On the Liouville property for sublaplacians

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1 Introduction

The Liouville theorem for harmonic functions states that a solution u of

$$u \geq 0,\ \Delta u = 0 \text{ in } \mathbb{R}^N$$

is a constant. This classical result has been extended to non-negative solutions of semilinear elliptic equations in $\mathbb{R}^N$ or in half-spaces by B.Gidas and J.Spruck [?]. For the case of the whole space they proved that the unique solution of

$$u \geq 0,\ \Delta u + Cu^\alpha = 0 \text{ in } \mathbb{R}^N$$

is $u \equiv 0$, provided $1 < \alpha < \frac{N+2}{N-2}$ and C is a strictly positive constant.

The Liouville property is more delicate to establish for semilinear elliptic equations or inequalities of the form

$$u \geq 0,\ \Delta u + h(x)u^\alpha \leq 0 \text{ in } \Sigma,$$

where Σ is a cone in $\mathbb{R}^N$ and $h \geq 0$ is a function which may vanish on the boundary of Σ. Liouville type theorems in this case have been established recently by H.Berestycki, L.Nirenberg and the first author. In the paper [?] they obtained, by a simpler method than in [?], a general result in this direction under some conditions relating the exponent α, the rate of growth of h at infinity, the opening of the cone Σ and the space dimension N. In the special case $\Sigma = \{x = (x_1, ..., x_N) \in \mathbb{R}^N \ : x_N > 0\}$ and $h(x) = x_N$, the above mentioned theorem states that the unique solution of

$$u \geq 0, \ \Delta u + x_N u^\alpha \leq 0 \ \text{ in } \Sigma$$

is $u \equiv 0$, provided $1 < \alpha < \frac{N+2}{N-1}$.

In [?] and [?] these non-existence results have been applied to show via a blow-up analysis the validity, under restrictions on α dictated by the Liouville theorems, of a priori estimates in the sup norm for all solutions $(u, \tau) \geq 0$ of the problem

$$\begin{array}{l} u \geq 0, \ \Delta u + a(x)u^\alpha + \tau = 0 \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{array}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $\tau \in \mathbb{R}$. These estimates allow to prove, via the Leray-Schauder degree theory, the existence of non-trivial solutions of the Dirichlet problem

$$\begin{array}{l} u \geq 0, \ \Delta u + a(x)u^\alpha = 0 \ \text{ in } \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{array}$$

even when the weight a may change sign in Ω (see [2] for such indefinite type problems).

The approach of [?], which works for general second order uniformly elliptic operators in non divergence form, has been adapted by I. Birindelli and the present authors to deal with the semilinear operator $\Delta_{H^n} u + a(\xi)u^\alpha$.
Here, Δ_{H^n} is the second order degenerate elliptic operator

$$\Delta_{H^n} = \sum_{i=1}^{2n} (\frac{\partial^2}{\partial \xi_i^2} + 4\xi_i^2 \frac{\partial^2}{\partial \xi_{2n+1}^2}) + 4 \sum_{i=1}^{n} (\xi_{i+n} \frac{\partial^2}{\partial \xi_i \partial \xi_{2n+1}} - \xi_i \frac{\partial^2}{\partial \xi_{i+n} \partial \xi_{2n+1}}) \tag{1.1}$$

acting on functions $u = u(\xi)$ where $\xi = (\xi_1, \cdots, \xi_{2n}, \xi_{2n+1}) \in \mathbb{R}^{2n+1}$.
In [?] and [?], the results described above for the case of the Laplace operator have been indeed extended to the operator in (1.1) under some pseudo-convexity condition on $\partial\Omega$ which allows to manage the extra difficulties posed by the presence of *characteristic points*.

The basic idea in [?] and [?] is to look at the Kohn Laplacian Δ_{H^n} as a *sublaplacian* on $\mathbb{R}^{2n+1}$ endowed with the Heisenberg group action

$$\xi \circ \eta = (\xi_1 + \eta_1, \ldots, \xi_{2n} + \eta_{2n}, \xi_{2n+1} + \eta_{2n+1} + 2\sum_{i=1}^{n}(\xi_{i+n}\eta_i - \xi_i\eta_{i+n})) \ .$$

By this we mean that Kohn Laplacian in (1.1) can be expressed as $\Delta_{H^n} = \sum_{i=1}^{2n} X_i^2$, with

$$X_i = \frac{\partial}{\partial \xi_i} + 2\xi_{i+n}\frac{\partial}{\partial \xi_{2n+1}} \quad , X_{i+n} = \frac{\partial}{\partial \xi_{i+n}} - 2\xi_i\frac{\partial}{\partial \xi_{2n+1}} \tag{1.2}$$

for $i = 1, \ldots, n$. This observation allows to exploit conveniently the scaling properties of the fields X_i and of the operator Δ_{H^n} with respect to the *anistropic dilations*

$$\delta_\lambda(\xi) = (\lambda\xi_1, \ldots, \lambda\xi_{2n}, \lambda^2\xi_{2n+1}) \ \ (\lambda > 0 \ \ )$$

and the action of Δ_{H^n} on functions depending only on the *homogeneous norm*

$$\rho(\xi) = ((\sum_{i=1}^{2n} \xi_i^2)^2 + \xi_{2n+1}^2)^{\frac{1}{4}} \ . \tag{1.3}$$

Liouville theorems, a priori estimates and the existence of non trivial solutions in Hölder-Stein spaces for the Dirichlet problem

$$\Delta_{H^n} u + a(\xi)u^\alpha \ = 0 \ \ , \ \ u = 0 \text{ on } \partial\Omega$$

are therefore obtained in the above mentioned papers under a restriction on the exponent α depending on the *homogeneous dimension* $Q = 2n + 2$ of the Heisenberg group rather than on its linear dimension $N = 2n + 1$.

The ideas and methods outlined above for the case of Δ_{H^n} can be generalized to sublaplacians L of the form $L = \sum_{i=1}^{n_1} X_i^2$ where the first order differential operators X_i in the preceding generate the whole Lie algebra of left-invariant vectorfields on a nilpotent, stratified Lie group $(G, \circ)$, see Section 2 for a quick review of the basic notions and terminology.

In Section 3 of the present paper, which originates from the graduate dissertation of the second author [?], we propose some abstract results of Liouville type for operators L as above, both in the linear and the semilinear case. The final Section 4 is devoted to the study of the semilinear Liouville property for some second order degenerate elliptic operator which do not fit in the abstract setting of Section 2 , the main example being the Grushin operator which is defined on $\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^q$ by

$$\sum_{i=1}^{p} \frac{\partial^2}{\partial {x_i}^2} + |x|^{2k} \sum_{i=1}^{q} \frac{\partial^2}{\partial y_i^2}$$

where $k \in \mathbb{N}$ and $\xi = (x_1, \ldots, x_p, y_1, \ldots, y_q)$ is the typical point of $\mathbb{R}^N$.

Let us mention finally that different aspects of semilinear subelliptic problems have been investigated in [?] and, more recently, in [?, ?, ?, ?, ?, ?, ?]. Liouville type theorems for linear Fuchsian or weighted elliptic operators have been established in [?, ?, ?].

2 Sublaplacians on stratified Lie groups

In this section we recall briefly a few notions which are relevant to the analysis on Lie groups and some fundamental properties of sublaplacians on stratified, nilpotent Lie groups. For more details, see. e.g. [?, ?].

2.1 Stratified nilpotent Lie groups

Let $\mathcal{G}$ be a real finite dimensional Lie algebra, i. e. a vector space on $\mathbb{R}$ with a *Lie bracket* $[\cdot\,,\cdot]$, that is a bilinear map from $\mathcal{G} \times \mathcal{G}$ into $\mathcal{G}$ which is alternating

$$[X, Y] = -[Y, X] \text{ for all } X, Y \in \mathcal{G} \tag{2.1}$$

and satisfies the *Jacobi identity*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \text{ for all } X,\ Y,\ Z \ \in \mathcal{G}. \tag{2.2}$$

$\mathcal{G}$ is called *$m-$nilpotent and stratified* if it can be decomposed as a direct sum of subspaces satisfying

$$\begin{gathered} \mathcal{G} = V_1 \oplus V_2 \ldots \oplus V_m \text{ with } \dim V_j = n_j \\ [V_1, V_j] \subset V_{j+1} \text{ if } 1 \leq j < m;\ [V_1, V_m] = \{0\}. \end{gathered} \tag{2.3}$$

Therefore, V_1 generates, by means of the Lie bracket $[\cdot,\cdot]$, $\mathcal{G}$ as a Lie algebra.

Let $(G,\circ)$ be the simply connected Lie group associated to the Lie algebra $\mathcal{G} = (G,[\cdot,\cdot])$ as follows:

$$G = \mathbb{R}^N \text{with } N = \sum_{j=1}^{m} n_j \quad ,$$

equipped with the group action $\circ$ defined by the Campbell-Hausdorff formula, namely

$$\eta \circ \xi = \eta + \xi + \frac{1}{2}[\eta,\xi] + \frac{1}{12}[\eta,[\eta,\xi]] + \frac{1}{12}[\xi,[\xi,\eta]] + \dots \tag{2.4}$$

(for the other terms see e.g.[**?**]). Note that, in view of the nilpotency of $\mathcal{G}$, in the right hand side there is only a finite sum of terms involving commutators of ξ and η of lenght less than m.
Observe that the group law (**??**) makes $G = \mathbb{R}^N$ a Lie group whose Lie algebra of left-invariant vectorfields $Lie(G)$ coincides with $\mathcal{G}$. Recall that the Lie algebra $Lie(G)$ is the algebra of left-invariant vectorfields Y which satisfy

$$Yf(\eta \circ \xi) = (Yf)(\eta \circ \xi) \quad ,$$

for every smooth function f, equipped with the bracket $[[X,Y]] = XY - YX$.

Let $e_1,\dots,e_{n_1}$ be the canonical basis of the subspace $\mathbb{R}^{n_1}$ of G; then as a basis of the Lie algebra $\mathcal{G} = Lie(G)$ we can choose the vectorfields $X_1,\dots,X_{n_1}$ defined for smooth f by

$$X_i(f)(\xi) = \lim_{t\to 0+} \frac{f(\xi \circ te_i) - f(\xi)}{t}, \ \xi \in G. \tag{2.5}$$

Since V_1 generates $\mathcal{G}$ as a Lie algebra we can define recursively, for $j = 1,\dots,m$, and $i = 1,\dots,n_j$, a basis $\{X_{i,j}\}$ of V_j as

$$\begin{aligned} &X_{i,1} = X_i \quad (i = 1,\dots,n_1) \\ &X_\alpha = [X_{i_1},[X_{i_2},\dots,[X_{i_{j-1}},X_{i_j}]]\dots], \end{aligned}$$

with $\alpha = (i_1,\dots,i_j)$ multi-index of length j and $X_{i_k} \in \{X_1,\dots,X_{n_1}\}$.

In terms of the decomposition $G = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2} \oplus \dots \oplus \mathbb{R}^{n_m}$ one defines then a one - parameter group of *dilations* δ_λ on G by setting for

$$\xi = \xi_1 + \xi_2 + \dots + \xi_m, \quad (\xi_j \in \mathbb{R}^{n_j}) \quad ,$$

$$\delta_\lambda(\xi) = \sum_{j=1}^{m} \lambda^j \xi_j. \tag{2.6}$$

Observe that, for any $\xi \in G$, the Jacobian of the map $\xi \longrightarrow \delta_\lambda(\xi)$ equals λ^Q, where

$$Q = \sum_{j=1}^{m} j\, n_j \quad . \tag{2.7}$$

The integer Q is the *homogeneous dimension* of G. Observe that the linear dimension of G is $N = \sum_{j=1}^{m} n_j$; hence $Q \geq N$ and equality holds only in the trivial case $m = 1$ and $G = \mathbb{R}^{n_1}$.

Let us recall that a *dilation - homogeneous norm* on G is, by definition, a mapping $\xi \to \rho(\xi)$ from G to $\mathbb{R}^+$ such that:

$$\begin{aligned}
&i)\ \xi \to \rho(\xi) \text{ is continuous on } G \text{ and smooth on } G \setminus \{0\}\\
&ii)\ \rho(\xi) = 0 \text{ if and only if } \xi = 0\\
&iii)\ \rho(\xi) = \rho(-\xi)\\
&iv)\ \rho(\delta_\lambda(\xi)) = \lambda\rho(\xi) \quad \text{for each } \lambda > 0.
\end{aligned} \tag{2.8}$$

All homogeneous norms on G are equivalent; moreover they satisfy the triangle inequality

$$\rho(\xi \circ \eta) \leq C_0(\rho(\xi) + \rho(\eta)) \quad \text{for all } \xi, \eta \in G$$

for some constant $C_0 \geq 1$. For a given homogeneous norm and positive real R, the *Koranyi ball* centered at 0 is the set

$$B(0, R) = \{\xi \in G\ : \rho(\xi) < R\}.$$

These balls form, for $R > 0$, a fundamental system of neighborhoods of the origin in $(G, \circ)$. Through the group law $\circ$ one defines then the distance between $\xi, \eta \in G$ by the position

$$d(\xi, \eta) = \rho(\eta^{-1} \circ \xi)\ ,$$

where η^{-1} is the inverse of η with respect to $\circ$, i.e. $\eta^{-1} = -\eta$. The Koranyi ball of radius R centered at η is defined accordingly.

It is important to point out that the Lebesgue measure is invariant for the group action and that the volumes scale as R^Q.
More precisely, if $|E|$ denotes the N - dimensional Lebesgue measure (recall that $N = \sum_{j=1}^{m} n_j$), we have

$$|B(\eta, R)| = |B(0, R)| = |B(0, 1)|R^Q.$$

as a consequence of (**??**) and (**??**).

2.2 Sublaplacians

Let us come back now to the vectorfields X_i $(i = 1, \dots, n_1)$ defined in (**??**). The first remark is that X_i are 1 - homogeneous with respect to the dilations δ_λ, i.e.

$$X_i f(\delta_\lambda(\xi)) = \lambda (X_i f)(\delta_\lambda \xi) \tag{2.9}$$

Indeed, from the definition **??** of X_i we have

$$X_i f(\delta_\lambda(\xi)) = \lim_{t \to 0+} \frac{f(\delta_\lambda \xi \circ \lambda t e_i) - f(\delta_\lambda \xi)}{t}$$

Setting $\tau = t\lambda$, the righ-hand side of the preceding is

$$\lambda \lim_{\tau \to 0+} \frac{f(\delta_\lambda \xi \circ \tau e_i) - f(\delta_\lambda \xi)}{\tau} = \lambda (X_i f)(\delta_\lambda(\xi))$$

In a similar way one can check that the vectorfields of V_j are homogenous of degree j, that is

$$X_{i,j} f(\delta_\lambda(\xi)) = \lambda^j (X_{i,j} f)(\delta_\lambda \xi) \ \ , \ \forall i = 1, \dots, n_j \tag{2.10}$$

Let us make now some simple remarks on the representation of the vectorfields X_i as first order partial differential operators. If one chooses $(e_1, \dots, e_{n_1}, \dots, e_N)$ as the canonical basis of $G = \mathbb{R}^N$, then each X_i $(i = 1, \dots, n_1)$ can be expressed in terms of the partial derivatives $\frac{\partial}{\partial x_j}$ as

$$X_i = \sum_{j=1}^{N} \sigma_{ij}(x) \frac{\partial}{\partial x_j} \tag{2.11}$$

Here, $\sigma(x) = (\sigma_{ij}(x))$ is a $n_1 \times N$ matrix of the form

$$\sigma = (I_{\mathbb{R}^{n_1}}, \ Q_2(x) \ , \ \dots, \ Q_m(x)) \tag{2.12}$$

where $I_{\mathbb{R}^{n_1}}$ denotes the identity on $\mathbb{R}^{n_1}$ and $Q_j(x)$ are $n_1 \times n_j$ matrices $(j = 2, \dots, m)$. As a consequence of (**??**) one has

$$\frac{\partial}{\partial x_j} \sigma_{ij}(x) = 0 \quad \text{for } j = 1, \dots, N. \tag{2.13}$$

The *sublaplacian* L on the group G is defined then on smooth functions u by

$$Lu = \sum_{i=1}^{n_1} X_i^2 u \tag{2.14}$$

Observe that L is $2-$homogeneous with respect to the dilations δ_λ since the X_i's are $1-$homogeneous; moreover, L is left-invariant with respect to the group action $\circ$, since the X_i's are such.

In view of the preceding discussion, L is a second order partial differential operator; as a consequence of (**??**) it can be expressed in divergence form as

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) \ , \tag{2.15}$$

where $A(x) = (a_{ij}(x)) = \sigma^T(x)\sigma(x)$ is a positive semidefinite $N \times N$ matrix. When $m = 1$ the sublaplacian L coincides with the Laplace operator

$$\Delta = \sum_{i,j=1}^{N} \frac{\partial^2}{\partial {x_j}^2}$$

On the other hand, as soon as $m \geq 2$, the matrix σ has a non trivial kernel. The sublaplacian L is therefore no more uniformly elliptic but only degenerate elliptic and, more precisely, a second order operator with non-negative characteristic form according to [**?**]. Nevertheless, the stratification condition implies that the fields X_i $(i = 1, \dots, n_1)$ satisfy the *Hörmander condition*

$$Lie(G) = \mathcal{G} \ \ . \tag{2.16}$$

As a consequence of (**??**), L is *subelliptic* (see [**?**]). Let us just mention here that this implies the validity of Bony's Maximum Principle (see [**?**]).
In the sequel we will use the notation $\nabla_L u = (X_1 u, \dots, X_{n_1} u)$.

Let us conclude this section by two basic examples.

2.3 Examples

Example 1. Take $\mathcal{G} = \mathbb{R}^N$ with the trivial Lie bracket $[X, Y] = 0$ for all X, Y and stratification $V_1 = \mathbb{R}^N, V_2 = \{0\}$. The dilation and the homogeneous norm in this case are, of course, isotropic. They are given, respectively, by

$$\delta_\lambda(\xi) = (\lambda\xi_1, \ldots, \lambda\xi_N)\,; \quad \rho(\xi) = (\sum_{i=1}^{N} \xi_i^2)^{\frac{1}{2}}$$

The homogeneous dimension is N, the fields X_i are the partial derivatives and the sublaplacian is the standard Laplacian Δ.

Example 2. Take $\mathcal{G} = \mathbb{R}^{2n+1}$ $(n \geq 1)$ with the Lie bracket $[X, Y] = XY - YX$ and the stratification $G = \mathbb{R}^{2n} \oplus \mathbb{R}^1$. The homogeneous dimension in this case is then $Q = 2n + 2$. The dilation and the homogeneous norm on G are, respectively,

$$\delta_\lambda(\xi) = (\lambda\xi_1, \ldots, \lambda\xi_{2n}, \lambda^2\xi_{2n+1})\,; \quad \rho(\xi) = ((\sum_{i=1}^{2n} \xi_i^2)^2 + \xi_{2n+1}^2)^{\frac{1}{4}}$$

It is easy to check that the group action $\circ$ defined in (**??**) is

$$\eta \circ \xi = (\xi_1 + \eta_1, \ldots, \xi_{2n} + \eta_{2n}, \xi_{2n+1} + \eta_{2n+1} + 2\sum_{i=1}^{n} (\xi_i\eta_{i+n} - \xi_{i+n}\eta_i)). \tag{2.17}$$

From this it follows that the fields X_i are given in this case by (1.2) and the sublaplacian associated with the Heisenberg group $H^n = (\mathbb{R}^{2n+1}, \circ)$ is therefore given by (1.1).

3 The Liouville property for sublaplacians on nilpotent stratified groups

3.1 The linear case

This section is devoted to the generalization to sublaplacians L of the well-known Liouville property valid for the Laplace operator. Indeed, we prove that bounded $L-$harmonic functions on stratified groups G are necessarily constant.

Let $L = \sum_{i=1}^{n_1} X_i^2$ be the sublaplacian on the stratified group $(G, \circ)$. A function u is $L-$harmonic on G if

$$u \in \Gamma^2(G),\ Lu = 0 \text{ in } G$$

where $\Gamma^2(G)$ is the space of functions $u : G \to \mathbb{R}$ such that

$$u, X_i u \in L^\infty(G) \cap C(G)$$

and

$$\sup_{\xi,\eta} \frac{|X_i u(\eta \circ \xi) + X_i u(\eta \circ \xi^{-1}) - 2X_i u(\eta)|}{\rho(\xi)} < \infty$$

for $i = 1, \ldots, n_1$.

The basic tool in our proof of the linear Liouville theorem is the following *mean value* property for $L-$harmonic functions:

$$v(\xi) = \frac{C_Q}{R^Q} \int_{B_L(\xi,R)} v(\eta) |\nabla_L d_L(\xi,\eta)|^2 d\eta \,, \tag{3.1}$$

where $d_L(\xi,\eta) := \rho_L(\xi^{-1} \circ \eta)$, C_Q is a suitable constant and $B_L(\xi,R)$ denotes the Koranyi ball associated to an appropriate $C^\infty(G \setminus \{0\})$ homogeneous norm $\rho_L(\cdot)$. Note that

$$\rho_L^2(\xi^{-1} \circ \eta) \approx \Gamma^Q(\xi,\eta)$$

where Γ is the fundamental solution of L (see [**?**, **?**, **?**]).

Theorem 3.1 *Let $L = \sum_{i=1}^{n_1} X_i^2$ be the sublaplacian on the nilpotent stratified group G. If u is $L-$harmonic on G , then u is a constant.*

Proof. As a consequence of (**??**) the vectorfields $X_{i,m}$ commute with X_i for $i = 1, \ldots, n_1$. Hence the sublaplacian L satisfies:

$$X_{i,m} L u = X_{i,m} \sum_{j=1}^{n_1} X_j^2 u = \sum_{j=1}^{n_1} X_j^2 X_{i,m} u.$$

Consequently, if u is $L-$harmonic the same is true for $X_{i,m} u$ $(i = 1, \ldots n_m)$. Therefore, by the mean value formula (**??**) applied to $X_{i,m} u$ we get

$$X_{i,m} u = \frac{C_Q}{R^Q} \int_{B_L(\xi,R)} X_{i,m} u(\eta) |\nabla_L d_L|^2 d\eta \tag{3.2}$$

Integrating by parts the right-hand side of (**??**), we obtain:

$$\begin{aligned} X_{i,m} u = \; & -\tfrac{C_Q}{R^Q} \textstyle\int_{B_L(\xi,R)} u(\eta) X_{i,m} \left(|\nabla_L d_L|^2\right) d\eta \\ & + \tfrac{C_Q}{R^Q} \textstyle\int_{\partial B_L(\xi,R)} u |\nabla_L d_L|^2 \frac{X_{i,m} d_L}{|\nabla d_L|} d\Sigma. \end{aligned}$$

Here ∇ denotes the usual gradient; observe also that $\nu = \frac{\nabla d_L}{|\nabla d_L|}$ is the normal vector to ∂B_L.

Since the X_i are $1-$homogeneous with respect to the intrinsic dilation, see (**??**), and are left-invariant with respect to the group action $\circ$, it follows that $X_{i,m}$ is homogeneous of degree m with respect to δ_λ and left-invariant with respect to $\circ$. The previous remark, together with the fact that d_L is homogenous of degree 1 with respect to δ_λ, provide the following estimates:

$$|X_{i,m}(d_L)| \le \frac{C}{d_L^{m-1}}\,, \quad |X_{i,m}\nabla_L d_L| \le \frac{C}{d_L^m} \tag{3.3}$$

Indeed, for the first estimate in (**??**) observe that

$$X_{i,m}(d_L(\eta)) = \frac{X_{i,m}(d_L(\frac{\eta}{\rho_L(\eta)}))}{\rho_L^{m-1}(\eta)}$$

and that $|X_{i,m}(d_L(\frac{\eta}{\rho_L(\eta)}))|$ is bounded since d_L is C^∞ on $\partial B_L(0,1)$. The second estimate is achieved by using the same argument and the $1-$homogeneity of ∇_L. Moreover, $X_{i,m}(|\nabla_L d_L|^2) = 2X_{i,m}\nabla_L d_L \cdot \nabla_L d_L$.

Hence,

$$|X_{i,m}u(\xi)| \le \frac{C_Q}{R^{Q+m}}||u||_{L^\infty}R^Q + \frac{C_Q}{R^{Q+m-1}}||u||_{L^\infty}R^{Q-1} \le \frac{C_Q}{R^m}||u||_{L^\infty}$$

for every $\xi \in G$.
Therefore, letting $R \to \infty$, one deduces that

$$X_{i,m}u \equiv 0 \quad \text{in } G \quad \text{for every } i = 1,\dots,n_m. \tag{3.4}$$

Now, from the stratification of G it follows that $X_{i,m-1}u$ is also $L-$harmonic. Indeed, for $k = 1,\dots n_1$,

$$[X_{i,m-1}u, X_k u] = X_\alpha u \quad \text{for some } \alpha \text{ with } |\alpha| = m \tag{3.5}$$

Thus, being $X_{i,m}$ a basis of V_m, (**??**) yields $X_\alpha u \equiv 0$ in G. Repeating the same argument and using the fact that the vectorfields $X_{i,j}$ form a basis of V_j and are $j-$homogeneous with respect to δ_λ, see (**??**), one finally obtains that

$$X_{i,j}u \equiv 0 \quad \text{for } j = 1,\dots,m,\ i = 1,\dots,n_j.$$

Consequently, from the Hörmander condition *span* $(X_{i,j}) = Lie(X_i) = \mathcal{G}$, we deduce that $\nabla u = 0$ in G and the claim is proved. $\square$

3.2 The semilinear case

In this section we prove a Liouville theorem for nonnegative solutions of semilinear equations associated to sublaplacians L on stratified groups G.
The proof, which is inspired from [?], relies in particular on the behaviour of the operator L defined in (??) on functions which are radial with respect to the homogeneous norm $\rho_L(\cdot)$, see Section 3.1. From now we shall write, for simplicity, $\rho_L = \rho$.
One can easily check by a direct computation using (??) that the following holds for smooth $f : \mathbb{R} \to \mathbb{R}$ and $\rho \neq 0$

$$Lf(\rho) = f''(\rho)|\nabla_L \rho|^2 + f'(\rho)L\rho \tag{3.6}$$

As recalled in the previous section, $\rho^{2-Q} \approx \Gamma$, where Γ is the fundamental solution of L. Therefore, using (??) with $f(\rho) = \rho^{2-Q}$, one finds that

$$0 = L\rho^{2-Q} = (2-Q)(1-Q)\rho^{-Q}|\nabla_L \rho|^2 + (2-Q)\rho^{1-Q}L\rho$$

for $\rho \neq 0$. Hence,

$$L\rho = (Q-1)\rho^{-1}|\nabla_L \rho|^2$$

yielding to the following radial expression of L :

$$Lf(\rho) = |\nabla_L \rho|^2[f''(\rho) + f'(\rho)\frac{Q-1}{\rho}] \ .$$

Let us observe that $\nabla_L \rho$ is homogeneous of degree zero and therefore is bounded in G; the same is true for $\rho L\rho$. In the sequel we will use the notation $\psi(\rho) = |\nabla_L \rho|^2$.

Theorem 3.2 *Suppose that $u \in \Gamma^2_{loc}(G) \cap C(G)$ satisfies*

$$u \geq 0, \;\; Lu(\xi) + k(\xi)u^\alpha \leq 0 \quad \textit{in } G \tag{3.7}$$

where k is a continuous nonnegative function such that

$$k(\xi) \geq K\rho^\gamma(\xi)$$

for sufficiently large $\rho(\xi)$ and for some $K > 0$, $\gamma > -2$. Then $u \equiv 0$, provided $1 < \alpha \leq \frac{Q+\gamma}{Q-2}$.

Proof. For each $R > 0$ consider a cut-off function ϕ_R such that

$$\begin{cases} \phi_R(\rho) := \phi(\dfrac{\rho}{R}) \text{ with } \phi \in C^\infty[0, +\infty) \\ 0 \le \phi \le 1, \ \phi \equiv 1 \text{ in } [0, \frac{1}{2}], \ \phi \equiv 0 \text{ in } [1, +\infty), \\ -\dfrac{C}{R} \le \dfrac{\partial \phi_R}{\partial \rho} \le 0, |\dfrac{\partial^2 \phi_R}{\partial \rho^2}| \le \dfrac{C}{R^2} \quad \text{for some constant } C > 0. \end{cases} \tag{3.8}$$

Set then

$$I_R := \int_G k(\xi) u^\alpha {\phi_R}^\beta d\xi = \int_{B_L(0,R)} k(\xi) u^\alpha \phi_R^\beta d\xi \tag{3.9}$$

where $\frac{1}{\beta} = 1 - \frac{1}{\alpha}$. Observe that $I_R \ge 0$ and that (**??**) implies

$$I_R \le -\int_{B_L(0,R)} Lu \phi_R^\beta \, d\xi$$

Therefore, an integration by parts yields:

$$\begin{aligned} &I_R \le -\int_{B_L(0,R)} uL(\phi_R^\beta) d\xi + \int_{\partial B_L(0,R)} u \nabla_L(\phi_R^\beta) \cdot \nu_L d\Sigma \\ &-\int_{\partial B_L(0,R)} \phi_R^\beta \nabla_L u \cdot \nu_L d\Sigma \le -\int_{B_L(0,R)} uL(\phi_R^\beta) d\xi \\ &+\int_{\partial B_L(0,R)} u\beta \phi_R^{\beta-1} \phi_R' \nabla_L \rho \cdot \nu_L d\Sigma \le -\int_{B_L(0,R)} uL(\phi_R^\beta) d\xi, \end{aligned} \tag{3.10}$$

where $\nu_L(\xi) = \sigma(\xi)\nu(\xi)$, ν being the exterior normal to ∂B_L , see (**??**), and $d\Sigma$ denotes the $(N-1)$ - dimensional Hausdorff measure.

On the other hand, (**??**) implies

$$L\phi_R^\beta = \psi \frac{\partial^2}{\partial \rho^2} \phi_R^\beta + \rho L \rho \frac{\partial}{\partial \rho} \phi_R^\beta \quad .$$

Thus, by the assumptions made on ϕ_R and taking (**??**) into account, we find, for $\Sigma_R : = B_L(0,R) \setminus B_L(0, \frac{R}{2})$,

$$\begin{aligned} &I_R \le -\int_{\Sigma_R} u[\beta \phi_R^{\beta-1} \phi_R'' \psi + \beta \phi_R^{\beta-1} \phi_R' L\rho] d\xi \\ &\le \frac{C}{R^2} \int_{\Sigma_R} u \phi_R^{\beta-1} d\xi \end{aligned}$$

since ψ and $\rho L\rho$ are bounded. Then, by Hölder inequality,

$$I_R \leq \frac{C}{R^2}[\int_{\Sigma_R} u^\alpha \rho^\gamma \phi_R^{(\beta-1)\alpha} d\xi]^{\frac{1}{\alpha}}[\int_{\Sigma_R} \rho^{\frac{-\gamma\beta}{\alpha}} d\xi]^{\frac{1}{\beta}}.$$

Choosing $R > 0$ sufficiently large so that $k \geq K\rho^\gamma$ in Σ_R, we obtain

$$I_R \leq C[\int_{\Sigma_R} u^\alpha k \phi_R^\beta \, d\xi]^{\frac{1}{\alpha}} \; R^{(\frac{-\gamma}{\alpha}+\frac{Q}{\beta}-2)} \tag{3.11}$$

Therefore, for large R,

$$??I_R^{1-\frac{1}{\alpha}} \leq C \; R^{(\frac{-\gamma}{\alpha}+\frac{Q}{\beta}-2)} \tag{3.12}$$

Letting $R \to \infty$ in the above we conclude that, if $1 < \alpha < \frac{Q+\gamma}{Q-2}$, then

$$I := \lim_{R\to\infty} I_R = \int_G k u^\alpha d\xi = 0 \quad .$$

This implies $u \equiv 0$ outside a large ball since k is strictly positive there. Choose now $\overline{R} > 0$ in such a way that $k > 0$ for $\rho \geq \overline{R}$. Then, as proved above, $u \equiv 0$ on $G \setminus B_L(0, \overline{R})$.

Hence u satisfies:

$$\begin{cases} u \geq 0 \;\text{ in } B_L(0, \overline{R}+\delta) \\ Lu \leq 0 \;\text{ in } B_L(0, \overline{R}+\delta) \\ u \equiv 0 \;\text{ for } \overline{R} \leq \rho \leq \overline{R}+\delta \end{cases} \tag{3.13}$$

for some $\delta > 0$. Therefore, by the Bony's Maximum Principle, see [?], u has to be identically zero on G since u is not strictly positive in view of the last condition in (??).

Consider now the case $\alpha = \frac{Q+\gamma}{Q-2}$. In this case, from (??) we deduce that I_R is uniformly bounded with respect to R. Moreover, since $R \to I_R$ is increasing the integral on the right - hand side of (??), which coincides with $I_R - I_{\frac{R}{2}}$, goes to zero as R tends to infinity. This implies $I = 0$ and we conclude as before. $\square$

Remark 3.1 The claim of Theorem **??** holds under the less restrictive assumption that, for some $K > 0$ and $\gamma > -2$, $k(\xi) \geq K\psi\rho^\gamma(\xi)$ for sufficiently large $\rho(\xi)$. The proof is similar but one has to take into account that $\rho L\rho = (Q-1)\psi$ and also that ψ vanishes, by its very definition, on the characteristics points of the Koranyi's ball which are, by the way, a set of $N-$dimensional measure equal to zero (see [?]).

Remark 3.2 The exponent $\frac{Q+\gamma}{Q-2}$ in Theorem **??** is optimal. To see this, observe first that in view of (**??**) the function $u(\rho) = (1+\rho^2)^{-\frac{p}{2}}$ satisfies

$$Lu + p(Q-p-2)\psi(1+\rho^2)^{-(\frac{p}{2}+1)} \leq 0 \ .$$

Thus, were $\alpha > \frac{Q+\gamma}{Q-2}$, one could choose p such that

$$Q-2 > p \quad , \quad \alpha\frac{p}{2} - \frac{\gamma}{2} \geq \frac{p}{2} + 1 \ \ .$$

Therefore, setting $v = Cu$ with $C = (p(Q-p-2))^{\frac{1}{\alpha-1}}$, one obtains easily that

$$-Lv \geq \psi\,(p(Q-p-2))^{\frac{\alpha}{\alpha-1}}\,(1+\rho^2)^{-\alpha\frac{p}{2}+\frac{\gamma}{2}} \geq \psi\rho^\gamma v^\alpha \ \ .$$

4 Other Liouville type results

Here we prove some semilinear Liouville type results like those of the previous section for some degenerate elliptic second order operators of the form

$$L = \sum_{i=1}^{N} X_i^2 \tag{4.1}$$

which are 2−homogeneous with respect to a family of dilations but do not fit in the setting of Section 3 since they are not left-invariant with respect to any group action on $\mathbb{R}^N$.

The first example we consider is the Grushin operator L defined on $\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^q$ by

$$L = \sum_{i=1}^{p} \frac{\partial^2}{\partial {x_i}^2} + |x|^{2k} \sum_{i=1}^{q} \frac{\partial^2}{\partial y_i^2} \tag{4.2}$$

where $k \in \mathbb{N}$ and $(x,y) = (x_1,\ldots,x_p,y_1,\ldots,y_q)$ denotes the typical point of $\mathbb{R}^N$. This operator may be written in the form (**??**) by choosing

$$X_i = \frac{\partial}{\partial x_i} \ \text{ for } i = 1,\ldots,p$$
$$X_{i+p} = |x|^k \frac{\partial}{\partial y_i} \ \text{ for } i = 1,\ldots,q.$$

It is easy to check that L satisfies the Hörmander condition (**??**) since the X_i generate $\mathbb{R}^N$ by commutators of lenght $\leq k$. It is also easy to realize that the Lie algebra generated by X_i for $k > 1$ has no constant dimension.
However, for the dilation

$$\delta_\lambda \xi = (\lambda x, \lambda^{k+1} y) \tag{4.3}$$

we have

$$X_i u(\delta_\lambda \xi) = \lambda (X_i u)(\delta_\lambda \xi) \ \ (i = 1, \ldots, p \ \ )$$
$$X_{i+p} u(\delta_\lambda \xi) = |x|^k \lambda^{k+1} \frac{\partial u}{\partial y_i} = \lambda (X_{i+p} u)(\delta_\lambda \xi) \ \ (i = 1, \ldots, q)$$

Hence, L is $2-$homogeneous with respect to (**??**). Moreover, the norm

$$\rho(\xi) = |x| + |y|^{\frac{1}{k+1}} \ , \tag{4.4}$$

where $\xi = (x, y)$ and $|\cdot|$ denotes the euclidean norm on $\mathbb{R}^N$, is $1-$homogeneous with respect to the dilation in (**??**).
It follows that N-dimensional measure of the ball

$$\Omega_R = B_p(0, R) \times B_q(0, R^{k+1})$$

associated with (**??**) (here B_p denotes the euclidean ball of $\mathbb{R}^p$) is proportional to R^Q, with

$$Q = p + (k+1)q = N + kq \ .$$

Theorem 4.1 *Let u be a solution of*

$$u \geq 0 \ \ , \ \ \sum_{i=1}^{p} \frac{\partial^2 u}{\partial {x_i}^2} + |x|^{2k} \sum_{i=1}^{q} \frac{\partial^2 u}{\partial y_i^2} + u^\alpha \leq 0 \quad in \ \mathbb{R}^N \tag{4.5}$$

Then $u \equiv 0$, provided that $k > 1$ and $1 < \alpha \leq \frac{Q}{Q-2}$.

Proof of Theorem ?? Set $\Omega_R := B_p(0, R) \times B_q(0, R^{k+1})$. Let φ_R and θ_R be the cut-off functions satisfying, for some constant $C > 0$,

$$\begin{cases} \varphi_R(r) := \varphi(\dfrac{r}{R}); \ \theta_R(s) := \theta(\dfrac{s}{R^{k+1}}) \text{ with } \varphi, \theta \in C^\infty[0, +\infty), \\ 0 \leq \varphi \leq 1, \ 0 \leq \theta \leq 1, \ \varphi = \theta \equiv 1 \text{ in } [0, \frac{1}{2}], \ \varphi \equiv \theta \equiv 0 \text{ in } [1, +\infty), \\ -\dfrac{C}{R} \leq \dfrac{\partial \varphi_R}{\partial r} \leq 0, \ -\dfrac{C}{R^{k+1}} \leq \dfrac{\partial \theta_R}{\partial s} \leq 0, \ \ |\dfrac{\partial^2 \varphi_R}{\partial r^2}| \leq \dfrac{C}{R^2} \, , \ \ |\dfrac{\partial^2 \theta_R}{\partial s^2}| \leq \dfrac{C}{R^{2(k+1)}} \, , \end{cases} \tag{4.6}$$

where $r = |x|$ and $s = |y|$. Let us set then, for $\frac{1}{\beta} = 1 - \frac{1}{\alpha}$,

$$I_R := \int_{\mathbb{R}^N} u^\alpha (\theta_R \varphi_R)^\beta d\xi \tag{4.7}$$

From (**??**) we obtain

$$0 \le I_R = -\int_{\Omega_R} uL[(\theta_R\varphi_R)^\beta]dxdy + \int_{\partial\Omega_R} u\beta(\theta_R\varphi_R)^{\beta-1}\nabla_L(\theta_R\varphi_R)\cdot\nu_L d\Sigma \ \ , \tag{4.8}$$

where $\nabla_L u = (X_1 u, \ldots, X_N u)$ and $\nu_L = (\nu_1, \ldots, \nu_p, |x|^k\nu_{p+1}, \ldots, |x|^k\nu_N)$, ν being the exterior normal to $\partial\Omega_R$.
On the other hand, simple computations show that

$$\begin{array}{ll} L\varphi_R = \Delta_p\varphi_R) \quad , & L\theta_R = |x|^{2k}\Delta_q\theta_R \ , \\ \nabla_L\varphi_R = (\nabla_x\varphi_R, 0) \quad , & \nabla_L\theta_R = (0, |x|^k\nabla_y\theta_R) \end{array}$$

where $\Delta_p,\ \Delta_q$ denote the Laplacians on $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively.
The integral on the boundary in (**??**) vanishes since $\nabla_L\varphi_R\cdot\nabla_L\theta_R = \theta_R\varphi_R = 0$ on $\partial\Omega_R$ and $\beta > 1$. Therefore, by the properties of φ_R and θ_R and setting $\Sigma_R := \Omega_R \setminus (B_p(0, \frac{R}{2}) \times B_q(0, \frac{R^{k+1}}{2}))$, we obtain

$$\begin{aligned} I_R &\le -\int_{\Sigma_R} u\beta[\varphi_R^\beta\theta_R^{\beta-1}|x|^{2k}(\theta_R'' + \frac{q-1}{s}\theta_R') + \varphi_R^{\beta-1}\theta_R^\beta(\varphi_R'' + \frac{p-1}{r}\varphi_R')]d\xi \\ &\le \frac{C}{R^2}\int_{\Sigma_R} u\beta\varphi_R^\beta\theta_R^{\beta-1}d\xi. \end{aligned}$$

By the Hölder inequality then

$$I_R \le \frac{C}{R^2}[\int_{\Sigma_R} u^\alpha(\varphi_R\theta_R)^\beta dxdy]^{\frac{1}{\alpha}}[\int_{\Sigma_R} dxdy]^{\frac{1}{\beta}} \tag{4.9}$$

yielding

$$I_R^{\frac{1}{\beta}} \le CR^{Q-2-\frac{Q}{\alpha}}.$$

At this point the claim follows by the same arguments as in the proof of Theorem 3.2. $\square$

The next result concerns the $k-$dimensional $(1 < k < N)$ Laplace operator on $\mathbb{R}^N$, that is

$$\Delta_k = \sum_{i=1}^k \frac{\partial^2}{\partial x_i^2}$$

This example shows that subellipticity is not necessary to obtain semilinear Liouville type results. The main ingredients in the proof are again the $2-$homogeneity of the operator with respect to a suitable family of dilations and that the balls associated with an appropriately defined distance invade the whole space as the radius diverges.

The result is as follows:

Theorem 4.2 *Let $u \in C^2$ be a solution of*

$$u \geq 0, \quad \Delta_k u + u^\alpha \leq 0 \quad \text{in } \mathbb{R}^N \tag{4.10}$$

If $k > 2$ and $1 < \alpha < \frac{k}{k-2}$, then $u \equiv 0$. The same conclusion holds if $k = 2$ and $\alpha > 1$.

Proof of Theorem ??

The proof is similar to that of Theorem **??**. The first observation is that, for every $\epsilon > 0$, the operator Δ_k is $2-$homogeneous with respect to the following dilations:

$$\delta_\lambda(\xi) = \delta_\lambda(x_1, \ldots, x_N) = (\lambda x_1, \ldots, \lambda x_k, \lambda^\epsilon x_{k+1}, \ldots, \lambda^\epsilon x_N) \tag{4.11}$$

since Δ_k does not act on the variables x_j for $j = k+1, \ldots, N$.
As in the proof of Theorem **??** one considers then the sets

$$B_k(0, R) \times B_{N-k}(0, R^\epsilon)$$

where B_j denotes the $j-$dimensional euclidean ball. Set $\xi = (x, y)$ with $x = (x_1, \ldots, x_k)$ and $y = (x_{k+1}, \ldots x_N)$ and consider the same cut-off functions φ_R , θ_R defined in (**??**) with $k+1$ replaced by ϵ.
Proceeding as in the proof of Theorem **??** one shows then that the integral

$$I_R := \int_{\mathbb{R}^N} u^\alpha \theta_R {\varphi_R}^\beta d\xi \ , \ (\frac{1}{\alpha} + \frac{1}{\beta} = 1) \ ,$$

satisfies

$$I_R^{\frac{1}{\beta}} \leq C R^{\epsilon(N-k)+k-2-\frac{k}{\alpha}} \tag{4.12}$$

Let $k > 2$. By assumption, $\alpha < \frac{k}{k-2}$; hence one can choose $\epsilon > 0$ so small that $\alpha < \frac{k}{k-2+\epsilon(N-k)}$. Thus (**??**) implies that I_R goes to zero as $R \to \infty$.

In the case $k = 2$, for every $\alpha > 1$ there exists $\epsilon > 0$ such that $\alpha < \frac{2}{\epsilon(N-2)}$ and we conclude again from (**??**) that $I_R \to 0$ as $R \to \infty$.
On the other hand, $B_{N-k}(0, R^\epsilon) \times B_k(0, R)$ invade $\mathbb{R}^N$ as R goes to infinity. Hence

$$I_R \to \int_{\mathbb{R}^N} u^\alpha d\xi$$

as $R \to \infty$ and this implies $u \equiv 0$. $\Box$

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