HYPERPLANE ARRANGEMENTS AND HOLONOMY EQUATIONS

C. De Concini and C. Procesi

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0 Introduction This paper stems from our attempt to understand Drinfeld's construction (cf. [Dr2]) of special solutions of the Knizhnik-Zamolodchikov equation (cf. [K-Z]) with some prescribed asymptotic behavior and its consequences for some universal constructions associated to braiding: universal unipotent monodromy representations of braid groups, the construction of a universal Vassiliev invariant for knots, braided categories etc..

The K-Z connection is a special flat meromorphic connection on $\mathbb{C}^n$ with simple poles on a family of hyperplanes. It turns out that the prescription of the asymptotic behavior for such connections is controlled by the geometry of a suitable modification of $\mathbb{C}^n$ in which the union of the polar hyperplanes is replaced by a divisor with normal crossings (cf. [Ke]).

In the process of developing this geometry we realized that our constructions could be developed more generally for subspace arrangements and the more general results obtained are contained in our paper: "A wonderful model for the complement of subspaces" [D-P1] of which this is partly a continuation although most of the general results of that paper are not needed. In the first section we recall a few definitions and facts which in this paper we only need in the special case of a complement of hyperplanes.

The main part of this paper is devoted to discuss the holonomy equation, the universal unipotent monodromy representations of braid groups and the computation of the universal Vassiliev invariant. In this part we try to give a uniform geometric approach to several constructions which are discussed in the literature with a more categorical point of view. Let us explain in more detail our results.

To a family of hyperplanes there is an associated formal connection and equation of parallel transport generalizing the Kniznik-Zamolodchikov equation (K-Z for short).

For the K-Z equation one considers the 1-form:

$$\Omega := \frac{1}{2\pi i}\sum_{i<j} t_{ij}\, d\log(z_i - z_j). \tag{0.1}$$

defined on the set of $n-$tuples of distinct complex numbers $z_1, \dots, z_n$, and the associated equation of parallel transport:

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$$dG = \Omega G, \text{ or } (d - \Omega)G = 0 \tag{0.2}$$

Here the elements t_{ij} can either be taken as linear operators or as generators of a formal graded Lie algebra. The main assumption is that the connection should be flat, i.e. that $\Omega \wedge \Omega = 0$. This requirement is equivalent to some well known commutation relations among the elements t_{ij} which we will presently recall in a more general context [Kh1].

In general one should consider a family of hyperplanes with linear equations $\alpha_i = 0$ and the form $\Omega = \frac{1}{2\pi i}\sum_i t_i d\log(\alpha_i)$. The form is defined on the open set $\mathcal{A}$ of $\mathbb{C}^n$ complement of the union of hyperplanes of equations $\alpha_i = 0$. Its flatness is equivalent to the commutation relations of the holonomy algebra introduced by Chen, which is the free algebra in the variables t_i modulo the following relations.

For any 2-dimensional subspace W spanned by some of the linear functions α_i we have:

$$[\sum_{j|\alpha_j \in W} t_j, t_h] = 0, \ \forall h|\alpha_h \in W. \tag{0.3}$$

Thus we will seek solutions G, to 0.2, as functions on the universal covering space $\tilde{\mathcal{A}}$ of $\mathcal{A}$, with values in the completion $\hat{R}$ of the graded algebra R, given by the generators t_i and the relations 0.3.

R is the universal enveloping algebra of the Lie algebra L given by the same generators and relations.

Since a global solution on $\tilde{\mathcal{A}}$ is uniquely determined once we choose a local solution a strategy is to construct a solution by prescribing its asymptotic behavior in a suitable region. This is done in the following way. Suppose we choose a neighborhood V of 0 in $\mathbb{C}^n$ and a holomorphic map of V in $\mathbb{C}^n$ which maps the complement of the coordinate hyperplanes $u_i = 0$ in the complement of the hyperplanes $\alpha_j = 0$ so that in these coordinates we have:

$$\alpha_j = \prod u_k^{m_{j,k}} p_j(u_1, \dots, u_n)$$

with $m_{j,k} \geq 0$ and $p_j(u_1, \dots, u_n)$ holomorphic in a neighborhood of 0 with $p_j(0) \neq 0$.

In this case we can rewrite the form Ω in these coordinates as

$$\frac{1}{2\pi i}\sum_j t_j d\log(\prod u_k^{m_{j,k}} p_j(u)) = \frac{1}{2\pi i}\sum_{j,k} t_j m_{j,k} d\log(u_k) + \Omega_0 = \sum_{k=1}^n A_k d\log(u_k) + \Omega_0$$

where the elements A_k are computed from the formula and Ω_0 is holomorphic around 0. The element A_k is called the *residue* at $u_k = 0$.

The flatness of the connection implies that the elements A_k commute and by recursive approximation one can compute a solution of equation 0.2 of the form

$$f(u_1, \dots, u_n)\prod_{k=1}^n u_k^{A_k}$$

with $f(u_1,\dots,u_n)$ holomorphic around 0 and $f(0)=1$. Of course in this solution there is a part which is polydromous, as it should be due to the logarithmic singularity in the equation. The study of the polydromy can to some extent be reduced to one dimensional computations using the fact that, if we single out one coordinate, say u_n the function $G' := f(u_1,\dots,u_{n-1},0)\prod_{k=1}^{n-1} u_k^{A_k}$ in the $n-1$ variables $u_1,\dots,u_{n-1}$ satisfies the differential equation $dG' = \Omega' G'$ where $\Omega' := \sum_{k=1}^{n-1} A_k d\log(u_k) + \Omega_0(u_1,\dots,u_{n-1},0)$.

With this discussion in mind it is clear that it is quite useful to construct a model of the open set $\mathcal{A}$ of the type considered by Hironaka. We want a smooth variety Z proper over $\mathbb{C}^n$, containing $\mathcal{A}$ as open set and such that its complement is a union of smooth divisors with normal crossings. Choosing a point of intersection of n such divisors and local equations u_i for the divisors furnishes the map described before and hence a solution of the associated KZ with prescribed asymptotic behavior. This is essentially what Drinfeld does in his paper [Dr2] although he prefers to use the language of real variables. The coordinates that he chooses to define the solutions $W_i, i=1,\dots,5$ for the KZ equation in 4 variables can indeed be interpreted as coordinates in open charts of such a model. The comparison between various solutions with different asymptotic behaviors has then important implications, which in the case of Drinfeld connect with the theory of quantum groups and the monodromy representation of the connection introduced by Chen and Khono.

The set of invertible solutions on the universal covering space $\tilde{\mathcal{A}}$ of $\mathcal{A}$ is a principal homogeneous space on the group $\hat{R}^*$ of invertible elements of $\hat{R}$. In the case of a root system the generalized braid group B, thought as group of covering transformations of $\tilde{\mathcal{A}}$ over $\mathcal{A}/W$, acts in a natural way on $\hat{R}^*$ by the W action on the elements t_α. The group B also acts on the set of solutions of the holonomy equation by the formula $(bG)(P) := b(G(b^{-1}P))$. Since two solutions differ by a multiplicative constant one obtains, for each solution G, a monodromy representation ϕ_G of B in the semidirect product $\hat{R}^* \ltimes W$. Two solutions give rise to two conjugate representations. In fact one can restrict to solutions with value in the *pro-Lie group* of group like elements (since R is a Hopf algebra).

We study explicitly the monodromy representations associated to the special solutions of the holonomy equation, defined by the asymptotic boundary conditions, in the case of root systems and prove that explicit formulas can be found only depending on the Drinfeld's associator $\Phi(A,B)$ (cf. [Dr2]) in cases A_n, D_n, E_6 while for the other cases one has to introduce a new basic associator which arises from the monodromy picture of B_2.

Let us use as notation t_i, s_i for the residue and the simple reflection associated to the simple root α_i indexed by a node i of the Dynkin diagram.

The points in the boundary around which we want to study asymptotic behavior of solutions are classified combinatorially by *maximal nested sets* of connected subdiagrams of the Dynkin diagram. These are collection of subdiagrams simply obtained by removing a node in the Dynkin diagram, taking the connected components of the resulting diagram and iterating the procedure on each such component.

The strategy is then the following. For each node i on the Dynkin diagram we can select a maximal nested set in which the node i appears as an element. The associated solution has the property that for it the monodromy of T_i is simply given by $e^{\frac{t_i}{2}} s_i$. Then two solutions associated to two different maximal nested sets differ by a multiplicative constant which, in sufficiently many cases, can be expressed in terms of the basic associators computed in suitable residues which can be simply described.

One should remark then that the monodromy representations obtained by these special solutions take value in the semidirect product of W with the much smaller and geometric subgroup of *group like* elements of $\hat{R}$, i.e. the elements a for which $\Delta(a) = a \otimes a$ where Δ is the comultiplication of R with its natural Hopf algebra structure given by its presentation as universal enveloping algebra.

The case A_n has a special interest since the monodromy formulas are directly connected to the theory of braided categories and the universal Vassiliev invariant. Here we give a more geometric (and simpler) proof of the equivalence between the construction of the universal Vassiliev invariant by Kontsevich integral and the one by the category of tangles and infinitesimal braided categories (introduced in [Ca],[Pi],[L-M]) developed by Le and Murakani. Our proof is based on the idea that the Konsevitch integral associated to a knot constructed by closing a braid can be embedded in the asymptotic behavior of one of the special solutions of the holonomy equation, a behavior which can be reconstructed from the monodromy formula.

Finally we may remark that the model of the root space that we construct is the total space of a line bundle over a smooth projective variety. Its real points are naturally decomposed as a $W-$equivariant polytope translating the closure of the Weyl chamber which is homeomorphic to a ball. This seems to be the correct generalization, with its complete structure of faces in all dimensions, of Bar Natan associahedron [B-N2] and should be compared with Kapranov's permutoassociahedron [Kp] (from which our model seems to be obtained contracting some faces).

1. Hyperplane configurations

1.1 In a finite dimensional complex vector space V consider a finite family of linear hyperplanes H_i. In the dual space V^* the orthogonal of H_i is a line. We may fix a non zero vector in each of these lines and obtain a finite set $X \subset V^* - \{0\}$ of pairwise linearly independent vectors. The elements of X are a set of defining equations for the hyperplane arrangement. We use X as indexing set of the hyperplanes and for $x \in X$ let $H_x := \{u \in V | < x|u >= 0\}$.

To this family one can associate several geometric objects. We let $\mathcal{A} := V - \cup_{x \in X} H_x$ be the complement of the union of the H_x. Consider also the family $\mathcal{C}$ of all subspaces $< A >$ generated by subsets $A \subset X$.

In [D-P1] we have introduced several notions for a family of subspaces closed under sum, in the present case they have a simple interpretation which can also be taken as definition.

Two notions introduced in [D-P1] are those of the family $\mathcal{F}$ of irreducibles of $\mathcal{C}$ and nested sets relative to $\mathcal{F}$.

We can define irreducible elements from a combinatorial point of view.

Definition. 1) A subset A of X will be called *complete* if $A =< A > \cap X$. In general the set $< A > \cap X$ is called the *completion* of A and denoted by $\overline{A}$.

2) We say that a set $Y \subset V^* - \{0\}$ is *reducible* if there exists two non empty subsets Y_1, Y_2 in Y such that $Y = Y_1 \cup Y_2$ and $< Y >=< Y_1 > \oplus < Y_2 >$. Otherwise we say that it is *irreducible.*

The definition has been set so that the family $\mathcal{C}$ of subspaces spanned by subsets of X is in 1-1 correspondence (preserving inclusions) with complete sets in X.

Thus we will use freely this identification. Moreover we will denote by $\mathcal{F}$ the subset of $\mathcal{C}$ formed by the complete sets which are irreducible.

Given two complete sets $A_1, A_2 \subset X$ the complete set corresponding to $< A_1 > + < A_2 >$ is the completion $\overline{A_1 \cup A_2}$. If A is a complete set in X, then $A = A_1 \cup A_2$ is a decomposition if and only if $< A >=< A_1 > \oplus < A_2 >$ is a decomposition according to [D-P1] 2.1 (and A_1, A_2 are necessarily complete).

Every complete set A is decomposed canonically as the disjoint union of the maximal complete and irreducible subsets contained in itself.

The combinatorics is all encoded in the *matroid* structure of X, i.e. in the information of which subsets of X are linearly dependent. This information is also contained in the function $d(A) := dim(< A >)$. The theory of [D-P1] §2 can be recast as follows:

Theorem. *1) Given a set $Z \subset V - \{0\}$ There exists a unique decomposition*

$$Z = \cup_{i=1}^n Z_i$$

into disjoint irreducible subsets such that

$$< Z >= \oplus_{i=1}^n < Z_i > .$$

(The irreducible decomposition of Z.)

2) If $A \subset Z$ is irreducible, then A is contained in exactly one of the Z_i's.

3) If A is irreducible every element $x \in A$ is dependent on $A - \{x\}$.

Proof. The fact that Z has an irreducible decomposition follows by induction. The uniqueness follows from part 2). This in turn is clear from $< A >= \oplus < A \cap Z_i >$. Part 3 is clear. $\square$

We have also defined nested sets $\mathcal{S}$ in $\mathcal{C}$ (or relative to X). A nested set is a set $F_1, \dots, F_k$ of irreducible subsets, such that whenever we choose out of them a subset formed by pairwise non comparable elements these are the elements of the irreducible decomposition of their union (i.e. their union is complete and it spans the direct sum of the spans of the given sets).

Let $n = dim < X >$ be the dimension of the span of all vectors in X. For a subset G of V not reduced to 0 we let $p_{\mathcal{S}}(G)$ to be the minimum $A \in \mathcal{S}$ with $G \subset < A >$. We let $p_{\mathcal{S}}(G) = V$ if there is no such A.

Proposition.

(1) *Let $\mathcal{S}$ be a maximal nested set in $\mathcal{C}$. If $A \in \mathcal{S}$ and $A_1, \dots, A_k \in \mathcal{S}$ are maximal in A we have $\sum_{i=1}^k dim(A_i) = dim(A) - 1$.*
(2) *Every maximal nested set in $\mathcal{C}$ has n elements.*

Proof. (1) By definition A is spanned by a subset of vectors in X and $\oplus_{i=1}^k A_i$ is properly contained in A, hence there is an $x \in X \cap A$ and $x \notin \oplus_{i=1}^k A_i$. We have thus $p_{\mathcal{S}}(x) = A$ (cf. [D-P1] 1.2) and applying Proposition [D-P1] 2.4 (2) we deduce that $A = Kx + \oplus_{i=1}^k A_i$, (1) follows.

(2) Let A_i denote the set of maximal elements of $\mathcal{S}$. By Remark 2.4 [D-P1], the family $\mathcal{S}$ is the union of the families $\mathcal{S}_i$ each maximal nested in A_i, and the direct sum of the A_i is $< X >$. We may work by induction on the dimension of $< X >$ and see that we are reduced to the case $< X > \in \mathcal{S}$. Then we can apply part (1) to the maximal elements A_i properly contained in $< X >$, $\mathcal{S}$ is the union of $< X >$ and the maximal nested sets $\mathcal{S}_i$ contained in A_i thus $|\mathcal{S}| = 1 + \sum_i |\mathcal{S}_i| = 1 + \sum_i dim(A_i) = dim < X >$ by induction. $\square$

In [D-P1] we have constructed a canonical variety associated to the family of irreducibles in $\mathcal{C}$ we recall the construction here.

Let $V_{\mathcal{F}} := \cup_{A \in \mathcal{F}} A^{\perp}$ be the union of all the subspaces $A^{\perp}$, $A \in \mathcal{F}$, it is easily seen that $V_{\mathcal{F}} = \cup_{x \in X} H_x$ and so $\mathcal{A}$ is the open set of V complement of $V_{\mathcal{F}}$.

Denote by $\mathbb{P}_A = \mathbb{P}(V/A^{\perp})$ the projective space of lines in $V/A^{\perp}$.

By construction the rational map $\pi_A : V \to V/A^{\perp} \to \mathbb{P}_A$ is defined outside $A^{\perp}$ and thus we have a regular morphism $\mathcal{A} \to \prod_{A \in \mathcal{F}} \mathbb{P}_A$, its graph is a closed subset of $\mathcal{A}_{\mathcal{F}} \times \prod_{A \in \mathcal{F}} \mathbb{P}_A$ which embeds as open set into $V \times \prod_{A \in \mathcal{F}} \mathbb{P}_A$. Finally we have an embedding:

$$\rho : \mathcal{A}_{\mathcal{F}} \to V \times \prod_{A \in \mathcal{F}} \mathbb{P}_A$$

as a locally closed subset.

Definition. We let Y_X to be the closure of the image of $\mathcal{A}$ under ρ.

If we fix a complement U' to $< X >^{\perp}$, we see immediately, from the construction of Y_X, that this variety is the product of U' with the variety Y'_X associated to

X thought as a subset of $< X >$. Thus we will assume that $V^* =< X >$ (or $\cap_{x\in X} H_x = 0$).

Given a maximal nested set $\mathcal{S}$ we have seen in the previous Proposition that if $A \in \mathcal{S}$ and $A_1, \dots, A_k \in \mathcal{S}$ are maximal in A we have $\sum_{i=1}^k dim(A_i) = dim(A) - 1$. In particular we may select an element $x_A \in A - \cup_i A_i$ and we see that, for every $A \in \mathcal{S}$ the set $b_A := \{x_B | B \in \mathcal{S},\ B \subset A\}$ is a basis of A.

This is what we have called in [D-P1] 1.3 an adapted basis (notice that in this case every element of the basis is marked). In a nested set the elements $C \in \mathcal{S}$ properly containing a non maximal element A are linearly ordered and we have denoted by $c(A)$ the minimum of these elements, then one sets $u_A = x_A/x_{c(A)}$ if A is not maximal and $u_A = x_A$ otherwise.

¿From the Theory developed in [D-P1] we recall that the functions u_A can be considered as local coordinates in an open set of Y_X, which is denoted by $\mathcal{U}_{\mathcal{S}}$, where $u_A = 0$ defines an irreducible boundary divisor D_A. The divisors $D_A,\ A \in \mathcal{S}$ intersect transversally in a single point which we will denote by $P_{\mathcal{S}}$.

Lemma 3.1 of [D-P1] describes the actual form of the open set and in this context takes a simpler form, moreover the open set $\mathcal{U}_{\mathcal{S}}$ is independent of b and will be denoted by $\mathcal{U}_{\mathcal{S}}$:

Lemma.

(1) *If $\mathcal{S}$ is maximal nested, given any $x \in X$, let $A = p_{\mathcal{S}}(x) \in \mathcal{S}$. Then $x = x_A P_x(u_B)$, P_x is a polynomial depending only on the variables $u_B,\ B < A$, P_x does not vanish in 0.*

(2) *The open set $\mathcal{U}_{\mathcal{S}}$ is independent of b.*

Proof. (1) The first part is like in [D-P1] 3.1. The non vanishing follows from the same reasoning as in Proposition 1.1 (1).

(2) It is enough to see this when we change a single basis element, say x_A is replaced by a new element x'_A which necessarily satisfies $A = p_{\mathcal{S}}(x'_A)$. Then let us indicate by u'_B the new coordinates.

Clearly $u'_B = u_B$ unless, either $A = B$ or $A = c(B)$. In the first case $u'_A = u_A P_{x'_A}$ in the second case $u'_B = u_B P_{x'_A}^{-1}$. As for the polynomials $P_x := x/x_B$ ($B = p_{\mathcal{S}}(x)$), it also depends on the basis but in the new basis $P_x = P'_x$ unless $B = A$ in which case $P'_x = P_x P_{x'_A}$ and the claim is clear. $\square$

The open set $\mathcal{U}_{\mathcal{S}}$ is thus defined, in the space $\mathbb{C}^{\mathcal{S}}$, as complement of the union of the hypersurfaces of equations $P_x(u_B) = 0$.

The main theorem of [D-P1] in this case is (assume X irreducible for simplicity):

Theorem.

(1) *Y_X is a smooth variety covered by the open sets $\mathcal{U}_{\mathcal{S}}$ as $\mathcal{S}$ varies over all maximal nested sets of irreducibles in X.*

(2) *Y_X is the total space of a line bundle over a smooth projective variety $\overline{Y}_X$. There is a canonical map from $\overline{Y}_X$ to $\mathbb{P}(V)$ such that Y_X is the pullback of the tautological bundle.*

(3) *The proper map $Y_X \to V$ is an isomorphism outside the open set $\mathcal{A}$ complement of the union of the hyperplanes $H_x,\ x \in X$. The complement of $\mathcal{A}$ in Y_X is a divisor with normal crossings whose irreducible components D_B*

are smooth and indexed by the irreducibles $B \in \mathcal{F}$. The 0 section of the line bundle corresponds to the whole set X.

(4) *A family $\{D_B\}_{B\in\mathcal{T}}$ of boundary divisors has a non empty intersection if and only if $\mathcal{T}$ is nested. In this case the intersection is transversal (hence smooth) and irreducible.*

(5) *Y_X can be obtained from V by a sequence of blow ups along smooth centers which are proper transforms of subspaces intersections of some of the H_x.*

Let us finally make an observation on the computation of the cohomology of Y_X (this will not be used in this paper). In the hyperplane case the cohomology computations are rather easier and one gets (notations as in in section 5 in [D-P1]).

Proposition. *Let $\mathcal{T}$ be a nested set in X. Consider the subvariety $D_{\mathcal{T}}$ intersection of the divisors D_C, $C \in \mathcal{T}$. Then $H^*(D_{\mathcal{T}}, \mathbb{Z})$ is the algebra generated by the classes c_A (induced by the duals of the divisors D_A) for $A \in \mathcal{F}$ with relations*

$$\prod_{A\in\mathcal{M}} c_A, \quad \textit{if} \quad \mathcal{M}\cup\mathcal{T} \quad \textit{is not nested}$$

$$\sum_{x\in A} c_A \quad \textit{for all} \quad x \in X.$$

Proof. The proof is similar to that given in section 5 in [D-P1] and we leave it to the reader. □

1.2 It is useful to discuss a simple generation of maximal nested sets which reveals some combinatorial properties which will be needed in section 2.

Remark that, one can define a nested family in a finite set to be a family of subsets which are disjoint when non comparable. The two following facts are easily verified

Lemma.

(1) *A maximal nested family $\mathcal{S}$ in a set B with n elements has n elements.*

(2) *One can identify the set B with $1, 2, \dots, n$ in such a way that each element $A_i \in \mathcal{S}$ is a segment $[i, j] := \{k | i \le k \le j\}$.*

□

We go back to the family $\mathcal{C}$ whose elements are spanned by vectors in X. Clearly a maximal flag in $\mathcal{C}$ is necessarily the flag $C_h :=< x_1, x_2, \dots, x_h >$ associated to an (ordered) basis $x_1, x_2, \dots, x_n$ extracted from X.

One can thus apply Theorem 2.4 of [D-P1] and see that a maximal nested set $\mathcal{S}$ is generated by such a basis taking successively the decomposition of the spaces C_h. In this case though, by Proposition 1.1 (2) a maximal nested set in C_h has h elements, thus in the decomposition of C_{h+1} we find exactly one new element A_{h+1} of the nested set which has not been listed before in the decomposition of the previous spaces C_i, $i \le h$.

It is easily seen that we may take, for each h, the vector x_h as vector of an adapted basis marked by A_h.

The total ordering A_i coming from a flag is a refinement of the inclusion order and has the property that $dim(\sum_{i\leq h} A_i) = h$. Conversely, if $A_1, \dots , A_n$ is a total ordering on a maximal nested set $\mathcal{S}$, such that $dim(\sum_{i\leq h} A_i) = h$, the subspaces $C'_h := \sum_{i\leq h} A_i$ form a maximal flag from which $\mathcal{S}$ is deduced.

The condition $dim(\sum_{i\leq h} A_i) = h$ is equivalent to

$$\sum_{A_j \text{ maximal among } A_1,\dots,A_h} dim(A_j) = h.$$

We may define a total ordering of $A_1, \dots , A_h$ as *compatible* if it satisfies the previous conditions.

When we choose an adapted basis for the maximal nested set, the basis is indexed by the elements of $\mathcal{S}$ and so we order it as $\mathcal{S}$, then the flag associated to the adapted basis generates under decomposition the nested set $\mathcal{S}$.

Proposition.

(1) *The decomposition map induces a bijective correspondence between maximal flags in $\mathcal{C}$ and maximal nested sets with a compatible ordering.*
(2) *If $\mathcal{S}$ is a maximal nested set in $\mathcal{C}$ and b is an adapted basis the sets b_A, $A \in \mathcal{S}$ form a maximal nested family in the set b.*

□

A simple way to visualize maximal nested sets of segments in $1, 2, \dots , n$ is by *nested parentheses* i.e. sequences $(()(()))(())()$.

For combinatorial reasons it is useful to introduce the *flag graph*. One considers as vertices all maximal flags and joins two vertices with an edge if the two maximal flags differ only in one position.

Similarly we consider maximal nested sets as vertices of a graph joining two maximal nested sets by an edge if they differ by one element.

Theorem. *The graphs of flags and of maximal nested sets are connected.*

Proof. It is enough to prove this for the flag graph. Two maximal flags can be given by two ordered adapted bases. Let them be $v_1, v_2, \dots , v_n$; $w_1, w_2, \dots , w_n$.

We give an explicit algorithm. At the i^{th} step suppose we have already replaced i vectors v_j with $w_1, w_2, \dots , w_i$ and assume that the two maximal flags associated to the bases $v_1, v_2, \dots , v_n$; $w_1, w_2, \dots , w_i, v_{k_1}, \dots , v_{k_{n-i}}$ are already connected. Now we want to insert w_{i+1} in the basis $w_1, w_2, \dots , w_i, v_{k_1}, \dots , v_{k_{n-i}}$ taking out one of the v_{k_j}, for this it is enough to select the minimal j for which w_{i+1} is dependent on $w_1, w_2, \dots , w_i, v_{k_1}, \dots , v_{k_j}$. By this procedure the flag and hence the nested set does not change. Now we proceed to place w_{i+1} in the $i + 1$ position and we can do this by a sequence of transpositions in successive places. Each such transposition generates two bases with associated flags connected by an edge. □

2 Drinfeld's theory

2.1 Let us briefly explain Drinfeld's theory in an algebraic geometric language.

We discuss the formal connection with form $\Omega := \sum t_x d\log(x)\ |x \in X$ where t_x lie in a (possibly) non commutative algebra which we will specify presently. For every complete set $A \subset X$ set $t_A := \sum_{x \in A} t_x$.

According to [Kh1] the flatness condition for this connection is the following. For every complete set $A \subset X$ spanning a 2 dimensional space one must impose the relation

$$[t_A, t_x] = 0,\ \forall x \in A. \tag{2.1.1}$$

The free non commutative algebra in the t_x modulo the relations 2.1.1 is called by Chen [Ch] the *holonomy algebra*, we will denote it by M. We think of Ω as the *matrix* of a connection with values in M. The analogue of the K-Z equation is $dG = \Omega G$ and will be called the holonomy equation.

In order to perform the necessary analysis we have to introduce the algebra $\hat{M}$, completion of M with respect to the natural gradation $deg(t_x) = 1$.

The solutions of the holonomy equation are holomorphic (in their domain of definition) functions with values in $\hat{M}$, there is no essential restriction in assuming that the constant term of such solutions be non 0, in this case two solutions G_1, G_2 are related by $G_2 = G_1 K$ with $K \in \hat{M}$ an invertible constant.

The analytic computations performed with functions with values in $\hat{M}$ are all justified by the fact that $\hat{M}$ is the inverse limit of the finite dimensional algebras M/J_n where J_n is the ideal of the elements of degree $\geq n$. So a function is determined by a sequence of compatible functions in the finite dimensional algebras M/J_n.

In fact one may take a more geometric approach. Since M is the enveloping algebra of a Lie algebra L it has a Hopf algebra structure with comultiplication Δ and antipode S defined by:

$$\Delta(t_i) := t_i \otimes 1 + 1 \otimes t_i;\ S(t_i) := -t_i.$$

In $\hat{M}$ we consider the *group like elements* i.e. the elements g with $\Delta(g) = g \otimes g$. We restrict to the ones which have constant term 1 and they are of the form e^a where a is a primitive element and in the completion $\hat{L}$ of the Lie algebra L. These elements form a group $\hat{U}$ with Lie algebra $\hat{L}$.

Lemma. *A solution $G(s)$ of the holonomy equation takes all its values in $\hat{U}$ if and only if it takes a value in $\hat{U}$ in a point.*

Proof. $\Delta(\Omega) = \Omega \otimes 1 + 1 \otimes \Omega$ hence, if $G(s)$ is a solution of the holonomy equation we have $d[G(s) \otimes G(s)] = \Delta(\Omega)[G(s) \otimes G(s)]$. Also $d\Delta(G(s)) = \Delta(\Omega)\Delta(G(s))$ thus $G(s) \otimes G(s)$, $\Delta(G(s))$ are two solutions of the holonomy equation associated to $\Delta(\Omega)$. It follows that $G(s) \otimes G(s) = \Delta(G(s))K$ for some constant K, thus if in a point p_0 we have $G(p_0) \otimes G(p_0) = \Delta(G(p_0))$ the constant K equals 1. $\square$

We will call such solutions **unipotent**. They form a principal homogeneous space under $\hat{U}$.

Our first goal will be to show the existence of special solutions of the holonomy equation, indexed by maximal nested sets $\mathcal{S}$. For such sets the intersection $\cap_{A\in\mathcal{S}} D_A$ is reduced to a single point, which we will denote by $P_{\mathcal{S}}$, the seeked for solution is characterized by the asymptotic behavior around the point $P_{\mathcal{S}}$ in the compactification.

In what follows we shall denote by x^t, for t an element in the holonomy algebra of degree one and $x \in X$, the power series $e^{t\log(x)}$. This will be well defined either on the universal covering space of $\mathcal{A}$ or in a region in which a suitable determination of $\log(x)$ has been chosen. We shall specify the choice when needed.

Let us first of all discuss the 2-dimensional case.

If $X = \{x, y\}$ then X is reducible, the algebra M is commutative and we have a canonical solution $G = x^{t_x} y^{t_y}$.

Otherwise X is irreducible, $T := \sum_{x\in X} t_x$ is in the center of the algebra M, and a maximal nested set with an adapted basis is just the choice of an ordered basis $x_B, x_X \in X$. Setting $u := u_B = x_B/x_X,\ v = x_X$, for each $x \in X,\ x \neq x_B$, we have that $x = \alpha_x v(1 + c_x u)$ and

$$\Omega = \frac{Tdv}{v} + \frac{t_{x_B} du}{u} + (\sum_{x\neq x_B} \frac{c_x t_x}{1 + c_x u})du,$$

a solution to $dG = \Omega G$ can always be written in the form Fv^T where (v^T is central and) F is solution of

$$\frac{dF}{du} = (\frac{t_{x_B}}{u} + (\sum_x \frac{c_x t_x}{1 + c_x u}))F \tag{2.1.2}$$

as remarked by Drinfeld there is a unique solution to this equation in a neighborhood of 0 of the form $f(u)u^{t_{x_B}}$ with f holomorphic and $f(0) = 1$, which is thus associated to the choice of our ordered basis.

By making suitable cuts on $\mathcal{A}$ in real codimension one, we can obtain a simply connected dense open set on which all of these solutions extend. By selecting a section of this open set in the universal covering space these functions extend to global solutions in this space.

The constants relating two such solutions are thus elements in $\hat{M}$ which depend on the c_x as parameters (and also on the choice of the cuts).

Of particular importance is the case of 3 vectors which we can normalize to be $x, y, x + y$. This corresponds to the root system A_2 and it is conventional to write the form as

$$t_{12} d\log(z_1 - z_2) + t_{13} d\log(z_1 - z_3) + t_{23} d\log(z_2 - z_3).$$

(Setting $x = z_1 - z_2 = \alpha_1,\ y = z_2 - z_3 = \alpha_2$ the two simple roots). In this case, up to the symmetry given by the action of the the Weyl group S_3, one is reduced to the two maximal nested sets associated to x, y respectively and with adapted basis $x, x+y;\ y, x+y$. The corresponding coordinates u are $u = x/x+y;\ 1-u = y/x+y$. The holonomy form is

$$Td\log(x + y) + t_{12} d\log(u) + t_{23} d\log(1 - u).$$

By 2.1.2 we are reduced to the study of the two asymptotic solutions

$$G_0 = f(u)u^{t_{12}},\ G_1 = g(u)(1-u)^{t_{23}},\ f(0) = g(1) = 1$$

of the equation

$$\frac{dF}{du} = \frac{1}{2\pi i}\left(\frac{A}{u} + \frac{B}{1-u}\right)F$$

around 0 and 1 (with $A = 2\pi i t_{12}$ and $B = 2\pi i t_{23}$).

The corresponding constant $\Phi(A, B)$, defined by the property $G_0\Phi(A, B) = G_1$, on the open interval $(0, 1)$ is a formal power series in the free algebra generated by A, B and a fundamental object in Drinfeld's theory. One verifies easily that it is a group like element and that the two special solutions are unipotent.

Let us pass to the general case. We fix a set X of vectors in V^* and a building set $\mathcal{G}$ in the family $\mathcal{C}$ of subspaces generated out of X. From now on we shall consider $\mathcal{G}$-nested sets and often call them nested. These notions are slightly more general than the ones used up to now of irreducibles and are useful for inductions. We refer to [D-P1] 2.3 for details.

For a maximal nested set $\mathcal{S}$ and a $B \in \mathcal{S}$, let $R_B := \sum_{p(x)=B} t_x$, we have that $t_B = \sum_{x\in B} t_x = \sum_{C\in\mathcal{S}, C\subset B} R_C$ and hence

$$\sum_{B\in\mathcal{S}} R_B d\log x_B = \sum_{B\in\mathcal{S}} t_B d\log u_B \tag{2.1.3}$$

$$\prod_{A\in\mathcal{S}} u_A^{t_A} = \prod_{A\in\mathcal{S}} u_A^{\sum_{C\subseteq A} R_C} = \prod_{C\in\mathcal{S}}\Big(\prod_{A\supseteq C} u_A\Big)^{R_C} = \prod_{C\in\mathcal{S}} x_C^{R_C}.$$

Theorem.

(1) *Let $A \in \mathcal{G}$ then the residue of Ω on the boundary divisor D_A in the model $Y_{\mathcal{G}}$ is t_A. If $\{A, B\}$ is nested $[t_A, t_B] = 0$.*

(2) *Given a maximal nested set $\mathcal{S}$ and an adapted basis x_B, G is a solution of the holonomy equation $dG = \Omega G$ if and only if $F := G\prod_{A\in\mathcal{S}} x_A^{-R_A}$ is a solution of the equation*

$$dF = \Big[\sum_{A\in\mathcal{S}} R_A d\log x_A, F\Big] + \Omega_{\mathcal{S}}F = \Big[\sum_{A\in\mathcal{S}} t_A d\log u_A, F\Big] + \Omega_{\mathcal{S}}F. \tag{2.1.4}$$

where

$$\Omega_{\mathcal{S}} = \Omega - \sum_{A\in\mathcal{S}} R_A d\log x_A = \sum_x t_x d\log P_x \tag{2.1.5}$$

is holomorphic on $\mathcal{U}_{\mathcal{S}}$.

Proof. 1) Consider a maximal nested set $\mathcal{S}$ containing A with an adapted basis x_B and its coordinates $u_B = x_B/x_{c(B)}$.

¿From Lemma 1.1, for every $x \in X$ with $A = p(x)$, we have that $x = x_A P_x = \prod_{B \supset A} u_B P_x$ and $d\log(x) = \sum_{B \supset p(x)} \frac{du_B}{u_B} + d\log\, P_x$ then

$$\Omega = \sum_x t_x (\sum_{B \supset p(x)} \frac{du_B}{u_B} + d\log\, P_x) = \tag{2.1.6}$$

$$\sum_B (\sum_{x | B \supset p(x)} t_x) \frac{du_B}{u_B} + \Omega_{\mathcal{S}} = \sum_B t_B \frac{du_B}{u_B} + \Omega_{\mathcal{S}}$$

where $\Omega_{\mathcal{S}} = \sum_x t_x d\log\, P_x$ is holomorphic on $\mathcal{U}_{\mathcal{S}}$ by definition. The computation of the residue follows. As for the commutation property, every nested set is contained in a maximal one and, for a maximal nested set, the commutation of the residues follows immediately from the flatness of Ω and the formula 2.1.6.

2) The equivalence of the two equations is a simple computation due to the fact that the R_A commute. The second part follows from 2.1.3, 2.1.6 and the definition of $\mathcal{U}_{\mathcal{S}}$. □

We shall refer to the equation 2.1.4 as the holonomy equation localized in $\mathcal{U}_{\mathcal{S}}$ (notice that it depends on the choice of the x_A's). If A is maximal, then t_A lies in the center of the holonomy algebra and $\Omega_{\mathcal{S}}$ does not depend on u_A. Therefore any solution of 2.1.4 is independent of the coordinates u_A for A maximal.

Corollary. *Fix a maximal nested set $\mathcal{S}$ with an adapted basis x_A, and a simply connected open set $\mathcal{B}$, contained in $\mathcal{U}_{\mathcal{S}}$ and containing the point $P_{\mathcal{S}}$ (of coordinates $u_A = 0$ cf. 1.1).*

1. *There exists a uniquely defined holomorphic function $F_{\mathcal{S}}$ on $\mathcal{B}$, such that $F(P_{\mathcal{S}}) = 1$ and for every determination of $\log(x_A)$, $A \in \mathcal{S}$, the function $G_{\mathcal{S}} = F_{\mathcal{S}} \prod_{A \in \mathcal{S}} x_A^{R_A} = F_{\mathcal{S}} \prod_{A \in \mathcal{S}} u_A^{t_A}$ is a solution of the holonomy equation.*
2. *The function $F_{\mathcal{S}}$ is the unique solution of the differential equation 2.1.4 with the given initial condition.*
3. *$G_{\mathcal{S}}$ is unipotent.*

Proof. 1), 2) This result is essentially contained in Whittaker Watson [W-W], Lappo Danilevsky [La] or Drinfeld [Dr2]. We offer a slightly different proof.

Our statements are equivalent to prove that there is a unique solution $F_{\mathcal{S}}$ with the given initial condition of the equation 2.1.4.

Remark that a solution to the equation $dG = \Omega G$ can be constructed inductively for each degree. If we have the degree i part G_i, $i \leq n$ we have the equation $dG_{n+1} = \Omega G_n$ which is compatible by the flatness condition $\Omega \wedge \Omega = 0$.

Setting $\Omega_{\mathcal{S}} = \sum_{A \in \mathcal{S}} P_A du_A$ and writing $F_{\mathcal{S}} = \sum_{i=0}^{\infty} F_i$ (F_i of degree i) with $F_0(P_{\mathcal{S}}) = 1$, $F_i(P_{\mathcal{S}}) = 0$, $i > 0$, the equation is equivalent to the system

$$dF_{i+1} = [\sum_{A \in \mathcal{S}} t_A d\log u_A, F_i] + \Omega_{\mathcal{S}} F_i = \sum_{A \in \mathcal{S}} ([t_A d\log u_A, F_i] + P_A F_i du_A) \tag{2.1.7}$$

with the given initial conditions.

To start $dF_0 = 0$ implies $F_0 = 1$ and $[\sum_{A \in \mathcal{S}} R_A d\log x_A, F_0] = 0$. Assume inductively that $[\sum_{A \in \mathcal{S}} t_A d\log u_A, F_i] + \Omega_{\mathcal{S}} F_i$ is holomorphic. The compatibility of 2.1.4 is clearly equivalent to the flatness of Ω. We claim that, if we choose F_{i+1} solution of 2.1.7 with the initial condition $F_{i+1}(P_{\mathcal{S}}) = 0$, $[\sum_{A\in\mathcal{S}} t_A d\log u_A, F_{i+1}]$ is again holomorphic. To see this it is enough to show that $[t_A, F_{i+1}]$ is divisible by u_A or that $[t_A, F_{i+1}]$ vanishes on $u_A = 0$. Since $[t_A, F_{i+1}]$ vanishes in the point $P_{\mathcal{S}}$ of coordinates $u_B = 0$ it suffices to show that $[t_A, F_{i+1}]$ is constant on $u_A = 0$ or that $[t_A, dF_{i+1}]_{u_A=0} = 0$.

Now $[t_A, dF_{i+1}] = [\sum_{B\in\mathcal{S}} t_B d\log u_B, [t_A, F_i]] + [t_A, \Omega_{\mathcal{S}}]F_i + \Omega_{\mathcal{S}}[t_A, F_i]$. Setting $u_A = 0$ and applying induction we see that all terms vanish except possibly $[t_A, \Omega_{\mathcal{S}}]F_i$. By the flatness condition $\Omega\wedge\Omega = 0$, we get $[\frac{t_A}{u_A} + P_A, \frac{t_B}{u_B} + P_B] = 0, \forall B \neq A$. Multiplying by u_A ,we get $[t_A, \frac{t_B}{u_B} + P_B]_{u_A=0} = 0$ and so $[t_A, \Omega_{\mathcal{S}}]_{u_A=0} = 0$.

3) Remark that $\Delta F_{\mathcal{S}} \prod u_A^{t_A} = (\Delta F_{\mathcal{S}}) \prod u_A^{t_A\otimes 1 + 1\otimes t_A}$ and $\Delta F_{\mathcal{S}}(0) = 1\otimes 1$.

It follows that $\Delta G_{\mathcal{S}} = G_{\mathcal{S}} \otimes G_{\mathcal{S}}$ by the uniqueness of the solution with given asymptotic behavior. $\square$

Remark. Notice that a locally defined solution of the holonomy equation gives rise to infinitely many solutions on the universal cover of $\mathcal{A}$ which are in a unique orbit under the action of the group of deck transformations. All the solutions $G_{\mathcal{S}}$ for a given maximal nested set $\mathcal{S}$ give rise to the same orbit of solutions on the universal cover.

2.2 The computations of the next 3 sections can be viewed as a direct and explicit approach in our special case to the groupoid monodromy picture treated by Deligne in [De 2].

Let $A \subset X$ be a complete set.

Consider the space $V/A^\perp$. The elements of A induce a hyperplane configuration in $V/A^\perp$ and A can be considered as a subset of the dual of $V/A^\perp$. Consider the form $\Omega^A := \sum_{x\in A} t_x d\log x$. Notice that Ω_A is the pull-back of the holonomy form Ω'_A in $V/A^\perp$ associated to the hyperplane configuration in $V/A^\perp$ induced by A and with the same residues.

Suppose now that $\mathcal{S}$ is a maximal nested set and $A \in \mathcal{S}$. Let $\mathcal{U}_{A,\mathcal{S}}$ denote the subspace of $\mathcal{U}_{\mathcal{S}}$ defined by the vanishing of the coordinates u_C for $C \not\subset A$. Let us denote on this space the remaining coordinates by $\overline{u}_B$, $B \subsetneq A$. If we take the open set $\mathcal{U}_{\mathcal{S}_A} \to V/A^\perp$ with coordinates u_B, $B \subseteq A$ we can define a function $\gamma : \mathcal{U}_{\mathcal{S}_A} \to \mathcal{U}_{A,\mathcal{S}}$ mapping the point of coordinates u_B, $B \subseteq A$ to the point of coordinates $\overline{u}_B = u_B$ for $B \subsetneq A$.

Lemma. *If $x \notin A$, the restriction of the polynomial P_x to the subspace $\mathcal{U}_{A,\mathcal{S}}$ is the non zero constant $P_x(0)$.*

Proof. By assumption $p(x) \not\subset A$, thus we have two cases. If $p(x)$ is not comparable to A the polynomial P_x depends only on the coordinates which are set to zero.

If $p(x) \supsetneq A$ consider the monomials in P_x in which a coordinate u_C, $C \subseteq A$ appear. By the construction of P_x (see Lemma 1.1 and [D-P1] 3.1), the same monomial contains also the variable u_A which is set equal to zero. $\square$

Corollary. *The restriction of the form $\Omega_{\mathcal{S}}$ to the subspace $\mathcal{U}_{A,\mathcal{S}}$ equals the restriction of the form $\Omega_{\mathcal{S}}^A := \sum_{x\in A} t_x d\log P_x$.* $\square$

Let $F_{\mathcal{S}}$ be the solution of the holonomy equation localized in $\mathcal{U}_{\mathcal{S}}$. Consider next the function $F_A(u_C)$, $C \subsetneq A$ defined by setting equal to 0 all the variables

u_B, $B \not\subset A$ in $F_{\mathcal{S}}$. This is a function on $\mathcal{U}_{A,\mathcal{S}}$. Composing with γ, we can consider it as a function on $\mathcal{U}_{\mathcal{S}_A}$. We claim that

Proposition. *The function F_A is the unique solution of the holonomy equation (for the form Ω'_A) localized in $\mathcal{U}_{\mathcal{S}_A}$ such that $F_A(P_{\mathcal{S}_A}) = 1$.*

Proof. 1) We have already remarked that since t_A lies in the center of the algebra generated by the elements t_x with $x \in A$, the holonomy equation (for the form Ω'_A) localized in $\mathcal{U}_{\mathcal{S}_A}$ equals

$$dF = [\sum_{B \in \mathcal{S}_A - \{A\}} t_B d\log u_B, F] + \Omega^A_{\mathcal{S}} F.$$

Write $\Omega_{\mathcal{S}} := \sum_{B \in \mathcal{S}} P_B du_B$, then the holonomy equation for Ω localized in $\mathcal{U}_{\mathcal{S}}$ is equivalent to the system of p.d.e

$$\frac{\partial F}{\partial u_B} = [\frac{t_B}{u_B}, F] + P_B F,$$

for B not a maximal element in $\mathcal{S}$ and $F_{\mathcal{S}}$ satisfies this system. By the previous corollary it is then clear that F_A satisfies the analogous system for the form Ω'_A.
□

2.3 Consider now an element $x \in X$ and $\Omega_x := \sum_{y \neq x} t_y d\log y$. This form is also defined on the hyperplane $x = 0$ and we denote by Ω^0_x its restriction to this hyperplane. We also restrict the linear functions $y \in X$ to $x = 0$; they define some hyperplanes β_i in $x = 0$ corresponding to the family $\mathcal{C}_{2,x}$ of 2 dimensional subspaces spanned by vectors in X and containing x. Define $\bar{t}_W = \sum_{y \in W,\ y \neq x} t_y = t_W - t_x$. Select, for each $W \in \mathcal{C}_{2,x}$ a generator y_W of $W/Kx \subset V^*/Kx$, let $\overline{X}$ be the set of these elements. We have then that the restriction of Ω_x to $x = 0$ is $\Omega^0_x = \sum_{W \in \mathcal{C}_{2,x}} \bar{t}_W d\log y_W$.

Now we claim that the form Ω^0_x is flat. If U is a 3 dimensional subspace containing x we must show that $t_U - t_x = \sum_{W \subset U,\ W \in \mathcal{C}_{2,x}} \bar{t}_W$ commutes with all the elements $\bar{t}_W$. We know that t_U commutes with t_W and with t_x by nestedness hence the claim.

Remark now that for any subspace M containing x we have, as in 2.1

$$t_{M/Kx} = \sum_{W \in \mathcal{C}_{2,x}, W \subset M} \bar{t}_W = t_M - t_x.$$

For uniformity of notations we set $\bar{t}_M = t_{M/Kx}$ if $x \in M$, $\bar{t}_M = t_M$ otherwise.

We fix now a maximal nested set $\mathcal{S}$ containing $\{x\}$. For the induced configuration on the hyperplane $x = 0$ we have, as in [D-P1] 2.3, an induced building set $\overline{\mathcal{G}}$ and a maximal $\overline{\mathcal{G}}$-nested set $\mathcal{S}'$, formed by the subsets $\overline{B} := \overline{X} \cap (< B > + Kx/Kx)$, $B \in \mathcal{S}$, $B \neq \{x\}$.

As in Corollary 2.1 consider the solution $F_{\mathcal{S}}$ of the holonomy equation localized at $\mathcal{U}_{\mathcal{S}}$.

Let us observe that, by [D-P1] 1.1.1, the divisor D_x in $Y_{\mathcal{G}}$ is isomorphic to the variety $Y_{\overline{\mathcal{G}}}$; moreover it is easily verified that the subset of $\mathcal{U}_{\mathcal{S}}$ where $u_x = 0$ (which is the intersection of $Y_{\overline{\mathcal{G}}}$ with $\mathcal{U}_{\mathcal{S}}$) can be naturally identified to the open set $\mathcal{U}_{\mathcal{S}'}$ where b' is the adapted basis induced by $b - \{x\}$ in V^*/Kx.

Proposition. *The function $F_{\mathcal{S}}$, restricted to $u_x = 0$ is the solution $F_{\mathcal{S}'}$ on $x = 0$ of the holonomy equation for the form Ω_x^0, localized in $\mathcal{U}_{\mathcal{S}'}$.*

Proof. We have already remarked in the proof of Corollary 2.1 that

$$[\frac{t_x}{x}, F_{\mathcal{S}}] = (\prod_{B \supsetneq \{x\}} u_B)^{-1} [\frac{t_x}{u_x}, F_{\mathcal{S}}]$$

is holomorphic in a neighborhood meeting $u_x = 0$ and so in this neighborhood

$$[\frac{t_x}{x}, F_{\mathcal{S}}] dx|_{x=0} = 0.$$

We deduce that so $F_{\mathcal{S}}$, restricted to $x = 0$, is a solution of the holonomy equation associated to Ω_x^0 localized in $\mathcal{U}_{\mathcal{S}'}$. From the explicit form of this solution it follows that it has the required initial condition. $\square$

2.4 Now we can proceed by induction and get a more general result. Suppose that $\mathcal{S}$ is a $\mathcal{G}$-nested set with adapted basis b. Let $\mathcal{T} \subset \mathcal{S}$ be a subset containing a unique maximal element A and such that if $C = \cup_{B \in \mathcal{T} - \{A\}} B$, $\dim \frac{<A>}{<C>} \geq 2$.

Consider the set $A - C$. Given $x, y \in A - C$ we say that $x \sim y$ if x and y define the same line in $\frac{<A>}{<C>}$. This is an equivalence relation and defines a set of distinct lines in $\frac{<A>}{<C>}$ and hence a configuration of hyperplanes in $C^\perp / A^\perp$. We choose in every equivalence class a vector x and denote by $\overline{x}$ its image in $\frac{<A>}{<C>}$ in such a way that if the equivalence class contains a necessarily unique, vector in our adapted basis this will be the chosen vector.

In this way we get a set of mutually non proportional vectors $X_{A,C} \subset \frac{<A>}{<C>}$, and a map $\pi : A - C \to X_{A,C}$. It is easy to see that, if we set $\overline{\mathcal{G}} = \{\overline{D} \subset X_{A,C} | \overline{D} = \pi(D \cap (A - C))$ with $D \subset \mathcal{G}\}$, this is a building family relative to $X_{A,C}$ and that the image $\overline{\mathcal{S}}$, of the maximal $\mathcal{G}$-nested set $\mathcal{S}$ in $\overline{\mathcal{G}}$ is a maximal $\overline{\mathcal{G}}$-nested set with adapted basis $\overline{b}$, given by the vectors $\overline{x}_B$ with $B \subset A$ but $B \not\subset C$.

Give $\overline{x} \in X_{A,C}$, we set $t_{\overline{x}} = \sum_{y \sim x} t_y$. Consider the holonomy form

$$\Omega_A^C = \sum_{\overline{x} \in X_{A,C}} t_{\overline{x}} d \log \overline{x}$$

as a form on $C^\perp / A^\perp - \cup_{\overline{x} \in X_{A,C}} \{\overline{x} = 0\}$.

Lemma. *Let $\mathcal{U}$ be the family of elements in $\mathcal{S}$ which are either not properly contained in A or contained in C then the coefficients $t_{\overline{x}}$ of the form Ω_A^C commute with the residue t_B, $B \in \mathcal{U}$.*

Proof. Given any element $\overline{x}$ the subspace spanned by C and all the elements similar to x decomposes in pieces which together with $\mathcal{U}$ form a nested set, hence the claim. $\square$

Let $\mathcal{U}_{A,\mathcal{S}}^C$ denote the subspace of $\mathcal{U}_{\mathcal{S}}$ defined by the vanishing of the coordinates u_D for $D \not\subset A$ and $D \subseteq C$. Let us denote on this space the remaining coordinates by $\overline{u}_B$, $B \subsetneq A$, $B \nsubseteq C$. If we take the open set $\mathcal{U}_{\overline{\mathcal{S}}} \to C^\perp / A^\perp$ with coordinates

u_B, $B \subseteq A$, $B \nsubseteq C$, we can define a function $\gamma : \mathcal{U}_{\overline{\mathcal{S}}} \to \mathcal{U}^C_{A,\mathcal{S}}$ mapping the point of coordinates u_B, $B \subseteq A$, $B \nsubseteq C$ to the point of coordinates $\overline{u}_B = u_B$ for $B \subsetneq A$, $B \nsubseteq C$.

Let $F_{\mathcal{S}}$ be the solution of the holonomy equation localized in $\mathcal{U}_{\mathcal{S}}$. Consider next the function $F^C_A(u_B)$ with $B \subsetneq A$ and $B \nsubseteq C$ defined by setting equal to 0 all the variables u_D, $D \not\subset A$ or $D \subseteq C$ in $F_{\mathcal{S}}$. This is a function on $\mathcal{U}^C_{A,\mathcal{S}}$. Composing with γ, we can consider it as a function on $\mathcal{U}_{\mathcal{S}_A}$. Using together Propositions 2.2 and 2.3, we get

Theorem.

(1) *Ω^C_A is flat.*
(2) *The function F^C_A is the unique solution of the holonomy equation (for the form Ω^C_A) localized in $\mathcal{U}_{\overline{\mathcal{S}}}$, with $F^C_A(D_{\overline{\mathcal{S}}}) = 1$.*

Proof. Easy induction left to the reader. □

3 The monodromy representation for root systems

3.1 We consider now the special case of a root system Φ, i.e. we take as hyperplanes the (complex) zeroes of the roots of a finite root system.

For properties of root systems we refer to [Bo]. Here we shall introduce some of the facts which we are going to use in the sequel.

By definition Φ is a finite subset spanning a real Euclidean space $\mathbb{E}$. Using the scalar product on $\mathbb{E}$ we identify each root α with a linear function on the complexification V of $\mathbb{E}$. Thus the hyperplane it defines is the complexification of the real hyperplane in $\mathbb{E}$ orthogonal to α.

We choose a set Φ_+ of positive roots and denote by $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$ the corresponding set of simple roots. We denote by W the Weyl group of Φ which is generated by the simple reflections s_i with respect to the hyperplanes orthogonal to the simple roots.

The Weyl group W preserves Φ, hence W acts on the canonical model associated to the family of irreducibles. We shall denote this model by Y_Φ.

In this case it is convenient to give a more geometric description of nested sets. We leave the details to the reader.

Since everything is defined over $\mathbb{R}$ we can use nested sets of subspaces of $\mathbb{E}$. First of all we should remark that:

Lemma.

(1) *The decomposition of a root system Φ in irreducible root systems is the same as the abstract decomposition into irreducible subsets.*
(2) *Each irreducible complete set is W equivalent to a subroot system generated by a connected set of simple roots.*

Proof. (1) If we decompose Φ in irreducible root systems Φ_i then each Φ_i spans a subspace $< \Phi_i >$ and the $< \Phi_i >$ form direct sum. Conversely if we can decompose $\Phi = X_1 \cup X_2$ with $< X_1 > \oplus < X_2 >$ we must have that $\alpha + \beta$ is never a root if $\alpha \in X_1$, $\beta \in X_2$. This implies that X_1, X_2 are orthogonal and Φ is reduced.

(2) Given a subspace U of $\mathbb{E}$ spanned by roots the set $\Phi_U := \Phi \cap U$ is also a root system. One may choose a suitable set of simple roots Δ_1 for Φ so that $\Delta_U := \Delta_1 \cap U$ is a set of simple roots for Φ_U. Since any two set of simple roots are conjugate under W, we deduce the claim. □

Thus, up to W equivalence the complete irreducible subsets are classified by connected subdiagrams of the Dynkin diagram. To make the statement more precise one has to specify when two irreducible subdiagrams generate W equivalent root systems. This requires a case by case analysis.

Proposition.

(1) *Any nested set is W equivalent to a nested set in which every subspace $A \in \mathcal{S}$ is spanned by simple roots. We call these* **fundamental nested sets**.
(2) *A maximal fundamental nested set is obtained by choosing a maximal flag in the set of vertices of the Dynkin diagram (or equivalently a total ordering of the simple roots) and, for each set of the flag its decomposition is the decomposition into connected components of the subdiagram having as vertices the given set.*

Proof. 1) We can assume that $\mathcal{S}$ is maximal and Φ is irreducible hence $\Phi \in \mathcal{S}$. Let $A_1, \dots, A_k$ be maximal in $\mathcal{S} - \Phi$. Consider $A = \cup A_i$.

By Lemma 3.1, $< A > \cap \Phi$ is a root system with irreducible components the A_i and moreover by the previous remarks, up to W equivalence we may assume that each A_i is spanned by $\Delta \cap A_i$. Now look separately at the nested sets in each A_i. By induction they are equivalent to fundamental nested sets up to W_i equivalence, where W_i is the Weyl group of A_i and the claim follows.

2) is clear by the previous Lemma. □

¿From the previous facts we see that to a fundamental nested set is associated its set of supports in the Dynkin diagram which we shall continue to call nested.

Remark that a set of supports is nested if and only if, given a family D_i of pairwise incomparable supports in this set, they are the connected components of $\cup D_i$.

Given a fundamental maximal nested set $\mathcal{S}$, for every $A \in \mathcal{S}$ set x_A equal to the sum of simple roots corresponding to vertices in A.

The choice of the x_A's defines a basis adapted to $\mathcal{S}$ and formed of positive roots.

¿From now on we will always assume that this is the choice of adapted basis and we will drop the b in the notation $\mathcal{U}_{\mathcal{S}}$ for the associated open set writing simply $\mathcal{U}_{\mathcal{S}}$.

The case A_n can be pictorially described in a different way, which is the one used by Drinfeld.

The fundamental maximal nested sets of A_n can be described by the non associative monomials in the ordered $n+1$ variables x_i. In other words we take a formal monomial $x_1 x_2 \dots x_{n+1}$ and make it non associative by a choice of parentheses e.g $((x_1 x_2)x_3)(x_4 x_5)$. The way to which we associate to the nested sets the monomials is the following. The positive roots of A_n can be described as the linear functions $z_i - z_j, 1 \leq i < j \leq n+1$ and each such root is the longest root of the subdiagram joining the nodes between $i, j-1$. To each support in a nested set one can choose its longest root and put a pair of parentheses in the positions $(x_i \dots x_j)$, it is easy to see that this establishes the required correspondence and these longest roots are an adapted basis.

Due to the relevance of this case, which is directly connected with the Kniznik-Zamolodchikov equation, we explicit it further.

Recall that one can encode a non commutative monomial with a planar rooted binary tree.

It is a tree Γ with root v_0, if it is not reduced to the root from the root we have exactly 2 branches Γ_s, Γ_d, the left and right branch. The non associative monomial M_Γ relative to Γ is v_0 if Γ reduces to v_0, otherwise $M_\Gamma := (M_{\Gamma_s})(M_{\Gamma_d})$ recursively. Such a tree appears also as a partially ordered set P. The non minimal elements (nodes of the tree) correspond to the elements of the nested set. The minimal elements associated to a monomial in n variables are n and come indexed naturally as $1, 2, \dots, n$. Each node γ corresponds to a submonomial $x_i x_{i+1} \dots x_j$. We associate to γ the root $z_\gamma := z_i - z_j$, and we get in this way an adapted basis. If $z_h - z_k$ is a positive root ($h < k$), the minimal γ in the nested set containing $z_h - z_k$ is found by taking the maximum of h, k in the partially ordered set P. Let γ be such a maximum with associated root $z_i - z_j$. The expression of $z_h - z_k$ in the adapted basis is recursively computed as follows.

Let Γ_1, Γ_2 be the left and right branch on γ, necessarily $h \in \Gamma_1,\ k \in \Gamma_2$. If neither h nor k exhaust the corresponding branch the adapted basis elements of Γ_1, Γ_2 are of type $z_{\Gamma_1} := z_i - z_u,\ z_{\Gamma_2} := z_v - z_j$. Write $z_h - z_k = z_i - z_j + (z_j - z_k) - (z_i - z_h)$, then i, h lie in the branch Γ_1 and j, k in the branch Γ_2 and one proceeds recursively. The other cases are similar.

As for the coordinates u_A they can be easily read off from the non associative monomial, A corresponds to a pair of matching parentheses, $c(A)$ to the innermost pair of matching parentheses containing A and $u_A = \frac{x_i - x_j}{x_h - x_k}$ if $(x_i \dots x_j)$ is the monomial enclosed by the parentheses A and $(x_h \dots x_k)$ is the monomial enclosed by the parentheses $c(A)$.

Let $\mathcal{S}$ be maximal fundamental nested set. We use the roots x_A defined in 3.1, the notations $c(A), p(\alpha)$ are as in [D-P1] 1.2,1.3, the polynomials $P_\alpha = \frac{\alpha}{p(\alpha)}$ are defined as in Lemma 1.1. Notice that $p(\gamma)$ is the minimal element in $\mathcal{S}$ containing the support of γ.

Theorem.

(1) *The map p establishes a one to one correspondence between the set of simple roots and $\mathcal{S}$ and we denote by α_D the simple root corresponding to $D \in \mathcal{S}$. Moreover given $A \in \mathcal{S}$, the maximal elements in $\mathcal{S}$ contained in A are the connected components of the diagram $A - \{\alpha_A\}$ and, choosing a compatible total ordering of $\mathcal{S}$ the corresponding ordering on Δ defines a flag whose decomposition is $\mathcal{S}$.*

(2) *For every simple root α with $A = p(\alpha)$ we have:*

$$\alpha = x_A - \sum_{B \in \mathcal{S} | c(B) = A} x_B,\ P_\alpha = 1 - \sum_{B \in \mathcal{S} | c(B) = A} u_B. \tag{3.1.1}$$

(3) *If a positive root γ is a sum $\alpha + \beta$ of two positive roots then for one of the 2, say α, we have $A := p(\gamma) = p(\alpha)$ while $B := p(\beta) \leq A$ and $P_\gamma =$*

$P_\alpha + P_\beta \prod_{B\le D<A} u_D$. *If* $\gamma = \sum_{D\subseteq p(\gamma)} n_D\alpha_D$ *then*

$$P_\gamma = \sum_{D\subset p(\gamma)} n_D \prod_{D\subseteq H\subsetneq p(\gamma)} u_H P_{\alpha_D}. \tag{3.1.2}$$

Proof. (1),(2). If α is a simple root, $A = p(\alpha)$, then α is a node of A and the elements $B\in\mathcal{S}|c(B) = A$ are the connected components of $A-\{\alpha\}$. From this it is clear that A determines α and, by the definition of the elements x_B, the formulas 3.1.1 follow. The statement on the order is easily verified.

(3),(4). We prove (4), (3) is similar and easier. $P_\gamma = \frac{\gamma}{x_{p(\gamma)}} = \sum_{D\subset p(\gamma)} n_D\frac{\alpha_D}{x_{p(\gamma)}} = \sum_{D\subset p(\gamma)} n_D\frac{\alpha_D}{x_D}\frac{x_D}{x_{p(\gamma)}}$ hence 3.1.2. □

Remark. The connected components of $A-\alpha_A$ are usually 1 or 2 (or 0 if A is reduced to α_A), unless A is a diagram of type D, E and α_D corresponds to the branching node. In this case we have 3 connected components. Thus the polynomials P_α for α a simple root can be of the form $1,\ 1-u,\ 1-u-v,\ 1-u-v-w$.

We want to analyze the graph structure of fundamental nested sets.

Let $\mathcal{S}$ be a fundamental maximal nested set $B\in\mathcal{S}$ a non maximal element, remove B from $\mathcal{S}$ and then replace it with another element B' to get a new fundamental maximal nested set $\mathcal{T}$ joined to $\mathcal{S}$ by an edge. This is a subgraph of the graph of nested sets defined in 1.2.

Let $A = c(B)$ and α,β the two simple roots associated to A, B. The only possibility to perform the previous construction is clearly to remove β from A and let B' be the connected component of the resulting diagram containing α. In $\mathcal{T}$ the roots α,β are associated to B', A respectively, otherwise the correspondence between simple roots and elements of the nested set remains unchanged.

Remark. We can describe $\mathcal{S}$ by a total ordering on Δ in which β,α are consecutive. Then $\mathcal{T}$ is obtained by exchanging the order of these two roots. Thus, repeating the argument in Theorem 1.2 we get that the subgraph of fundamental maximal nested sets is connected.

Let thus $\mathcal{S},\mathcal{T}$ A, B, B',α,β be as before. Call u_D, u'_D the two associated coordinates and $c(E), c'(E)$ the successor map in the two nested sets ([D-P1] 1.4), to distinguish also the map p and the polynomials P_α we will use $p, p'; P_\alpha, P'_\alpha$ in the two cases. We have $A = c(B) = c'(B')$.

One easily sees that, if $D := B\cap B'$ is not empty it is a connected component of $A-\{\alpha,\beta\}$, and that $B\cup B' = A$. Thus $D\in\mathcal{S}\cap\mathcal{T}$ and $x_A + x_D = x_B + x_{B'}$ (with $x_D = u_D = 0$ if D is empty). From this we deduce:

$$u_B + u'_{B'} = 1 + u_D u_B = 1 + u'_D u'_{B'}. \tag{3.1.3}$$

For all other $E\in\mathcal{S}\cap\mathcal{T}$ we have $u_E = u'_E$ unless we are in one of the following 3 cases:

$$\begin{aligned} &c(E) = A,\ c'(E) = B', && u_E = u'_E u'_{B'}\\ &c(E) = B,\ c'(E) = A, && u'_E = u_E u_{B'}\\ &\qquad E = D && u_D u_B = u'_D u'_{B'} \end{aligned} \tag{3.1.4}$$

Proposition.

(1) *For every positive root α we have $p(\alpha) = p'(\alpha)$ unless we are in one of the following two cases. $p(\alpha) = B$, $p'(\alpha) = A$ or $p(\alpha) = A$, $p'(\alpha) = B'$. In these cases we have $P_\alpha = P'_\alpha u_B$ and $P'_\alpha = P_\alpha u'_{B'}$ respectively.*

(2) *In particular $P'_{x_B} = u_B^{-1}, P_{x_{B'}} = u_{B'}^{-1}$.*

(3) *The intersection $\mathcal{U}_\mathcal{S} \cap \mathcal{U}_\mathcal{T}$ is the open set of $\mathcal{U}_\mathcal{S}$ where $u_B \neq 0$ (resp. of $\mathcal{U}_\mathcal{T}$ where $u'_{B'} \neq 0$).*

Proof. The first statement is a simple case analysis. Since $P_{x_B} = 1$ we have $1 = P'_{x_B} u_B$, thus u_B is invertible on $\mathcal{U}_\mathcal{S} \cap \mathcal{U}_\mathcal{T}$, similarly for $u'_{B'}$. The explicit formulas for the coordinate change and for the inequalities P_α, P'_α defining the 2 open sets imply the claims. $\square$

3.2 In this section we describe the real picture. Denote by $Y_\mathbb{R}$ the set of real points of Y_Φ. The next proposition is left to the reader. It can be proved, for instance, from the description of Y_Φ as successive blow ups of subvarieties defined over $\mathbb{R}$.

Proposition. *$Y_\mathbb{R}$ is a real analytic manifold.*

$Y_\mathbb{R}$ is the total space of a real line bundle over a compact analytic manifold Π_Φ consisting of the real points of the divisor D_Φ.

The union of the Weyl chambers $\mathbb{E}^0$ is open dense in $Y_\mathbb{R}$ and moreover the group $\mathbb{R}^*$ of homotheties on $\mathbb{E}^0$ is identified to the homotheties in the line bundle. $\square$

For every nested set $\mathcal{R}$ the real points of the subvariety $D_\mathcal{R}$ form an analytic submanifold $D_{\mathbb{R},\mathcal{R}}$. In the same way we get submanifolds of Π_Φ.

Let us denote by C the fundamental Weyl chamber, C' its image in the real projective space, which is naturally an open set in Π_Φ. $\overline{C}, \overline{C}'$ their closures in $Y_\mathbb{R}$, Π_Φ respectively.

Let $\mathcal{S}$ be a fundamental maximal nested set, $\mathcal{U}_\mathcal{S}$ the associated open set and $\mathcal{U}_\mathcal{S}^\mathbb{R}$ the real points in this set.

Define the open set

$$C_\mathcal{S} := \{p \in \mathcal{U}_\mathcal{S}^\mathbb{R} | \forall \alpha \in \Phi_+,\ P_\alpha(p) > 0\} \tag{3.2.1}$$

and in it the closed subset $C_\mathcal{S}^+$ where all the coordinates u_A are non negative. First remark that:

Lemma.

(1) *For a point in $\mathbb{R}^\mathcal{S}$ with all the coordinate u_A non negative the inequalities $P_\alpha(p) > 0$ where α runs on the set of simple roots imply the inequalities $P_\alpha(p) > 0$ for all positive roots.*

(2) *The closure of the Weyl chamber C in the open set $\mathcal{U}_\mathcal{S}$ is the set $C_\mathcal{S}^+$ which is described in coordinates by:*

$$u_A \geq 0, \sum_{c(B)=A} u_B < 1. \tag{3.2.2}$$

Proof. (1) This is immediate from 3.1.2.

(2) From the formula 3.1.1 and the previous discussion it follows that, in the coordinates u_A the Weyl chamber is described clearly by $u_B > 0, \sum_{c(B)=A} u_B < 1$. In $\mathcal{U}_{\mathcal{S}}^{\mathbb{R}}$ the polynomials P_α are non zero by assumption, hence in the closure of C the inequalities 3.2.2 still hold. On the other hand on the set defined by 3.2.2 in $\mathbb{R}^{\mathcal{S}}$ the polynomial P_α are strictly positive, hence non 0, for all positive roots and so this set is contained in $\mathcal{U}_{\mathcal{S}}^{\mathbb{R}}$. $\square$

Let now $\mathcal{R} \subset \mathcal{S}$. Define

$$F_{\mathcal{R},\mathcal{S}} := \{p \in C_{\mathcal{S}}^{+} | u_A = 0 \text{ if and only if } A \in \mathcal{R}\}.$$

Thus this set has coordinates u_B, $B \in \mathcal{S} - \mathcal{R}$ with the description:

$$u_B > 0, \sum_{c(B)=A,\ B \in \mathcal{S}-\mathcal{R}} u_B < 1, \forall A \in \mathcal{S}. \tag{3.2.3}$$

We notice that $F_{\mathcal{R},\mathcal{S}}$ is an open set in the submanifold $D_{\mathbb{R},\mathcal{R}}$, the coordinates u_D, $D \in \mathcal{S} - \mathcal{R}$ give it the structure of an open convex polyhedron.

Let us denote by $D^0_{\mathbb{R},\mathcal{R}}$ the open part of $D_{\mathbb{R},\mathcal{R}}$ obtained by removing all the submanifolds $D_{\mathbb{R},\mathcal{R}'}$ as $\mathcal{R}'$ runs over all nested sets properly containing $\mathcal{R}$, we have.

Theorem.

1. $F_{\mathcal{R},\mathcal{S}} = \overline{C} \cap D^0_{\mathbb{R},\mathcal{R}}$ *is independent of* $\mathcal{S}$ *and it will be denoted by* $F_{\mathcal{R}}$.
2. $\overline{C} = \cup_{\mathcal{R}} F_{\mathcal{R}}$ *a disjoint union.*
3. *The closure* $\overline{F}_{\mathcal{R}}$ *of* $F_{\mathcal{R}}$ *is the (disjoint) union* $\cup_{\mathcal{M} \supset \mathcal{R}} F_{\mathcal{M}}$.
4. $F_{\mathcal{R}}$ *is a connected component of* $D^0_{\mathbb{R},\mathcal{R}}$.
5. $\overline{C}$ *is homeomorphic to a convex rational polyhedron generated (minimally) by its vertices corresponding to the points* $P_{\mathcal{S}}$ *as* $\mathcal{S}$ *runs over the maximal fundamental nested set.*

Proof. (1) By the connectedness of the subgraph, to prove that $F_{\mathcal{R},\mathcal{S}}$ does not depend on the maximal set $\mathcal{S}$ it suffices to prove that $F_{\mathcal{R},\mathcal{S}} = F_{\mathcal{R},\mathcal{T}}$ where $\mathcal{S}, \mathcal{T}$ are connected by an edge. Use the notations and results of 3.1. By definition on $F_{\mathcal{R},\mathcal{S}}$ all coordinates u_D, $D \notin \mathcal{R}$ (resp. u'_D, $D \notin \mathcal{R}$) are non zero. It follows that $F_{\mathcal{R},\mathcal{S}}$ is contained in $\mathcal{U}_{\mathcal{S}} \cap \mathcal{U}_{\mathcal{T}}$ which proves the first claim.

(2) is a special case of (3), for this remark that every nested set can be extended to a maximal nested set, so take $\mathcal{S} \supset \mathcal{M} \supset \mathcal{R}$ maximal, in the open set $\mathcal{U}_{\mathcal{S}}$ it is clear that $\overline{F}_{\mathcal{R}} \cap \mathcal{U}_{\mathcal{S}} = \cup_{\mathcal{S} \supset \mathcal{M} \supset \mathcal{R}} F_{\mathcal{M}}$. Thus it suffices to show that $\cup_{\mathcal{M} \supset \mathcal{R}} F_{\mathcal{M}}$ is closed. In turn this is implied by the statement (2). To prove that $\cup_{\mathcal{R}} F_{\mathcal{R}}$ is closed one can use the real analytic version of the valuative criterion (the curve selection lemma []). One takes a real analytic curve $\rho(t)$ which for $t \neq 0$ lies in the interior of the Weyl chamber and has a limit as $t \to 0$ on the boundary. One computes the orders of vanishing of the positive roots on this curve and deduces a fundamental nested set, the limit then in Y_Φ exists and lies in the corresponding face $F_{\mathcal{R}}$. (4) Since $Y_\Phi = \cup_{w \in W} w(\overline{C})$ and also $F_{\mathcal{R}}$ is open in $D^0_{\mathbb{R},\mathcal{R}}$ the submanifolds $D_{\mathbb{R},\mathcal{R}}$ are permuted by W.

We claim that $W(F_{\mathcal{R}})$ is the open part of the union of the submanifolds $D_{\mathbb{R},\mathcal{R}'}$ where $\mathcal{R}'$ is any nested set (not necessarily fundamental) W conjugate to $\mathcal{R}$, obtained removing the union of the manifolds $D_{\mathbb{R},\mathcal{R}''}$ of smaller dimension contained in this union.

We must have that $D^0_{\mathbb{R},\mathcal{R}} = \cup_{w|w(\mathcal{R})=\mathcal{R}} w(F_{\mathcal{R}})$. Thus it suffices to show that, if $w(\mathcal{R}) = \mathcal{R}$ then either $w(F_{\mathcal{R}}) = F_{\mathcal{R}}$ or $w(F_{\mathcal{R}}) \cap F_{\mathcal{R}} = \emptyset$. Now if $w(F_{\mathcal{R}}) \cap F_{\mathcal{R}}$ is not empty we easily see that $w(F_{\mathcal{R}})$ meets $\overline{C}$ in $w(F_{\mathcal{R}}) \cap F_{\mathcal{R}}$ which is thus open and closed in $w(F_{\mathcal{R}})$. By connectedness we get the equality.

(5) By the description of Y_X as obtained by a succession of blow ups it follows that $\overline{C}$ is obtained by a standard simplex by a sequence or real blow-ups of faces (in each step one easily verifies that the intersection of the subvariety to be blown up with the closure of C is a face). Now for such a sequence one construct a sequence of corresponding blow ups of the standard projective space, thought of as standard projective toric variety having the standard simplex as image of the moment map and also as closure of the positive real part of the torus. Then to this sequence the blow ups of the toric variety is associated a sequence of blow ups of the closure of the positive real part of the torus along a face. The resulting polyhedron is homeomorphic to its image under a suitable moment map and the claim follows from the theory of toric varieties. □

(We thank Bob MacPherson for pointing out (5)).

We work out the 1 dimensional case. Let $\mathcal{R} = \mathcal{S} - B = \mathcal{T} - B'$, consider the 1-dimensional subvariety $D_{\mathcal{R}}$. On this variety consider the two functions $u_B := x_B/x_A$, $u_{B'} = x_{B'}/x_A$ which are coordinates respectively around the points $P_{\mathcal{S}}, P_{\mathcal{T}}$. From 3.1.3 and the fact that u_D vanishes we deduce the relation $u_{B'} = 1 - u_B$ on $D_{\mathcal{R}}$.

Now we claim that:

Lemma. *The segment* $(0, 1)$ *(in either coordinate* $u_{B'}, u_B$*) is contained in the intersection* $\mathcal{U}_{\mathcal{S}} \cap \mathcal{U}_{\mathcal{T}}$ *and equals* $F_{\mathcal{R}}$*, its extremes are the points* $P_{\mathcal{S}}, P_{\mathcal{T}}$*.*

Proof. The explicit description of $F_{\mathcal{R}}$ given through 3.1.4 insures the claim. □

We have thus given to Π_Φ the structure of polyhedron, generalizing the construction of Bar-Natan [B-N2], we propose to call it the *root-associahedron* since for A_3 it is the union of pentagons for the different associations of a monomial in 4 variables. We leave to the reader (hint: take out 3 nodes from A_n) to see how the 2 dimensional faces behave and why in case A_n one also obtains exhagons etc..

3.3 For a root system take as holonomy form:

$$\frac{1}{2\pi i}\sum_{\alpha\in\Phi^+} t_\alpha d\log(\alpha) = \frac{1}{4\pi i}\sum_{\alpha\in\Phi} t_\alpha d\log(\alpha),\ (t_\alpha = t_{-\alpha}). \tag{3.3.1}$$

The algebra M generated by the t_α with their commutation relations 2.1.1 inherits an action of the Weyl group W setting $w(t_\alpha) = t_{w(\alpha)}$.

Let $\tilde{\mathcal{A}}$ be the universal covering space of the complement $\mathcal{A}$ of the root hyperplanes. Since W acts freely on $\mathcal{A}$ the composed map $\tilde{\mathcal{A}} \to \mathcal{A} \to \mathcal{A}/W$ is also the universal cover of $\mathcal{A}/W$. It is known that $\tilde{\mathcal{A}}$ is contractible so that $\mathcal{A}, \mathcal{A}/W$ are

$K(\pi,1)$ [Br2],[De2]. The homotopy group of $\mathcal{A}/W$ is called the generalized Braid group B (cf [Br2]), the fibration $\mathcal{A}\to\mathcal{A}/W$ with fiber W induces an exact sequence $1\to P\to B\to W\to 1$. The group P is called the generalized pure braid group, and it is the homotopy group of $\mathcal{A}$. We shall recall presently a presentation of B due to Brieskorn [Br2] by some canonical generators T_i which lift the simple reflections s_i.

Call $\tilde{\Omega}$ the pullback of the holonomy form (with values in M) to $\tilde{\mathcal{A}}$.

Lemma. *Under the two actions, the geometric action of B as group of deck transformations and the action of B via W on the algebra M, the form $\tilde{\Omega}$ is equivariant.*

Proof. Clear by construction. □

It follows that, if $G(P)$ is a non zero function on $\tilde{\mathcal{A}}$ (with values in $\hat{M}$), solution of the holonomy equation $dG=\tilde{\Omega}G$, then the transformed function $b(G(b^{-1}P)),\ b\in B$ is also a solution. Clearly then $b(G(b^{-1}P))=G(P)\rho_G(b)$ where $\rho_G(b)$ is a constant (in the multiplicative group $\hat{M}^*$ of the completion of M).

Form the semidirect product $\hat{M}^*\ltimes W$, if $b\in B$ denote by $\overline{b}$ the class of b in the Weyl group. In $\hat{M}^*\ltimes W$ we also consider the subgroup $\hat{U}^*\ltimes W$ of group like elements.

Theorem. *The map*

$$\phi_G(b):=b\to\rho_G(b)\overline{b} \tag{3.3.2}$$

is a homomorphism. If G is unipotent it takes values in $\hat{U}^*\ltimes W$.

Fix a different solution $G'=GK$ with $K\in\hat{M}^*$ (a constant). Then

$$\phi_{G'}(b)=K^{-1}\phi_G(b)K$$

a conjugate representation.

We call the functions ρ_G,ϕ_G respectively the **monodromy factor** *and the* **monodromy representation** *associated to the solution G.*

Proof. Both statements are essentially immediate. For the first remark that, if G is unipotent so is $b(G(b^{-1}P)$ and hence also $\rho_G(b)$.

For the second let us explicit

(3.3.3)
$$\begin{aligned}G(P)\rho_G(b)b(K)&=b(G(b^{-1}P)K)=b(G'(b^{-1}P))=G'(P)\rho_{G'}(b)=\\&=G(P)K\rho_{G'}(b),\ \text{hence}\quad \rho_G(b)b(K)=K\rho_{G'}(b),\end{aligned}$$

from which the formula $\phi_{G'}(b)=K^{-1}\phi_G(b)K$ follows. □

In the case A_n this is the monodromy representation associated to the construction of Khono [Kh1], which Drinfeld [Dr1] uses to give one of the various constructions of quantum groups.

We want to extend solutions of the holonomy equations on the fundamental Weyl chamber to solutions on $\tilde{\mathcal{A}}$.

Following [Br2] we can, once we have fixed a point $p \in C$ give explicit generators T_i for B whose images in W are the fundamental reflections s_i. For this take a point p in the Weyl chamber and decompose it as $p = a + b$ where a is a multiple of α_i and b is orthogonal to α_i, define the curve $\gamma_i : [0,1] \to \mathcal{A}$ by $\gamma_i(t) = exp(\pi\sqrt{-1}t)a + b$. Clearly $\gamma_i(0) = p$ while $\gamma_i(1) = s_i(p)$ so that the image of γ_i in $\mathcal{A}/W$ defines a loop. Its image in $\pi_1(\mathcal{A}/W, p)$ is the generator T_i. Notice that, since the Weyl chamber is contractible we have canonical identifications of all the homotopy groups $\pi_1(\mathcal{A}/W, p)$, $p \in C$ preserving the corresponding generators T_i.

Consider the real points E_0 of $\mathcal{A}$, E_0 is the disjoint union of the Weyl chambers each of which is contractible. Hence the preimage of E_0, under the covering map $\pi : \tilde{\mathcal{A}} \to \mathcal{A}$, is the disjoint union of its connected components which are permuted in a simply transitive way by the group of deck transformations B'. In particular we take C to be a component lying over the fundamental Weyl chamber, and identify $\pi^{-1}(E_0)$ with $B' \times C$ with action on the left factor. Having fixed C the group B' is identified with the homotopy groups $\pi_1(\mathcal{A}/W, p)$ in a way independent of $p \in C$.

3.5 As in section 7 we get a solution $G_{\mathcal{S}}$ of the holonomy equation, which is well defined on C. Having embedded C in $\tilde{\mathcal{A}}$, $G_{\mathcal{S}}$ extends uniquely to a global solution on $\tilde{\mathcal{A}}$ which we will continue to denote by $G_{\mathcal{S}}$. We will denote by $\rho_{\mathcal{S}}$ the monodromy factor on B associated to the solution $G_{\mathcal{S}}$.

Let α_i be a simple root with $\{\alpha_i\} \in \mathcal{S}$. Then we claim:

Proposition. $\rho_{\mathcal{S}}(T_{\alpha_i}) = e^{\frac{t_{\alpha_i}}{2}}$.

Proof. Given any function which is positive on C we can clearly define its logarithm on C by taking its standard determination.

We want now to rewrite the solution $G_{\mathcal{S}} = F_{\mathcal{S}}(u_A) \prod_{A \in \mathcal{S}} u_A^{t_A}$ on C using new coordinates.

We define a new basis for V by taking, for $A \in \mathcal{S}$, $y_A = \frac{x_A + s_i(x_A)}{2} = x_A - hx_{\alpha_i}$ with $h = \frac{(x_A, x_{\alpha_i})}{(x_{\alpha_i}, x_{\alpha_i})}$ if $A \neq \{\alpha_i\}$ and $y_{\alpha_i} = x_{\alpha_i} = \alpha_i$. Notice that since x_A is a positive root if $A \neq \{\alpha_i\}$, also $s_i(x_A)$ is a positive root. It follows that y_A is a real positive function on C. We also set $v_A = \frac{y_A}{y_{c(A)}}$, with the usual convention that $y_{c(A)} = 1$ if A is maximal. Notice that s_i fixes v_A for each $A \neq \{\alpha_i\}$ while $s_i v_{\alpha_i} = -v_{\alpha_i}$.

By the definition of v_A we immediately deduce that for each $A \in \mathcal{S}$, $u_A = v_A H_A$ where H_A is a rational function in the variables v_B with $B \leq A$, such that $H_A(0) = 1$. It follows that the v_A's define coordinates in a neighborhood of $D_{\mathcal{S}}$ in $\mathcal{U}_{\mathcal{S}}$. We fix for such a neighborhood the polydisc $\mathcal{P}$ defined by $|v_A| < \varepsilon$ for each $A \in \mathcal{S}$ with ε a sufficiently small positive real number.

We remark that the intersection $\mathcal{P}^0$ of $\mathcal{P}$ with $\mathcal{A}$ i.e. the set of points where the coordinates u_A are not zero clearly coincides with the set where of coordinates v_A are not zero. Furthermore $\mathcal{P}$ is stable under the action of s_i and that s_i acts as the reflection with respect to $v_{\alpha_i} = 0$. Notice also that given a point $P \in C \cap \mathcal{P}$ the curve $\gamma_{i,P}(t)$ is contained in $\mathcal{P}$ and one has $v_A(\gamma_{i,P}(t)) = v_A(P)$ if $A \neq \{\alpha_i\}$, $v_{\alpha_i}(\gamma_{i,P}(t)) = e^{\pi\sqrt{-1}t} v_{\alpha_i}(P)$. This immediately implies that the connected component of $\pi^{-1}(\mathcal{P}^0) \subset \tilde{\mathcal{A}}$ meeting C is T_i-stable.

Consider now the solution $G_{\mathcal{S}}$ on C. Substituting these new coordinates we get

$$G_{\mathcal{S}} = F_{\mathcal{S}}(u_A)\prod_{A\in\mathcal{S}}(v_A H_A)^{\frac{t_A}{2\pi i}} = \overline{F_{\mathcal{S}}(v_A)}\prod_{A\in\mathcal{S}}(v_A)^{\frac{t_A}{2\pi i}}$$

where the function $\overline{F}_{\mathcal{S}}(v_A) = F_{\mathcal{S}}(u_A)\prod_{A\in\mathcal{S}}(H_A)^{\frac{t_A}{2\pi i}}$ extends to a holomorphic function on $\mathcal{P}$ with values in $\hat{M}$ (notice that we have used the fact that the t_A's commute).

If we rewrite, repeating the computations of 2.1, the holonomy form Ω with the v_A's we get, $\Omega = \sum_{A\in\mathcal{S}}\frac{t_A}{2\pi i}d\log v_A + \overline{\Omega}_{\mathcal{S}}$ where $\overline{\Omega}_{\mathcal{S}}$ is holomorphic in zero. Notice that the W-equivariance of Ω and the invariance of the polar part $\sum_{A\in\mathcal{S}} t_A d\log v_A$ under s_i clearly imply the equivariance of $\overline{\Omega}_{\mathcal{S}}$ under the action if s_i.

Now since $\overline{F}_{\mathcal{S}}(v_A)$ is the unique solution of the localized equation

$$dF = [\sum_{A\in\mathcal{S}}\frac{t_A}{2\pi i}d\log v_A, F] + \overline{\Omega}_{\mathcal{S}}F \tag{3.5.1}$$

which is holomorphic in $\mathcal{P}$ and with $\overline{F}_{\mathcal{S}}(D_{\mathcal{S}}) = 1$, by the equivariance of this equation under s_i we deduce that $\overline{F}_{\mathcal{S}}$ is s_i-equivariant.

From this, if we denote also by $\overline{F}_{\mathcal{S}}$, the unique global solution of 3.5.1 defined on $\tilde{\mathcal{A}}$ and restricting to $\overline{F}_{\mathcal{S}}\circ\pi$ on the connected component of $\pi^{-1}\mathcal{P}$ meeting C, we deduce for a point $P\in\mathcal{P}\cap C$, $T_{\alpha_i}\overline{F}_{\mathcal{S}}(T_{\alpha_i}^{-1}(P)) = s_i\overline{F}_{\mathcal{S}}(s_iP) = \overline{F}_{\mathcal{S}}(P)$.

This implies that the computation of the monodromy factor $\rho_{\mathcal{S}}(T_{\alpha_i})$ is reduced to the computation of the monodromy factor of the function $\prod_{A\in\mathcal{S}}(v_A)^{\frac{t_A}{2\pi i}}$ relative to T_{α_i}. This can be performed in $\mathcal{P}^0$ and reduces to the usual monodromy of logarithms. $\square$

3.6 Let us now give two maximal fundamental nested sets $\mathcal{S},\mathcal{T}$, we have the two associated solutions $G_{\mathcal{S}}, G_{\mathcal{T}}$. By the general theory there is an invertible multiplicative constant, which we will denote by $k_{\mathcal{S},\mathcal{T}}$ such that $G_{\mathcal{S}}k_{\mathcal{S},\mathcal{T}} = G_{\mathcal{T}}$. We want to discuss now the value of this constant.

First of all we may remark that, from Theorem 1.2, one can reduce the computation of this constant to the case in which $\mathcal{S},\mathcal{T}$ differ only by one element. We can thus assume that there is an irreducible subdiagram A of the Dynkin diagram and 2 nodes i,j in A such that $\mathcal{S},\mathcal{T}$ contain A and differ only as $\mathcal{S}$ contains the decomposition of $A-\{i\}$ while $\mathcal{T}$ contains the decomposition of $A-\{j\}$. Otherwise setting $C = A-\{i,j\}$ we are in the situation of 2.4. Setting $\mathcal{U} := \mathcal{S}\cap\mathcal{T}$ we have that $\mathcal{S} = \mathcal{U}\cup\{B\}$, $\mathcal{T} = \mathcal{U}\cup\{B'\}$ for two irreducible subdiagrams B, B'. Denote by x, y the restriction of the two simple roots α_i, α_j to the induced two dimensional space and $\overline{\mathcal{S}},\overline{\mathcal{T}}$ be the nested sets in the 2 dimensional space, associated to x, y and adapted bases $x, x+y$, and $y, x+y$ resp..

We have two associated solutions of the holonomy equation with prescribed asymptotic behavior in 0,1. They are related by a constant $k_{\overline{\mathcal{S}},\overline{\mathcal{T}}}$.

Write by u_D, u'_D the two sets of coordinates, and by R_D, R'_D the two sets of residues. Notice that $u_D = u'_D$, $R_D = R'_D$ for all the elements $D\in\mathcal{U}-\{A\}$. Moreover $R_A + R_B = R'_A + R'_{B'}$ since it equals the sum t_α over all positive roots with support in A but not in C, let us indicate this sum by T. The two solutions can be written as:

$$G_{\mathcal{S}} = F_{\mathcal{S}} u_B^{R_B} x_A^T \prod_{D \in \mathcal{U} - \{A\}} x_D^{R_D}, \ G_{\mathcal{T}} = F_{\mathcal{T}} {u'_{B'}}^{R_{B'}} x_A^T \prod_{D \in \mathcal{U} - \{A\}} x_D^{R_D}.$$

In order to compute the constant we remark that $F_{\mathcal{S}} u_B^{R_B}, \ F_{\mathcal{T}} {u'_{B'}}^{R_{B'}}$ are solutions of the same differential equation

$$(3.6.1) \qquad dF = -FM + \Omega F \text{ where } M = T d\log(x_A) + \sum_{D \in \mathcal{U} - \{A\}} R_D d\log(x_D).$$

¿From this equation it is clear that the degree 0 part of any solution $H = \sum H_i$ of 3.6.1, expanded according to the degree of the holonomy algebra, has H_0 constant, moreover if $H_0 = 0$ by an easy induction on the degrees (since M, Ω have degree 1) we get $dF = 0$ hence F constant.

Now by Theorem 2.4 we know that the two functions $F_{\mathcal{S}} u_B^{R_B}, \ F_{\mathcal{T}} {u'}_{B'}^{R_{B'}}$ restricted to the segment $(0,1)$ where the $u_D = 0, D \in \mathcal{U}$ satisfy the same holonomy equation and thus differ by a constant K in the algebra generated by the residues of this equation. By Lemma 2.4 these residues commute with the elements $T, R_D, \ D \in \mathcal{U} - \{A\}$ hence $F_{\mathcal{S}} u_B^{R_B}, \ F_{\mathcal{T}} {u'_{B'}}^{R_{B'}} K$ satisfy the equation 3.6.1 coincide in at least a point and thus by the previous analysis coincide everywhere. Since K commutes with $x_A^T \prod_{D \in \mathcal{U} - \{A\}} x_D^{R_D}$ the same is true for $G_{\mathcal{S}}, \ G_{\mathcal{T}}$.

We finally have:

Theorem. *We have the equality:*

$$k_{\overline{\mathcal{S}}, \overline{\mathcal{T}}} = k_{\mathcal{S}, \mathcal{T}}.$$

3.7 We are now going to see how using the above results, can at least in principle determine the monodromy representation associated to the holonomy form Ω. We shall assume that our root systems are irreducible.

First of all let us consider the rank two case. Denote by α_1, α_2 the two simple roots and assume that α_1 is a long root. In this case we have two fundamental maximal nested sets $\mathcal{S}$ and $\mathcal{T}$ associated to the two orderings α_1, α_2 and α_2, α_1 respectively. Taking as coordinates $u := \frac{\alpha_1}{\alpha_1 + \alpha_2}$ and $v := \alpha_1 + \alpha_2$ we deduce from our previous analysis the two functions $G_{\mathcal{S}}(u) = F_{\mathcal{S}}(u) u^{\frac{t_{\alpha_1}}{2\pi i}}$ and $G_{\mathcal{T}}(u) = F_{\mathcal{T}}(u)(1-u)^{\frac{t_{\alpha_2}}{2\pi i}}$ which are solutions of the equations

$$G' = \frac{1}{2\pi i}\left(\frac{t_{\alpha_1}}{u} + \frac{t_{\alpha_2}}{1-u}\right) G \qquad \text{for type } A_2,$$

$$G' = \frac{1}{2\pi i}\left(\frac{t_{\alpha_1}}{u} + \frac{t_{\alpha_2}}{1-u} + \frac{t_{\alpha_1 + 2\alpha_2}}{2-u}\right) G \qquad \text{for type } B_2,$$

$$G' = \frac{1}{2\pi i}\left(\frac{t_{\alpha_1}}{u} + \frac{t_{\alpha_2}}{1-u} + \frac{t_{\alpha_1+2\alpha_2}}{2-u} + \frac{t_{\alpha_1+3\alpha_2}}{3-2u} + \frac{t_{2\alpha_1+3\alpha_2}}{3-u}\right)G \qquad \text{for type } G_2,$$

defined in the open interval $(0,1)$, with $F_{\mathcal{S}}(u)$ analytic in $[0,1)$ with $F_{\mathcal{S}}(0)=1$ and $F_{\mathcal{T}}(u)$ analytic in $(0,1]$ with $F_{\mathcal{T}}(1)=1$. Thus the constant $\kappa_{\mathcal{S},\mathcal{T}}$ equals $G_{\mathcal{S}}^{-1}G_{\mathcal{T}}$.

We shall presently see that for any root system we can choose a suitable fundamental nested set $\mathcal{S}$ such that the monodromy representation relative to the fundamental solution $G_{\mathcal{S}}$ can be expressed in terms of the constants obtained in the rank two case. Furthermore we shall show that except for the case G_2 only the constant obtained for types A_2 and B_2 can be used.

We have seen above that any maximal fundamental nested set is associated to a total ordering on the set Δ of simple roots. Furthermore two such orderings give rise to the same fundamental maximal nested set if and only if they can be obtained one from the other by a sequence of steps consisting in exchanging two consecutive simple roots α and β such that if D is the subdiagram of the Dynkin diagram whose vertices correspond to the simple roots which are less than or equal to β in the chosen ordering, then α and β belong to different connected components of D. Using this observation we can consider the graph Γ, whose vertices are the total orderings on Δ and whose edges are given by pairs of of vertices differing by the exchange of two consecutive simple roots. It is then clear that the graph of fundamental maximal nested sets considered in section 3.1 is obtained from Γ by collapsing all the edges connecting two orderings giving rise to the same fundamental maximal nested set.

Given a vertex $\gamma = \{\alpha_{i_1}, \ldots, \alpha_{i_{h-1}}, \alpha_{i_h}, \ldots, \alpha_{i_r}\}$ of Γ we shall denote by γ_h the set of simple roots $(\alpha_{i_1}, \ldots, \alpha_{i_{h-1}}, \alpha_{i_h})$, by Φ_h^γ the corresponding root system and by $\mathbb{E}_h^\gamma$ the euclidean space spanned by γ_h.

Definition. Let γ be a vertex in Γ. Let γ' be the new vertex obtained from γ by exchanging $\alpha_{i_{h-1}}$ with α_{i_h} so that (γ,γ') is a edge.

We shall say that this edge is of type $A_1 \times A_1$ (resp. A_2, B_2, G_2) if the configuration of lines in $\mathbb{E}_h/\mathbb{E}_{h-2}$ orthogonal to a vector in the image of $\Phi_h - \Phi_{h-2}$ is the configuration of a root system of type $A_1 \times A_1$ (resp. A_2, B_2, G_2).

.

Lemma. *Let (γ,γ') be an edge in Γ as in the above proposition. Let D be the connected component of the Dynkin diagram of Φ_h containing the vertex corresponding to α_{i_h}. Set θ_D equal to the highest root for the root system $\Phi_D \subseteq \Phi_h$ corresponding to D. Set c_{h-1} and c_h equal to the coefficient of $\alpha_{i_{h-1}}$ and α_{i_h} in the expression of θ_D as a linear combination of the simple roots in γ_h.*

Then if $c_{h-1}c_h = 0$ (resp. 1, 2), (γ,γ') is of type $A_1 \times A_1$ (resp. A_2, B_2). Furthermore (γ,γ') is of type $A_1 \times A_1$ if and only if γ and γ' define the same fundamental maximal nested set.

Proof. The two roots $\alpha_{i_{h-1}}, \alpha_{i_h}$ give a basis in $\mathbb{E}_h/\mathbb{E}_{h-2}$ so that we get coordinates (x,y) in this space.

Clearly $c_{h-1}c_h = 0$ if and only if $\alpha_{i_{h-1}}$ is not in the support of θ_D. It follows that γ and γ' define the same fundamental maximal nested set and furthermore if $\alpha \in \Phi_h - \Phi_{h-2}$ its image in $\mathbb{E}_h/\mathbb{E}_{h-2}$ either spans the line $x=0$ or the line $y=0$, proving the claim.

Suppose now $c_{h-1}c_h \neq 0$. Then any root in $\Phi_h - \Phi_{h-2}$ has support in D and γ and γ' define two different fundamental maximal nested sets.

If $c_{h-1}c_h = 1$ then the only possible vectors which are image of positive roots in $\Phi_h - \Phi_{h-2}$ are the vectors $(1,0)$ which is the image of $\alpha_{i_{h-1}}$, $(0,1)$ which is the image of α_{i_h} and $(1,1)$ which is the image of θ_D and our claim follows.

Finally if $c_{h-1}c_h = 2$, we can assume $c_{h-1} = 2$ and $c_h = 1$. Then the only possible vectors which are image of positive roots in $\Phi_h - \Phi_{h-2}$ are the vectors $(1,0)$ which is the image of $\alpha_{i_{h-1}}$, $(0,1)$ which is the image of α_{i_h}, $(1,1)$ which is the image of x_D, $(2,1)$ which is the image of θ_D and eventually $(2,0)$ which defines the same line as $(1,0)$. Again our claim follows. $\square$

We can now prove

Theorem.

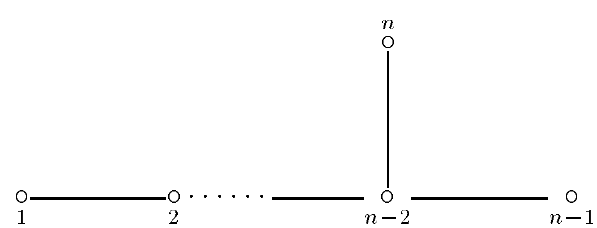
(1) *If the root system Φ is of types A_n, D_n or E_6 there is a connected subgraph $\tilde{\Gamma}$ in Γ such that for any simple root α there exists a vertex $\gamma \in \tilde{\Gamma}$ with $\gamma_1 = \{\alpha\}$ and each edge in $\tilde{\Gamma}$ is of type $A_1 \times A_1$ or A_2.*

(2) *If the root system Φ is of types B_n, C_n, F_4 or $E_{7,8}$ there is a connected subgraph $\tilde{\Gamma}$ in Γ such that for any simple root α there exists a vertex $\gamma \in \tilde{\Gamma}$ with $\gamma_1 = \{\alpha\}$ and each edge in $\tilde{\Gamma}$ is of type $A_1 \times A_1$ or A_2 or B_2.*

Proof. We shall prove the first statement, the proof of the second being similar.

If Φ is of type A_n when we express any positive root as a linear combination of simple roots, its coefficients are either 0 or 1. So our statement follows immediately from the previous Lemma for Γ itself.

Consider now the Dynkin diagram for type D_n



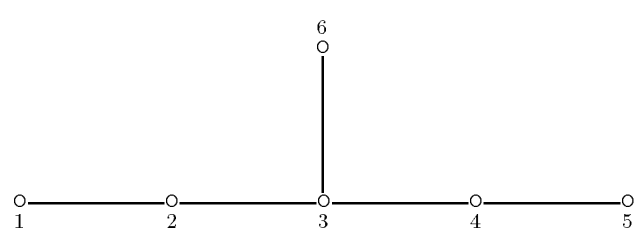
For simplicity we shall denote the simple roots by their labels in the diagram.

Recall that the highest root in this case has coordinates $(1, 2, \dots, 2, 1, 1)$ so that the two vertices $\{1, 2, \dots, n-1, n\}$ and $\{1, 2, \dots, n, n-1\}$ give a A_2 edge.

On the other hand the two full subgraphs Γ_1 and Γ_2 consisting respectively of vertices ending with the root of label n and $n-1$ are the graphs of a system of type A_{n-1} and hence by the above remark their edges are all of type $A_1 \times A_1$ or A_2.

We can now take for $\tilde{\Gamma}$ the graph obtained from the disjoint union of Γ_1 and Γ_2 by adding the A_2 edge $(\{1, 2, \dots, n-1, n\}, \{1, 2, \dots, n, n-1\})$.

For type E_6 consider the Dynkin diagram



and recall that the highest root has coordinates $(1,2,3,2,1,2)$.

We deduce that the vertices $\{3,6,2,4,1,5\}$ and $\{3,6,2,4,5,1\}$ are connected by a A_2 edge. Reasoning exactly as above we are reduced to exhibit, in the two full subgraphs of type D_5 consisting of vertices whose last root has either label 1 or 5, two connected subgraphs Γ_1 and Γ_2 containing respectively the two above vertices and with all the required properties. Notice the since the edge $(\{3,6,2,4,5,1\},\{3,6,4,2,5,1\})$ is a A_2 edge we are reduced to a completely symmetric situation so that we shall treat only the case of vertices with last label equal to 5.

By the result for D_n proven above there is such a good graph containing the vertex $\{3,2,1,6,4,5\}$.

Consider now the sequence of vertices

$$\{3,6,2,4,1,5\},\{3,6,2,1,4,5\},\{3,2,6,1,4,5\},\{3,2,1,6,4,5\}.$$

Each consecutive pair is connected by a A_2 edge and we are done. $\square$

Example: For A_n choose as maximal nested set $\mathcal{S} := \{A_1 \subset A_2 \subset \ldots A_n\}$ in which A_i is the subdiagram generated by the simple roots $\alpha_j = z_j - z_{j+1},\ j \le i$, as adapted basis we have the highest roots $x_{A_i} := z_1 - z_{i+1} = \sum_{k\le i} \alpha_i$. This is associated to the standard ordering of the roots $\alpha_1\ \alpha_2\ \ldots\alpha_n$.

For these choices we want to compute the monodromy representation. We will do it inductively.

In order to compute the monodromy representation for the element $T_i,\ i \le n-1$ one can reorder the sequence by exchanging α_{i-1} with α_i, the new nested set $\mathcal{S}^i$ has α_i as isolated node and hence $\rho_{\mathcal{S}^i}(T_i) = e^{\frac{t_{i\ i+1}}{2}}$. As for the constant $k_{\mathcal{S},\mathcal{S}^i}$ it is $\Phi(A,B)$ where A,B are computed by restricting the roots with support in A_i and not in A_{i-2}. We get $A = \sum_{k<i-1} t_{k\ i},\ B = t_{i\ i+1}$. We deduce:

Theorem. *The monodromy factor ρ of the braid group B_n given by the nested set $A_1 \subset A_2 \subset \ldots A_n$ is:*

(3.7.1)
$$\rho(T_k) = \Phi(\sum_{i<k} t_{ik}, t_{k(k+1)})^{-1} e^{\frac{t_{k\ k+1}}{2}} (k\ k+1) \Phi(\sum_{i<k} t_{ik}, t_{k(k+1)}) (k\ k+1)^{-1} =$$
$$\Phi(\sum_{i<k} t_{ik}, t_{k(k+1)})^{-1} e^{\frac{t_{k\ k+1}}{2}} \Phi(\sum_{i<k} t_{i\ k+1}, t_{k(k+1)}),\ k \le n.$$

$\square$

Of course the fact that from formula 3.7.1 one obtains a representation of the braid group can be translated in identities for the elements Φ, these are essentially the identities of Drinfeld. The statement of the theorem appears in the more abstract context of braided categories [Ca],[Pi].

4 The universal Vassiliev invariant

4.1 In this section we want to relate our constructions with the universal Vassiliev invariants constructed by Kontsevich (by analytic means) [K] and Cartier [Ca], Piunikhin [Pi] in a categorical or combinatorial way. The fact that these two constructions produce equal invariants has been proved by Le and Murakani [L-M] by a rather direct and difficult analytic computation, here we will compare them passing through the monodromy representation and obtaining a simpler proof.

Let $\tilde{\mathcal{A}}$ denote the universal covering space of $\mathbb{C}^n - \Delta$, where Δ is the big diagonal.

Remark. Let $b \in B_n$ and P a point in the universal covering space $\tilde{\mathcal{A}}$, for any solution G of the K-Z equation $G(b^{-1}P)G(P)^{-1}$ is independent of G and equals the value of the integral of the connection on a path joining P with $b^{-1}P$, moreover $G(b^{-1}P) = b^{-1}(G(P)\rho(b))$ where $\rho(b)$ is the monodromy factor associated to the solution G. In particular, if G_0 is the solution with $G_0(P) = 1$ then $G(b^{-1}P)G(P)^{-1} = G_0(b^{-1}P)$ equals $b^{-1}(\rho_{G_0}(b))$.

Consider now the nonassociative monomial M_n in $2n$ variables denoted in order

$$x_1, \dots, x_n, y_1, \dots, y_n$$

and constructed by induction as $M_1 = (x_1 y_1)$ and M_n is obtained from M_{n-1} substituting to x_{n-1} the expression $(x_{n-1}(x_n y_n))$. Let $\mathcal{S}$ be the associated nested set, $G := G_{\mathcal{S}}$ be the associated solution of the K-Z equation and ρ the corresponding monodromy factor.

We need first of all to compute the residues A_i relative to the associator M_n. To make the computations clear let us draw the planar graph relative to the associator M_4.

fig.1

Use the convention to number the indices of the first n variables x_i as i and the ones of the variables y_j as $\overline{j}$. Thus $1, 2, \dots, i, \dots, n, \overline{n}, \dots, \overline{i}, \dots, \overline{2}, \overline{1}$ are the indices of the nodes of the associated Dynkin diagram of type A_{n-1}. A connected subdiagram corresponds to a segment of this totally ordered set. For such a segment S let $t_S := \sum_{h<k,\ h,k\in S} t_{hk}$.

Notice that, as nested set of subdiagrams of the Dynkin diagram, we have that $\mathcal{S}$ is formed by the segments with endpoints $i\bar{i},\ 1 \leq i \leq n$ and the ones with endpoints $i\overline{i+1},\ 1 \leq i \leq n-1$.

We may then write the coordinates in the corresponding open set $\mathcal{U}_{\mathcal{S}}$ as

$$u_i := \frac{x_i - y_{i+1}}{x_i - y_i},\ i = 1, \ldots, n-1;\ u_{\bar{i}} := \frac{x_i - y_i}{x_{i-1} - y_i},\ i = 2, \ldots, n,\ u_{\bar{1}} := x_1 - y_1.$$

From the definition of $\mathcal{S}$ it is clear that the $2n-1$ residues are of two types:

$$t_{S_i},\ \text{where } S_i = i, \ldots, n, \bar{n}, \ldots, \bar{i},\ t_{R_i},\ \text{where } R_i = i, \ldots, n, \bar{n}, \ldots, \overline{i+1}.$$

Let R_{2n} be the algebra generated by the elements t_{hk} for the given Dynkin diagram A_{2n-1}, let furthermore σ be a full cycle on n-elements, by abuse of notation we think of σ as being also a permutation on $1, 2, \ldots, i, \ldots, n, \bar{n}, \ldots, \bar{i}, \ldots, \bar{2}, \bar{1}$ which is the identity on $\bar{n}, \ldots, \bar{i}, \ldots, \bar{2}, \bar{1}$.

In the theory of Vassiliev invariants we have an algebra A introduced by several authors spanned by chord diagrams in a circle modulo 2 types of relations: isolated chords and 4 term relations (cf. [B-N1]).

Consider the map $T_\sigma : R_{2n} \to A$ obtained as follows. Draw $2n$ vertical strands indexed by the indices $1, 2, \ldots, i, \ldots, n, \bar{n}, \ldots, \bar{i}, \ldots, \bar{2}, \bar{1}$, think of the first n as oriented upwards and the last n downwards. Consider each t_{hk} as the chord between the strands hk if they are equioriented, otherwise as minus the same chord. A monomial in the t_{hk} can be considered as a chord diagram descending sequence of the corresponding chords (with sign). The map T_σ consists in closing the $2n$ strands so that they will form an oriented circle, by joining the bottom end of the $i\underline{i}$ strands with a half-circle and similarly the top of $\sigma(i)$ with $\bar{i}$, e.g. $\sigma = (123)$

fig.2

We want to consider now any element $u \in R_{2n}$.

Lemma. $T_\sigma(ut_{h\overline{k}}) = -T_\sigma(ut_{hk}), T_\sigma(ut_{\overline{hk}}) = T_\sigma(ut_{hk})$.

$T_\sigma(t_{h\overline{k}}u) = -T_\sigma(t_{h\sigma^{-1}(k)}u), T_\sigma(t_{\overline{hk}}u) = T_\sigma(t_{\sigma^{-1}(h)\sigma^{-1}(k)}u)$.

$T_\sigma(ut_{S_i}) = T_\sigma(\sigma^{-1}(t_{S_i})u) = T_\sigma(ut_{R_i}) = T_\sigma(\sigma^{-1}(t_{R_i})u) = 0$.

Proof. The first 4 relations are clear by the geometric construction of equivalence of chord diagrams. We have $t_{S_i} = \sum_{i\leq h<k\leq n}(t_{hk} + t_{h\overline{k}} + t_{k\overline{h}} + t_{\overline{kh}}) + \sum_{i\leq h\leq n} t_{h\overline{h}}$. Now use $T_\sigma(ut_{h\overline{k}}) = -T_\sigma(ut_{hk})$ and $T_\sigma(ut_{h\overline{h}}) = 0$ by the relation of isolated chords and the claim follows. The second claim is similar, from $T_\sigma(t_{i\overline{\sigma(j)}}u) = -T_\sigma(t_{ij}u)$ we deduce

$$T_\sigma((\sigma^{-1}(t_{S_i})u) = T_\sigma(\sum_{i\leq h<k\leq n}(t_{\sigma^{-1}(h)\sigma^{-1}(k)} + t_{\sigma^{-1}(h)\overline{k}} + t_{\sigma^{-1}(k)\overline{h}} + t_{\overline{kh}}) + \sum_{i\leq h\leq n} t_{\sigma^{-1}(h)\overline{h}})u)$$

again using the first part. Next

$$T_\sigma(t_{\sigma^{-1}(h)\sigma^{-1}(k)} + t_{\sigma^{-1}(h)\overline{k}} + t_{\sigma^{-1}(k)\overline{h}} + t_{\overline{kh})} =$$
$$T_\sigma(t_{\sigma^{-1}(h)\sigma^{-1}(k)} - t_{\sigma^{-1}(h)\sigma^{-1}(k)} - t_{\sigma^{-1}(k)\sigma^{-1}(h)} + t_{\sigma^{-1}(k)\sigma^{-1}(h)}) = 0$$

while $T_\sigma(t_{\sigma^{-1}(h)\overline{h}})u = 0$ since it contains an isolated chord. For t_{R_i} the arguments are similar, let us do the first. $t_{R_i} = t_{S_{i+1}} + \sum_{i<k\leq n}(t_{ik} + t_{i\overline{k}})$ so again $T_\sigma(u(\sum_{i<k\leq n} t_{ik} + t_{i\overline{k}})) = T_\sigma(u(\sum_{i<k\leq n} t_{ik} - t_{ik})) = 0$.

Let $b \in B_n$ be an element which induces a full cycle in S_n so that the associated link is a knot. Denote by $\overline{b}$ the braid on $2n$ strands equal to b on the first n strands and identity on the others. Let $\rho(\overline{b})$ be the monodromy factor associated to the solution $G_{\mathcal{S}}$ of the K-Z equation previously introduced. As in the previous discussion we may construct $T_\sigma(\sigma^{-1}(\rho(\overline{b})))$ to be the element in the Konsevitch Bar-Natan algebra associated to $\sigma^{-1}(\rho(\overline{b}))$ by closing the braid. We want to prove now that:

Theorem. $T_\sigma(\sigma^{-1}(\rho(\overline{b})))$ *equals the value of the Konsevitch integral for the corresponding knot.*

Proof. When we close such a braid we get a knot $\mathcal{C}$ as in fig.1. of which we compute the corresponding Konsevitch integral $T(\mathcal{C})$ as element of the Konsevitch Bar-Natan algebra.

We can use vertical and horizontal movements (which do not change this integral cf. B-N) and deform the knot to the curve $\overline{\mathcal{C}}$ as in fig. 4, fig.3 fig. 4

which can be thought as the support of a path $\phi : (0,1) \to \tilde{\mathcal{A}}$ which has limit end points n-coinciding coordinates. The different order of vanishing of the difference of the coordinates is expressed by requiring that this path extends in the affine chart $\mathcal{U}_{\mathcal{S}}$ having as limit in $0, 1$, the point with coordinates 0, and that (by symmetry of closing the braid), for ϵ small we have that $\phi(1-\epsilon) = \overline{b}^{-1}\phi(\epsilon)$.

By definition, and the remark at the beginning, the Konsevitch integral $T(\mathcal{C})$ is the limit, as ϵ tends to 0, of the class in the Konsevitch Bar-Natan algebra of $G(\phi(1-\epsilon))G(\phi(\epsilon))^{-1}$ for any given solution of the K-Z equation (here the limit is to be understood separately in each degree). Now we choose as solution the G associated to M_n and the nested set $\mathcal{S}$.

We have

$$G(\phi(1-\epsilon))G(\phi(\epsilon))^{-1} = G(\overline{b}^{-1}\phi(\epsilon))G(\phi(\epsilon))^{-1} = \overline{b}^{-1}(G(\phi(\epsilon))\rho(\overline{b}))G(\phi(\epsilon))^{-1}$$

furthermore write in the u coordinates of $\mathcal{U}_{\mathcal{S}}$, $G(u) = F(u)\prod_i u_i^{A_i}$ with $F(0) = 1$. Recall that, up to a factor of $2\pi i$ the elements A_i are the elements t_{S_i}, t_{R_i} used in the previous Lemma. Expand

$$F = 1 + F_0, F^{-1} = 1 + F_1,$$
$$\prod_i u_i^{A_i} = \prod_i e^{log(u_i)A_i}, (\prod_i u_i^{A_i})^{-1} = \prod_i e^{-log(u_i)A_i}$$

Let us denote by abuse of notations, $u_i(\epsilon), F_0(\epsilon), F_1(\epsilon)$ the value of these functions on the point $\phi(\epsilon)$ we have:

$$G(\phi(1-\epsilon))G(\phi(\epsilon))^{-1} =$$
$$\sigma^{-1}((1+F_0(\epsilon))\prod_i(u_i(\epsilon))^{A_i}\rho(\overline{b}))\prod_i(u_i(\epsilon))^{-A_i}(1+F_1(\epsilon))$$

thus

$$T_\sigma(G(\phi(1-\epsilon))G(\phi(\epsilon))^{-1}) = T_\sigma(\sigma^{-1}(\prod_i (u_i(\epsilon))^{A_i}\rho(\overline{b}))\prod_i (u_i(\epsilon))^{-A_i})+$$
$$T_\sigma(\sigma^{-1}(F_0(\epsilon))\prod_i (u_i(\epsilon))^{A_i}\rho(\overline{b}))\prod_i (u_i(\epsilon))^{-A_i})(1+F_1)+$$
$$\sigma^{-1}(\prod_i (u_i(\epsilon))^{A_i}\rho(\overline{b}))\prod_i (u_i(\epsilon))^{-A_i})F_1)$$

we claim that by Lemma $T_\sigma(\sigma^{-1}(\prod_i (u_i(\epsilon))^{A_i}\rho(\overline{b}))\prod_i (u_i(\epsilon))^{-A_i}) = T_\sigma(\sigma^{-1}(\rho(\overline{b})))$ while

$$\lim_{\epsilon\to 0}\sigma^{-1}(F_0(\epsilon))\prod_i (u_i(\epsilon))^{A_i}\rho(\overline{b}))\prod_i (u_i(\epsilon))^{-A_i})(1+F_1)+$$
$$\sigma^{-1}(\prod_i (u_i(\epsilon))^{A_i}\rho(\overline{b}))\prod_i (u_i(\epsilon))^{-A_i})F_1 = 0$$

(the limit is separately in each degree), in fact each coefficient is a sum of terms of type $p(u)\prod_i (log(u_i)^{h_i})$ where p is a polynomial vanishing in 0 and so the claim.

The definitions taken imply the theorem. □

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Scuola Normale Superiore.
E-mail address: deconcin@ux1sns.sns.it

Università di Roma "La Sapienza".
E-mail address: claudio@mat.uniroma1.it