MAKING CONICAL COMPACTIFICATIONS INTO WONDERFUL ONES

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Introduction.

Let M be a complex manifold. Suppose we are given a compactification X of M that is "conical" (def. 1 below), which means roughly that every point on X has a neighborhood in which $X - M$ has the structure of a cone. In this paper we construct a "minimal wonderful" compactification $\tilde{X}$ of X. The compactification $\tilde{X}$ is wonderful ([D-P2]) in the sense that $\tilde{X} - M$ is a divisor with normal crossings in $\tilde{X}$. Furthermore, $\tilde{X}$ is minimal among the wonderful compactifications obtained from X by blowing up along strata.

Several interesting compactifications in the literature turn out to be examples of the minimal wonderful compactification $\tilde{X}$ constructed in this paper. These include some compactifications of symmetric varieties of [D-P2],[D-P3], the compactifications of configuration spaces of [F-M], and the compactifications of arrangements of complements of linear subspaces of [D-P1]. See section 4 for these examples.

1. Conical stratifications

1.1 Let X be a smooth $n-$dimensional analytic manifold. A *stratification* of X is a decomposition $X = \bigcup_\alpha S_\alpha$ of X, where S_α is a locally finite family of locally closed disjoint analytic submanifolds of X called the *strata*, such that the closure $\overline{S}_\alpha$ of each stratum is a union of strata. This condition implies that the set of strata forms a poset: $S_\alpha \geq S_\beta$ if and only if S_β lies in the closure of S_α. If X is connected, then there is a unique open dense stratum in X called the *generic* stratum.

If $Y \to X$ is a submersion of complex manifolds (i.e. a map with surjective differential), a stratification of X determines an *induced stratification* of Y whose strata are the inverse images of the strata of X. If X and Y are both stratified, then there is a *product stratification* of $X \times Y$ whose strata are products of strata of X with strata of Y. The *trivial stratification* of X is the stratification with one stratum, namely X itself.

Assumption. For convenience, we will assume that all of the strata S_α in X are connected. Further, we will assume that all S_α are locally connected relative to X, i.e. every point $x \in X$ has some neighborhood U such that $S_\alpha \cap U$ is connected.

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The assumption that the strata are connected involves no loss of generality, as we can always stratify X by the connected components of a given stratification. The assumption that the strata are locally connected relative to X is for convenience of exposition. It may be dropped as explained in §3.1.

We want to define the notion of a conical stratification. Consider the case $X = \mathbb{C}^n$, with the group $\mathbb{C}^*$ acting by scalar multiplication.

We say that $\mathbb{C}^n$ is *stratified as a cone* if $\{0\}$ is a stratum and all strata are $\mathbb{C}^*$ stable.

A *disk stratified as a cone* is an open ball D of radius r centered at 0 in $\mathbb{C}^n$ whose stratification is induced from a stratification of $\mathbb{C}^n$ as a cone.

More generally we will also need to consider a *closed cone* X in $\mathbb{C}^n$, i.e. a closed $\mathbb{C}^*$ stable analytic (hence algebraic by G.A.G.A.) subvariety of $\mathbb{C}^n$.

A *local cone stratified as a cone* is the intersection of a closed cone with an open ball D of radius r centered at 0 in $\mathbb{C}^n$, whose stratification is induced from a stratification of $\mathbb{C}^n$ as a cone.

Remarks.

(1) Stratifying $\mathbb{C}^n$ as a cone is equivalent to stratifying the complex projective space $\mathbb{P}^{n-1}$. Given a stratification of $\mathbb{P}^{n-1}$, stratify $\mathbb{C}^n - \{0\}$ by the stratification induced by projection $\mathbb{C}^n - \{0\} \to \mathbb{P}^{n-1}$, and add the stratum $\{0\}$. We call the stratification of $\mathbb{P}^n$ the *projectivization* of the stratification of $\mathbb{C}^n$.

(2) Similarly, if D is a disk stratified as a cone, $D - \{0\}$ fibers over $\mathbb{P}^{n-1}$ with fibers punctured disks. The conical stratification of D is again induced by one in $\mathbb{P}^{n-1}$.

Definition. A stratification of a smooth variety X is *conical* if, given any point $x \in X$ there exists an open neighborhood A of x and an analytic isomorphism of A with a product of two disks D_T, D_N such that the stratification induced in A is the product of the trivial stratification of D_T with a stratification of D_N as a cone. In this situation, we call D_T a tangent disk and D_N a normal disk at x.

For a possibly singular variety X a stratification is *conical* if, given any point $x \in X$ there exists an open neighborhood A of x and an analytic isomorphism of A with a product of a disk D_T with a local cone C_N such that the stratification induced in A is the product of the trivial stratification of D_T with a stratification of C_N as a cone. In this situation, we call D_T a tangent disk and C_N a normal cone at x.

A prime example of a conical stratification is a wonderful stratification:

Definition.

A stratification of X is *wonderful* if X is smooth and there is a divisor with normal crossings $D = \bigcup D_i$, where D_i are nonsingular irreducible divisors meeting transversely, such that two points are in the same stratum if and only if they are contained in the same set of irreducible divisors.

More generally, one could consider stratifications that are locally wonderful. See §3.2.

Conical stratifications are natural and well behaved mathematical objects, as indicated in the following propositions:

Proposition.

(1) *The product stratification on $X \times Y$ is conical if and only if the stratifications of X and Y are both conical.*
(2) *A conical stratification is locally finite, i.e. any $x \in X$ has a neighborhood U that is met by finitely many strata of X. So if X is compact, the component stratification is finite.*
(3) *A stratification of $\mathbb{C}^n$ as a cone is conical if and only if the associated stratification of the complex projective space $\mathbb{P}^{n-1}$ is conical.*
(4) *The closure of each stratum of a conical stratification is an analytic variety (possibly singular), with an induced conical stratification. The strata of a conical stratification of $\mathbb{P}^{n-1}$ are algebraic subvarieties.*

Proof. For the "if" part of (1), observe that if $\mathbb{C}^n$ and $\mathbb{C}^m$ are both stratified as cones, then the product stratification of $\mathbb{C}^n \times \mathbb{C}^m$ is stable under the product action of $\mathbb{C}^* \times \mathbb{C}^*$ and thus under the diagonal action. For the converse, first we consider the case where the stratification of Y is trivial, i.e. it has only one stratum. Assume that the product stratification of $X \times Y$ is conical, and choose $x \in X$. We want to find tangent and normal disks D_T, D_N with the defining property at $x \in X$. Choose an arbitrary $y \in Y$. Let $\tilde{D}_T, \tilde{D}_N$ be tangent and normal disks for (x, y). $\tilde{D}_T \cap (X \times y)$ is a complex manifold because the map to Y has maximal rank. For the same reasons the map from $(\tilde{D}_T \cap (X \times y)) \times \tilde{D}_N$ to X induced by projection is a local isomorphism. For the general case, one reduces to the previous case by considering the top stratum of the Y times the stratification of X.

(2) We use induction on the dimension. It is clear that the first part of the statement implies the second. Applying the second part of the statement to $\mathbb{P}^k$ (which is compact) for $k < n$, we get the first part for varieties X of dimension n, using tangent and normal disks (where $k+1$ is the dimension of the normal disk).

(3) Follows from (1) and the fact that $\mathbb{C}^n - 0$ is a $\mathbb{C}^*$ bundle over $\mathbb{P}^{n-1}$ with local trivializations near each point in $\mathbb{P}^{n-1}$.

(4) We use induction on dimension. Suppose that we know the first part for varieties of dimension $n-1$. Then the closures of the strata of a conical stratification of $\mathbb{P}^{n-1}$ are analytic subvarieties of $\mathbb{P}^{n-1}$, so by GAGA they're algebraic, so the strata themselves are algebraic. Suppose we know the second part for $\mathbb{P}^{n-1}$. The cone over an algebraic variety is algebraic. Let X have dimension n. Using tangent an normal disks, we get that the closure of each stratum in X is algebraic in local coordinates, so it's analytic.

1.2 Normal bundles to strata. For any stratum S of a conical stratification of a smooth variety X, there is a stratification of the normal bundle $T_S X$ called the *normal stratification* constructed as follows:

For $x \in S$, choose tangent and normal D_T and D_N disks for a neighborhood A of x, and an embedding $D_N \subset \mathbb{C}^m$ as a round disk centered at 0, where $\mathbb{C}^m$ is stratified as a cone. This determines an identification of the restriction of $T_S X$ to

$A \cap S$ with

$$D_T \times T_0 D_N = D_T \times T_0 C^m = D_T \times \mathbb{C}^m.$$

We stratify the restriction of $T_S X$ to $A \cap S$ by the product of the trivial stratification of D_T with the stratification of $\mathbb{C}^m$. We must show that it is independent of the choices involved.

Proposition.

(1) *The stratification of the restriction of $T_S X$ to A defined above is independent of the choice of the tangent and normal disks, and is independent of the choice of embedding of the normal disk in $\mathbb{C}^m$.*

(2) *For any two points x and y in the stratum S, there is an isomorphism of the fibers of $T_S X$ at x and y as vector spaces that identifies their stratifications induced from the stratification of $T_S X$.*

(3) *For any two points x and y in the same stratum S, if D_T^x and D_N^x are normal and tangential disks near x and D_T^y and D_N^y are normal and tangential disks near y, then the stratifications of D_N^x and D_N^y are identical, when restricted to possibly smaller neighborhoods of 0 in each disk.*

Proof. We will construct the stratification of $T_S X$ by a process that is independent of the choice of normal and tangential disks. Let Z be a closed union of strata of X. Then the corresponding closed union of strata in $T_S X$ is the normal cone to S along Z determined by the following property:

Let $f : C \to X$ be any analytic curve parameterized by an open set $C \subset \mathbb{C}$ such that f maps $0 \in C$ to S and maps C to Z. The derivative of f at 0 projects to a vector in the normal cone to S along Z. The individual strata of the stratification of $T_S X$ can be reconstructed by a process of taking differences.

If X is not smooth the previous analysis extends but we have to replace the normal bundle with a normal *cone* bundle.

Remark. Any conical stratification is Whitney. This is because a stratification of a disk D_n conically induced from a Whitney stratification of $\mathbb{P}^{n-1}$ is Whitney, and the product in $D_N \times D_T$ of a Whitney stratification of D_N with a trivial stratification of D_T is Whitney. The Whitney conditions will not be used explicitly in this paper.

2. The minimal wonderful blowup.

2.1. Irreducible strata. We come now to the main definition.

Definition.

(1) Suppose that X has a conical stratification. A stratum S is *reducible* if for any (and hence every) point $x \in S$ there is a neighborhood A of x with a decomposition $D = D_T \times D_N^1 \times D_N^2$ into disks (or local cones in the singular case), where D_N^1 and D_N^2 are of positive dimension, such that the induced stratification of A is the product of the trivial stratification of D_T with conical stratifications of D_N^1 and D_N^2. Otherwise S is *irreducible.*

If 0 is reducible in one of the normal disks, say D_N^2, we can further factor a smaller neighborhood of 0 in D_N^2 as $\tilde{D}_N^2 \times D_N^3$. Proceeding in this way, we can finally write find a neighborhood of x with an isomorphism $D = D_T \times D_N^1 \times D_N^2 \times \cdots \times D_N^{k-1} \times D_N^k$ where the origin is irreducible in each D_N^i. We will call such neighborhood a *complete product neighborhood.*

We want to show that the list of stratified disks D_N^i obtained in this way is unique. In fact, they are characterized as the normal disks to a list of strata called the factors of S:

Definition. Given any stratum S, the *factors* of S are the irreducible strata A such that $A \geq S$ and there is no irreducible stratum T with $A > T \geq S$.

Lemma. *If we have a product stratification $X \times Y$ then an irreducible stratum is either of the form $S \times Y$ with S irreducible in X or $X \times T$ with T irreducible in Y.*

As a consequence we have:

Proposition. (Unique factorization of conical stratifications) *Assume that $D = D_T \times D_N^1 \times D_N^2 \times \cdots \times D_N^{k-1} \times D_N^k$ with the D_N^i irreducible in the origin. Any irreducible stratum of the induced stratification of D is of the form $D_T \times D_N^1 \times D_N^2 \times D_N^{i-1} \times S \times D_N^{i+1} \dots D_N^{k-1} \times D_N^k$ where S is an irreducible stratum in D_N^i. In particular, the factors of $D = D_T \times 0_1 \times 0_2 \times \cdots \times 0_k$ are the strata of the form $D_T \times D_N^1 \times D_N^2 \times D_N^{i-1} \times O_i \times D_N^{i+1} \dots D_N^{k-1} \times D_N^k$ where O_i is the origin in D_N^i.*

Remark. Let S be any stratum and let U be a neighborhood of A that only intersects strata $\geq S$. Then within U, the closures of the factors of S are smooth, they intersect transversely, and their intersection is S.

2.2 Blowups. From now on, until 3.3 we shall discuss the case in which X is smooth.

Recall the blow-up construction: If $Y \subset X$ is a closed nonsingular analytic subvariety, then the *blow-up of X along Y* is a a smooth analytic variety $Bl_Y(X)$ equipped with a map $\pi : Bl_Y(X) \to X$.

We first consider the case that X is $\mathbb{C}^n$ and Y is the origin. Then $BL_Y(X) = Bl_0(\mathbb{C}^n)$ is nothing else than the total space of the tautological bundle of $\mathbb{P}^{n-1}$. The projection $Bl_0\mathbb{C}^n \to (\mathbb{C}^n)$ takes the complement of the 0 section isomorphically to $\mathbb{C}^n - 0$, and takes the 0 section to $0 \in \mathbb{C}$. The 0 section is itself isomorphic to $\mathbb{P}^{n-1}$.

Definition. Suppose that $\mathbb{C}^n$ is conically stratified as a cone. Then the *induced stratification* of $Bl_0(\mathbb{C}^n)$ is constructed as follows: On $Bl_0(\mathbb{C}^n) - \pi^{-1}0 = \mathbb{C}^n - 0$ is stratified the same way $\mathbb{C}^n - 0$ is, and $\pi^{-1} = \mathbb{P}^{n-1}$ is stratified by the projectivization of the stratification of $\mathbb{C}^n$

Lemma. *The induced stratification of $Bl_0(\mathbb{C}^n)$ is conical. The irreducible strata of this stratification are the irreducible strata in $\mathbb{C}^n - 0$ together with the 0 section of the line bundle.*

Proof. Each point of the 0 section has a product neighborhood (by trivializing the line bundle) where the induced stratification is the product stratification. So the claim follows.

Let D be the unit ball in $\mathbb{C}^n$, stratified as a cone. Then Bl_0D is the unit disk bundle in the tautological bundle over $\mathbb{P}^{n-1}$. It is stratified by restricting the stratification of $Bl_0\mathbb{C}^n$, we shall often use the notation $\tau D := Bl_0D$.

2.3 Blowing up an irreducible stratum.

Definition. A stratum S is called *minimal irreducible* if it is irreducible, and every stratum $< S$ is reducible.

Proposition. *Suppose that S is a minimal irreducible stratum of a conical stratification of X.*

(1) *The closure Y of S is nonsingular.*
(2) *Every point $y \in Y$ has neighborhood D with a product stratification $D_Y \times D_N$, where D_N is a complex analytic ball stratified as a cone, and $Y \cap D = D_Y \times 0$.*
(3) *The blowup $Bl_Y X$ of X along Y has an induced stratification characterized by the property that, over the product neighborhood $D = D_Y \times D_N$ it coincides with the product stratification $Bl_{Y\cap D} D = D_Y \times \tau D_N$, where $\tau D_N = Bl_0 D_N$ is stratified as in the Lemma of §2.2.*
(4) *The induced stratification of $Bl_Y X$ is conical.*
(5) *The irreducible strata of $Bl_Y X$ are the irreducible strata of $Bl_Y X - \pi^{-1} Y = X - Y$ together with the generic stratum of $\pi^{-1} Y$.*

Proof. Take a point $y \in Y - S$. Then y must lie in some reducible stratum T. Take a complete product neighborhood D of y, namely $D = D_T \times D_N^1 \times D_N^2 \times \cdots \times D_N^{k-1} \times D_N^k$ with the D_N^i irreducible in the origin. Now, since S is irreducible, we know that for some i, $S \cap D$ is an irreducible stratum R in D_N^i times the generic stratum in the other factors. We assert that R is the zero stratum in D_N^i. If not, the 0 stratum in D_N^i times the generic stratum in the other factors would be an irreducible stratum in the closure of S, contradicting the minimality of S. Now take D_N to be D_N^i and take D_Y to be the product of the other factors. This proves the first two statements.

For the third statement, the difficulty is to show that locally the stratification of the exceptional divisor of $Bl_Y X$ is independent of the product structure chosen on the neighborhood. We do this by giving an invariant description of the induced stratification of the exceptional divisor, which is the projectivization of the bundle $T_Y X$ to Y. The normal bundle $T_Y X$ has a canonical stratification derived from that of D_N. It also has the stratification obtained by taking inverse images of the strata of Y. The induced stratification is the common refinement of these two stratifications.

The fourth statement follows from the fact that conical stratifications are preserved under products. The last statement follows from the unique factorization of stratifications.

2.4. Construction of the minimal wonderful stratification.

The result of the last section can be paraphrased as follows: Blowing up X along the closure Y of a minimal irreducible stratum S preserves the number of irreducible strata, preserves the codimension of all of them except for S, and replaces S by an irreducible stratum of codimension 1. This represents progress towards producing a wonderful blowup for the following reason:

Proposition. *A conical stratification is wonderful if and only if the irreducible strata are all of codimension 1.*

Proof. It is clear that a wonderful stratification has the given property. Conversely, suppose that the irreducible strata are all of codimension 1. For any point $x \in X$, choose a complete product neighborhood of x. All of the normal cones C_N^i are one dimensional hence smooth and X is smooth, by the unique factorization result of §2.1. Therefore, the stratification is wonderful, at least locally. The global result follows from the fact that the strata are locally connected.

Definition. The *minimal wonderful blowup* $\tilde{X}$ of X is the result of repeatedly blowing up the irreducible strata of minimal dimension, until all irreducible strata are of codimension 1. (Irreducible strata of minimal dimension are, of course, minimal irreducible strata.)

More precisely, let m be the number of irreducible strata of X. Let $X_0 = X$, and for all $1 \leq i \leq n$, let X_i be the result of blowing up of X_{i-1} along a the closure of an irreducible stratum of minimal dimension of X_{i-1}, with the induced stratification. Then $X_m = \tilde{X}$.

This procedure is always possible because every conical stratification has at least one minimal irreducible stratum. (Take an irreducible stratum of smallest dimension, for example.)

Proposition. *The minimal irreducible compactification is canonically independent of the choices involved: Suppose $\pi_1 : \tilde{X}_1$ and $\pi_2 : \tilde{X}_2$ are minimal irreducible compactifications constructed with from two different sequences of choices. Then $\pi_2^{-1}\pi_1$ takes the generic stratum of $\tilde{X}_1$ isomorphically to the generic stratum of $\tilde{X}_2$. This extends uniquely to an isomorphism $\tilde{X}_1 \to \tilde{X}_2$.*

Proof. Suppose that $Y_1, Y_2, \ldots, Y_j$ are the closures of the k dimensional irreducible strata of X, and every other irreducible stratum has dimension $> k$. Let $\tilde{X}$ be the result of blowing up first along Y_1, then along the proper transform of Y_2, and so on. More precisely $Bl_Y(X)$, is constructed iteratively as follows: let X_0 be X itself. Suppose that X_i has been constructed, and that it maps to X via the composition $X_i \to X_{i-1} \to \cdots \to X_0 = X$. Then the proper transform of Y_{i+1} in X_i is nonsingular. X_{i+1} is the blow up X_i along the proper transform of Y_{i+1}.

Then X_{i+1} maps to X_i and hence to X via the composition $X_{i+1} \to X_i \to \cdots \to X_0 = X$.

Then $X_j = \tilde{X}$.) We need to show that $\tilde{X}$ is independent of the ordering of the varieties $Y_1, Y_2, \ldots, Y_j$.

First, blowing up is a local notion, in the sense that if a smooth variety X has an open covering U_i and Y is a smooth subvariety then the varieties $Bl_Y U_i$ form an open covering of $Bl_Y X$, so it is enough to prove it locally.

Second, for $x \in X$, take a complete product neighborhood $D = D_T \times D_N^1 \times D_N^2 \times \cdots \times D_N^{k-1} \times D_N^k$ of x. Consider the varieties Y_i which have a non empty intersection with D, we may assume (renumbering them) that they are the first h and that, after a change of numbering of the normal disks, $Y_i \cap D = D_T \times D_N^1 \times \cdots \times 0_i \times \cdots \times D_N^k$, where 0_i is the zero point inside the i-th disk. This follows form §2.1. Then the part of $\tilde{X}$ over D is

$$D_T \times \tau D_N^1 \times \tau D_N^2 \times \cdots \times \tau D_N^h \times D_N^{h+1} \times \ldots D_N^{k-1} \times D_N^k$$

(remember that τD is the blow up of a disk at the origin). This is visibly independent of the ordering.

Moreover the same argument implies that this independence is also valid as stratified space in case the Y_i are the closures of the minimal irreducibles since locally the stratification we obtain is always the product stratification of this expression.

Remark In fact it is not essential to blow up first the closures of the irreducible strata of minimal dimension but it suffices to do it for all the minimal irreducible strata.

Question. *For a sequence of positive integers $V = v_1, v_2, \dots, v_n$, let I_V be the sheaf of ideals of functions on X that vanish to degree $\geq v_i$ on each irreducible stratum X_i of codimension i. Then for $v_1 << v_2 << \cdots << v_n$, the blowup of X at this I_V is the wonderful blowup.*

3. Formal properties of the wonderful blowup. The divisors (closures of codimension one strata) of the minimal wonderful compactification $\tilde{X}$ of X are clearly in one to one correspondence with the irreducible strata of X. Since $\tilde{X}$ is wonderful, any other stratum is an intersection of some set of divisors, so it will correspond to some subset of the set of irreducible strata. The subsets that occur are the nested subsets, as described below.

3.1 Nested subsets. A *flag of strata* or simply a flag, is a sequence $S_1, \dots, S_k$ of strata such that, for each i we have $S_i \leq S_{i+1}$.

We now give two definitions of a *nested set* of irreducible strata.

Definition. A set $\mathcal{S}$ of irreducible strata is called *nested* if it satisfies one of the following two equivalent properties:

(1) There exists a flag of strata $\mathcal{F}$ such that the elements in $\mathcal{S}$ are all irreducible factors of elements of $\mathcal{F}$.
(2) Let $A_1, \dots, A_k$ be the minimal elements of $\mathcal{S}$ and let $\mathcal{S}_i$ be the set of elements in $\mathcal{S}$ that are $> A_i$. Then $A_1, \dots, A_k$ are all factors of some single stratum, and $\mathcal{S}_i$ is nested, as defined by induction.

Given a flag of strata, we can associate to this the set of all factors of all the strata in the flag. We call this the *factorization* of the flag.

Theorem.

(1) *The irreducible divisors of the boundary divisor of the wonderful blowup are indexed by the irreducible strata of X. The irreducible divisor is the closure of the inverse image of the corresponding irreducible stratum.*
(2) *The family of divisors indexed by some set $\mathcal{T}$ of irreducible strata has nonempty intersection if and only if the set $\mathcal{T}$ is nested.*
(3) *The strata of the wonderful blowup are indexed by the nested sets. The order of strata is the opposite to the inclusion ordering of nested sets.*

Proof. This is by induction based on the basic step of blowing up the closure of the minimal irreducibles.

A maximal set of divisors with nonempty intersection corresponds to the factors of a maximal flag.

Proposition. *The number of factors of a maximal flag equals the length of the flag.*

Proof. By induction, take a minimal stratum S_0 in the flag and its decomposition in a point $D_T \times \prod_i D_N^i$, a minimal stratum over S_0, locally in this disk, is of the form $D_T \times_{i \neq j} O_i \times X_j$ where X_j is minimal in D_N^j. In particular S_1 is of this form for some j_1. Of the factors of S_0 there is only one which does not appear by induction in the factorization of the flag starting at S_1 that is $D_T \times_{i \neq j} D_N^i \times O_{j_1}$.

3.2. Strata that aren't locally connected. If we drop the simplifying assumption that strata of the conical stratification of X are locally connected, there will still be a minimal wonderful blowup $\tilde{X}$. However, the strata of the wonderful blowup will not necessarily be locally connected. (So the irreducible divisors in the wonderful blowup will not necessarily be smooth: they will themselves have singularities with normal crossings.)

Consider a complete product neighborhood $D = D_T \times D_N^1 \times D_N^2 \times \cdots \times D_N^{k-1} \times D_N^k$ of a point $x \in X$. Let $S_i \subset D$ be the product of the zero stratum in D_N^i with the generic stratum in the other factors. Failure of local connectedness allows the possibility that more than one of the S_i may be the intersection of D with the same stratum of X.

Now, as in the proof of proposition 2.4, suppose that the smallest irreducible strata have dimension k. We want to blow up the closure of S_1, then the proper transform of the closure of S_2, and up to S_j, where $D_N^1, \ldots, D_N^j$ are the $n-k$ dimensional disks. Globally, this is impossible since the stratum intersecting D in S_1 may also intersect D in another S_i. However locally within D, we can do this, and the result is canonically independent of the choices. The result is obtained by doing these blowups locally, and patching them together globally.

3.3 Singular varieties We want to sketch the Theory in the case of a conical stratification of a singular variety.

The blow up construction is quite general and it can be applied when one has an ideal sheaf $\mathcal{I}$ in an analytic variety X. The blow up is defined as $Proj(\oplus_{k=0}^{\infty} \mathcal{I}^k$.

Now if we consider a closed cone C in $\mathbb{C}^n$, which is the affine cone associated to a projective variety $\overline{C}$, and we blow up C in the origin, the resulting variety is the restriction to $\overline{C}$ of the tautological bundle on the projective space. For a local cone C we shall again denote by τC the resulting blow up, one should remark that now the stratification of τC is such that the normal cones are all of smaller dimension than C.

Proposition 2.3 needs several changes. One has to use again the fact that blowing up is a local notion and moreover that, if a variety is a product $X \times Y$ and the ideal sheaf being blown up is the ideal sheaf of the subvariety $X \times O$ with O a point in Y, the resulting blow up is $X \times Bl_O Y$.

This, when applied to a closure of an ireducible stratum produces, in the blown up variety a stratification where the minimal irreducible strata have normal cones of smaller dimension.

This fact implies that after a sequence of blow ups all normal cones become disks and the final model is smooth and wonderful.

4. A wider class of wonderful compactifications.

The previous notions are related to the ideas of *building sets* introduced in [D-P1] which provide a finite family of wonderful models of which the one constructed before is in fact really minimal.

Definition. Let $\mathcal{W}$ be a set of strata $\geq S$. We say that S is *factored* by $\mathcal{W}$ if for any (and hence every) point $x \in S_\alpha$ there is a neighborhood A of x with the a decomposition $D = D_T \times D_N^1 \times \cdots \times D_N^m$, where D_N^i is stratified as a cone, and the strata in $\mathcal{W}$ intersected with A are the products of 0 in one normal disk D_N^i with the open dense stratum in the other normal disks.

Additionally, a stratum S is factored by the set consisting of itself.

It follows that each stratum in $\mathcal{W}$ has a nonsingular closure in some neighborhood of S, and these closures intersect transversely in S.

Definition.

(1) A set $\mathcal{G}$ of strata is a *building set* if any stratum S is factored by the minimal elements in $\mathcal{G}$ which are $\geq S$. (Remark: this condition is always satisfied if S is an element of $\mathcal{G}$.)

(2) Given a stratum S, a minimal element $G \in \mathcal{G}$ with $G \geq S$ is called a $\mathcal{G}$ *factor* of S.

(3) Let $\mathcal{G}$ be a building set. A set $\mathcal{T} \subset \mathcal{G}$ of strata is called **nested with respect to** $\mathcal{G}$ if it satisfies one of the following two equivalent properties:
   a) There exists a flag of strata $\mathcal{F}$ such that the elements in $\mathcal{T}$ are all $\mathcal{G}$-factors of elements of $\mathcal{F}$.
   b) Let $A_1, \ldots, A_k$ be the minimal elements of $\mathcal{T}$ and let $\mathcal{T}_i$ be the set of elements in $\mathcal{T}$ that are $> A_i$. Then $A_1, \ldots, A_k$ are all $\mathcal{G}$- factors of some single stratum, and $\mathcal{T}_i$ is nested, as defined by induction.

Examples: The set of all irreducible strata is a building set: it is the smallest one. The set of all strata is a building set: it is, quite naturally, the largest one. See [DeConcini- Procesi] for a procedure for enlarging a building set.

One can use building sets, rather than just the set of irreducible strata, to construct non minimal wonderful blowups. The procedure follows verbatim that already developed, one first proves that the closure Y of a minimal element S of a building set $\mathcal{G}$ is smooth, blows up such a variety and constructs in the blow up the induced building set which contains all the strata in $\mathcal{G}$ different from S and hence disjoint from Y and the open stratum in the exceptional divisor of the blow up, one verifies that this is again a building set and proceeds until all the elements of the building set are divisors. Let us denote by $X_\mathcal{G}$ the wonderful model so obtained.

It is easy to see that given two building sets $\mathcal{G}_i$, $\mathcal{G}_2$ then there is a (unique) map $X_{\mathcal{G}_1} \to X_{\mathcal{G}_2}$ if and only if $\mathcal{G}_1 \supset \mathcal{G}_2$ and that every building set contains all the irreducible strata, thus

Corollary. *The minimal wonderful blowup is the minimal blowup among the set of blowups constructed from building sets.*

As for the minimal wonderful model one has that the irreducible divisors in $X_\mathcal{G}$ are indexed by the elements of $\mathcal{G}$ and that a set of them has non empty intersection if and only if the corresponding elements of $\mathcal{G}$ are $\mathcal{G}$- nested.

5. Examples.

In this section, we consider several examples of complex manifolds M that have an obvious "naive" compactification X, with an obvious stratification that is conical. In the examples considered here, the minimal wonderful compactification $\tilde{X}$ was already known, and it turned out to be useful.

5.1 Configurations spaces. Let Z be a compact complex manifold and let M be the $N-$th configurations space of Z: a point in Z is an ordered set of N distinct points in Z. A naive compactification of M is the N-fold cartesian product $\{z_1, z_2, \dots, z_N | z_i \in Z\}$. It has a stratification $X = \bigcup_\alpha S_\alpha$, where α is a partition of the set $\{1, 2, \dots, N\}$ into subsets and S_α is defined by the condition that $z_i = z_j$ if i and j are in the same subset. This stratification is conical since being conical is a local property and locally this is a special case of the next example, which is clearly conical. The minimal wonderful compactification $\tilde{X}$ of M is the compactification constructed in [F-M].

5.2 Complements of configurations of linear subspaces. Let X be a complex vector space, let L_i be a collection of proper distinct linear subspaces of X, and let M be the complement $X - \bigcup_i L_i$. Then X itself is a naive compactification of M. It has a stratification $X = \bigcup_\alpha S_\alpha$, where α is a subset of the L_i and S_α consists of points in the intersection of all L_i in α that do not lie in any smaller intersection of the linear subspaces. This stratification is conical, and the minimal wonderful compactification $\tilde{X}$ of M is the compactification constructed in [D-P1].

5.3 Complete quadrics, projectivities and null-correlations. Let M be the variety of nonsingular quadric hypersurfaces in $\mathbb{P}^n$. It is compactified by the variety X of all quadric hypersurfaces in $\mathbb{P}^n$, which is stratified by the rank of the symmetric matrix giving the quadratic equation. This is a conical stratification. The minimal wonderful compactification $\tilde{X}$ is the variety of complete quadrics, as in [D- P2],[D-P3]. In this case, all strata of X are irreducible and all subsets are nested.

Similarly we can consider the projective space associated to the linear space of $n \times m$ matrices resp. $n \times n$ skew symmetric matrices stratified by rank. The open stratum is classically called the space of projectivities resp. the space of null correlations. In these cases also, the stratification is conical, all strata are irreducible and all sets of strata nested. The resulting minimal wonderful compactifications are the space of complete projectivities resp. complete null-correlations.

The way to prove that the stratifications are conical is to consider the group of symmetries of the stratification that for symmetric or skew symmetric $n \times n$ matrices is the general linear group $GL(n, \mathbb{C})$ while for the $n \times m$ matrices is the product $GL(n, \mathbb{C}) \times GL(m, \mathbb{C})$. In every case the strata are the orbits. So to verify the conical property we can do it in just one point p in each orbit. One can always then choose a normal slice that is stable under some subgroup of the stabilizer of p, which acts as scalars on this normal slice giving rise to the conical structure.

In these examples, it is interesting to analyze the action of the group of symmetries also on the wonderful blow-up. One sees that again the strata are orbits.

Remark Usually the construction of a wonderful model is a first step in some geometric program (relative to intersection theory or Hodge theory for eaxmple) on the variety or the open stratum in consideration.

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