ARITHMETIC PROPERTIES OF THE COHOMOLOGY OF ARTIN GROUPS

C. De Concini*, C. Procesi*, M. Salvetti*, F. Stumbo*

February 1999

Introduction. Let $R := \mathbb{Q}[q, q^{-1}]$ be the ring of rational Laurent polynomials in one variable q. Let $Br(n)$ be the Artin braid group with n strings and let R_q be the $Br(n)$-module given by the action over R defined by mapping each standard generator of $Br(n)$ to the multiplication by $-q$. In [DPS] the first three authors computed the cohomology of $Br(n)$ with coefficients in R_q. All cohomology modules are cyclotomic fields (or zero) and the table $(H^i(Br(j); R_q))_{i,j}$ has a very interesting arithmetic behaviour.

Here we consider all other Artin groups, associated to finite irreducible Coxeter groups of type $\mathbf{B}_n$ ($= \mathbf{C}_n$), $\mathbf{D}_n$, more all exceptional cases. We compute the cohomology of these groups with coefficients in the same module R_q, where the action is still given by $q-$multiplication. The results can be expressed similarly to case $\mathbf{A}_n$. Let $\mu_d(q)$ denote the cyclotomic polynomial of primitive $d-$roots of 1 and let

$$\{d\} := \mathbb{Q}[q, q^{-1}]/\mu_d(q) = \mathbb{Q}[q]/\mu_d(q)$$

be the associated cyclotomic field, thought as $R-$module.

Then in case $\mathbf{B}_n$ we have

Theorem (case $\boldsymbol{B}_n$).

$$\begin{aligned} H^{m-2s+1}(BO_m, R_q) &= \oplus_{r \le \frac{m}{2s},\ r|m}\{2r\}, && (s > 1) \\ H^{m-2(s-1)}(BO_m, R_q) &= \oplus_{r \le \frac{m-1}{2(s-1)},\ r|m}\{2r\}, && (s > 1) \\ H^m(BO_m, R_q) &= \oplus_{r|m}\{2r\} \end{aligned}$$

In case $\mathbf{D}_n$ the algebraic complex which compute cohomology splits as a direct sum of two subcomplexes, which are the invariant and anti-invariant parts with respect to a suitable involution. The invariant subcomplex is isomorphic to the complex of case $\mathbf{A}_{n-1}$. We obtain

*Partially supported by M.U.R.S.T. 40%

Theorem (case D_n). *With the convention*

$$\{h\} = \begin{cases} R/(\mu_h) & \text{if } h \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

we have that

$$H^m(D_m) = \left\{ \frac{[2(m-1)]}{[m-1]} \right\} \bigoplus_{\substack{h \mid m \\ 2 \mid h \\ 2 < h}} \{h\}$$

$$H^{m-2s}(D_m) = \bigoplus_{0 \le k} \left\{ \frac{2(m-1)}{2s+2k+1} \right\} \bigoplus_{k \mid \frac{m}{2}} \{2k\} \bigoplus_{2 \mid \frac{m}{s+1}} \left\{ \frac{m}{s+1} \right\} \bigoplus \left\{ \frac{m-1}{s} \right\} \qquad \text{for } s > 0$$

$$H^{m-2s+1}(D_m) = \bigoplus_{\substack{2 < h \le \frac{m-3}{s} \\ 2 \mid h \\ P_m(h)}} \{h\} \bigoplus_{\substack{2 \mid \frac{m}{s} \\ 2 < \frac{m}{s}}} \left\{ \frac{m}{s} \right\} \bigoplus \left\{ \frac{m}{s} \right\} \qquad \text{for } s > 0$$

The methods of proof are similar to case $\mathbf{A}_n$: we consider a filtration of subcomplexes of the algebraic complex, coming from [S] (see also [DS]), which computes the cohomology of Artin groups. The cohomologies of these subcomplexes are related in short exact sequences with the cohomology of Artin groups of type **A** and different ranks. We look at boundary operators in the associated long exact sequences and we determine the conditions in order that such boundaries vanish. Then, starting from below, we use induction to recover the cohomology of the Artin group.

We decided to skip some technical details which are very similar to case $\mathbf{A}_n$, putting the precise reference for them.

For exceptional cases we developed a suitable computer program: we present here a table containing all their cohomology groups. This is included here by completeness: a similar table was already given in [S], but it contained some misprints.

The integer cohomology of Artin groups for the series **B** and **D** was computed in [G].

Our results in case of infinite series agree with stability results which were found in [DS1].

The "top" cohomology of Artin groups was obtained in a more general form in [DSS].

1. Strings (case B_n)

Let us start recalling some notation from [DPS]. As usual, set

$$[n] := \frac{q^n - 1}{q - 1}, \quad [n]! := \prod_{i=1}^{n} [i], \ \left[{n \atop k}\right] := \frac{[n]!}{[k]![n-k]!}$$

as the $q-$analogue of the numbers n, $n!$, $\binom{n}{k}$.

So

$$[n] = \prod_{d|n,\ d\neq 1} \mu_d(q)$$

and

$$R/([n]) = \oplus_{d|n,\ d\neq 1}\{d\}.$$

We start establishing some notations slightly different from the usual.

We will denote a subdiagram of B_m by its characteristic function which we display as a sequence of $0, 1$; in short, a *string of length* m the only proviso is that we box the last node of the Dynkin diagram as in $01\boxed{1}$.

This allows us to multiply strings which is useful for some cochain computations. We let the degree of a string to be the number of $1's$ in the string.

For a string s we define $s! = \prod_i [k_i + 1]![2h]!!$ where the numbers k_i are the lengths of the maximal substrings of s which are formed entirely by 1's and are not boxed and h the length of the maximal string of 1 containing $\boxed{1}$.

Next let us recall, with a different notation, the basic complex C^*_m of Salvetti [S] (see also [DS] for variations and generalizations).

C^k_m has as basis the strings B of length m and degree $k = |B|$.

The coboundary $d : C^*_m \to C^*_m$ can be defined on the strings by the following rules:

(1.1)
$$d((A0)(0B)) = d(A0)0B + (-1)^{|A|} A0d(0B).$$

(1.2)
$$d(A101B) := d(A1)01B + (-1)^{|A|+1} A01d(1B) + (-1)^{|A|+1} \frac{A111B!}{A101B!} A111B.$$

(1.3)
$$d1^k = 0, \quad d01^k = [k+2]1^k\boxed{1}, \quad d1^k0 = (-1)^k[k+2]1^k\boxed{1}, \quad d01^k0 = [k+2](1^k\boxed{1}0 + (-1)^k 01^{k+1})$$

$$d1^{k-1}\boxed{1} = 0, \quad d01^{k-1}\boxed{1} = \frac{[2(k+1)]!!}{[2k]!!} 1^k\boxed{1}$$

The main fact we shall need from [S] is that, for all $t \geq 0$,

$$H^t C_m = H^t(BO_{m+1}, R_q).$$

Using this we shall from now on compute the cohomology of the complex C_m.

In order to do induction we shall consider the following subcomplexes and maps.

Take the map

$$\pi_m := \pi : C_m \to C_{m-1}, \ \pi(A\boxed{0}) := A; \ \pi(A\boxed{1}) := 0$$

π is clearly a homomorphism of complexes with kernel the subcomplex F_m^1 : having as basis all strings $A1$ ending with 1.

We continue in this way getting maps

$$\pi_m[k] := \pi : F_m^k \to C_{m-k-1}[k], \ \pi(A01^{k-1}\boxed{1}) := A; \ \pi(A1^k\boxed{1}) := 0$$

which induce isomorphisms of complexes:

$$F_m^k / F_m^{k+1} = C_{m-k-1}[k], \ k = 0, \dots, m-2$$

up to F_m^{m-1} which has as basis $b_{m+1} = 01^{m-2}\boxed{1}, 1^{m-1}\boxed{1} = c_{m+1} = v_{m+1}(1)$ with differential $db_{m+1} = \frac{[2m]!!}{[2(m-1)]!!} c_{m+1} = [2m] c_{m+1}$.

The starting point of the computation is of course the computation of the cohomology of F_m^{m-1} which is clearly 0 except for

$$H^m(F_m^{m-1}) = \mathbb{Q}[q, q^{-1}][v_{m+1}(1)] = R/([2m]) = \oplus_{h|2m}\{h\} \tag{1.4}$$

We need to do induction and thus it is again necessary to describe also the cohomology of the complexes F_m^k.

2. Euler Poincaré series

In order to motivate our next computations let us immediately deduce the computation of the Euler Poincaré characteristic.

Proposition.

$$\chi_m := (-1)^m([2m]-[m]),$$

dim. We have the recursive formulas:

$$\chi_m := \chi(F_m^{m-1}) + \sum_{k=0}^{m-2} \chi(F_m^k/F_m^{k+1}) = (-1)^m[2m] + \sum_{k=0}^{m-2}(-1)^k \chi(C_{m-k-1}) =$$

$$(-1)^m[2m] + \sum_{k=0}^{m-2}(-1)^k \chi_{m-k-1} = (-1)^m([2m]-[m])$$

□

Remark that the previous argument brings also to

$$\chi(F_m^k) = (-1)^m([2m]-[m-k]).$$

3. Preparation for the Main Theorem

We repeat the argument in type A_n. Consider the exact sequence of complexes

$$0 \to F_m^{k+1} \to F_m^k \to C_{m-k-1}[k] \to 0$$

and look at a typical piece of the corresponding long exact sequence:

$$\to H^{t-1}C_{m-k-1}[k] \xrightarrow{\delta} H^t F_m^{k+1} \to H^t F_m^k \xrightarrow{\pi} H^t C_{m-k-1}[k] \to H^{t+1}F_m^{k+1} \to \tag{3.1}$$

We know the terms

$$H^{t-1}C_{m-k-1}[k] = H^{t-k-1}C_{m-k-1}, \quad H^t C_{m-k-1}[k] = H^{t-k}C_{m-k-1}$$

which are either 0 or irreducibles of type $\{h\}$ generated by the appropriate cohomology classes. Therefore we can analyze the change in the module structure from $H^t F_m^{k+1}$ to $H^t F_m^k$.

The main point is the computation of the connecting homomorphisms, that we separate in two cases:

Case A) $H^{m-2s}(C_{m-k-1})[k] \xrightarrow{\delta} H^{m-2s+1}(F_m^{k+1})$.

Case B) $H^{m-2(s-1)-1}(C_{m-k-1})[k] \xrightarrow{\delta} H^{m-2(s-1)}(F_m^{k+1})$.

3.2 Case A). We may assume that $H^{m-2s}(C_{m-k-1})[k] \neq 0$ then, by induction,

$$H^{m-2s}(C_{m-k-1})[k] = H^{m-2s-k}(C_{m-k-1}) = H^{(m-k-1)-2s+1}(C_{m-k-1}) = \{h\}$$

if and only if

$$sh = m-(k+1), \quad h \quad \text{is an integer}$$

The module $\{h\}$ is irreducible and generated by the class of $z_h(s)$.

3.3 Lemma. *If $h := \frac{m-(k+1)}{s}$ is an integer then, $\{h\} = H^{m-2s}(C_{m-k-1})[k]$ generated by the class of $z_h(s)$ and:*

a) If $h|2(k+1)$ (or $h|2m$) the map δ is equal to 0.

b) If $h \not| 2(k+1)$, (or $h \not| 2m$), the map δ is different from 0.

Proof. In order to compute the transform of $z_h(s)$ under the connecting homomorphism we lift $z_h(s)$ to $z_h(s)01^{k-1}\boxed{1} \in F_m^k$ and compute $d(z_h(s)01^{k-1}\boxed{1}) \in F_m^{k+1}$.

We perform the computation as follows; first, compare $[h]z_h(s)01^{k-1}\boxed{1} = d(w_h^s)01^{k-1}\boxed{1}$ with $d(w_h^s 01^{k-1}\boxed{1})$. By formula 1.1 we have

$$d(w_h^s 01^{k-1}\boxed{1}) = d(w_h^s)01^{k-1}\boxed{1} + (-1)^{hs}[2(k+1)]w_h^s 1^k\boxed{1}.$$

a) If $h|2(k+1)$ then

$$d(z_h(s)01^{k-1}\boxed{1}) = (-1)^{hs}\frac{[2(k+1)]}{[h]}d(w_h^s 1^k\boxed{1})$$

is a coboundary in F_m^{k+1}.

b) Assume now that $h \not| 2(k+1)$. We have

$$d(w_h^s 1^k\boxed{1}) = \{[h](\sum_{j=0}^{s-2}(-1)^{hj}w_h^j z_h w_h^{s-j-1}1^k\boxed{1}) + (-1)^{hs}\frac{[2(h+k)]!!}{[h-1]![2(k+1)]!!}w_h^{s-1}01^{h+k-1}\boxed{1}\} =$$

$$\{[h]z_h(s)1^k\boxed{1} + (-1)^{hs}(\frac{[2(h+k)]!!}{[h-1]![2(k+1)]!!} - [h])w_h^{s-1}01^{h+k-1}\boxed{1}\}.$$

Since h does not divide $2(k+1)$ we have that μ_h divides $\frac{[2(h+k)]!!}{[h-1]![2(k+1)]!!} = \frac{[2(k+h)]\ldots[2(k+2)]}{[h-1]!}$, hence we can define the cocycle

$$z_{h,k+1}(s) := \mu_h^{-1}d(w_h^s 1^k\boxed{1}) \in F_m^{k+1,s(h-2)+2+k}$$

and

$$\frac{[h]}{\mu_h} d(z_h(s)01^{k-1}\boxed{1}) = (-1)^{hs}[2(k+1)]z_{h,k+1}(s)$$

We claim that $[2(k+1)][z_{h,k+1}(s)]$ is a non zero element in $H^{m-2s+1}(F_m^{k+1})$.

The cohomology class of $z_{h,k+1}(s)$ is non zero since it projects to the cohomology class of $v_h(s)$ in $H^{m-2s+1}(C_{m-(k+1)-1}[k])$, which is non 0 and generates a module isomorphic to $\{h\}$.

Therefore the claim follows since $h \not| 2(k+1)$, $[2(k+1)]$ and $[h]/\mu_h$ are not zero in $\{h\}$.

$\square$

3.4 Case B). We assume that $H^{m-2(s-1)-1}(C_{m-k-1})[k] \neq 0$ then:

$$H^{m-2(s-1)-1}(C_{m-k-1})[k] = H^{(m-k-1)-2(s-1)}(C_{m-k-1}) = \{h\}$$

if and only if $\quad sh = m-k = (m-k-1)+1, \quad h \quad$ is an integer.

The module $\{h\}$ is irreducible and generated by the class of $v_h(s)$.

3.5 Lemma. *If $h := \frac{m-k}{s}$ is an integer so that $H^{m-2(s-1)-1}(C_{m-k-1})[k] = \{h\}$:*

a) If h is even or it is odd and does not divide k, the map δ is equal to 0.

b) If h is odd and divides k (or divides m) the map δ is different from 0.

Proof. As we have done in Lemma 3.4, in order to compute the image of the cohomology class of $v_h(s)$ under the connecting homomorphism:

$$H^{m-2(s-1)-1}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2(s-1)}F_m^{k+1}$$

we lift $v_h(s)$ to $v_h(s)01^{k-1}\boxed{1} \in F_m^k$ and compute $d(v_h(s)01^{k-1}\boxed{1}) \in F_m^{k+1}$.

We have

$$d(w_h^s 1^{k-1}\boxed{1}) = \{[h]v_h(s)01^{k-1}\boxed{1} + (-1)^{hs}(\frac{[2(h+k-1)]!!}{[h-1]![2k]!!} w_h^{s-1}01^{h+k-1}\}$$

a) If h does not divide $2k$ then $[h]$ divides $\frac{[2(h+k-1)]!!}{[h-1]![2k]!!}$ and so $v_h(s)$ lifts in F_m^k to the cocycle $[h]^{-1}d(w_h^s 1^{k-1}\boxed{1})$ and thus $\delta = 0$.

b) If h divides $2k$ thus we can write $v_h(s)01^{k-1}\boxed{1}$ as

$$[h]^{-1}(d(w_h^s 1^{k-1}\boxed{1}) - (-1)^{hs}\frac{[2(h+k-1)]!!}{[h-1]![2k]!!} w_h^{s-1}01^{h+k-2}\boxed{1})$$

when we apply the differential to this lift we get up to a sign

$$[h]^{-1}\frac{[2(h+k-1)]!!}{[h-1]![2k]!!}d(w_h^{s-1}01^{h+k-2}\boxed{1}) = \frac{[2(h+k-1)]!!}{[h-1]![2k]!!}(z_h(s-1)01^{h+k-2}\boxed{1}\pm\frac{[2(h+k)]}{[h]}w_h^{s-1}1^{h+k-1}\boxed{1})$$

We set

$$v_{h,k+1}(s) := z_h(s-1)01^{h+k-2}\boxed{1}\pm\frac{[2(h+k)]}{[h]}w_h^{s-1}1^{h+k-1}\boxed{1}$$

Observe that this element lies in F^{h+k-1} and its image in $C_{m-(h+k-1)-1} = C_{m-(h+k)}$ is exactly $z_h(s-1)$. By induction $z_h(s-1)$ gives a non 0 cohomology class in $H^{(s-1)(h-2)+1}C_{m-(h+k-1)-1}$ generating a module isomorphic to $\{h\}$. Hence the cohomology class of $v_{h,k+1}(s)$ in $H^{m-2s-1}F^{h+k-1}$ is non zero and the module it generates has a factor isomorphic to $\{h\}$.

In order to show that the element $z_h(s-1)$ gives a non 0 cohomology class in $H^{m-2(s-1)}F^{k+1}$ we prove that the contribution of the factor $\{h\}$ to $H^{m-2(s-1)}F_m^{h+k-1}$ does not disappear when passing to $H^{m-2(s-1)}F_m^{k+1}$.

For this let us remark that whenever we pass from a level $r+1$ of filtration to r we have the exact sequence:

$$H^{m-2s+1}(C_{m-r-1}[r]) \xrightarrow{\delta} H^{m-2(s-1)}(F_m^{r+1}) \to H^{m-2(s-1)}(F_m^r) \xrightarrow{\pi} H^{m-2(s-1)}(C_{m-r-1}[r])$$

so an irreducible factor in $H^{m-2s}(F_m^{r+1})$ of type $\{h\}$ can disappear in $H^{m-2s}(F_m^r)$ only if $H^{m-2s-1}(C_{m-r-1}[r]) = \{h\}$ and $\delta \neq 0$. In particular only when $h := \frac{m-r}{s}$ and thus in the inclusion $H^{m-2(s-1)}F_m^{h+k-1} \xrightarrow{i} H^{m-2(s-1)}F_m^{k+1}$ the factor $\{h\}$ is not cancelled.

In fact we have shown that, in $H^{m-2(s-1)}F_m^{k+1}$, the element $[z_h(s-1)]$ generates a submodule with a factor of type $\{h\}$.

To finish we verify that if h is odd then $\frac{[2(h+k-1)]!!}{[h-1]![2k]!!} \neq 0$ in $\{h\}$.

In fact

$$\frac{[2(h+k-1)]!!}{[h-1]![2k]!!} = \frac{[2(k+h-1)][2(k+h-2)]\ldots[2(k+1)]}{[h-1]!}$$

and since h divides $2k$ it must divide k and its next multiple is $k+h$ and so the claim follows. Assume now $h = 2r$ and r divides k, the factor $2(k+r)$ gives 0 in $\{h\}$ finishing the proof of the Lemma. □

Summarizing:

Case A) $H^{m-2s}(C_{m-k-1})[k] \xrightarrow{\delta} H^{m-2s+1}(F_m^{k+1})$.

If $h := \frac{m-(k+1)}{s}$ is an integer, then $\{h\} = H^{m-2s}(C_{m-k-1})[k]$

a) If $h|2m$ the map δ is equal to 0.

b) If $h \not| 2m$, the map δ is different from 0.

Case B) $H^{m-2(s-1)-1}(C_{m-k-1})[k] \xrightarrow{\delta} H^{m-2(s-1)}(F_m^{k+1})$.

If $h := \frac{m-k}{s}$ is an integer, then $H^{m-2(s-1)-1}(C_{m-k-1})[k] = \{h\}$:

a) If h is even or it is odd and does not divide m, the map δ is equal to 0.

b) If h is odd and divides m the map δ is different from 0.

$$
\begin{array}{r}
H^{m-2s}(C_{m-k-1}[k]) \xrightarrow{\delta} H^{m-2s+1}(F_m^{k+1}) \rightarrow \\
H^{m-2s+1}(F_m^k) \xrightarrow{\pi} H^{m-2(s-1)-1}(C_{m-k-1}[k]) \xrightarrow{\delta'} H^{m-2(s-1)}(F_m^{k+1})
\end{array}
$$

Let $A(s,k)$ be the class of $H^{m-2s+1}(F_m^k)$ in the Grothendieck group then:
If $h := \frac{m-(k+1)}{s}$ is an integer and does not divide $2m$ then,
From $A(s,k+1)$ to $A(s,k)$ we have that $\{h\}$ is deleted.

If $g := \frac{m-k}{s}$ is an integer and g is even or it is odd and does not divide m, passing from $A(s,k+1)$ to $A(s,k)$ we have that $\{g\}$ is added.

Conclusion:
i) h odd and h does not divide m, if $m-hs \geq k$ then $\{h\}$ has been added but if $m-hs \geq k+1$ then $\{h\}$ has been also deleted so remain only the odd h which do not divide m and with $m-hs=k$ or $h := \frac{m-k}{s}$ if an odd integer and $h \not| m$.

ii) If g is even then $m-sg \geq k$ implies that it is added but if g does not divide $2m$ then if $m-gs \geq k+1$ it has also been deleted.

So add all even $g \le \frac{m-k}{s}$ and $g|2m$ and all even g with $g \not|2m$ and $\frac{m-(k+1)}{s} < g \le \frac{m-k}{s}$ thus $g = \frac{m-k}{s}$ if even and does not divide $2m$.

$$H^{m-2s+1}(F_m^k) = \begin{cases} \oplus_{r \le \frac{m-k}{2s},\ r|m}\{2r\} & \text{unless} \quad h := \frac{m-k}{s} \quad \text{is an integer and} \quad h \not| m \\ \oplus_{r \le \frac{m-k}{2s},\ r|m}\{2r\} \oplus \{h\} & \text{if} \quad h := \frac{m-k}{s} \quad \text{is an integer and} \quad h \not| m \end{cases}$$

$$H^{m-2(s-1)-1}(C_{m-k-1}[k]) \xrightarrow{\delta} H^{m-2(s-1)}(F_m^{k+1}) \to$$
$$H^{m-2(s-1)}(F_m^k) \xrightarrow{\pi} H^{m-2(s-1)}(C_{m-k-1}[k]) \xrightarrow{\delta'} H^{m-2(s-1)+1}(F_m^{k+1})$$

Let $B(s,k)$ be the class of $H^{m-2(s-1)}(F_m^k)$ in the Grothendieck group then:

If $h := \frac{m-k}{s}$ is an integer, h is odd and divides m, we have that $\{h\}$ is deleted from $B(s,k+1)$ to $B(s,k)$.

If $g := \frac{m-(k+1)}{s-1}$ is an integer if $g|2m$, from $B(s,k+1)$ to $B(s,k)$ we have that $\{g\}$ is added.

Conclusion:

$h|m$ odd, if $m - h(s-1) \ge k+1$ then $\{h\}$ has been added, but if $m - hs \ge k$ then $\{h\}$ has been also deleted so remain only the odd $h|m$ with $m - hs = k$ or $h := \frac{m-k}{s}$ if an odd integer.

Or the odd divisors of m which appear in $B(s,k)$ are subject to

$$\frac{m-k}{s} < h \le \frac{m-(k+1)}{s-1}.$$

The even divisors of $2m$ which appear in $B(s,k)$ are subject to

$$m - (s-1)g \ge k+1.$$

$$H^{m-2(s-1)}(F_m^k) = \oplus_{r \le \frac{m-(k+1)}{2(s-1)},\ r|m}\{2r\} \oplus_{\frac{m-k}{s} < 2r+1 \le \frac{m-(k+1)}{s-1},\ 2r+1|m} \{2r+1\}$$

For $s = 1$ nothing is ever added but only the odd divisors h of m with $m - hs \ge k$ have been deleted.

4. The Theorem

Theorem. *A)*

$$H^{m-2s+1}(F_m^k) = \begin{cases} \oplus_{r\le \frac{m-k}{2s},\ r|m}\{2r\} & \textit{unless} \quad h := \frac{m-k}{s} \quad \textit{is an integer and} \quad h \not| m \\ \oplus_{r\le \frac{m-k}{2s},\ r|m}\{2r\} \oplus \{h\} & \textit{if} \quad h := \frac{m-k}{s} \quad \textit{is an integer and} \quad h \not| m \end{cases}$$

B)

$$H^{m-2(s-1)}(F_m^k) = \oplus_{r\le \frac{m-(k+1)}{2(s-1)},\ r|m}\{2r\} \oplus_{\frac{m-k}{s}<2r+1\le \frac{m-(k+1)}{s-1},\ 2r+1|m}\{2r+1\}$$

$k = 0$:

A)

$$H^{m-2s+1}(BO_m) = \oplus_{r\le \frac{m}{2s},\ r|m}\{2r\}$$

B)

$$H^{m-2(s-1)}(BO_m) = \oplus_{r\le \frac{m-1}{2(s-1)},\ r|m}\{2r\}$$

$$H^{m}(BO_m) = \oplus_{r|m}\{2r\}$$

5. Strings (case D_n)

We finally consider the case D_m. It is generated, in the natural way, by $s_1, \dots, s_m$ with the relations $s_i^2 = 1$, $(s_i, s_{i+1})^3 = 1$ for $i = 1, \dots, m-1$, $(s_{m-2}s_m)^3 = 1$ and $s_is_j = s_js_i$ in the other cases.

Let $C(D_m)$ be the complex for its cohomology as defined in [S]. As usual, we consider its basis given by the strings of 0's and 1's with m elements. We define an involution of $C(D_m)$ as follows:

$$\sigma(A00) = A00 \qquad \sigma(A11) = -A11$$

$$\sigma(A01) = A10 \qquad \sigma(A10) = A01$$

This involution is compatible with the differential, so that

$$C(D_m) = K_n \bigoplus P_m$$

where K_m is the sub-complex of the invariants and P_m is that of the anti-invariants. It is easy to see that $K_m = C(A_{m-1}) = C_{m-1}$, the complex for A_{m-1}, which we already know. So we must compute P_m, which has a basis given by $\{1/2(A01 - A10),\ A11\}$.

To compute the coboundary we need the formulas

$$d1^{m-2}00 = (-1)^{m-2}[m](1^{m-1}0 + 1^{m-2}01)$$

$$d01^{m-1}0 = -d1^{m-2}01 = (-1)^{m-1}\frac{[2(m-1)]!!}{[m-1]!}1^m$$

$$d01^{m-1} = \frac{[2(m-1)][m]}{[m-1]}1^m$$

The other cases are computed in the same way as for A_m. We define $G_m^1 = P_m$, $G_m^i =< A1^i >$ for $i = 2, \dots, m-1$. Moreover, let F_{m-1}^1 be the first step of the filtration for the case A_{m-1} (see [DPS]).

As usual, we consider the map

$$\pi = \pi_m : G_m^i \to C_{m-i-1}[i] \qquad (i > 1)$$

given by $\pi(A01^i) = A$, $\pi(A1^{i+1}) = 0$, while for $i = 1$ we define $\pi : G_m^1 \to F_{m-1}^1$ by $\pi(A11) = 0$, $\pi(1/2(A01 - A10) = A1$.

With these definitions, we have that $G_m^1/G_m^2 = F_{m-1}^1$ and, for $1 < 1 < m-1$, we get $G_m^i/G_m^{i+1} = C_{m-i-1}[i]$. Finally, G_m^{m-1} is the complex

$$\delta : R \to R$$

translated by $m-1$, where δ is the multiplication by $\frac{[2(m-1)][m]}{[m-1]}$.

The starting point of our computation is

$$H^m(G_m^{m-1}) = R/\left(\frac{[2(m-1)][m]}{[m-1]}\right).$$

2. Euler-Poincaré series

Proposition We have

$$\chi(D_m) = (-1)^m\frac{[2(m-1)][m]}{[m-1]} + (-1)^{m-1}[m-2]$$

Proof. We refer to [DPS] for the computation of $\chi(A_j)$ and $\chi(F_k^l)$. Then

$$\begin{aligned}
\chi(D_m) &= \chi(K_m) + \chi(G_m^1) = \chi(K_m) + \chi(G_m^1/G_m^2) + \chi(G_m^1) \\
&= \chi(A_{m-1}) + \chi(F_{m-1}^1[1]) + \sum_{i=2}^{m-2}\chi(G_m^i/G_m^{i+1}) + \chi(G_m^{m-1}) \\
&= (-1)^{m-1}([m]-[m-1]) + (-1)^m([m]-[m-2]) \\
&\quad + (-1)^{m+1}\sum_{k=1}^{m-2}(-1)^k\chi(A_k) + (-1)^m\frac{[2(m-1)][m]}{[m-1]} \\
&= (-1)^{m-1}([m]-[m-1]) + (-1)^m([m]-[m-2]) \\
&\quad + (-1)^{m+1}[m-1] + (-1)^m\frac{[2(m-1)][m]}{[m-1]} \qquad \square
\end{aligned}$$

3. Preparation for the main theorem

Our aim is to compute the cohomology of G^1_m knowing that of G^{m-1}_m, so we consider the exact sequence of complexes $0 \to G^{k+1}_m \to G^k_m \to G^{k+1}_m/G^k_m \to 0$.

¿From this sequence we get a long exact sequence, of which we need to compute the connecting homomorphism, to be able to deduce the cohomology of G^k_m.

We split this in two cases:

Case A) $\delta : H^{m-2s}(G^{k+1}_m/G^k_m) \to H^{m-2s+1}(G^{k+1}_m)$

Case B) $\delta : H^{m-2(s-1)-1}(G^{k+1}_m/G^k_m) \to H^{m-2(s-1)}(G^{k+1}_m)$

Moreover, since the term on the right is not described in a uniform way, we need to distinguish the subcases $k = 1$ and $k > 1$.

We begin with $k > 1$, case A).

Let us define $P_m(h) = \{h|m \bigvee (h|2(m-1) \bigwedge h \not| (m-1))\}$.

Lemma 1 Let $k > 1$ and

$$d : H^{m-2s}C_{m-k-1}[k] \to H^{m-2s+1}G^{k+1}_m.$$

Then

$$d \neq 0 \Leftrightarrow 1 < h = \frac{m-k-1}{s} \in N \text{ e } \sim P_m(h).$$

Proof. First one has

$$H^{m-2s}C_{m-k-1}[k] = H^{m-2s-k}C_{m-k-1} = H^{(m-k-1)-2s+1}C_{m-k-1}$$

and this does not vanish iff $1 < h = \frac{m-k-1}{s} \in N$; in which case it is isomorphic to $\{h\}$ and generated by the class of $z_h(s)$. So let us assume $1 < h = \frac{m-k-1}{s} \in N$. $z_h(s)$ lifts to $z_h(s)01^k \in G^k_m$ and $[h]z_h(s)01^k = d(w^s_h)01^k$. Now

$$d(w^s_h 01^k) = d(w^s_h)01^k + (-1)^{hs}\frac{[2k][k+1]}{[k]}w^s_h 1^{k+1}$$

from which

$$[h]d(z_h(s)01^k) = (-1)^{hs+1}\frac{[2k][k+1]}{[k]}d(w^s_h 1^{k+1}).$$

With these assumptions, $\mu_h | \frac{[2k][k+1]}{[k]} \Leftrightarrow P_{k+1}(h) \Leftrightarrow P_m(h)$. If this happens then $\mu_h^{-1}[h]d(z_h(s)01^k)$ is a coboundary in G^{k+1}_m so $d = 0$.

Assume now $\sim P_m(h)$. Then

$$[h]d(z_h(s)01^k) = (-1)^{hs+1}\frac{[2k][k+1]}{[k]}d(w^s_h 1^{k+1})$$

$$= (-1)^{hs+1}\frac{[2k][k+1]}{[k]}\left\{[h]\left(\sum_{j=0}^{s-2}(-1)^{hj}w^j_h z_h w^{s-j-1}_h 1^{k+1} + (-1)^{h(s-1)}w^{s-1}_h 1^{h-1}01^{k+1}\right)\right.$$

$$+(-1)^{hs}\frac{[2(k+h-1)]!![k+h]}{[h-1]![2k]!![k+1]}w_h^{s-1}01^{h+k}\Bigg\}$$

$$= (-1)^{hs+1}\frac{[2k][k+1]}{[k]}\Bigg\{[h]z_h(s)1^{k+1}+(-1)^{hs}\left(\frac{[2(k+h-1)]!![k+h]}{[h-1]![2k]!![k+1]}-[h]\right)w_h^{s-1}01^{h+k}\Bigg\}$$

$$= (-1)^{hs+1}\Bigg\{\frac{[2k][k+1]}{[k]}[h]z_h(s)1^{k+1}+(-1)^{hs}\left(\frac{[2(k+h-1)]!![k+h]}{[h-1]![2(k-1)]!![k]}-\frac{[2k][k+1]}{[k]}[h]\right)w_h^{s-1}01^{h+k}\Bigg\}.$$

Since $\sim P_m(h)$, the previous expression, divided for μ_h, projects on

$$\frac{[2k][k+1][h]}{[k]\mu_h}\overline{v_h(s)}$$

in

$$H^{m-2s+1}C_{m-(k+1)-1}[k+1] = \{h\} = \langle\overline{v_h(s)}\rangle$$

so it has a non vanishing value in this last module; so $d \neq 0$. □

Now we study case B).

Define $Q_m(h) = \{h|m \bigwedge (2 \not| h \bigvee h = 2)\}$.

Lemma 2 Let $k > 1$ and

$$d : H^{m-2(s-1)-1}C_{m-k-1}[k] \to H^{m-2(s-1)}G_m^{k+1}.$$

Then

$$d \neq 0 \Leftrightarrow 1 < h = \frac{m-k}{s} \in N \text{ and } Q_m h.$$

Proof. One has

$$H^{m-2(s-1)-1}C_{m-k-1}[k] = H^{(m-k-1)-2(s-1)}C_{m-k-1} = \{h\} = \langle\overline{v_h(s)}\rangle$$

iff $1 < h = \frac{m-k}{s} \in N$.

Lifting $v_h(s)$ to $v_h(s)01^k \in G_m^k$ then $w_h^s 1^k = w_h^{s-1}b_h 01^k$ and

$$d(w_h^s 1^k) = d(w_h^{s-1}b_h 01^k) = d(w_h^{s-1}b_h)01^k+(-1)^{hs}\gamma w_h^{s-1}01^{k+h-1} = [h]v_h(s)01^k+(-1)^{hs}\gamma w_h^{s-1}01^{k+h-1}$$

where we set

$$\gamma = \frac{[2(k+h-2)]!![k+h-1]}{[h-1]![2(k-1)]!![k].}$$

It is easy to see that $\mu_h|\gamma \Leftrightarrow \sim Q_k(h) \Leftrightarrow \sim Q_m(h)$. If $\mu_h|\gamma$, then

$$\mu_h^{-1}[h]d(v_h(s)01^k) = (-1)^{hs+1}\mu_h^{-1}\gamma d(w_h^{s-1}01^{k+h-1})$$

and one concludes as in lemma 1.

So assume $\mu_h \not| \gamma$.

$$[h]d(v_h(s)01^k) = (-1)^{hs+1}\gamma d(w_h^{s-1}01^{k+h-1}) =$$

$$= (-1)^{hs+1}\gamma\left\{[h]z_h(s-1)01^{k+h-1} + (-1)^{h(s-1)}\frac{[2(k+h-1)][k+h]}{[k_h-1]}w_h^{s-1}1^{k+h}\right\}.$$

Setting $\gamma\prime = \gamma\frac{[2(k+h-1)][k+h]}{[k+h-1]}$, one has that $\mu_h|\gamma\prime$. Therefore,

$$\mu_h^{-1}[h]d(v_h(s)01^k) = (-1)^{hs+1}\gamma\frac{[h]}{\mu_h}z_h(s-1)01^{k+h-1} + (-1)^h\frac{\gamma\prime}{\mu_h}w_h^{s-1}1^{k+h}.$$

This element belongs to G_m^{k+h-1} and its image in the module $H^{(s-1)(h-2)+1}C_{m-(k+h-1)-1}$ is, up to sign, $\mu_h^{-1}[h]\gamma z_h(s-1)$, which does not vanish by case A_n and by $\mu_h \not| \gamma$.

So the cohomology class of $\mu_h^{-1}[h]d(v_h(s)01^k)$ does not vanish in $H^{m-2(s-1)}G_m^{k+h-1}$ and generates a module isomorphic to $\{h\}$. Let us see that this submodule does not disappear passing from $H^{m-2(s-1)}G_m^{k+h-1}$ to $H^{m-2(s-1)}G_m^{k+1}$. Consider

$$H^{m-2s+1}C_{m-r-1}[r] \to H^{m-2(s-1)}G_m^{r+1} \to H^{m-2(s-1)}G_m^r \to H^{m-2(s-1)}C_{m-r-1}[r]$$

with $r = k+h-1, k+h-2, \dots, k+2, k+1$.

Since

$$H^{m-2s+1}C_{m-r-1}[r] = H^{m-2s+1-r}C_{m-r-1} = H^{(m-r-1)-2(s-1)}C_{m-r-1}$$

which is equal to $\{\frac{(m-r-1)-1+1}{s}\} = \{\frac{m-r}{s}\}$, when this is integer, then $d \neq 0$ □

We are now ready to compute the cohomology of G_n^k for $k > 1$. We begin with

Lemma 3

$$H^m G_m^{m-1} = \frac{R}{\left(\frac{[m][2(m-1)]}{[m-1]}\right)}$$

$$H^m G_m^k = \left\{\frac{[2(m-1)]}{[m-1]}\right\} \bigoplus_{\substack{h|m \\ 2|h \\ 2<h}} \{h\} \bigoplus_{\substack{h|m \\ 2\not|h \\ h>m-k}} \{h\} \qquad 1 < k < m-1$$

Proof. First formula derives immediately by definitions. The second one follows by induction because $H^mG_m^{k+1} \to H^mG_m^k$ is always surjective and $d: H_{m-1}C_{m-k-1}[k] \to H^mG^{k+1}$ is zero iff it holds $Q_m(h), h = m-k$; therefore, 2 and all odd divisors of m, strictly less than m, cancel in increasing order. □

We use the convention

$$\left\{\frac{h}{s}\right\} = \begin{cases} 0 & \text{if } s \not| h \text{ or } s = h \\ \left\{\frac{h}{s}\right\} & \text{otherwise} \end{cases}$$

Lemma 4

$$H^{m-2(s-1)-1}G_m^{m-k} = \frac{\bigoplus_{\substack{2s \le h \le k \\ \sim Q_m\left(\frac{h}{s}\right)}} \left\{\frac{h}{s}\right\}}{\bigoplus_{\substack{2s \le h \le k-1 \\ \sim P_m\left(\frac{h}{s}\right)}} \left\{\frac{h}{s}\right\}} \qquad k = 1, \dots, m-2$$

Proof. If $k = 1$ there is nothing to proof.
Consider now the following part of the long exact sequence:

$$H^{m-2s}C_k[m-(k+1)] \to H^{m-2(s-1)-1}G_m^{m-k} \to H^{m-2(s-1)-1}G_m^{m-(k+1)} \to H^{m-2(s-1)-1}C_k[m-(k+1)] \to$$

The right term does not vanish iff $\frac{k+1}{s}$ is integer bigger than 1;in this case it holds $\left\{\frac{k+1}{s}\right\}$ and its coboundary is zero iff $\sim Q_m\left(\frac{k+1}{s}\right)$.

Similarly, the left term does not vanish iff $\frac{k}{s}$ is integer bigger than 1, and it holds $\left\{\frac{k}{s}\right\}$ with zero coboundary iff $\sim P_m\left(\frac{k}{s}\right)$. But notice that $\sim P_m\left(\frac{k}{s}\right) \Rightarrow \sim Q_m\left(\frac{k}{s}\right)$. So by induction everything follows. □

If h/s equals 2 or an odd number then $\sim P_m\left(\frac{h}{s}\right) \Leftrightarrow \sim Q_m\left(\frac{h}{s}\right)$, while if it is even bigger than 2 then $\sim Q_m\left(\frac{h}{s}\right)$ always holds. Hence we can write formula in lemma 4 without fractions as follows:

$$H^{m-2(s-1)-1}G_m^{m-k} = \bigoplus_{\substack{2 < h \le \frac{k-1}{s} \\ 2 | h \\ P_m(h)}} \{h\} \bigoplus_{\sim Q_m\left(\frac{k}{s}\right)} \left\{\frac{k}{s}\right\}$$

Lemma 5 For $s > 0$ and $k = 1, \dots, m-2$ one has

$$H^{m-2s}G_m^{m-k} = \frac{\bigoplus_{\substack{2s \le h < k \\ P_m\left(\frac{h}{s}\right)}} \left\{\frac{h}{s}\right\}}{\bigoplus_{\substack{2s \le h \le k \\ Q_m\left(\frac{h}{s+1}\right)}} \left\{\frac{h}{s+1}\right\}}$$

Proof. If k=1 there is nothing to prove. Otherwise, consider the following part of the long exact sequence

$$H^{m-2s-1}C_k[m-(k+1)] \to H^{m-2s}G_m^{m-k} \to H^{m-2s}G_m^{m-(k+1)} \to H^{m-2s}C_k[m-(k+1)] \to$$

The right term does not vanish iff $\frac{k}{s}$ is integer bigger than 1, when it holds $\{\frac{k}{s}\}$, with non zero coboundary iff it holds $P_m(\frac{k}{s})$. The left term does not vanish iff $\frac{k+1}{s+1}$ is integer bigger than 1, when it holds $\{\frac{k+1}{s+1}\}$, with non zero coboundary iff it holds $Q_m(\frac{k+1}{s+1})$. We have to prove that if $(k+1)/(s+1) = l$ is an integer bigger than 1 and it holds $Q_m(l)$ then there exists an integer $2s \leq h < k$ such that $l = h/s$ and it holds $P_m(l)$. Set $h = sl$. Then $2s \leq h = sl = s(k+1)/(s+1) = k+1-(k+1)/(s+1) = k+1-l < k$. Since $Q_m(l) \Rightarrow l|m \Rightarrow P_m(l)$ we can apply induction and conclude. $\square$

In this case also we can write formula in lemma 5 without fractions:

$$H^{m-2s}G_m^{m-k} = \bigoplus_{\substack{2<h\leq \frac{k}{s+1} \\ h\not| m \bigwedge h|2(m-1) \bigwedge h\not| m-1 \\ \text{oppure} \\ h|m \bigwedge 2|h}} \{h\} \bigoplus_{\substack{\frac{k}{s+1}<h<\frac{k}{s} \\ P_m(h)}} \{h\}$$

It remains to compute $H^iG_m^1$, by using the exact sequence

$$0 \to G_m^2 \to G_m^1 \to F_{m-1}^1 \to 0.$$

¿From case A_n we have $H^mF_{m-1}^1 = 0$ and, for $s > 0$

$$H^{m-2s}F_{m-1}^1 = \{h\} \quad \text{if } 2 < h = \frac{m-2}{s} \in N$$
$$H^{m-2s+1}F_{m-1}^1 = \{h\} \quad \text{if } 2 < h = \frac{m}{s} \in N$$

(the other cohomology is zero).

In the first case, $\{h\}$ is generated by the cohomology class of

$$d(z_h(s)0) \;=\; (-1)^{hs+1}[2]z_h(s)1 - ([h+1]-[2])w_h^{s-1}01^h,$$

while in the second one it is generated by $v_h(s)$.

The cohomology of G_2 is expressed by the following formulas:

$$H^mG_m^2 = \left\{\frac{[2(m-1)]}{[m-1]}\right\} \bigoplus_{\substack{h|m \\ 2|h \\ 2<h}} \{h\} \bigoplus_{2\not| m} \{m\}$$

$$H^{m-2s}G_m^2 = \bigoplus_{\substack{2<h\leq \frac{m-2}{s+1} \\ h\not| m \bigwedge h|2(m-1) \bigwedge h\not| m-1 \\ \text{oppure} \\ h|m \bigwedge 2|h}} \{h\} \bigoplus_{\substack{\frac{m-2}{s+1}<h<\frac{m-2}{s} \\ P_m(h)}} \{h\} \qquad \text{for } s > 0$$

$$H^{m-2s+1}G_m^2 = \bigoplus_{\substack{2<h\leq \frac{m-3}{s} \\ 2|h \\ P_m(h)}} \{h\} \bigoplus \left\{\frac{m-2}{s}\right\}$$

Notice that H^0, H^1 and H^2 both of G^1_m and F^1_{m-1} are zero, so $\{\frac{m}{s+1}\}$ always appears in $H^{m-2s}G^2_m$. So we can suppose $i > 2$. Moreover we can assume $h > 2$ since $\{2\}$ does not appear in any of the previous expressions.

Now we compute the connecting homomorphism. Again, we begin by the case A).

Lemma 6

$$d : H^{m-2s}F^1_{m-1} \to H^{m-2s+1}G^2_m$$

is always injective.

Proof. The left module is $\{h\}$, with $h = \frac{m-2}{s}$, which appears on the right also, generated by the class of

$$(-1)^{hs+1}[2]z_h(s)1 - ([h+1]-[2])w_h^{s-1}01^h.$$

This lift to

$$\frac{1}{2}((-1)^{hs+1}[2]z_h(s)(10-01) - ([h+1]-[2])w_h^{s-1}01^{h-1}(10-01)) =$$

$$= \frac{1}{2}((-1)^{hs+1}[2]z_h(s-1)w_h(10-01) + (-1)^{h+1}[2]w_h^{s-1}1^{h-1}0(10-01)+$$

$$-[2]w_h^{s-1}01^{h-1}(10-01) - ([h+1]-[2])w_h^{s-1}01^{h-1}(10-01))$$

$$= \frac{1}{2}((-1)^{hs+1}[2]z_h(s-1)w_h(10-01)+(-1)^{h+1}[2]w_h^{s-1}1^{h-1}0(10-01)-[h+1]w_h^{s-1}01^{h-1}(10-01))$$

of which we have to compute the boundary. Since the result is in G^2_m, it is sufficient to determine the components which end by 11. One obtains for the boundary the expression:

$$-[2]^2z_h(s-1)w_h11 + (-1)^{h(s-1)+1}[2]^2w_h^{s-1}1^{h-1}011(-1)^{hs+1}\frac{[2h]!![h+1]}{[h]!}w_h^{s-1}01^{h-1}11 =$$

$$= -[2]^2v_h(s)011 + (-1)^{hs+1}\frac{[2h]!![h+1]}{[h]!}w_h^{s-1}01^{h-1}11$$

This element is taken by the map $H^{m-2s+1}G^2_m \to H^{m-2s+1}C_{m-3}[2]$ into the generator $v_h(s)$ of $H^{m-2s+1}C_{m-3}[2] = \{h\}$. So $d \neq 0$. □

Let us consider now the boundary

$$d : H^{m-2s+1}F^1_{m-1} \to H^{m-2s+2}G^2_m$$

Set $h = \frac{m}{s}$, when this is an integer bigger than 2.

And now case B).

Lemma 7 The boundary

$$d : H^{m-2s+1}F^1_{m-1} \to H^{m-2s+2}G^2_m$$

does not vanish iff $h = \frac{m}{s}$ is odd.

Proof. $\{h\} = H^{m-2s+1}F^1_{m-1}$ is generated by the class of $v_h(s)$. This last lifts in G^1_m, to

$$\frac{1}{2}(z_h(s-1)01^{h-3}(10-01) + (-1)^{h(s-1)}w_h^{s-1}1^{h-2}(10-01))$$

(if $s=1$, it lifts to $1^{m-2}(10-01)$). Computing the boundary one obtains:

$$(-1)^{hs}\frac{[2(h-2)]!![h-1]}{[h-1]!}z_h(s-1)01^{h-1} + (-1)^{h-1}\frac{[2(h-1)]!![h]}{[h]!}w_h^{s-1}1^h \qquad (*)$$

(which reduces to $(-1)^{m-1}\frac{[2(m-1)]!![m]}{[m]!}1^m$ if $s=1$).

It is easy to see that $\mu_h \not| \frac{[2(h-2)]!![h-1]}{[h-1]!}$ iff h is odd. If h is odd, (*) gives a non zero class in $H^{m-2s+2}G^{h-1}_m$ because it projects to a non zero multiple of the generator $z_h(s-1)$ of $H^{m-2s+2}C_{m-r}[r-1]$. We show that when passing from $H^{m-2s+2}G^r_m$ to $H^{m-2s+2}G^{r-1}_m$, $2 < r \le h-1$, this class remains different from 0.

In fact, consider the exact sequence

$$H^{m-2s+2}G^r_m \to H^{m-2s+2}G^{r-1}_m \to H^{m-2s+2}C_{m-r}[r-1]$$

We know that $H^{m-2s+2}C_{m-r}[r-1] = \{\frac{m-r}{s-1}\}$. Since in the given interval $h \ne \frac{m-r}{s-1}$, the factor $\{h\}$ in $H^{m-2s+2}G^{r-1}_m$ derives from an analog factor in $H^{m-2s+2}G^r_m$, so we have done.

If h is even, (*) is divisible by μ_h so it gives the zero class in $H^{m-2s+2}G^{h-1}_m$, and therefore in $H^{m-2s+2}G^2_m$.

If $s=1$ one concludes with an analog argument. □

The main theorem

Putting together the previous results we get the cohomology of G^1_m:

Proposition.

$$H^m G^1_m = \left\{\frac{[2(m-1)]}{[m-1]}\right\} \bigoplus_{\substack{h \mid m \\ 2 \mid h \\ 2<h}} \{h\} \qquad (= H^m D_m)$$

$$H^{m-2s} G^1_m = \bigoplus_{\substack{2<h\leq \frac{m-2}{s+1} \\ h\nmid m \bigwedge h \mid 2(m-1) \bigwedge h \nmid m-1 \\ \text{or} \\ h \mid m \bigwedge 2 \mid h}} \{h\} \bigoplus_{\substack{\frac{m-2}{s+1}<h<\frac{m-2}{s} \\ h \mid 2(m-1) \bigwedge h\nmid m-1}} \{h\} \bigoplus_{2 \mid \frac{m}{s+1}} \left\{\frac{m}{s+1}\right\} \qquad \text{for } s>0$$

$$H^{m-2s+1} G^1_m = \bigoplus_{\substack{2<h\leq \frac{m-3}{s} \\ 2 \mid h \\ P_m(h)}} \{h\} \bigoplus_{\substack{2 \mid \frac{m}{s} \\ 2<\frac{m}{s}}} \left\{\frac{m}{s}\right\}$$

and now, adding the cohomology of A_{m-1}, we get the main theorem:

Theorem. With the usual convention that

$$\{h\} = \begin{cases} R/(\mu_h) & \text{if } h \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

we have that

$$H^m(D_m) = \left\{\frac{[2(m-1)]}{[m-1]}\right\} \bigoplus_{\substack{h \mid m \\ 2 \mid h \\ 2<h}} \{h\}$$

$$H^{m-2s}(D_m) = \bigoplus_{\substack{2<h\leq \frac{m-2}{s+1} \\ h\nmid m \bigwedge h \mid 2(m-1) \bigwedge h \nmid m-1 \\ \text{or} \\ h \mid m \bigwedge 2 \mid h}} \{h\} \bigoplus_{\substack{\frac{m-2}{s+1}<h<\frac{m-2}{s} \\ h \mid 2(m-1) \bigwedge h\nmid m-1}} \{h\} \bigoplus_{2 \mid \frac{m}{s+1}} \left\{\frac{m}{s+1}\right\} \bigoplus \left\{\frac{m-1}{s}\right\} \qquad \text{for } s>0$$

$$H^{m-2s+1}(D_m) = \bigoplus_{\substack{2<h\leq \frac{m-3}{s} \\ 2 \mid h \\ P_m(h)}} \{h\} \bigoplus_{\substack{2 \mid \frac{m}{s} \\ 2<\frac{m}{s}}} \left\{\frac{m}{s}\right\} \bigoplus \left\{\frac{m}{s}\right\}$$

Remark. Notice that $H^i(D_m),\ i<m$, is not always cyclic, as it was the case for both A_m and B_m.

6. The exceptional cases

	$\mathbf{I}_2(m)$	$\mathbf{H}_3$	$\mathbf{H}_4$	$\mathbf{F}_4$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
H^0	0	0	0	0	0	0	0
H^1	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2	φ_2
H^2	$[m]$	0	0	φ_2	0	0	0
H^3		$\varphi_2\varphi_6\varphi_{10}$	0	$\varphi_2\varphi_3\varphi_6$	0	0	0
H^4			$[60]/\varphi_{60}$	$\varphi_2\varphi_3\varphi_4\varphi_6\varphi_8\varphi_{12}$	0	0	0
H^5					$\varphi_6\varphi_8$	φ_6	φ_2
H^6					$\varphi_3\varphi_6\varphi_9\varphi_{12}$	φ_6	0
H^7						$\varphi_2\varphi_6\varphi_{14}\varphi_{18}$	$\varphi_8\varphi_{12}$
H^8							p

Table 4

where $p=\varphi_2\varphi_3\varphi_4\varphi_5\varphi_6\varphi_8\varphi_{10}\varphi_{12}\varphi_{15}\varphi_{20}\varphi_{24}\varphi_{30}$.

References

[DPS] C. De Concini, C. Procesi, M. Salvetti, *Arithmetic properties of the cohomology of braid groups*, to appear.

[DS] C. De Concini, M. Salvetti, *Cohomology of Artin groups*, Math. Res. Letters **3** (1996), 293–297.

[DS1] C. De Concini, M. Salvetti, *Stability for the Cohomology of Artin Groups*, to appear in Advan. Math..

[DSS] C. De Concini, M. Salvetti, F. Stumbo, *The top cohomology of Artin groups with coefficients in rank 1 local systems over $\mathbb{Z}$*, Topology and its Appl. **78** (1997), 5–20.

[G] Goryunov, V.V., *Cohomologies of groups of braid of series C and D and certain stratifications*, Funk. Anal. Priloz. **12,2** (1978), 138–140.

[S] Salvetti, M., *The homotopy type of Artin groups*, Math. Res. Letters **1** (1994), 565–577.