Area preserving curve shortening flows: from phase transitions to image processing

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Abstract

Some known models in phase separation theory (Hele-Shaw, nonlocal mean curvature motion) and their approximated phase field models (Cahn–Hilliard, nonlocal Allen-Cahn) are used to generate planar curve evolution without shrinkage, with application to shape recovery. This turns out to be a level set approach to an area preserving geometric flow, in the spirit of Sapiro and Tannenbaum [36]. We discuss the theoretical validation of this method, together with the results of some numerical experiments.

1 Introduction

Mathematical models based on planar curve evolution have recently been studied in Computer Vision with application to shape recovery and analysis as well as to image segmentation [21, 22, 23, 35, 36]. The curves represent the contours of objects in a grey level image and the idea is to use a geometric flow to reduce a given initial set of curves to a form which is more manageable for pattern recognition and interpretation. The curve evolution has to be designed in order to yield both noise suppression in the initial data and a representation of the object contours across different scales. The time variable in the evolution corresponds to scale in the representation.

Let $C(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ denote a family of simple closed curves, where the variable p parametrizes the curves and t is *time* or *scale*. A type of evolution that has been thoroughly studied both in the mathematical literature [16, 18, 19, 20] and in Computer Vision [22, 23, 35] is the *Euclidean curve shortening flow* (also called the *geometric heat flow*) :

$$\frac{\partial C}{\partial t} = \kappa \mathcal{N} \ , \tag{1.1}$$

where κ and $\mathcal{N}$ are, respectively, the Euclidean curvature and the inward unit normal of C. Equation (1.1) defines an Euclidean curve shortening flow in the sense that the Euclidean perimeter shrinks as quickly as possible when the curve evolves according to (1.1).

The geometric heat flow yields an Euclidean invariant multi–scale representation of closed curves which satisfies many useful properties (recursivity, locality, grey-scale invariance, conservation of inclusions, ...) for shape recovery and analysis of images [23, 27, 3]. After a small time the evolving curve still mantains fine details of the object contour. The multi–scale representation is then obtained by means of the removal of small-scale features from the contours as time increases. Furthermore, the curve shortening effect yields curve smoothing so that the noise present in the initial data is suppressed.

A major drawback in the application of (1.1) to image processing is the well-known fact [18, 20] that a planar embedded curve which evolves according to the geometric heat flow (1.1) becomes more and more circular as t increases and eventually shrinks to a point in finite time. Such a shrinking effect is indeed an undesirable feature in image processing applications: since the geometric heat flow is the fastest way to shrink the length of a curve [17], the evolving contour will diverge rapidly from the desired shape.

In [17] , Gage proposed two possible corrections to the geometric heat flow in order to avoid the shrinking effect. The first flow is defined by subtracting the component of the length gradient which lies parallel to the area gradient. The evolution is defined then by :

$$\frac{\partial C}{\partial t} = \left(\kappa - \frac{2\pi}{\mathcal{L}}\right)\mathcal{N} \; . \tag{1.2}$$

Alternatively, he proposed to magnify the plane by a homothety simultaneously with the evolution, that is :

$$\frac{\partial C}{\partial t} = \left(\kappa - \frac{\pi\rho}{\mathcal{A}}\right)\mathcal{N} + \alpha\mathcal{T} \; , \tag{1.3}$$

where ρ is the support function of the curve ($\rho = - < C, \mathcal{N} >$) and $\alpha\mathcal{T}$ is a tangential component which does not change the shape. The resulting flows are curve shortening motions which preserve the area of the region enclosed by the curve.

The area-preserving geometric flow (1.3) was applied by Sapiro and Tannenbaum [36] to the problem of shape recovery in image processing in order to overcome the above mentioned undesirable effects of the geometric heat flow. In the study of these flows it is convenient, for theoretical as well as numerical reasons, to adopt the so called *level set approach* , that is to embed the original curve flows in the level set evolution of an evolving surface [24, 31, 11, 15]. If the surface is denoted by $z = \phi(x,t)$, $x \in \mathbf{R}^2$ then the evolving curve is represented at all times by its zero level set $\Gamma_t = \{x \in \mathbf{R}^2 : \; \phi(x,t) = 0\}$. In the level set method, if the surface evolves according to the parabolic partial differential equation

$$\frac{\partial \phi}{\partial t} = |\nabla\phi| \mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) \; , \tag{1.4}$$

(the *mean curvature motion equation*) it is well-known that the sets Γ_t evolve according to (1.1) , no matter how we choose the surface $\phi(x,0)$ detecting the initial set Γ_0. This formulation of (1.1) as a partial differential equation allows for singularities to form in the evolution, and the generalized solution of (1.4) can be found in the framework of viscosity solutions theory [11, 15]. An important reason for which the level set representation of curve evolution was introduced in computer vision [24] is that it automatically handles changes in the contour topology during the evolution. Several simultaneously evolving curves can easily split and merge so that an unknown number of object boundaries can be detected in the image. In this case, a suitable attraction term is introduced in equation (1.4) in order to drive the curves towards the boundaries of the objects of interest in the image [8, 24].

The motion by mean curvature defined by (1.4) arises as an asymptotic limit of phase field equations. More precisely, let us consider the Allen–Cahn equation of the phase transition theory [2]

$$\frac{\partial u}{\partial t} = \Delta u - \frac{1}{\varepsilon^2}\omega'(u) \ , \tag{1.5}$$

where $\varepsilon > 0$, u is an order parameter, and ω is a double-well potential, for instance $\omega(u) = \frac{1}{2}(1-u^2)^2$, the minima of which correspond to two different phases. It may be shown [5, 12, 14] that, as $\varepsilon \to 0$, u tends to either one of these minima, in two regions separated by a sharp interface which moves by its mean curvature. Then, in two dimensions, as $\varepsilon \to 0$, the interface corresponds to the zero level set of the solution u and it consists of a set of curves which evolves according to equation (1.1).

The numerical solution of the Allen–Cahn equation can also be considered as a tool for the effective computation of the geometric heat flow which handles changes of topology during the evolution [28]. This approach has been applied to the problem of shape recovery by Nochetto and Verdi [29].

In this paper (see Section 2) we consider some models of phase separation which generate an area–preserving evolution of the interfaces and we suggest the application of such models to the problem of shape recovery in image processing. First we consider the Hele–Shaw (or Mullins-Sekerka) model. In the Hele–Shaw free boundary problem (see [9]) , a function μ is harmonic on both sides of an interface while two boundary conditions are required on the interface: the jump of $\nabla\mu$ in the normal direction is proportional to the normal velocity of the interface, and the boundary value of μ is proportional to the mean curvature. If the interface is a planar closed curve the problem then consists in finding the motion of such a curve.

It turns out that this motion is a curve shortening motion which does not change the area of the region enclosed by the curve. Hence the resulting flow can be used as an alternative approach in order to obtain a model of area-preserving curve evolution without shrinkage for shape recovery purposes.

Another model is given by the non–local mean curvature flow where the interface Γ evolves with normal velocity given by

$$V = \kappa - \frac{1}{\mid \Gamma \mid} \int_{\Gamma} \kappa dS \ , \tag{1.6}$$

$\mid \Gamma \mid$ denoting the total area of the evolving front. In the case of a single curve, the motion defined by (1.6) reduces to the first area–preserving flow of Gage [17] mentioned before.

The direct numerical computation of the previous flows is not an easy task for image processing applications where, in general, several object contours have to be simultaneously detected and then the simultaneous tracking of an unknown number of evolving curves is required.
A phase field approach seems to present, as we suggest in Section 3, some interesting features in order to overcome these computational difficulties. More precisely, we propose the use of the Cahn–Hilliard equation and the nonlocal Allen–Cahn equation, respectively, as computational tools for the recovery of shapes from images.

Let us consider for example the Cahn–Hilliard equation of phase transitions theory:

$$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon} \Delta(\omega'(u) - \epsilon^2 \Delta u) \quad \text{ in } \Omega, \tag{1.7}$$

where $\varepsilon > 0$ is a small parameter and ω is a double–well potential as before. It has been shown formally by Pego [32], and then rigorously by Alikakos, Bates and Chen [1, 10] that, as $\varepsilon \to 0$, the zero level sets of the solution of equation (1.7) evolve according to the Hele–Shaw flow. Hence the solution of the Cahn–Hilliard equation yields asymptotically the desired area preserving curve shortening flow.

Similarly, for the nonlocal Allen–Cahn equation:

$$\frac{\partial u}{\partial t} = \Delta u - \frac{1}{\varepsilon^2} \omega'(u) + \frac{1}{\varepsilon^2 \mid \Omega \mid} \int_{\Omega} \omega'(u) dx \ , \tag{1.8}$$

it has been formally shown by Rubinstein and Sternberg [33], and rigorously proved by Bronsard and Stoth [6] in the case of radially simmetric solutions in a bounded spherically simmetric domain Ω that, as $\varepsilon \to 0$, the interfaces move according to the nonlocal mean curvature flow given by (1.6).

Therefore, the nonlocal flows obtained as asymptotic limits of the phase field equations (1.7), (1.8) should regularize the contours of an image without

pushing the evolving curves far away from the desired object boundaries, because the limit flows are both curve shortening and area preserving.
It is important to remark that the last property has to be understood in a global sense, so that small "blobs" due to the presence of noise in the image do not persist during the evolution because of the area conservation. On the contrary, a coarsening process associated to the phase field equations forces, in general, finer structures to disappear as time evolves [33]. Consequently, the related models for shape recovery and analysis show an interesting noise suppression property which consists in a shrinking effect on small blobs.

Since the area preserving flow (1.3) has some nice scale-space properties, which are known to be important in shape recovery (see [3]), in the present paper we shall also discuss the analogous properties shared by the flows connected with the phase field equations.

Finally, in Section 4 we apply to the problem of shape recovery some numerical methods developed for the phase field equations and we show the results of numerical experiments on both synthetic and real images which seem to validate this approach.

2 Shape recovery by phase separation models

In this section we review some known properties of area-preserving curve shortening flows and of phase separation models which yield such flows for the interface, and we point out their relevance for shape recovery. Let us observe preminarily that the geometric heat flow (1.1) is the gradient flow of the length functional on the space of smooth plane curves endowed with the Euclidean metric. If $C(p,t)$ is a one-parameter family of curves with length $\mathcal{L}(t)$, then (see [17])

$$\frac{d\mathcal{L}}{dt} = -\int < \frac{\partial C}{\partial t}, \kappa\mathcal{N} > ds \ ,$$

where s denotes the arclength. So the geometric heat flow is the fastest way to shrink the length of a curve. To keep the area enclosed by the curve constant while decreasing the length, Gage [17] proposed to subtract the component of the length gradient which lies parallel to the area gradient. Since the first variation of the area is [17]

$$\frac{d\mathcal{A}}{dt} = -\int < \frac{\partial C}{\partial t}, \mathcal{N} > ds \ ,$$

and, for simple closed curves,

$$\int < \kappa\mathcal{N}, \mathcal{N} > ds = \int \kappa ds = 2\pi \ ,$$

then Gage proposed the geometric evolution equation

$$\frac{\partial C}{\partial t} = \left(\kappa - \frac{2\pi}{\mathcal{L}}\right)\mathcal{N} \ , \tag{2.1}$$

as the gradient flow of the length functional along curves which enclose a fixed area. It is worth to point out that this non local flow lacks some important properties like scale invariance and conservation of inclusions; observe in this respect that if two curves are tangent at some point and the inner curve is shorter than the outer one, then at some later time the regions bounded by the evolved curves will certainly overlap.

An alternative method proposed by Gage in [17] is to use the geometric heat flow (1.1) while simultaneously magnifying the plane by a homothety which keeps constant the enclosed area. The corresponding evolution equation becomes

$$\frac{\partial C}{\partial t} = \left(\kappa - \frac{\pi\rho}{\mathcal{A}}\right)\mathcal{N} + \alpha\mathcal{T} \ , \tag{2.2}$$

where $\rho = - < C, \mathcal{N} >$) is the support function of the curve and $\alpha\mathcal{T}$ is a tangential component which does not affect the shape. The solution curves to (1.2) or (1.3) have the same geometric properties. In [17] Gage proved that a convex curve in the plane which evolves according to each one of these flows remains convex and converges to a circle in the C^∞ metric. Since the flow defined by (1.3) and the geometric heat flow are related by dilations, then (1.3) satifies all the requirements of a multi–scale shape analysis and has been applied by Sapiro and Tannenbaum [36] to shape recovery in order to overcome the undesirable shrinking effect of the geometric heat flow.

A conceptually completely different way to generate flows having similar geometric properties is, not surprisingly, via the motion of interfaces in phase separation modelswhere conservation of mass enters as a constitutive law.

Let us consider for example the Hele–Shaw model: let Ω be a bounded and simply connected domain in $\mathbf{R}^2$ and let Γ_{00} be a finite set of non-intersecting smooth closed curves in Ω and consider the free boundary problem of finding a function $\mu(x,t)$, $x \in \Omega$, $t \in [0,T]$, and a free boundary $\Gamma = \cup_{0\le t\le T}(\Gamma_t \times \{t\})$ satisfying

$$\begin{aligned} \Delta\mu(\cdot,t) &= 0 \qquad &&\text{in} \qquad &&\Omega \setminus \Gamma_t,\ t \in [0,T], \\ \frac{\partial\mu}{\partial n} &= 0 &&\text{on} &&\partial\Omega \times [0,T], \\ \mu &= \lambda\kappa &&\text{on} &&\Gamma_t,\ t \in [0,T], \\ \frac{1}{2}\left[\frac{\partial\mu}{\partial n}\right]_{\Gamma_t} &= V &&\text{on} &&\Gamma_t,\ t \in [0,T], \\ \Gamma_0 &= \Gamma_{00} &&\text{on} &&\{t=0\}, \end{aligned} \tag{2.3}$$

where λ is a positive constant, n is the unit outward normal to $\partial\Omega$ or to Γ_t, κ and V are, respectively, the curvature and the normal velocity of Γ_t (with the sign convention that the curvature of a circle and the normal velocity of an expanding curve are positive),

$$\left[\frac{\partial\mu}{\partial n}\right]_{\Gamma_t} = \frac{\partial}{\partial n}\mu^+ - \frac{\partial}{\partial n}\mu^- \ , \tag{2.4}$$

and μ^+ and μ^- are the restrictions of μ on Ω_t^+ and Ω_t^- (respectively, the exterior and interior of Γ_t in Ω). In the following we always assume that the curves in Γ_t do not collide during the evolution.

Chen [9] established the local existence of a solution of the Hele–Shaw problem for an arbitrary (smooth) initial curve, and global existence of a solution when the initial curve is close to a circle.

The following two features of the solution of (2.3) are relevant for the application to the problem of shape recovery. Denote by $\mathcal{A}(t)$ and $\mathcal{L}(t)$ the area of Ω_t^- and the length of Γ_t, respectively. By using the divergence theorem it follows that (see [9, 32])

$$\frac{d\mathcal{A}}{dt} = -\int_{\Gamma_t} V ds = -\frac{1}{2}\int_{\Gamma_t}\left[\frac{\partial\mu}{\partial n}\right] ds = \frac{1}{2}\int_{\partial\Omega}\frac{\partial\mu}{\partial n} ds - \frac{1}{2}\int_{\Omega\setminus\Gamma_t}\Delta\mu \ dx = 0 \ ,$$

$$\frac{d\mathcal{L}}{dt} = -\int_{\Gamma_t}\kappa V ds = -\frac{1}{2\lambda}\int_{\Gamma_t}\mu\left[\frac{\partial\mu}{\partial n}\right] ds = -\frac{1}{2\lambda}\int_{\Omega\setminus\Gamma_t} |\ \nabla\mu\ |^2 \ dx \le 0 \ ,$$

since $\lambda > 0$. Therefore the Hele–Shaw flow preserves the area of the region enclosed by Γ_t while decreasing its length.

A different flow with similar features in phase separation theory is the nonlocal mean curvature flow (1.6). In fact, $\frac{d\mathcal{A}}{dt} = 0$ by definition, while

$$\frac{d\mathcal{L}}{dt} = -\int_{\Gamma_t} \kappa V ds = -\int_{\Gamma_t} \kappa^2 ds + \frac{1}{\mid \Gamma_t \mid} \left(\int_{\Gamma_t} \kappa ds \right)^2 \leq 0 \ .$$

If the front reduces to a single closed curve in the plane, (1.6) gives back the Gage flow (1.2). In the general case, which occurs in image processing, different flows evolve at different speeds keeping the total area fixed. A gradual coarsening process takes place since the only possible equilibrium state should be that of one or more disks of the same size. The flow (1.6) shares with the classical mean curvature flow the property that a convex initial curve in the plane remains convex for all time, in this case converging to a circle of the same area as $t \to \infty$. A remarkable difference, however, is that now a nonconvex curve evoloving through (1.6) can develop singularities in finite time (see [33]).

An important issue in the application of these models to the shape recovery of an image, given its intensity function g, is of course the choice of the initial set of curves $C(p, 0)$. A possible one, suggested by Shah [37], is

$$C(p, 0) = \{x \in \Omega : \ \Delta((G_\sigma * g)(x)) = 0\} \ , \tag{2.5}$$

where G_σ is the standard 2D Gaussian function with extent σ, and $*$ denotes the convolution operator. The computation of the zero–crossings of the image convolved with the Laplacian of a Gaussian function was originally proposed by Marr and Hildreth [26] as a theory of edge detection. It may be shown [38] that the set of zero–crossings consists of a collection of closed curves.

We then suggest to use the curve evolution generated by the Hele–Shaw (or the nonlocal mean curvature) flow starting with an initial set of curves obtained from equation (2.5) with a small value of σ. Indeed, for small σ the initial set turns out to be close to the "true" contours of the image, even if still noisy [26]. The Hele–Shaw flow should then regularize the contours without pushing the evolving curves far away from the true contours because the flow is both curve shortening and area preserving.

One could wonder if small "blobs" due to the presence of noise in the image datum g could persist during the evolution because of area conservation. This event is not possible since a noise suppression property due to a shrinking effect on small blobs takes place. More precisely, suppose that the initial set of curves Γ_{00} contains a small circle with radius r. Since $\mu(x,0)$ is harmonic in $\Omega \setminus \Gamma_0$, then $\mu(x,0)$ is constant inside the circle. Hence on the boundary of the circle we have :

$$\mu(x,0) = \frac{\lambda}{r} , \qquad V = \frac{1}{2}\frac{\partial}{\partial n}\mu^+ \quad \text{at} \quad t = 0. \tag{2.6}$$

By the maximum principle for harmonic functions, $\mu(x,0)$ takes then its maximum value on $\Gamma_0 \cup \partial\Omega$. If we assume that $\mu(x,0)$ takes the maximum value λ/r on the boundary of the small circle, then from the maximum principle it follows easily that on the boundary of the circle we have

$$\frac{\partial}{\partial n}\mu^+ < 0 \ .$$

Hence (2.6) implies $V < 0$ so that the circle shrinks.

A similar property holds for the nonlocal curvature flow: this is a well known phenomenon in phase transition theory related to coarsening processes in which finer structures disappear as time evolves. There is however a remarkable difference between the two flows: in the Hele–Shaw model even very small spheres close to extinction have a great effect on the global solution μ (see [33, 6]). This should be of some relevance in the processing of very noisy images.

Summing up, the propagation law for free boundaries that arises in the two phase separation models considered here exhibits the following properties which are useful for shape recovery:

- curve smoothing and multi-scale representation by means of curve shortening: suppression of noise and removal of small-scale features as time (i.e. scale) increases;
- preservation of area: the evolving contour does not shrink, then it diverges less rapidly from the desired shape than the geometric heat flow (1.1);
- simultaneous evolution of all the contours in the image with suppression of small contours due to the presence of noise.

In the implementation we start the evolution by using an initial set of curves obtained from equation (2.5) with a small value of σ. The set Γ_{00} contains a noisy version of the contours in the image plus a large number of small blobs caused by noise.
Because of the curve shortening effect after a small time the curvature of Γ_t is largest at the boundaries of the blobs. Then the coarsening process starts and the small blobs due to the presence of noise disappear. The surviving curves can be considered as the boundaries of the objects of interest. The curve shortening effect removes noise from those remaining boundaries and the further evolution yields a simultaneous multi–scale representation for all the shapes present in the image. As usual in shape recovery by means of curve evolution, the process has to be stopped after a suitable time in order to avoid losing information as t becomes larger and larger.

3 Shape recovery by phase field equations

In this section we shortly review the approximation of the Hele–Shaw model (2.3) by means of the Cahn–Hilliard equation ([32, 1, 10]) and that of the nonlocal curvature model by means of the nonlocal Allen–Cahn equation ([33, 6]). We discuss also how, basing on these results, equations (1.7) and (1.8) can be used as effective tools for the computation of shapes.

In [32] Pego showed by using formal matched asymptotic expansions that level sets of solutions to the Cahn–Hilliard equation tend to solutions of the Hele–Shaw problem as $\varepsilon \to 0$. Alikakos, Bates and Chen [1] then rigorously proved Pego's result under the assumption that classical solutions of the Hele–Shaw problem exist and, more recently, Chen [10] removed such a regularity hypothesis.

We rewrite the Cahn–Hilliard equation (1.7) as the system

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial t} &= \frac{1}{\varepsilon}\Delta v^\varepsilon && \text{in } \Omega \times (0,T], \\ v^\varepsilon &= \omega'(u^\varepsilon) - \varepsilon^2 \Delta u^\varepsilon && \text{in } \Omega \times (0,T], \end{aligned} \tag{3.1}$$

where ω is a double equal-well potential taking its global minimum value 0 at $u = \pm 1$. Equations in (3.1) are supplemented with initial and boundary conditions

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \qquad \text{on } \Omega \times \{0\},$$

$$\frac{\partial}{\partial n} v^\varepsilon(x,t) = \frac{\partial}{\partial n} u^\varepsilon(x,t) = 0 \qquad \text{on } \partial\Omega \times (0,T]. \tag{3.2}$$

The property of mass conservation holds as a direct consequence of (3.1) and (3.2):

$$\int_\Omega u^\varepsilon(x,t)dx = \int_\Omega u_0^\varepsilon(x)dx \quad \forall t \in (0,T]. \tag{3.3}$$

Assume now that $(u^\varepsilon, v^\varepsilon)$ is a solution to (3.1), (3.2), and consider the zero level set

$$\Gamma^\varepsilon := \{(x,t) \in \Omega \times (0,T] : \; u^\varepsilon(x,t) = 0\} = \bigcup_{0\le t\le T} (\Gamma_t^\varepsilon \times \{t\}) \ .$$

Let Γ_{00} be a smooth closed curve in Ω and assume that the Hele–Shaw problem (2.3) starting from Γ_{00} has a smooth solution

$$(\mu, \Gamma := \cup_{0\le t\le T}(\Gamma_t \times \{t\}))$$

in a time interval $[0,T]$ with $\Gamma_t \subset \Omega$ for all $t \in [0,T]$. By the results of [1] there exists a family of smooth initial data $\{u_0^\varepsilon(x)\}_{0<\varepsilon\le 1}$ which are uniformly bounded in $\varepsilon \in (0,1]$ such that if u^ε are the corresponding solutions of (3.1),(3.2) then

$$\begin{aligned} &\lim_{\varepsilon\to 0} u^\varepsilon(x,t) = \begin{cases} -1 & \text{if } (x,t) \in Q_0^-, \\ +1 & \text{if } (x,t) \in Q_0^+, \end{cases} \qquad \text{uniformly on compact subsets,} \\ &\lim_{\varepsilon\to 0}(\varepsilon\Delta u^\varepsilon - \frac{1}{\varepsilon}\omega'(u^\varepsilon))(x,t) = \mu(x,t) \quad \text{uniformly on } \overline{\Omega} \times (0,T], \end{aligned} \tag{3.4}$$

where Q_0^+ and Q_0^- denote respectively the exterior and interior of Γ^0 in Ω.

The above mentioned result shows that the zero level set of the solution to the Cahn–Hilliard equation approximates as $\varepsilon \to 0$ a free boundary that solves the Hele–Shaw problem. In particular the mass conservation law (3.3), taking (see [1]) into account (3.4), corresponds to the approximate conservation of the area of the region enclosed by the zero level set Γ^ε, which becomes exact as $\varepsilon \to 0$.

In a similar way, let us consider now the nonlocal Allen–Cahn equation

$$\begin{aligned} \tfrac{\partial u}{\partial t} = \Delta u - \tfrac{1}{\varepsilon^2}\omega'(u) + \tfrac{1}{\varepsilon^2|\Omega|}\textstyle\int_\Omega \omega'(u)dx \quad & \text{on } \partial\Omega \times (0,T], \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) \quad & \text{on } \Omega, \\ \tfrac{\partial}{\partial n} u^\varepsilon(x,t) = 0 \quad & \text{on } \partial\Omega \times (0,T], \end{aligned} \tag{3.5}$$

where the nonlocal term in the equation plays the role of a Lagrangian multiplier in order to preserve mass.
In [33] the method of matched asymptotic expansions has been used to show that as $\varepsilon \to 0$ the motion of the zero level sets of the solutions of (3.5) describes a coarsening process, with velocity given by equation (1.6), which preserves the total area inside the fronts . A rigorous proof of this convergence has been given, at least for the case of radially simmetric solutions of (3.5) in a bounded spherically simmetric domain Ω by Bronsard and Stoth [6].

It should be remarked (see [33]) that both the previous models can be interpreted as particular cases of the generalized viscous Cahn–Hilliard equation

$$\alpha u_t = \Delta[\omega'(u) - \beta\Delta u + \nu u_t],$$

introduced by Novick-Cohen [30] to take care of viscous effects in phase separation models: $\nu \to 0$ yields the Cahn–Hilliard equation, while $\alpha \to 0$ gives the nonlocal Allen–Cahn equation.

The convergence results reviewed in this section, together with the properties of the limit models already discussed in Section 2, suggested to us the use of the equations (3.1) or (3.5) to build a numerical algorithm for shape recovery in image segmentation applications: the zero level set of their solutions after a short time can be used to recover the boundaries of the objects of interest in an image.

An approach to shape recovery based on one of the two previous equations has actually several interesting properties. Indeed, many efficient numerical methods developed for both models in the context of phase transitions theory can be adapted to the problem of shape recovery. For instance, the second order splitting method for the Cahn–Hilliard equation proposed in [13] yields a more straightforward numerical scheme than the solution of the geometric equation (1.3), used by Sapiro and Tannenbaum, by means of the level set method. Moreover, the double well potential ω can be replaced for practical reasons by a double obstacle potential such as

$$\psi(s) := \begin{cases} 1 - s^2 & \text{if } s \in [-1,1], \\ +\infty & \text{if } s \notin [-1,1]. \end{cases} \tag{3.6}$$

With such a choice (see for example [4] for the Cahn–Hilliard model and [28] for the Allen–Cahn model), the essential properties of the flows are preserved while the solutions (which are now solutions of a double obstacle variational inequality) are forced to remain in the interval $[-1, 1]$ for all times; more than that, they differ from the extrema ± 1 only on a narrow transition region of size proportional to ε, so that the use of a dynamic mesh algorithm becomes possible [28].

We remark that this approach can be still considered as a level set method, since an arbitrary and apriori unknown number of contours corresponding to several objects can be simultaneously tracked and recovered from an image, even though of course we are not interested in changes of topology during the evolution of the curves that are object boundaries. Furthermore, the coarsening effect, which takes place in these models, makes the small contours due to the presence of noise vanish during the evolution.

Then an algorithm for shape recovery based on the previous phase field models should contain the following steps:

- choose an initial set of contours by means of a pre–segmentation of the image using, for instance, the zero–crossings method or suitable thresholds;
- compute an initial data for the phase field equation from the initial set of curves;
- compute a solution of the phase field equation by using a numerical method commonly used for these models in phase transitions theory;
- stop the computation at a suitable time t in order to save the information contained in the evolving contour (see the comment below);
- plot the zero level set of the solution at different times obtaining a multi–resolution representation of the shapes contained in the original image.

One way to implement the stopping criterion is to modify the evolution equation by introducing an interaction with the image intensity function g in order to moderate the geometric diffusion effect when the evolving curve drifts far away from the original shape. This interaction can be for instance introduced in the Cahn–Hilliard equation as in [8]:

$$\frac{\partial u}{\partial t} = -\frac{\Delta v}{1 + \nu |\nabla G_\sigma * g|^2} \ ,$$

where ν is a positive parameter. If ν is large enough the evolution is stopped near to the edges (large values of the gradient) of g. Of course such an interaction violates the area conservation law.

A different approach based on optimal control ideas has been proposed in [7].

A final remark is that most of the results discussed in the present paper generalize to any space dimension, so that the models of shape recovery based on the analogy with phase transitions can also be applied to recover surfaces which are the boundaries of objects in three–dimensional images.

4 Numerical experiments

5 Conclusion

References

[1] ALIKAKOS N., BATES P. & CHEN X. Convergence of the Cahn-Hilliard equation to the Hele-Shaw model. *Arch. Rational Mech. Anal.* **128**, (1994) 165–205.

[2] ALLEN S.M. & CAHN J.W. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Met.* **27**, (1979) 1085–1095.

[3] ALVAREZ L., GUICHARD F., LIONS P.L. & MOREL J.M. Axioms and fundamental equations of image processing. *Arch.Rational Mech.Anal.* **123**, (1993) 199–257.

[4] BLOWEY J.F. & ELLIOTT C.M. The Cahn–Hilliard gradient theory for phase separation with non–smooth free energy. Part II: Numerical analysis. *Euro. J. Appl. Math.* **3**, (1992) 147–179.

[5] BRONSARD L. & KOHN R.V. Motion by mean curvature as a singular limit of Ginzburg-Landau dynamics. *J. Diff. Eqs.* **90**, (1991) 211–237.

[6] BRONSARD L. & STOTH B. Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation. *SIAM J. Math. Anal.* **28**, (1997) 769–807.

[7] Capuzzo Dolcetta I. & Ferretti R. On an optimal stopping time formulation of adaptive image filtering. *MTNS98*,.

[8] Caselles V., Catté F., Coll T. & Dibos F. A geometric model for active contours in image processing. *Numer. Math.* **66**, (1993) 1–31.

[9] Chen. X. The Hele-Shaw problem and area-preserving curve-shortening motions. *Arch. Rational Mech. Anal.* **123**, (1993) 117–151.

[10] Chen. X. Global asymptotic limit of solutions of the Cahn-Hilliard equations. *J. Differential Geom.* **47**, (1996) 262–311.

[11] Chen Y.-G., Giga Y. & Goto S. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Diff. Geom.* **33**, (1991) 749–786.

[12] De Mottoni P. & Schatzman M. Evolution géométrique d'interfaces. *C.R. Acad. Sci. Paris* Sér. I Math. **309**, (1989) 453–458.

[13] Elliott C.M., French D.A. & Milner F.A. A second order splitting method for the Cahn-Hilliard equation. *Num. Math.* **54**, (1989) 575–590.

[14] Evans L.C., Soner H.M. & Souganidis P.E. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.* **45**, (1992) 1097–1123.

[15] Evans L.C. & Spruck J. Motion of level sets by mean curvature. *J. Diff. Geom.* **33**, (1991) 635–681.

[16] Gage M. Curve shortening makes convex curves circular. *Invent. Math.* **76**, (1984) 357–364.

[17] Gage M. On an area-preserving evolution equation for plane curves. *Contemp. Math.* **51**, (1986) 51–62.

[18] Gage M. & Hamilton R.S. The heat equation shrinking convex plane curves. *J. Differential Geometry* **23**, (1986) 69–96.

[19] Grayson M. The heat equation shrinks embedded plane curves to round points. *J. Differential Geometry* **26**, (1987) 285–314.

[20] Grayson M. Shortening embedded curves. *Annals of Mathematics* **129**, (1989) 71–111.

[21] Kichenassamy S., Kumar A., Olver P., Tannenbaum A. & Yezzi Jr. A. Conformal curvature flows: from phase transitions to active vision. *Arch. Rational Mech. Anal.* **134**, (1996) 275–301.

[22] Kimia B.B. & Siddiqi K. Geometric heat equation and nonlinear diffusion of shapes and images. *Computer Vision and Image Understanding* **64**, (1996) 305–322.

[23] Kimia B.B., Tannenbaum A. & Zucker S.W. Shapes, shocks, and deformations, I: the components of shape and the reaction-diffusion space. *Int. J. Comput. Vision* **15**, (1995) 189–224.

[24] Malladi R., Sethian J.A. & Vemuri B.C. Shape modeling with front propagation: a level set approach. *IEEE Trans. PAMI* **17**, (1995) 158–175.

[25] March R. & Dozio M. A variational method for the recovery of smooth boundaries. *Image and Vision Computing Journal* **15**, (1997) 705–712.

[26] Marr D. & Hildreth E. Theory of edge detection. *Proc. Roy. Soc. Lond. Ser. B* **207**, (1980) 187–217.

[27] Mokhatarian F. & Mackworth A. A theory of multi-scale, curvature based shape representation for planar curves. *IEEE Trans. PAMI* **14**, (1992) 789–805.

[28] Nochetto R.H., Paolini M. & Verdi C. A dynamic mesh algorithm for curvature dependent evolving interfaces *J. Comput. Phys.* **123**, (1996) 296–310

[29] Nochetto R.H. & Verdi C. Approximating curvature driven interfaces with applications to shape recovery. *Curvature flows and related topics (Levico, 1994), Gakuto Internat. Ser. Math. Sci. Appl.* **5**, (1995) 159–177.

[30] NOVICK-COHEN A. On the viscous Cahn-Hilliard equation. in: *Material Instabilities in Continuum Mechanics and Related Mathematical Problems*, J.Ball ed., Clarendon Press, Oxford (1988) 329–342.

[31] OSHER S.J. & SETHIAN J.A. Fronts propagation with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations. *J. Comp. Phys.* **79**, (1988) 12–49.

[32] PEGO R.L. Front migration in the nonlinear Cahn-Hilliard equation. *Proc. Roy. Soc. Lond. Ser. A* **422**, (1989) 261–278.

[33] RUBINSTEIN J. & STERNBERG P. Nonlocal reaction-diffusion equations and nucleation. *IMA J. of Appl. Math.* **48**, (1992) 249–264.

[34] RUBINSTEIN J., STERNBERG P. & KELLER J.B. Fast reaction, slow diffusion, and curve shortening. *SIAM J. Appl. Math.* **49**, (1989) 116–133.

[35] SAPIRO G. & TANNENBAUM A. Affine invariant scale-space. *Int. J. Comput. Vision* **11**, (1993) 25–44.

[36] SAPIRO G. & TANNENBAUM A. Area and lenght preserving geometric invariant scale-spaces. *IEEE Trans. PAMI* **17**, (1995) 67–72.

[37] SHAH J. Uses of elliptic approximations in computer vision. *Variational Methods for Discontinuous Structures*, R.Serapioni and F.Tomarelli eds., Birkhauser, Boston (1996) 19–34.

[38] TORRE V. & POGGIO T. On edge detection. *IEEE Trans. PAMI* **8**, (1986) 147–163.