ARITHMETIC PROPERTIES OF THE COHOMOLOGY OF BRAID GROUPS

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Introduction.

The cohomology of classical Artin's braid group B_m with trivial coefficients has been determined in the 70's with contributions from several people (see [A], [F], [C], [Se], [Va]). The general description is complicated (see for ex. [Vs; ch.1, §2]).

In this paper we are going to consider cohomology with cofficients in a very natural module over B_m, which always gives non trivial top-cohomology, and such that all cohomology has a very interesting and neat description.

Let R_q be the B_{m+1}-module given by the action over the ring

$$R := \mathbb{Q}[q, q^{-1}]$$

of rational Laurent polynomials in the variable q, which is defined by mapping each standard generator σ_i to the operator of multiplication by $-q$ (the shift of the index is because we prefer to speak of the cohomology of B_{m+1} as that of the Dynkin diagram A_m).

Denote by $\mu_d(q)$ the cyclotomic polynomial having as roots the primitive $d-$roots of 1 and let

$$\{d\} := \mathbb{Q}[q, q^{-1}]/\mu_d(q) = \mathbb{Q}[q]/\mu_d(q)$$

be the cyclotomic field of $d-$roots of 1, thought as $R-$module.

Then we obtain

Theorem. *If $hi = m + 1$ or $hi = m$ then*

$$H^{i(h-2)+1}(B_{m+1}, R_q) = \{h\} \quad .$$

All other cohomology is 0.

So in the two cases where we have

$$a) \qquad h := \frac{m+1}{i}, \qquad \text{resp.} \quad b) \qquad h := \frac{m}{i}, \qquad \text{is an integer}$$

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then

$$(0.1) \qquad a) \quad H^{m-2(i-1)}(B_{m+1}, R_q) = \{h\}; \qquad b) \quad H^{m-2i+1}(B_{m+1}, R_q) = \{h\}\ .$$

There is a simple way to display this information: consider the infinite strings

$$s_i := 2\ 0^i\ 3\ 0^i\ 4\ 0^i \dots h\ 0^i\ (h+1)\ 0^i \dots$$

and construct an infinite matrix built up by placing successively as *diagonals*

$$s_0, s_0, s_1, s_1, s_2, s_2, \dots, s_h, s_h, s_{h+1} s_{h+1}$$

as

$$
\begin{array}{cccccccccccccccccccc}
 & & & & & & & & & & & & & & & & & & & 21 \\
 & & & & & & & & & & & & & & & & & & 20 & 20 \\
 & & & & & & & & & & & & & & & & & 19 & 19 & 0 \\
 & & & & & & & & & & & & & & & & 18 & 18 & 10 & 10 \\
 & & & & & & & & & & & & & & & 17 & 17 & 0 & 0 & 7 \\
 & & & & & & & & & & & & & & 16 & 16 & 9 & 9 & 0 & 0 \\
 & & & & & & & & & & & & & 15 & 15 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & 14 & 14 & 8 & 8 & 6 & 6 & 5 & 5 \\
 & & & & & & & & & & & 13 & 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & 12 & 12 & 7 & 7 & 0 & 0 & 0 & 0 & 4 & 4 \\
 & & & & & & & & & 11 & 11 & 0 & 0 & 5 & 5 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & & 10 & 10 & 6 & 6 & 0 & 0 & 4 & 4 & 0 & 0 & 0 & 0 \\
 & & & & & & & 9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
 & & & & & & 8 & 8 & 5 & 5 & 4 & 4 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 \\
 & & & & & 7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\
 & & & & 6 & 6 & 4 & 4 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & 5 & 5 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 4 & 4 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}
$$

We can quickly read off the information on the cohomology from this array, as h in the i, j position (strating from $1, 1$) means that $H^j(B_{i+1}, R_q) = \{h\}$.

Notice again that in the $i^{th}-$column of the table appear all divisors of i and $i+1$: therefore there is an interesting connection between this cohomology of B_{i+1} and the *regular numbers* (see [Sp]) of the symmetric group Σ_{i+1}.

Notice also that, moving horizontally to the right, cohomology stabilizes to zero (clearly this agrees with stability results in [DS1]).

We want to stress that our result has been really "discovered" by performing a large set of explicit computer computations, which allowed to construct part of the above table.

Using previous constructions ([S], [DS], [DS2]), we developed some computer programs which algorithmically compute cohomology of any Artin group and Coxeter group.

We finish this introduction by recalling that, in the case $t = m$, our result has been obtained in a more general form in [DSS].

The methods developed in this paper work for general Artin groups, the explicit computations will appear in a forthcoming paper ([DPSS]).

1. Strings

Let us denote as usual by

$$[n] := \frac{q^n - 1}{q - 1}, \quad [n]! := \prod_{i=1}^{n} [i], \ \left[{n \atop k} \right] := \frac{[n]!}{[k]![n-k]!}$$

the $q-$analogue of the numbers $n, \ n!, \ \binom{n}{k}$.

Then

$$[n] = \prod_{d|n, \ d \neq 1} \mu_d(q)$$

and

$$R/([n]) = \oplus_{d|n, \ d\neq 1} \{d\}.$$

We start establishing some notations slightly different from the usual.

We will denote a subdiagram of A_m by its characteristic function which we display as a sequence of $0, 1$ in short a *string of length* m.

This allows us to multiply strings by juxtaposition which is useful for some cochain computations, we let the degree $|A|$ of a string A to be the number of $1's$ in A.

For a string s we define $s! = \prod_i [k_i + 1]!$ where the numbers k_i are the lengths of the maximal substrings of s which are formed entirely by 1's.

Next let us recall, with a different notation, the basic complex C_m^* of Salvetti [S] (see also [DS] for variations and generalizations).

C_m^k has as basis the strings B of length m and degree $k = |B|$.

The coboundary $d : C_m^* \to C_m^*$ can be defined on the strings by the following rules:

$$d((A0)(0B)) = d(A0)0B + (-1)^{|A|}A0d(0B). \tag{1.1}$$

$$d(A101B) := d(A1)01B + (-1)^{|A|+1}A10d(1B) + (-1)^{|A|+1}\frac{A111B!}{A101B!}A111B. \tag{1.2}$$

$$\begin{gathered} d1^k = 0, \quad d01^k = [k+2]1^{k+1}, \\ d1^k0 = (-1)^k[k+2]1^{k+1}, d01^k0 = [k+2](1^{k+1}0 + (-1)^k01^{k+1}) \end{gathered} \tag{1.3}$$

The main fact we shall need from [S] is that, for all $t \geq 0$,

$$H^tC_m = H^t(B_{m+1}, R_q).$$

Using this we shall from now on compute the cohomology of the complex C_m.

Remark It is interesting to observe that this complex is in fact defined also over $\mathbb{Z}[q]$ and that, setting $q = 0$, one obtains the oriented and augmented simplicial complex of the $n-1$-dimensional simplex.

We want now to define some basic cocycles. Take $h \geq 2$ and set:

$$\begin{gathered} w_h := 01^{h-2}0,\ dw_h = [h]z_h,\ z_h := 1^{h-1}0 + (-1)^h01^{h-1}, \\ b_h := 01^{h-2},\ db_h = [h]c_h,\ c_h := 1^{h-1}. \end{gathered} \tag{1.4}$$

We can now use the rule 1.1 and compute

$$d(w_h^i) = [h]\sum_{j=0}^{i-1}(-1)^{hj}w_h^jz_hw_h^{i-j-1},\ d(w_h^{i-1}b_h) =$$

$$= [h](\sum_{j=0}^{i-2}(-1)^{hj}w_h^jz_hw_h^{i-j-2}b_h + (-1)^{h(i-1)}w_h^{i-1}c_h),$$

Let us then define the cochains

$$z_h(i) := \sum_{j=0}^{i-1}(-1)^{hj}w_h^jz_hw_h^{i-j-1} \in C_{ih}^{i(h-2)+1},$$

$$v_h(i) := \sum_{j=0}^{i-2}(-1)^{hj}w_h^jz_hw_h^{i-j-2}b_h + (-1)^{h(i-1)}w_h^{i-1}c_h \in C_{ih-1}^{i(h-2)+1}$$

Remark that

$$v_h(i) = z_h(i-1)b_h + (-1)^{h(i-1)}w_h^{i-1}c_h, \quad z_h(i) = z_h(i-1)w_h + (-1)^{h(i-1)}w_h^{i-1}z_h$$

By the complex property we have that

1.5 Lemma. *$z_h(i)$, $v_h(i)$ are cocycles.*

By the explicit formulas we have that these cocycles multiplied by $[h]$ are coboundaries but in fact we shall see that also multiplied by $\mu_h(q)$ they are coboundaries.

Let us remark some numerology.

For the cocycle $z_h(i) \in C_{ih}^{i(h-2)+1}$, if we set $m = ih$, then

$$z_h(i) \in C_m^{m-2i+1}, \quad h = \frac{m}{i} \tag{1.6}$$

For the cocycle $v_h(i) \in C_{ih-1}^{i(h-2)+1}$, if we set $m = ih - 1$, then

$$v_h(i) \in C_m^{m-2(i-1)}, \quad h = \frac{m+1}{i} \tag{1.7}$$

Therefore, given m:

A) the cocycles $z_h(i)$ exist for those i for which $h := \frac{m}{i}$ is an integer.

B) the cocycles $v_h(i)$ exist for those i for which $h := \frac{m+1}{i}$ is an integer.

In order to do induction we shall consider the following subcomplexes and maps.

Take the map

$$\pi_m := \pi : C_m \to C_{m-1}, \ \pi(A0) := A; \ \pi(A1) := 0$$

π is the homomorphism of complexes induced by the natural inclusion of B_m into B_{m+1} and has kernel the subcomplex F_m^1 having as basis all strings $A1$ ending with 1.

Our first remark is the following

1.8 Lemma. *If $m = hi$ then, under the map $\pi : C_m \to C_{m-1}$, we have the formula*

$$\pi(w_h^i) = w_h^{i-1} b_h, \quad \pi(z_h(i)) = v_h(i) \tag{1.9}$$

We can iterate this construction and consider the map

$$\pi_m[1] := \pi : F_m^1 \to C_{m-2}[1], \ \pi(A01) := A; \ \pi(A11) := 0$$

Notice that a cochain of degree k in F_m^1 maps to one of degree $k-1$, for this we prefer to shift the complex C_{m-2} by 1.

Again the kernel of $\pi_m[1]$ is the subcomplex F_m^2 spanned by the strings $A11$.

We continue in this way getting maps

$$\pi_m[k] := \pi : F_m^k \to C_{m-k-1}[k],\ \pi(A01^k) := A;\ \pi(A1^{k+1}) := 0$$

which induce isomorphisms of complexes:

$$F_m^k/F_m^{k+1} = C_{m-k-1}[k],\ k = 0, \dots, m-2$$

up to F_m^{m-1} which has as basis $b_{m+1} = 01^{m-1}, 1^m = c_{m+1} = v_{m+1}(1)$ with differential $db_{m+1} = [m+1]c_{m+1}$.

The starting point of the computation is of course the computation of the cohomology of F_m^{m-1} which is clearly 0 except for

$$H^m F_m^{m-1} = R[v_{m+1}(1)] \cong R/([m+1]) = \oplus_{h|m+1}\{h\}, \quad h > 1. \tag{1.10}$$

We need to do induction and thus it is necessary to describe also the cohomology of the complexes F_m^k. In fact our computations were inspired by the analysis of the spectral sequence of this filtration, which however we have decided not to use.

2. Euler Poincaré series

In order to motivate our next computations let us immediately deduce the computation of the Euler Poincaré characteristic.

When we deal with a complex K of $\mathbb{Q}[q]$ modules with cohomology finite dimensional over $\mathbb{Q}$, it is convenient to associate to each cohomology group its class as an element of the Grothendieck group G_0 of finite dimensional torsion modules over $\mathbb{Q}[q]$.

By convention a polynomial $f(q)$ is identified to the class in the Grothendieck group of the module $\mathbb{Q}[q]/(f(q))$.

If $f(q)$ is not irreducible it corresponds to the *sum* of its irreducible factors and G_0 has, as basis, the irreducible polynomials.

This justifies our notation

$$\{d\} := \mathbb{Q}[q]/\mu_d(q) \in G_0.$$

Let us set for a complex K with torsion cohomology

$$\chi(K) := \sum(-1)^i[H^iK]$$

Proposition. *C_m has torsion cohomology and*

$$\chi_m := \chi(C_m) = (-1)^m([m+1]-[m]), \quad \sum_{i=1}^{k-1}(-1)^i\chi_i = [k]$$

Proof. By 1.10 and induction all factor complexes F_m^k/F_m^{k+1} have torsion cohomology from which the first statement follows.

Remark that $[1] = 0$ in the Grothendieck group and $\chi_1 = -[2]$. Now assume by induction that for each $k < m$, $\chi_k := (-1)^k([k+1]-[k])$. We have the recursive formulas:

$$\chi_m := \chi(F_m^{m-1}) + \sum_{k=0}^{m-2}\chi(F_m^k/F_m^{k+1}) = (-1)^m[m+1] + \sum_{k=0}^{m-2}(-1)^k\chi(C_{m-k-1}) =$$
$$(-1)^m[m+1] + \sum_{k=0}^{m-2}(-1)^k\chi_{m-k-1} = (-1)^m([m+1] - \sum_{i=1}^{m-1}(-1)^i\chi_i)$$

proving our claims. □

Remark that the previous argument brings also to

$$\chi(F_m^k) = (-1)^m([m+1]-[m-k]).$$

3. Preparation for the Main Theorem

We want to compute now the cohomology of the various F_m^k assuming by induction the main Theorem, for all $n < m$, in the following form:

Main Theorem.

$$(3.1) \qquad H^{n-2i+1}C_n = \begin{cases} 0 & \textit{if} \quad h := \frac{n}{i} \quad \textit{is not an integer.} \\ \{h\} & \textit{generated by} \quad [z_h(i)] \quad \textit{if} \quad h := \frac{n}{i} \quad \textit{is an integer.} \end{cases}$$

$$(3.2) \qquad H^{n-2(i-1)}C_n = \begin{cases} 0 & \textit{if} \quad h := \frac{n+1}{i} \quad \textit{is not an integer.} \\ \{h\} & \textit{generated by} \quad [v_h(i)] \quad \textit{if} \quad h := \frac{n+1}{i} \quad \textit{is an integer.} \end{cases}$$

We adopt thus the following strategy: consider the exact sequence of complexes

$$0 \to F_m^{k+1} \to F_m^k \to C_{m-k-1}[k] \to 0$$

and look at a typical piece of the corresponding long exact sequence:

$$\to H^{t-1}C_{m-k-1}[k] \xrightarrow{\delta} H^t F_m^{k+1} \to H^t F_m^k \xrightarrow{\pi} H^t C_{m-k-1}[k] \to H^{t+1} F_m^{k+1} \to .$$

Our inductive hypothesis gives us a precise description of the terms

$$H^{t-1}C_{m-k-1}[k] = H^{t-k-1}C_{m-k-1}, \quad H^t C_{m-k-1}[k] = H^{t-k}C_{m-k-1}$$

which by induction are either 0 or irreducibles of type $\{h\}$ generated by the appropriate cohomology classes. Therefore we can analyze the change in the module structure from $H^t F_m^{k+1}$ to $H^t F_m^k$.

It will turn out that the irreducible factors in the modules $H^t F_m^k$ appear with multiplicity 1.

Let us make the following remark. If we have an exact sequence of modules

$$0 \to A \to B \to C \to D \to 0$$

and, A, D are irreducible, $B = A \oplus E$ is semisimple and D does not appear as summand of B then C is semisimple and isomorphic to $E \oplus D$.

Since, if t is fixed, the non 0 modules in the list $H^t C_{m-k-1}[k]$ are all irreducible and non isomorphic, it follows by an easy induction that each $H^t F_m^k$ is semisimple and that, if the map δ is non 0, we delete from $H^t F_m^{k+1}$ an irreducible summand isomorphic to $H^{t-1}C_{m-k-1}[k]$, while, if $\pi \neq 0$, we add to $H^t F_m^{k+1}$ an irreducible summand isomorphic to $H^t C_{m-k-1}[k]$ in order to obtain $H^t F_m^k$.

We start, assuming the main Theorem for $n < m$, to show that the cohomology classes of $z_h(i),\ v_h(i)$ are non 0.

By induction, and Formula 1.9 we deduce that the elements $z_h(i)$ give non 0 cohomology classes.

We come now to the $v_h(i)$.

The element $v_{m+1}(1)$ generates the module $H^m F_m^{m-1} = R/([m+1]) = \oplus_{h|m+1}\{h\}$, we claim that the summand $\{m+1\}$ survives in any $H^m F_m^k$. Look at the exact sequence

$$H^{m-1}C_{m-k-1}[k] \to H^m F_m^{k+1} \xrightarrow{j} H^m F_m^k \to 0$$

where j is surjective by degree considerations.

By induction $H^{m-1}C_{m-k-1}[k] = \{m-k\}$ which is an irreducible different from $\{m+1\}$ and this suffices.

Look now at some other $v_h(i)$ with $hi = m+1$ and $i > 1$, which gives a cohomology class in degree $m - 2s,\ s = i - 1$.

The element $v_h(s+1) \in F_m^{h-2}$ and generates a non 0 cohomology class in F_m^{h-2} since, under the projection map $\pi : F_m^{h-2} \to C_{m-(h-2)-1}$

$$\pi(v_h(i)) = \pi(z_h(i-1)b_h \pm w_h^{i-1}c_h) = z_h(i-1)$$

by induction $z_h(i-1)$ generates a non 0 cohomology class in $C_{m-(h-2)-1}$.

We need to show that, under the maps induced by inclusions $F_m^{h-2} \subset F^r$, $\forall r \leq h-2$ this cohomology class remains non 0.

Look at a typical step

$$H^{m-2s-1}C_{m-k-1}[k] \to H^{m-2s}F_m^{k+1} \xrightarrow{j} H^{m-2s}F_m^k.$$

Again by induction $H^{m-2s-1}C_{m-k-1}[k]$ is either 0 or an irreducible module different from $\{h\}$, so this module cannot cancel an element which contains $\mu_h(q)$ torsion.

We shall finish the proof of the main Theorem as a consequence of the computation of the connecting homomorphisms, that we separate in two cases:

Case A) $H^{m-2s}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2s+1}F_m^{k+1}$.

Case B) $H^{m-2(s-1)-1}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2(s-1)}F_m^{k+1}$.

We shall make use of the following obvious identity:

$$[a][{a+b \atop a}] = [b+1][{a+b \atop a-1}]$$

3.3 Case A). We may assume that $H^{m-2s}C_{m-k-1}[k] \neq 0$ then, by induction,

$$H^{m-2s}C_{m-k-1}[k] = H^{m-2s-k}C_{m-k-1} = H^{(m-k-1)-2s+1}C_{m-k-1} = \{h\}$$

if and only if

$$sh = m-(k+1), \quad h \quad \text{is an integer}$$

The module $\{h\}$ is irreducible and generated by the class of $z_h(s)$.

3.4 Lemma. *If $h := \frac{m-(k+1)}{s}$ is an integer then, $\{h\} = H^{m-2s}C_{m-k-1}[k]$ generated by the class of $z_h(s)$ and:*

a) If $h | k+2$ the map δ is equal to 0.

b) If $h \not| k+2$, the map δ is different from 0.

Proof. In order to compute the transform of $z_h(s)$ under the connecting homomorphism we lift $z_h(s)$ to $z_h(s)01^k \in F_m^k$ and compute $d(z_h(s)01^k) \in F_m^{k+1}$.

We perform the computation as follows. First compare $[h]z_h(s)01^k = d(w_h^s)01^k$ with $d(w_h^s 01^k)$. By formula 1.1 we have

$$d(w_h^s 01^k) = d(w_h^s)01^k + (-1)^{hs}[k+2]w_h^s 1^{k+1}$$

hence

$$d(z_h(s)01^k) = (-1)^{hs+1}[h]^{-1}[k+2]d(w_h^s 1^{k+1})$$

a) If $h|k+2$ this implies that $d(z_h(s)01^k)$ is a coboundary in F_m^{k+1}.

b) Assume now that $h \not| k+2$. We have

$$d(w_h^s 1^{k+1}) = [h](\sum_{j=0}^{s-2}(-1)^{hj} w_h^j z_h w_h^{s-j-1} 1^{k+1} + (-1)^{h(s-1)} w_h^{s-1} 1^{h-1} 01^{k+1}) +$$

$$(-1)^{hs}\left[\begin{matrix} h+k+1 \\ h-1 \end{matrix}\right] w_h^{s-1} 01^{h+k} = [h]z_h(s)1^{k+1} + (-1)^{hs}(\left[\begin{matrix} h+k+1 \\ h-1 \end{matrix}\right] - [h]) w_h^{s-1} 01^{h+k}.$$

We deduce that

$$d(z_h(s)01^k) = (-1)^{hs+1}[k+2]z_h(s)1^{k+1} - (\left[\begin{matrix} h+k+1 \\ h \end{matrix}\right] - [k+2]) w_h^{s-1} 01^{h+k}\}$$

We claim that the class of this element is non zero in $H^{m-2s+1}F_m^{k+1}$.

To see this, notice that this class projects to the cohomology class of $[k+2]v_h(s)$ in $H^{m-2s+1}C_{m-(k+1)-1}[k]$ so that it suffices to show that this class is non zero. By induction the class of $v_h(s)$ is non 0 and generates a module isomorphic to $\{h\}$. Therefore the claim follows since $h \not| k+2$ and $[k+2] \neq 0$ in $\{h\}$.

□

3.5 Case B). We assume that $H^{m-2(s-1)-1}C_{m-k-1}[k] \neq 0$ then, by induction,

$$H^{m-2(s-1)-1}C_{m-k-1}[k] = H^{(m-k-1)-2(s-1)}C_{m-k-1} = \{h\}$$

if and only if $\quad sh = m-k = (m-k-1)+1, \quad h \quad$ is an integer.

The module $\{h\}$ is irreducible and, by induction, generated by the class of $v_h(s)$.

3.6 Lemma. *If $h := \frac{m-k}{s}$ is an integer, then $H^{m-2(s-1)-1}C_{m-k-1}[k] = \{h\}$ generated by the class of $v_h(s)$, and:*

a) If h does not divide $k+1$, the map δ is equal to 0.

b) If h divides $k+1$ the map δ is different from 0.

Proof. As we have done in Lemma 3.4, in order to compute the image of the cohomology class of $v_h(s)$ under the connecting homomorphism:

$$H^{m-2(s-1)-1}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2(s-1)}F_m^{k+1}$$

we lift $v_h(s)$ to $v_h(s)01^k \in F_m^k$ and compute $d(v_h(s)01^k) \in F_m^{k+1}$.

We have

$$d(w_h^s 1^k) = [h]v_h(s)01^k + (-1)^{hs}\left[\begin{matrix} h+k \\ h-1 \end{matrix}\right] w_h^{s-1}01^{h+k-1}$$

a) If h does not divide $k+1$ then $[k+1] \neq 0$ in $\{h\}$ so that it suffices to show that the image of the class of $[k+1]v_h(s)$ is killed by the connecting homomorphism δ.

Since

$$[k+1]v_h(s)01^k = [h]^{-1}[k+1]d(w_h^s 1^k) - (-1)^{hs}\left[\begin{matrix} h+k \\ h \end{matrix}\right] w_h^{s-1}01^{h+k-1}$$

$[k+1]v_h(s)$ lifts in F_m^k to the cocycle $[h]^{-1}([k+1]d(w_h^s 1^k)) \in F_m^k$ and thus $\delta = 0$.

b) If h divides $k+1$, apply the differential to

$$v_h(s)01^k = [h]^{-1}(d(w_h^s 1^k) - (-1)^{hs}\left[\begin{matrix} h+k \\ h-1 \end{matrix}\right] w_h^{s-1}01^{h+k-1})$$

and get, up to a sign

$$[h]^{-1}\left[\begin{matrix} h+k \\ h-1 \end{matrix}\right] d(w_h^{s-1}01^{h+k-1}) = \left[\begin{matrix} h+k \\ h-1 \end{matrix}\right](z_h(s-1)01^{h+k-1} \pm \frac{[h+k+1]}{[h]} w_h^{s-1}1^{h+k})$$

We set

$$v_{h,k+1}(s) := z_h(s-1)01^{h+k-1} \pm \frac{[h+k+1]}{[h]} w_h^{s-1}1^{h+k}$$

Observe that this element lies in F_m^{h+k-1} and its image in $C_{m-(h+k-1)-1} = C_{m-(h+k)}$ is exactly $z_h(s-1)$.

By induction, $z_h(s-1)$ gives a non 0 cohomology class in $H^{(s-1)(h-2)+1}C_{m-(h+k-1)-1}$ generating a module isomorphic to $\{h\}$. Hence the cohomology class of $v_{h,k+1}(s)$ in $H^{m-2(s-1)}F_m^{h+k-1}$ is non zero and the module it generates has a factor isomorphich to $\{h\}$.

In order to show that the element $z_h(s-1)$ in $H^{m-2(s-1)}F_m^{k+1}$ gives a non 0 cohomology class, we prove that the contribution of the factor $\{h\}$ to $H^{m-2(s-1)}F_m^{h+k-1}$ does not disappear when passing to $H^{m-2(s-1)}F_m^{k+1}$.

For this let us remark that whenever we pass from a level $r+1$ of filtration to r we have the exact sequence:

$$H^{m-2s+1}C_{m-r-1}[r] \xrightarrow{\delta} H^{m-2(s-1)}F_m^{r+1} \to H^{m-2(s-1)}F_m^r \xrightarrow{\pi} H^{m-2(s-1)}C_{m-r-1}[r]$$

so an irreducible factor in $H^{m-2(s-1)}F_m^{r+1}$ of type $\{h\}$ can disappear in $H^{m-2(s-1)}F_m^r$ only if $H^{m-2s+1}C_{m-r-1}[r] = \{h\}$ and $\delta \neq 0$, in particular only when $h := \frac{m-r}{s}$. Thus in the inclusion $H^{m-2(s-1)}F_m^{h+k-1} \xrightarrow{i} H^{m-2(s-1)}F_m^{k+1}$, the factor $\{h\}$ is not cancelled.

In fact we have shown that, in $H^{m-2(s-1)}F_m^{k+1}$ the element $[z_h(s-1)]$ generates a submodule with a factor of type $\{h\}$.

To finish we verify that $\left[{h+k \atop h-1}\right] \neq 0$ in $\{h\}$. In fact

$$\left[{h+k \atop h-1}\right] = \frac{[h+k][h+k-1]\dots[k+2]}{[h-1]!}$$

and since h divides $k+1$ its next multiple is $k+1+h$ and so the claim follows, finishing the proof of the Lemma. □

4. The Theorem

In order to prove the main theorem we need, to do induction, to prove a general statement on the cohomology of all the complexes F_m^k, precisely we have:

Theorem. *A)* $s \geq 1$ *, then*

$$H^{m-2s+1}F_m^k = \{h\}$$

if and only if $h := \frac{m-k}{s}$ *is an integer* ($h > 1$) *and* h *does not divide* $m+1$.

Otherwise $H^{m-2s+1}F_m^k = 0$.

B)

$$H^{m-2s}F_m^k = \oplus\{h\} \quad \textit{where} \quad h|m+1,\ sh \leq m-k-1 < (s+1)h-1$$

Before we start the proof of this Theorem let us remark that, when $k=0$, we obtain exactly the statement of the Main Theorem as in §3.

Proof. We want to prove this Theorem using, besides the main induction, also a decreasing induction on k, starting from $k = m-1$ in which case we have

$$H^{m-t}F_m^{m-1} = 0,\ \forall t \geq 1, \quad H^m F_m^{m-1} = \oplus_{h|m+1,\ h>1}\{h\}.$$

We start remarking that, for every $s \geq 0$ we have that $H^{m-2s}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2s+1}F_m^{k+1}$ is surjective.

By induction $H^{m-2s+1}F_m^{k+1}$ is not 0 and equal to $\{h\}$ if and only if

$$sh = m - (k+1), \quad h \quad \text{is an integer and} \quad h \not| m+1 \quad (\text{or} \quad h \not| k+2).$$

On the other hand again by induction $H^{m-2s}C_{m-k-1}[k] \neq 0$ and is $\{h\}$ if and only if

$$sh = m - (k+1), \quad h \quad \text{is an integer}$$

thus in particular if $0 \neq H^{m-2s+1}F_m^{k+1}$ then also $H^{m-2s}C_{m-k-1}[k] \neq 0$ and they are isomorphic as modules. Since this module $\{h\}$ is irreducible it is enough to show that the connecting homomorphism is non 0. This is the content of Lemma 3.4 part b).

A) From the previous analysis we get an injection

$$0 \to H^{m-2s+1}F_m^k \to H^{m-2s+1}C_{m-k-1}[k].$$

So $H^{m-2s+1}F_m^k = 0$ unless the following two conditions are satisfied:

$$H^{m-2s+1}C_{m-k-1}[k] = \{h\} \neq 0,$$

the connecting homomorphism

$$H^{m-2(s-1)-1}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2(s-1)}F_m^{k+1}$$

is 0.

The first condition is satisfied if $m-k = (m-k-1)+1 = sh$, h is an integer.

The second, by Lemma 3.6 a), when h does not divide $m+1$.

B) We pass to analyze $H^{m-2s}F_m^k$; we need the sequence:

$$H^{m-2s-1}C_{m-k-1}[k] \xrightarrow{\delta} H^{m-2s}F_m^{k+1} \to H^{m-2s}F_m^k \xrightarrow{\pi} H^{m-2s}C_{m-k-1}[k]$$

From Lemma 3.6 it follows that:

1) $\delta \neq 0$ when: $h := \frac{m-k}{s+1}$ is an integer and $h|m+1$, hence, using the inductive hypothesis:

$$H^{m-2s}F_m^{k+1}/Im(\delta) = \oplus\{h\}, \ sh < m-k-1 < (s+1)h - 1 \tag{4.1}$$

2) $\pi \neq 0$ when:

$$g := \frac{m-(k+1)}{s} \qquad \text{is an integer.}$$

and $g|m+1$.

In this case, notice that g does not appear in the list in 4.1 and therefore, in order to describe $H^{m-2s}F_m^k$ we just add it to the list getting:

$$H^{m-2s}F_m^k = \oplus\{h\}, \ sh \leq m-k-1 < (s+1)h - 1$$

which is our expected result and completes the proof of the Theorem.

$\square$

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