u^ϵ of problem $(\text{HJ})^\epsilon$ is determined by the effective Hamiltonian $\bar{H} : \mathbf{R}^{2N} \to \mathbf{R}$ which is defined through the cell problem

$$\text{(CP)} \qquad H(x, \xi, p + Dv(\xi)) = \lambda, \qquad \xi \in \mathbf{R}^N,$$

where (x, p) is fixed in $\mathbf{R}^{2N}$ and the unknown is the pair $(\lambda, v) \in \mathbf{R} \times \text{BUC}(\mathbf{R}^N)$. It is well-known (see [12, 9, 10]) that, although the function v is not determined uniquely by the cell problem, nonetheless there is a unique value $\lambda = \lambda(x, p)$ of the real number λ for which (CP) has a solution.

The effective Hamiltonian $\bar{H} : \mathbf{R}^{2N} \to \mathbf{R}$ is then defined by

$$\text{(2.6)} \qquad \bar{H}(x, p) = \lambda.$$

It is worth to observe here that $\bar{H}$ enjoys structural properties similar to those assumed on H; these insure, in particular, the comparison property between sub and supersolutions and the uniqueness of a bounded viscosity solutions to equation ($\overline{\text{HJ}}$). A standard way of constructing a solution of (CP) is to introduce, for $\gamma > 0$, the auxiliary equation

$$\text{(ACP)} \qquad \gamma v^\gamma(\xi) + H(x, \xi, p + Dv^\gamma(\xi)) = 0, \qquad \xi \in \mathbf{R}^N,$$

where (x, p) plays again the role of a parameter. It can be proved indeed (see [12]) that, under our assumptions, the limit as $\gamma \to 0^+$ of $-\gamma v^\gamma(\xi)$ does not depend on ξ and that a solution (λ, v) of problem (CP) is given by

$$\lambda = \lim_{\gamma \to 0+} -\gamma v^\gamma(\xi), \qquad v(\xi) = \lim_{\gamma \to 0^+} \left(v^\gamma(\xi) - \inf_{\mathbf{R}^N} v^\gamma \right).$$

We denote by $v^\gamma(\xi; x, p)$ the unique periodic solution of (ACP) in $\mathrm{BUC}(\mathbf{R}^N)$ and we establish next some estimates on v^γ that will be useful later on.

Lemma 2.3 *Assume that H satisfies* (H1). *Then there exists a constant C_4 such that the following estimates hold: for any $(\xi, x, p) \in \mathbf{R}^{3N}$,*

(a) $\quad - \sup_{\xi \in \mathbf{R}^N} H(x, \xi, p) \le \gamma v^\gamma(\xi; x, p) \le - \inf_{\xi \in \mathbf{R}^N} H(x, \xi, p),$

(b) $\quad |D_\xi v^\gamma(\xi; x, p)| \le C_4(1 + |p|),$

(c) $\quad \gamma |Dv^\gamma(\xi; x, p)| \le C_4,$

(d) $\quad |\gamma v^\gamma(\xi; x, p) + \bar{H}(x, p)| \le \gamma C_4(1 + |p|),$

(e) $\quad \|D\bar{H}\|_\infty \le C_4.$

Proof. The assertion (a) is an immediate consequence of the comparison property for equation (ACP) since, for each $\gamma > 0$, the constants $-\gamma^{-1} \inf_{\xi \in \mathbf{R}^N} H(x, \xi, p)$ and $-\gamma^{-1} \sup_{\xi \in \mathbf{R}^N} H(x, \xi, p)$ are, respectively, a supersolution and a subsolution of (ACP).

From (a) and (2.3) it follows that

$$\gamma \|v^\gamma\|_\infty \le \nu |p| + C_3$$

and, consequently, that $v(\xi) := v^\gamma(\xi; x, p)$ satisfies

$$-\nu|p| - C_3 + H(x, \xi, p + Dv(\xi)) \le 0, \qquad \xi \in \mathbf{R}^N. \tag{2.7}$$

On the other hand, by virtue of (2.3) we have

$$H(x, \xi, p + q) > \nu|p| + C_3$$

for all $x, \xi, p, q \in \mathbf{R}^N$ with $|q| > 2(|p| + C_3/\nu)$. This and (2.7) imply that v satisfies

$$|Dv(\xi)| \le 2\left(|p| + \frac{C_3}{\nu}\right), \qquad \xi \in \mathbf{R}^N, \tag{2.8}$$

for each $x, p \in \mathbf{R}^N$. This proves statement (b).

In order to prove (c), fix $x, p, h, k, l \in \mathbf{R}^N$ and observe that (2.4) yields

$$|H(x + h, \xi + k, p + q + l) - H(x, \xi, p + q)| \le C_3(|h| + |k| + |l|)$$

for all $\xi, q \in \mathbf{R}^N$. It is easy to check that $w(\xi) := v^\gamma(\xi + k; x + h, p + l)$ satisfies the inequality

$$\gamma w(\xi) + H(x, \xi, p + Dw(\xi)) \le C_3(|h| + |k| + |l|), \qquad \xi \in \mathbf{R}^N,$$

in the viscosity sense. By comparing the above with equation (ACP) we conclude that

$$\gamma w(\xi) \le \gamma v^\gamma(\xi; x, p) + C_3(|h| + |k| + |l|), \qquad \xi \in \mathbf{R}^N.$$

A similar argument shows that

$$\gamma w(\xi) \ge \gamma v^\gamma(\xi; x, p) - C_3(|h| + |k| + |l|);$$

it follows then that

$$\gamma|v^\gamma(\xi + k; x + h, p + l) - v^\gamma(\xi; x, p)| \le C_3(|h| + |k| + |l|), \qquad \xi \in \mathbf{R}^N,$$

which proves (c) .

In order to prove statement (d), let us fix $(x, p) \in \mathbf{R}^{2N}$ and set

$$v(\xi) = v^\gamma(\xi; x, p), \qquad M(p) = 2(|p| + C_3/\nu).$$

As a consequence of (2.8), for $\xi, \eta \in [0,1)^N$ we have

$$|v(\xi) - v(\eta)| \leq M(p)|\xi - \eta| \leq M(p)\sqrt{N}. \tag{2.9}$$

We claim next that

$$\mu := \gamma \sup_{\mathbf{R}^N} v \geq -\bar{H}(x,p).$$

To prove this claim, note first that w is a supersolution of

$$\mu + H(x, \xi, p + Dv(\xi)) = 0, \qquad \xi \in \mathbf{R}^N. \tag{2.10}$$

Were the claim false, then by comparison between equations (2.10) and (CP) we would conclude that $v \geq w$ on $\mathbf{R}^N$ for any solution $w \in$ BUC($\mathbf{R}^N$) of (CP). This leads to a contradiction, since if w is a solution of (CP) then so is $w + C$, for any $C \in \mathbf{R}$. Thus, by (2.9) and using the $\mathbf{Z}^N$-periodicity of v, we infer that

$$\gamma v(\xi) \geq \gamma \sup_{\mathbf{R}^N} v - \gamma M(p)\sqrt{N} \geq -\bar{H}(x,p) - \gamma M(p)\sqrt{N},$$

for all $\xi \in \mathbf{R}^N$. A similar consideration shows that

$$\gamma v(\xi) \leq -\bar{H}(x,p) + \gamma M(p)\sqrt{N}, \quad \forall \xi \in \mathbf{R}^N.$$

Therefore we have

$$|\gamma v^\gamma(\xi; x, p) + \bar{H}(x,p)| \leq 2\gamma\left(|p| + \frac{C_3}{\nu}\right)\sqrt{N}, \qquad \xi, x, p \in \mathbf{R}^N,$$

and (d) is proved.

In order to prove statement (e) it is enough to observe that (c) and (d) immediately yield

$$\bar{H}(x,p) - \bar{H}(y,q) \leq \gamma \|Dv^\gamma\|_\infty(|p-q| + |x-y|) \leq C_4(|p-q| + |x-y|).$$

This completes the proof. □

The final result of this section is about the subdifferential of the sum of two Lipschitz functions. Let us recall that the subdifferential of a continuous function φ at a point x is the set

$$D^-\varphi(x) = \left\{ q \in \mathbf{R}^N : \liminf_{y \to x} \frac{\varphi(y) - \varphi(x) - q \cdot (y - x)}{|x - y|} \geq 0 \right\}.$$

Lemma 2.4 *Let u and v be locally Lipschitz continuous functions on $\mathbf{R}^N$. If $0 \in \overline{D}^-(u+v)(x)$ at some point x then there exists $q \in \mathbf{R}^N$ such that*

$$q \in \overline{D}^- u(x) \quad \textit{and} \quad -q \in \overline{D}^- v(x).$$

Proof. We may assume that $x = 0$ and that $u + v$ has a strict minimum at 0 and we suppose first that $0 \in D^-(u+v)(x)$. Consider then, for $\alpha > 0$, the function

$$\Phi(x,y) = u(x) + v(y) + \frac{\alpha}{2}|x-y|^2$$

and let (x_α, y_α) be a minimum point for Φ on $B(0,1) \times B(0,1)$. It is easily seen that

$$x_\alpha \to 0, \qquad y_\alpha \to 0 \quad \text{as} \quad \alpha \to +\infty$$

and we can therefore assume that $|x_\alpha| < 1$ and $|y_\alpha| < 1$ for large enough α. By elementary subdifferential calculus (see [4]) , then

$$0 \in D_x^- \Phi(x_\alpha, y_\alpha) \bigcap D_y^- \Phi(x_\alpha, y_\alpha)$$

so that

$$\alpha(y_\alpha - x_\alpha) \in D^- u(x_\alpha), \qquad \alpha(x_\alpha - y_\alpha) \in D^- v(y_\alpha).$$

The minimality of (x_α, y_α) yields

$$\frac{\alpha}{2}|x-y|^2 \leq u(y_\alpha) - u(x_\alpha)$$

which implies, since u is locally Lipschitz, that for some constant C,

$$\alpha|x_\alpha - y_\alpha| \leq C.$$

Hence, there exists $q \in \mathbf{R}^N$ such that, for some sequence $\alpha_j \to +\infty$,

$$\alpha_j(y_{\alpha_j} - x_{\alpha_j}) \to q.$$

It is now easy to conclude by continuity that

$$q \in \overline{D}^- u(0), \qquad -q \in \overline{D}^- v(0). \quad \square$$

3 Estimates on the rate of convergence

This section is entirely devoted to the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let u^ϵ, u and $v^\gamma \equiv v^\gamma(\cdot; x, p)$ be, respectively, the solutions of equations

$$\text{(HJ)}^\epsilon \qquad u^\epsilon(x) + H\left(x, \frac{x}{\epsilon}, Du^\epsilon(x)\right) = 0, \qquad x \in \mathbf{R}^N,$$

$$(\overline{\text{HJ}}) \qquad u(x) + \bar{H}(x, Du(x)) = 0, \qquad x \in \mathbf{R}^N,$$

$$\text{(ACP)} \qquad \gamma v^\gamma(\xi) + H(x, \xi, p + Dv^\gamma(\xi)) = 0, \qquad \xi \in \mathbf{R}^N \ .$$

For ϵ, δ and β in $(0,1)$ we consider the auxiliary function

$$\Phi(x, y) = u^\epsilon(x) - u(y) - \epsilon v^\gamma\left(\frac{x}{\epsilon}; x, \frac{x-y}{\epsilon^\beta}\right) - \frac{|x-y|^2}{2\epsilon^\beta} - \frac{\delta}{2}|y|^2$$

on $\mathbf{R}^{2N}$, where $\gamma = \epsilon^\theta$ with some $\theta > 0$ which will be fixed later on. In view of Lemma 2.1, Lemma 2.3 (a) and (2.3) we have

$$\Phi(x, y) \le 2C_2 + \frac{\epsilon}{\gamma}\left(\frac{\nu|x-y|}{\epsilon^\beta} + C_3\right) - \frac{|x-y|^2}{2\epsilon^\beta} - \frac{\delta}{2}|y|^2 \ ,$$

for all $(x, y) \in \mathbf{R}^{2N}$. Hence, Φ attains a global maximum at some point $(\hat{x}, \hat{y}) \in \mathbf{R}^{2N}$ depending, of course, on the various parameters appearing in the definition of Φ. We claim now that if

$$(3.1) \qquad 0 < \theta < 1 - \beta,$$

then there exist constants L and M such that

$$(3.2) \qquad |\hat{y}| \le \frac{L}{\delta^{1/2}} \ , \qquad \frac{|\hat{x} - \hat{y}|}{\epsilon^\beta} \le M,$$

for every $\delta \in (0, \frac{1}{2})$ and $\epsilon, \beta \in (0, 1)$.

Indeed, the inequality $\Phi(\hat{x},\hat{y}) \geq \Phi(0,0)$ together with Lemma 2.1, Lemma 2.3 (a) and (2.3) yields

$$\frac{\delta}{2}|\hat{y}|^2 + \frac{|\hat{x}-\hat{y}|^2}{2\epsilon^\beta} \leq 4C_2 + 2\frac{\epsilon}{\gamma}\left(\nu\frac{|\hat{x}-\hat{y}|}{\epsilon^\beta} + C_3\right).$$

Therefore, using Young's inequality, we obtain

$$\frac{\delta}{2}|\hat{y}|^2 \leq 4C_2 + 2\nu^2\epsilon^{2-\beta-2\theta} + 2C_3\epsilon^{1-\theta}\ . \tag{3.3}$$

For $0 < \theta < 1 - \frac{\beta}{2}$, the above inequality yields

$$\frac{\delta}{2}|\hat{y}|^2 \leq 4C_2 + 2\nu^2 + 2C_3,$$

for all $\epsilon \in (0,1)$, and the first estimate in (3.2) is proved.

In order to complete the proof of (3.2), we observe first that the inequality $\Phi(\hat{x},\hat{y}) \geq \Phi(\hat{x},\hat{x})$ gives

$$\begin{aligned} \frac{|\hat{x}-\hat{y}|^2}{2\epsilon^\beta} \ \leq \ & u(\hat{x}) - u(\hat{y}) + \epsilon v^\gamma\left(\frac{\hat{x}}{\epsilon};\hat{x},0\right) - \epsilon v^\gamma\left(\frac{\hat{x}}{\epsilon};\hat{x},\frac{\hat{x}-\hat{y}}{\epsilon^\beta}\right) \\ & + \frac{\delta}{2}(|\hat{x}|^2 - |\hat{y}|^2). \end{aligned} \tag{3.4}$$

Observe also that Lemma 2.1 implies a Lipschitz estimate for u, namely

$$|u(\hat{x}) - u(\hat{y})| \leq C_2|\hat{x}-\hat{y}|,$$

and that, on the other hand, Lemma 2.3 (c) gives

$$\epsilon v^\gamma\left(\frac{\hat{x}}{\epsilon};\hat{x},0\right) - \epsilon v^\gamma\left(\frac{\hat{x}}{\epsilon};\hat{x},\frac{\hat{x}-\hat{y}}{\epsilon^\beta}\right) \leq C_4\frac{\epsilon}{\gamma}\frac{|\hat{x}-\hat{y}|}{\epsilon^\beta}\ .$$

Therefore (3.4) implies

$$\frac{|\hat{x}-\hat{y}|}{\epsilon^\beta} \leq 2C_2 + 2\delta|\hat{y}| + \delta|\hat{x}-\hat{y}| + 2C_4\epsilon^{1-\theta-\beta}$$

from which, taking the first estimate in (3.2) into account, we obtain

$$(1-\delta\epsilon^\beta)\frac{|\hat{x}-\hat{y}|}{\epsilon^\beta} \leq 2C_2 + 2L\delta^{1/2} + 2C_4\epsilon^{1-\theta-\beta}. \tag{3.5}$$

Choosing now $\delta \in (0, \frac{1}{2})$, so that $1 - \delta\epsilon^\beta > \frac{1}{2}$ for all $\epsilon \in (0,1)$, from inequality (3.5) we get

$$\frac{|\hat{x} - \hat{y}|}{\epsilon^\beta} \leq 4\left(C_2 + L + C_4\epsilon^{1-\theta-\beta}\right).$$

The right-hand side of the above is bounded by $4(C_2 + L + C_4)$ for $\delta \in (0, \frac{1}{2})$ and $\epsilon \in (0,1)$, provided θ satisfies (3.1). This completes the proof of estimate (3.2).

Henceforth we assume that (3.1) is satisfied. We then claim that there is a constant $C_5 > 0$ independent of ϵ, δ, β, and θ for which

$$u^\epsilon(\hat{x}) + \bar{H}\left(\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon^\beta}\right) \leq C_5\left(\epsilon^\theta + \epsilon^{1-\theta-\beta}\right). \tag{3.6}$$

To see this, we first note that we may assume, by adding to u^ϵ a smooth function vanishing together with its first derivatives at $\hat{x}$, that the function

$$x \mapsto u^\epsilon(x) - \epsilon v^\gamma\left(\frac{x}{\epsilon}; x, \frac{x - \hat{y}}{\epsilon^\beta}\right) - \frac{|x - \hat{y}|^2}{2\epsilon^\beta}$$

has a strict maximum at $\hat{x}$. Consider next, for $\alpha > 0$, the function

$$\Psi(x, y, z) = u^\epsilon(x) - \epsilon v^\gamma\left(y; z, \frac{z - \hat{y}}{\epsilon^\beta}\right) - \frac{|x - \hat{y}|^2}{2\epsilon^\beta} - \frac{\alpha}{2}\left(|\epsilon y - x|^2 + |z - x|^2\right)$$

and let $(x_\alpha, y_\alpha, z_\alpha)$ be a maximum point of Ψ on $E = B(\hat{x}, 1) \times B(\frac{\hat{x}}{\epsilon}, 1) \times B(\hat{x}, 1)$. The function

$$x \mapsto u^\epsilon(x) - \left[\frac{|x - \hat{y}|^2}{2\epsilon^\beta} + \frac{\alpha}{2}\left(|\epsilon y_\alpha - x|^2 + |z_\alpha - x|^2\right)\right]$$

has a maximum at x_α and, on the other hand, the function

$$y \mapsto v^\gamma\left(y; z_\alpha, \frac{z_\alpha - \hat{y}}{\epsilon^\beta}\right) + \frac{\alpha}{2\epsilon}|\epsilon y - x_\alpha|^2$$

has a minimum at y_α. Since, as $\alpha \to +\infty$,

$$x_\alpha \to \hat{x}, \quad y_\alpha \to \frac{\hat{x}}{\epsilon}, \quad z_\alpha \to \hat{x},$$

we may assume that for large enough α

$$|x_\alpha - \hat{x}| + \left| y_\alpha - \frac{\hat{x}}{\epsilon} \right| + |z_\alpha - \hat{x}| < 1.$$

Since u^ϵ and v^γ are, respectively, viscosity solutions of equations $(\mathrm{HJ})^\epsilon$ and (ACP) , we obtain

$$u^\epsilon(x_\alpha) + H\left(x_\alpha, \frac{x_\alpha}{\epsilon}, \frac{x_\alpha - \hat{y}}{\epsilon^\beta} + \alpha(x_\alpha - \epsilon y_\alpha) + \alpha(x_\alpha - z_\alpha)\right) \leq 0, \tag{3.7}$$

$$\gamma v^\gamma\left(y_\alpha; z_\alpha, \frac{z_\alpha - \hat{y}}{\epsilon^\beta}\right) + H\left(z_\alpha, y_\alpha, \frac{z_\alpha - \hat{y}}{\epsilon^\beta} + \alpha(x_\alpha - \epsilon y_\alpha)\right) \geq 0. \tag{3.8}$$

From the inequality $\Psi(x_\alpha, y_\alpha, z_\alpha) \geq \Psi(x_\alpha, y_\alpha, x_\alpha)$ it follows that

$$\frac{\alpha}{2}|z_\alpha - x_\alpha|^2 \leq \frac{\epsilon}{\gamma}\left[\gamma v^\gamma\left(y_\alpha; x_\alpha, \frac{x_\alpha - \hat{y}}{\epsilon^\beta}\right) - \gamma v^\gamma\left(y_\alpha; z_\alpha, \frac{z_\alpha - \hat{y}}{\epsilon^\beta}\right)\right].$$

Therefore, using the estimate (c) in Lemma 2.3, we deduce that

$$\alpha|z_\alpha - x_\alpha| \leq 2C_4\frac{\epsilon}{\gamma}\left(1 + \frac{1}{\epsilon^\beta}\right) \leq 4C_4\epsilon^{1-\theta-\beta}. \tag{3.9}$$

Estimate (d) in Lemma 2.3 and (3.8) yield

$$0 \leq \gamma C_4\left(|\hat{p}_\alpha| + 1\right) - \bar{H}\left(z_\alpha, \hat{p}_\alpha\right) + H\left(z_\alpha, y_\alpha, \hat{p}_\alpha + \alpha(x_\alpha - \epsilon y_\alpha)\right), \tag{3.10}$$

where

$$\hat{p}_\alpha = \frac{(z_\alpha - \hat{y})}{\epsilon^\beta}.$$

Thanks to (2.4), the above and inequality (3.7) imply

$$u^\epsilon(x_\alpha) + H\left(z_\alpha, \frac{x_\alpha}{\epsilon}, \frac{x_\alpha - \hat{y}}{\epsilon^\beta} + \alpha(x_\alpha - \epsilon y_\alpha)\right) \leq C_3(1+\alpha)|z_\alpha - x_\alpha|.$$

Hence, (3.9) yields

$$u^\epsilon(x_\alpha) + H\left(z_\alpha, \frac{x_\alpha}{\epsilon}, \frac{x_\alpha - \hat{y}}{\epsilon^\beta} + \alpha(x_\alpha - \epsilon y_\alpha)\right) \leq 4C_3C_4\left(1 + \frac{1}{\alpha}\right)\epsilon^{1-\theta-\beta}. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain then

$$\begin{aligned} 0 \ \leq \ & \gamma C_4 \left(\frac{|z_\alpha - \hat{y}|}{\epsilon^\beta} + 1 \right) - \bar{H} \left(z_\alpha, \frac{z_\alpha - \hat{y}}{\epsilon^\beta} \right) \\ & + 4C_3C_4 \left(1 + \frac{1}{\alpha} \right) \epsilon^{1-\theta-\beta} - u^\epsilon(x_\alpha). \end{aligned}$$

Sending now $\alpha \to +\infty$ and taking estimate (3.2) into account we obtain

$$u^\epsilon(\hat{x}) + \bar{H} \left(\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon^\beta} \right) \leq C_4(1 + M)\epsilon^\theta + 4C_3C_4\epsilon^{1-\theta-\beta},$$

and (3.6) is proved.

The next step of the proof is to show that for some constant $C_6 > 0$ independent of ϵ, β, δ and θ,

$$u(\hat{y}) + \bar{H} \left(\hat{x}, \frac{\hat{x} - \hat{y}}{\epsilon^\beta} \right) + C \left(\epsilon^\beta + \delta^{\frac{1}{2}} + \epsilon^{1-\theta-\beta} \right) \geq 0. \tag{3.12}$$

To see this, note that the Lipschitz continuous function

$$y \mapsto \phi(y) := u(y) + \frac{|\hat{x} - y|^2}{2\epsilon^\beta} + \epsilon v^\gamma \left(\frac{\hat{x}}{\epsilon}; \hat{x}, \frac{\hat{x} - y}{\epsilon^\beta} \right) + \frac{\delta}{2}|y|^2$$

has a minimum at $\hat{y}$ and, consequently, $0 \in D^-\phi(\hat{y})$. If we set

$$w(y) = \epsilon v^\gamma \left(\frac{\hat{x}}{\epsilon}; \hat{x}, \frac{\hat{x} - y}{\epsilon^\beta} \right) ; \qquad \tilde{u}(y) = u(y) + \frac{|\hat{x} - y|^2}{2\epsilon^\beta} + \frac{\delta}{2}|y|^2 ,$$

we see, in view of Lemma 2.4, that there is a point $\hat{q} \in \mathbf{R}^N$ such that

$$\hat{q} \in \overline{D}^- w(\hat{y}) , \quad -\hat{q} \in \overline{D}^- \tilde{u}(\hat{y}) = \overline{D}^- u(\hat{y}) + \frac{\hat{y} - \hat{x}}{\epsilon^\beta} + \delta\hat{y} . \tag{3.13}$$

Since u is a supersolution of equation ($\overline{\text{HJ}}$), by continuity we obtain

$$u(\hat{y}) + \bar{H} \left(\hat{y}, \frac{\hat{x} - \hat{y}}{\epsilon^\beta} - \delta\hat{y} - \hat{q} \right) \geq 0 .$$

Then, by Lemma 2.3 (e) and the estimates (3.2),

$$u(\hat{y}) + \bar{H}\left(\hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon^\beta}\right) \geq -C_4\left(M\epsilon^\beta + L\delta^{\frac{1}{2}} + |\hat{q}|\right). \tag{3.14}$$

It remains to estimate $|\hat{q}|$. Note for this purpose that

$$\left|\epsilon v^\gamma\left(\frac{\hat{x}}{\epsilon}; \hat{x}, \frac{\hat{x}-y}{\epsilon^\beta}\right) - \epsilon v^\gamma\left(\frac{\hat{x}}{\epsilon}; \hat{x}, \frac{\hat{x}-z}{\epsilon^\beta}\right)\right| \leq \frac{\epsilon}{\gamma}\|\gamma D v^\gamma\|_\infty \left|\frac{z-y}{\epsilon^\beta}\right|$$

and therefore, by Lemma 2.3 (c) and (3.13), we find

$$|\hat{q}| \leq C_4\frac{\epsilon}{\gamma}\epsilon^{-\beta} = C_4\epsilon^{1-\theta-\beta}.$$

From the above and (3.14), the estimate (3.12) follows.

We choose now

$$\delta^{1/2} \leq \epsilon^\theta, \qquad 0 < \delta < \frac{1}{2}.$$

The inequalities (3.6) and (3.12) yield, for some constant C_7,

$$u^\epsilon(\hat{x}) - u(\hat{y}) \leq C_7 E(\epsilon), \tag{3.15}$$

where

$$E(\epsilon) = \epsilon^\theta + \epsilon^\beta + \epsilon^{1-\beta-\theta}. \tag{3.16}$$

Note that, by the choice of $(\hat{x}, \hat{y})$,

$$u^\epsilon(x) - u(x) \leq \Phi(\hat{x}, \hat{y}) + \epsilon v^\gamma\left(\frac{x}{\epsilon}; x, 0\right) + \frac{\delta}{2}|x|^2 \tag{3.17}$$

for all $x \in \mathbf{R}^N$. Now, taking Lemma 2.3 (a) and (3.15) into account, we obtain

$$\begin{aligned} u^\epsilon(x) - u(x) &\leq C_7 E(\epsilon) + \epsilon\left(v^\gamma\left(\frac{x}{\epsilon}; x, 0\right) - v^\gamma\left(\frac{\hat{x}}{\epsilon}; \hat{x}, \frac{\hat{x}-\hat{y}}{\epsilon^\beta}\right)\right) + \frac{\delta}{2}|x|^2 \\ &\leq C_7 E(\epsilon) + \frac{\epsilon}{\gamma}\left(\sup_{\xi\in\mathbf{R}^N} H\left(\hat{x}, \xi, \frac{\hat{x}-\hat{y}}{\epsilon^\beta}\right) - \inf_{\xi\in\mathbf{R}^N} H\left(\frac{x}{\epsilon}, \xi, 0\right)\right) + \frac{\delta}{2}|x|^2 \\ &\leq C_7 E(\epsilon) + \epsilon^{1-\theta}\left(\nu M + 2C_3\right) + \frac{\delta}{2}|x|^2 \end{aligned}$$

for all $x \in \mathbf{R}^N$. Hence, sending $\delta \to 0^+$, we see that for all $x \in \mathbf{R}^N$,

$$u^\epsilon(x) - u(x) \le (C_7 + \nu M + 2C_3)E(\epsilon).$$

By symmetry, the optimal choice of the parameters is

$$\theta = \beta = \frac{1}{3}.$$

Therefore we proved that for some constant $C > 0$,

$$\sup_{x \in \mathbf{R}^N} (u^\epsilon(x) - u(x)) \le C\epsilon^{1/3}.$$

Reversing the roles of u and u^ϵ, the opposite inequality is established by similar arguments and the proof of the theorem is complete. □

Proof of Theorem 1.2. The stategy for the proof is quite different from and much simpler than the above.

Note first that the effective Hamiltonian $\bar{H}(p) \equiv \bar{H}(x, p)$ is independent of x and equation ($\overline{\text{HJ}}$) in this case becomes

$$u(x) + \bar{H}(Du(x)) = 0, \quad x \in \mathbf{R}^N,$$

and that its solution is then $u(x) \equiv -\bar{H}(0)$. Let $v \in \text{BUC}(\mathbf{R}^N)$ be the solution of the cell problem

$$H(\xi, Dv(\xi)) = \bar{H}(0), \quad \xi \in \mathbf{R}^N,$$

where we have writen $H(\xi, p)$ for $H(x, \xi, p)$.

Define $w^\epsilon \in \text{BUC}(\mathbf{R}^N)$ by setting

$$w^\epsilon(x) = u(x) + \epsilon v\left(\frac{x}{\epsilon}\right).$$

It is then easy to check that w^ϵ is a viscosity solution of

$$w^\epsilon(x) + H\left(\frac{x}{\epsilon}, Dw^\epsilon(x)\right) = \epsilon v\left(\frac{x}{\epsilon}\right), \quad x \in \mathbf{R}^N.$$

Now, setting $M := \|v\|_\infty$, we see that the functions $w^\epsilon + \epsilon M$ and $w^\epsilon - \epsilon M$ are respectively a supersolution and a subsolution of $(\text{HJ})^\epsilon$. Hence, by comparison, we get

$$u^\epsilon(x) \le w^\epsilon(x) + \epsilon M; \qquad u^\epsilon(x) \ge w^\epsilon(x) - \epsilon M$$

for all $x \in \mathbf{R}^N$. Thus we obtain

$$\sup_{x \in \mathbf{R}^N} |u^\epsilon(x) - u(x)| \leq 2\epsilon M,$$

which conclude the proof. $\square$

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